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EQUIVARIANT CONSTRUCTIONS OF NONEQUIVARIANT SPECTRA

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In a conversation with me in the early 1970's, John emphasized his view that the existing constructions of the stable category "passed to homotopy much too quickly". His point was that, ideally, there ought to be a construction which results by passage to homotopy from a category of spectra and maps which enjoys the same kinds of closure properties under both limits and colimits and under both function objects and smash products as does the category of spaces.

Gaunce Lewis and I have since developed full details of just such a construction. More interestingly, our construction of the nonequivariant stable category readily generalizes to a construction of a good stable category of G-spectra for a compact Lie group G, stability meaning that one can desuspend by arbitrary representations of G. Since there are a great many new phenomena encountered in the equivariant setting, our full dress treatment [13] is quite lengthy. It breaks into two halves. The first, by

Lewis and myself, is aimed at equivariant applications. The second, by Lewis, Steinberger, and myself, is aimed at the exploitation of equivariant techniques for the construction of useful nonequivariant spectra. The purpose of this note is to give a brief summary of some of the main features of our work, with emphasis on the second half.

To give focus to the discussion, we state three theorems about the existence of nonequivariant spectra. For a based space Y and a subgroup π of the symmetric group Σ_j, the extended power $D_\pi Y$ is defined to be the half-smash product

$$E\pi \ltimes_\pi Y^{(j)} = E\pi \times_\pi Y^{(j)}/E\pi \times_\pi \{*\},$$

where $E\pi$ is a free contractible π-space and $Y^{(j)}$ is the j^{th} smash power of Y.

THEOREM A. *There is an extended power functor* $D_\pi E$ *on spectra* E *such that* $D_\pi \Sigma^\infty Y$ *is isomorphic to* $\Sigma^\infty D_\pi Y$ *for based spaces* Y.

Here Σ^∞ is the suspension functor from spaces to spectra.

The functor D_π was first constructed (although not fully understood) in 1976, and some of its early applications were announced in [14]. A great deal more has been done since, particularly by Bruner and McClure, and [4] gives details. Cohen [7], Jones and Wegmann [10,11], and Kuhn [12] have also exploited this functor, and Bruner's work in [4] played a central technical role in the original proof of Ravenel's nilpotency conjecture by Devinatz, Hopkins, and Smith [8,9].

The statement of Theorem A is incomplete. A full statement would explain that the spectrum level functor enjoys all of the good homological and homotopical properties of the space level functor, and then some. We give a slightly more complete (but somewhat vague) statement of our second theorem. Think of representations of G as real G-inner product spaces.

THEOREM B. *Let G be a compact Lie group. There exist spectra BG^{-V} for representations V of G and maps $f\colon BG^{-W} \to BG^{-V}$ for inclusions $V \subset W$ which satisfy the following properties.*

(1) *Under suitable orientability hypotheses, $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_V of degree $-\dim V$, and $f^* : H^*(BG^{-V}) \to H^*(BG^{-W})$ is the morphism of $H^*(BG)$-modules specified by $f^*(\iota_V) = \chi(W-V)\iota_W$, where $\chi(W-V)$ is the Euler class of the complement $W-V$ of V in W.*

(2) *For a split G-spectrum* k_G *with underlying nonequivariant spectrum* k, $k_*(BG^{-V})$ *is isomorphic to* $k_*^G(\Sigma^{-(V+A)}EG_+)$, *where* A *is the adjoint representation of* G. *Moreover, the diagram*

$$\begin{tikzcd} k_*(BG^{-W}) \arrow[rr, "f_*"] \arrow[d, phantom, "\cong" rotate=90] & & k_*(BG^{-V}) \arrow[d, phantom, "\cong" rotate=90] \\ k_*^G(\Sigma^{-(W+A)}EG_+) \arrow[r, "(1\wedge e)_*"] & k_*^G(\Sigma^{-(W+A)}EG_+\wedge S^{W-V}) \cong & k_*^G(\Sigma^{-(V+A)}EG_+) \end{tikzcd}$$

commutes, where $e\colon S^0 \to S^{W-V}$ *is the canonical inclusion.*

Here S^V denotes the 1-point compactification of V and EG_+ denotes the union of a free contractible G-space EG and a disjoint basepoint. Other terms will be explained in due course.

When G is finite, the spectra BG^{-V} play a basic role in Carlsson's proof of the Segal conjecture [6], and he gives an ad hoc construction adequate for his purposes. Property (2) for general theories k_*^G (as opposed to just stable homotopy) is needed for the generalization of Carlsson's work given in [5,15] and requires the more conceptual construction to be described here.

In fact, we have two apparently very different constructions of BG^{-V}. The most intuitive is obtained by regarding $-V$ as a virtual bundle over BG, namely the negative of the representation bundle $EG \times_G V \to BG$. As such, it is classified by a map

$$-V\colon BG \longrightarrow BO \times \{-n\} \subset BO \times Z, \quad n = \dim V.$$

As was first understood by Boardman [3], one can associate a Thom spectrum $M(\xi)$ to any map $\xi\colon Y \to BO \times Z$, and we define BG^{-V} to be $M(-V)$. A systematic study of such Thom spectra of maps defined on infinite complexes is given in [13, IX and X]. This definition makes the isomorphism of part (1) quite clear, the orientability hypothesis being the existence of a Thom class [13, X§5], but sheds little light on part (2). For that, a different, but equivalent, construction is appropriate.

Our second construction of BG^{-V} and our construction of $D_\pi E$ are both special cases of the "twisted half-smash product" in equivariant stable homotopy theory. For an unbased free G-space X and a based G-space Y, define

$$X \ltimes_G Y = X \times_G Y/X \times_G \{*\}.$$

THEOREM C. *Let* G *be a compact Lie group. There is a twisted half-smash product functor* $X \ltimes_G E$ *on* G*-spectra* E *such that* $X \ltimes_G \Sigma^\infty Y$ *is isomorphic to* $\Sigma^\infty(X \ltimes_G Y)$ *for based* G*-spaces* Y.

Here $\Sigma^\infty Y$ is the suspension G-spectrum of Y. Theorem A is obtained by taking $G = \pi$ and $X = E\pi$ and replacing E by the j-fold smash power of a nonequivariant spectrum. Theorem B is obtained by setting $BG^{-V} = EG \ltimes_G S^{-V}$, where S^{-V} is the (-V)-sphere

G-spectrum. For $V \subset W$, S^{-V} is equivalent to $S^{W-V} \wedge S^{-W}$, and we set

$$f = 1 \ltimes_G (e \wedge 1)\colon BG^{-W} = EG \ltimes_G (S^0 \wedge S^{-W}) \longrightarrow EG \ltimes_G (S^{W-V} \wedge S^{-W}) \simeq BG^{-V}.$$

We shall return to these special cases after saying just enough about the details to be able to explain the construction of $X \ltimes_G E$ in concrete space level terms.

For based G-spaces Y and Y', the smash product $Y \wedge Y'$ has diagonal G-action and the function space $F(Y,Y')$ of based maps $Y \to Y'$ has G-action by conjugation. We define suspension and loop G-spaces by $\Sigma^V Y = Y \wedge S^V$ and $\Omega^V Y = F(S^V, Y)$.

A G-universe is a countably infinite dimensional real G-inner product space U which is the sum of countably many copies of each of a set of irreducible representations. We require U to contain a trivial representation, so that the fixed point universe U^G is a copy of $\mathbb{R}^\infty$. We say that U is complete if it contains all irreducible representations. An indexing space in U is a finite dimensional sub G-inner product space. An indexing sequence $\{A_i\}$ in U is an expanding sequence of indexing spaces with $A_0 = \{0\}$ and $\cup A_i = U$. An indexing set $\mathscr{A}$ in U is a set of indexing spaces which contains an indexing sequence.

A G-prespectrum D indexed on $\mathscr{A}$ is a set of based G-spaces DV, $V \in \mathscr{A}$, and based G-maps $\sigma\colon \Sigma^{W-V} DV \to DW$ for

$V \subset W$ such that $\sigma = 1$ if $V = W$ and the evident transitivity diagrams commute. A map $f: D \to D'$ of G-prespectra is a set of maps $f: DV \to D'V$ such that $\sigma'\Sigma^{W-V}f = f\sigma$ for $V \subset W$. We say that D is an inclusion G-prepectrum if each adjoint map $\tilde{\sigma}: DV \to \Omega^{W-V}DW$ is an inclusion. We say that D is a G-spectrum if each $\tilde{\sigma}$ is a homeomorphism. We have categories

$$G\mathcal{P}\mathcal{A} \supset G\mathcal{Q}\mathcal{A} \supset G\mathcal{S}\mathcal{A}$$

of G-prespectra, inclusion G-prespectra, and G-spectra indexed on $\mathcal{A}$.

It is obvious that $G\mathcal{P}\mathcal{A}$ has arbitrary colimits and limits: we simply perform such constructions spacewise. It is very easy to see that $G\mathcal{S}\mathcal{A}$ is closed under limits. If we take the pullback of a diagram of spectra or take an (infinite) product of spectra, the result is still a spectrum. However, $G\mathcal{S}\mathcal{A}$ is not closed under colimits. Pushouts and wedges of spectra give prespectra which are generally not spectra (or even inclusion prespectra).

To remedy this defect, we observe that there is a "spectrification" functor $L: G\mathcal{P}\mathcal{A} \to G\mathcal{S}\mathcal{A}$ left adjoint to the forgetful functor $\ell: G\mathcal{S}\mathcal{A} \to G\mathcal{P}\mathcal{A}$. This is obtained in two steps. We first go from $G\mathcal{P}\mathcal{A}$ to $G\mathcal{Q}\mathcal{A}$ by a fairly unilluminating (transfinite) iteration of an image prespectrum functor or simply by categorical nonsense, quoting the Freyd adjoint functor theorem. We then go from $G\mathcal{Q}\mathcal{A}$ to $G\mathcal{S}\mathcal{A}$ by an obvious union construction, setting

$$(LD)(V) = \bigcup_{W \supset V} \Omega^{W-V} DW$$

for an inclusion prespectrum D. Now, to construct colimits in the category $G\mathcal{S}\mathcal{A}$, we simply apply the functor L to the prespectrum level colimits.

Similarly, to construct the smash product $E \wedge Y$ of a G-spectrum E and a G-space Y, we set $E \wedge Y = L(\ell E \wedge Y)$, where $(D \wedge Y)(V) = DV \wedge Y$ for a G-prespectrum D. Function G-spectra $F(Y,E)$ are defined directly by $F(Y,E)(V) = F(Y,EV)$, and the usual adjunctions hold. We now have cylinders $E \wedge I_+$ and thus a notion of homotopy. We say that a map $f: E \rightarrow E'$ of G-spectra is a weak equivalence if the H-fixed point map $f^H: (EV)^H \rightarrow (E'V)^H$ is a nonequivariant weak equivalence for each (closed) subgroup H of G and each $V \in \mathcal{A}$. The stable category $\bar{h}G\mathcal{S}\mathcal{A}$ is obtained from the homotopy category of G-spectra by adjoining formal inverses to the weak equivalences, so as to force the weak equivalences to become isomorphisms.

The book [13] begins with a preamble comparing the resulting construction of the nonequivariant stable category with the earlier constructions of Boardman [2] and Adams [1].

We obtain the suspension functor Σ^∞ from G-spaces to G-spectra by setting $\Sigma^\infty Y = L\{\Sigma^V Y\}$, where $\{\Sigma^V Y\}$ denotes the obvious inclusion prespectrum with V^{th} space $\Sigma^V Y$. More generally, for an indexing space Z, we define a shifted suspension functor $\Lambda^Z\Sigma^\infty Y$ from G-spaces to G-spectra by setting $\Lambda^Z\Sigma^\infty Y = L\{\Sigma^{V-Z}Y\}$, where $\{\Sigma^{V-Z}Y\}$ denotes the inclusion prespectrum with V^{th} space $\Sigma^{V-Z}Y$ if $V \supset Z$ and a point otherwise. The functor $\Lambda^Z\Sigma^\infty$ is left adjoint to the Z^{th} space functor, which assigns the space EZ to a spectrum E. The spectrum $\Lambda^Z\Sigma^\infty S^0$ is denoted S^{-Z} and called the $(-Z)$-sphere spectrum. The Z-sphere spectrum is just $\Sigma^\infty S^Z$. Since $U^G = \mathbb{R}^\infty$, we may use the indexing spaces $\mathbb{R}^n$ to obtain sphere spectra S^n for all integers n (assuming $\mathbb{R}^n \in \mathscr{A}$, as we may). For $H \subset G$ and $n \in Z$, we let $\pi_n^H(E)$ be the set of homotopy classes of G-maps $(G/H)_+ \wedge S^n \to E$. A key technical theorem (which is trivial in the nonequivariant case) asserts that $f: E \to E'$ is a weak equivalence if and only if $f_*: \pi_n^H(E) \to \pi_n^H(E')$ is an isomorphism for all H and n.

Given this much, it is entirely straightforward to develop a good theory of G-CW spectra, using spectrum level spheres $(G/H)_+ \wedge S^n$ as the domains of attaching maps of cells $(G/H)_+ \wedge CS^n$, where CE is the cone $E \wedge I$. In particular, it is easy to prove the stable cellular approximation and G-Whitehead theorems. The latter asserts that a weak equivalence between G-CW spectra is a G-homotopy equivalence. These results imply that the stable category is equivalent to the homotopy category of G-CW spectra and cellular maps.

The functor "$\Lambda^Z\Sigma^\infty$" above is the composite of Σ^∞ and a "shift desuspension" functor Λ^Z on G-spectra. The functor Λ^Z has an inverse shift suspension functor Λ_Z. The G-spectra $\Lambda^Z E$ and $\Omega^Z E$ have the same component G-spaces, but their structural homeomorphisms $\tilde{\sigma}$ differ by permutations of loop coordinates. It is easy to check that the functors Λ^Z and Ω^Z (on $\bar{h}G\mathcal{S}\mathcal{A}$) are naturally equivalent, and it follows adjointly that Λ_Z and Σ^Z are naturally equivalent. This implies that the functors Ω^Z and Σ^Z are adjoint equivalences, allowing us to view Ω^Z as a desuspension functor Σ^{-Z}. This argument is independent of the Freudenthal suspension theorem. It illustrates a thematic scheme of proof in [13]: use simple verifications with right adjoints to prove results about left adjoints. The remarkable efficacy of this scheme prevents the lack of good point set level control of the functor L from being a hindrance to proofs.

What we have said so far makes sense for any indexing set $\mathcal{A}$, and we shall exploit this freedom to use varying $\mathcal{A}$ below. It is also easy to check that isomorphic G-universes U give rise to equivalent categories $G\mathcal{S}U$, where $G\mathcal{S}U$ is defined using the canonical indexing set consisting of all indexing spaces in U. The comparison of $G\mathcal{S}U$ to $G\mathcal{S}U'$ for nonisomorphic universes U and U' is central to the theory. For a G-linear isometry $i: U \to U'$, there is a functor $i^*: G\mathcal{S}U' \to G\mathcal{S}U$ given by $(i^*E')(V) = E'(iV)$. That is, we ignore those indexing spaces not in the image of i. This functor has a left adjoint i_*. On the prespectrum level, $(i_*D)(V') = \Sigma^{V'-iV}DV$ where $V = i^{-1}(V')$; on the spectrum level, $i_*E = Li_*\ell E$.

A key case is the inclusion $i: U^G \to U$. We can only define orbit spectra and fixed point spectra directly when working in a trivial universe, such as U^G. Here we set $(D/G)(V) = DV/G$ on the prespectrum level and $E/G = L(\ell E/G)$ on the spectrum level and set $E^G(V) = (EV)^G$. The quickest way to construct a spectrum equivalent to $X \ltimes_G E$ for $E \in G\mathcal{S}U$ is to pass to orbits over G from the smash product $X_+ \wedge i^*E$. The trouble with this definition is that, while correct, it is useless for proving theorems, the problem being that the functor i^* is trivial to define but difficult to study. For example, it fails to preserve G-CW spectra.

To proceed further, we exploit the topology of the function G-space $\mathcal{I}(U,U')$ of linear isometries $U \to U'$, where U and U' are topologized as the unions of their finite dimensional subspaces.

For example, the definition of smash products runs as follows. For G-prespectra D and D' indexed on the set of all indexing spaces in U, we have an evident "external" smash product indexed on the set of indexing spaces of the form $V \oplus V'$ in $U \oplus U$; $D \wedge D'$ is specified by

$$(D\wedge D')(V \oplus V') = DV\wedge D'V',$$

with the obvious structural maps. For G-spectra E and E', we thus have the external smash product

$$E\wedge E' = L(\ell E\wedge \ell E').$$

To "internalize" the smash product, we choose a G-linear isometry $f\colon U \oplus U \to U$ and define the internal smash product of E and E' to be $f_*(E\wedge E')$. We then exploit the fact that $\mathcal{I}(U^j,U)$ is G-contractible for all j to prove that, after passage to the stable category, the resulting smash product is independent of the choice of f and is unital, associative, and commutative up to coherent natural isomorphism. It is also quite simple to give an explicit concrete definition of function G-spectra $F(E,E')$, such as dual G-spectra $D(E) = F(E,S^0)$. We can now redefine $\Sigma^{-Z}E$ to be $E\wedge S^{-Z}$; $\Sigma^{-Z}E$ is equivalent to $\Omega^Z E$ since $\Sigma^Z\Omega^Z E$ and $\Sigma^Z\Sigma^{-Z}E = E\wedge S^{-Z}\wedge S^Z$ are both equivalent to E.

Turning to twisted half smash products, we assume henceforward that the universe U is complete. This ensures that $\mathcal{I}(U,\mathbb{R}^\infty)$ is G-free and contractible, so that its orbit space is a classifying space for principal G-bundles. For an unbased free G-CW complex X, there is thus a G-map

$$\chi\colon X \longrightarrow \mathcal{I}(U,\mathbb{R}^\infty),$$

and χ is unique up to G-homotopy. We think of χ as a twisting function which intertwines the topology of X and the topology of the indexing spaces of G-spectra. For simplicity, we assume that X is finite. By compactness, this allows us to choose an indexing sequence $\{A_i\}$ in U and an indexing sequence $\{\mathbb{R}^{n_i}\}$ in $\mathbb{R}^\infty$ such that $\chi(X)(A_i) \subset \mathbb{R}^{n_i}$ for all $i \geq 0$. We then have G-bundle inclusions

$$\begin{tikzcd} X \times A_i \arrow[rr, "\tilde{\chi}"] \arrow[dr] & & X \times \mathbb{R}^{n_i} \arrow[dl] \\ & X & \end{tikzcd}$$

specified by $\tilde{\chi}(x,a) = (x,\chi(x)(a))$ for $x \in X$ and $a \in A_i$. Let T_i be the Thom complex of the complementary bundle (taken as the 1-point compactification of its total space). By elementary inspection of bundles, there are canonical G-homeomorphisms

(1) $$S^{A_i} \wedge T_i \cong X^+ \wedge S^{n_i}$$

and

(2) $$T_i \wedge S^{n_{i+1}-n_i} \cong T_{i+1} \wedge S^{A_{i+1}-A_i}.$$

For a G-prespectrum D indexed on $\{A_i\}$, define a prespectrum $X \ltimes_G D$ indexed on $\{\mathbb{R}^{n_i}\}$ by setting

(3) $$(X \ltimes_G D)(\mathbb{R}^{n_i}) = DA_i \wedge_G T_i.$$

The i^{th} structural map

(4) $$\sigma_i \colon \Sigma^{n_{i+1}-n_i}(DA_i \wedge_G T_i) \longrightarrow DA_{i+1} \wedge_G T_{i+1}$$

is defined by $\sigma_i(d \wedge (x,b) \wedge s) = \sigma(d \wedge a) \wedge (x,c)$, where $d \in DA_i$, $(x,b) \in T_i$ with $x \in X$ and $b \in \mathbb{R}^{n_i} - \chi(x)(A_i)$,

$s \in \mathbb{R}^{n_{i+1}-n_i}$, and $(x,b)\wedge s$ corresponds to $(x,c)\wedge a$ under the homeomorphism (2), so that $b + s = c + \chi(x)(a)$ in $(\mathbb{R}^{n_i}-\chi(x)(A_i))\oplus\mathbb{R}^{n_{i+1}-n_i}=(\mathbb{R}^{n_{i+1}}-\chi(x)(A_{i+1}))\oplus(\chi(x)(A_{i+1}-A_i))$.
For a G-spectrum E, define

$$\chi \ltimes_G E = L(\chi \ltimes_G \ell E).$$

After passage to the stable category, the resulting spectrum is independent of the choice of χ and is denoted $X \ltimes_G E$. To best handle infinite G-CW complexes X, one gives a more invariant reformulation of the definition above which allows one to construct $\chi \ltimes_G E$ by passage to colimits over the restrictions of χ to finite subcomplexes.

To check that $\chi \ltimes_G \Sigma^\infty Y$ is isomorphic to $\Sigma^\infty(X \ltimes_G Y)$ for a based G-space Y, observe that the homeomorphism (1) and the definition (3) give rise to a homeomorphism

$$(\chi \ltimes_G\{\Sigma^{A_i}Y\})(\mathbb{R}^{n_i}) \cong \Sigma^{n_i}(X \ltimes_G Y)$$

under which the structural map σ_i of (4) corresponds to the obvious identification. That is, we have an isomorphism

$$\chi \ltimes_G\{\Sigma^{A_i}Y\} \cong \{\Sigma^{n_i}(X \ltimes_G Y)\}$$

of prespectra indexed on $\{\mathbb{R}^{n_i}\}$. An easy formal argument (based on use of right adjoints) shows that

$$L(\chi \ltimes_G D) \cong L(\chi \ltimes_G \ell LD)$$

for G-prespectra D. Applied to $D = \{\Sigma^{A_i}Y\}$, this gives

$$\chi \ltimes_G \Sigma^\infty Y = L(\chi \ltimes_G \ell\Sigma^\infty Y) \cong L(\chi \ltimes_G \{\Sigma^{A_i}Y\}) \cong \Sigma^\infty(X \ltimes_G Y).$$

This proves Theorem C. We should explain why $X \ltimes_G E$ is equivalent to $X_+ \wedge_G i^* E$, $i\colon \mathbb{R}^\infty = U^G \subset U$, as claimed earlier. The principle is that "free G-spectra live in the trivial universe". One manifestation of this principle is that a free G-spectrum E indexed on U is isomorphic in $\bar{h}G\mathcal{S}U$ to $i_* E'$ for a free G-spectrum E' indexed on $\mathbb{R}^\infty$ and that E' is uniquely determined up to isomorphism in $\bar{h}G\mathcal{S}\mathbb{R}^\infty$. Now $\chi \ltimes_G E$ and $X_+ \wedge_G i^* E$ result by passage to orbits over G from free G-spectra $\chi \ltimes E$ and $X_+ \wedge i^* E$ indexed on $\mathbb{R}^\infty$, as it turns out that $i_*(\chi \ltimes E)$ and $i_*(X_+ \wedge i^* E)$ are both equivalent to the untwisted half-smash product $X_+ \wedge E$ indexed on U. For the first, using a more general definition of twisted half-smash products which allows non-trivial target universes, we find that $i_*(\chi \ltimes E) = (i \circ \chi) \ltimes E$, $i \circ \chi\colon X \to \mathcal{I}(U,U)$. We then see from the G-contractibility of $\mathcal{I}(U,U)$ that the twisted half-smash product $(i \circ \chi) \ltimes E$ is equivalent to the untwisted half-smash product $X_+ \wedge E$. For the second, $i_*(X_+ \wedge i^* E)$ is isomorphic to $X_+ \wedge i_* i^* E$, and the natural evaluation map $i_* i^* E \to E$ induces an isomorphism on the (nonequivariant) homotopy groups π_*. Just as on the space level, it follows that this map becomes a weak G-equivalence when smashed with the free G-space X_+.

To prove Theorem A, we take $U = (\mathbb{R}^\infty)^j$, with $\pi \subset \Sigma_j$ acting by permutations. For a spectrum E indexed on $\mathbb{R}^\infty$, the j-fold external smash power $E^{(j)}$ is a π-spectrum indexed on U, and we set $D_\pi E = E\pi \ltimes_\pi E^{(j)}$. For a based space Y, $(\Sigma^\infty Y)^{(j)}$ and $\Sigma^\infty(Y^{(j)})$ are isomorphic π-spectra indexed on U, and the relation $D_\pi \Sigma^\infty Y \cong \Sigma^\infty D_\pi Y$ follows.

As said before, we take $BG^{-V} = EG \ltimes_G S^{-V}$ to prove Theorem B. We sketch how (1) and (2) of that theorem follow from this definition. There is a general twisted diagonal map

$$\delta\colon X \ltimes_G (E \wedge E) \longrightarrow (X \ltimes_G E) \wedge (X \ltimes_G E').$$

With $X = EG$, $E = S^0$ and $E' = S^{-V}$, it specializes to give a coaction

$$\delta\colon BG^{-V} \longrightarrow \Sigma^\infty BG_+ \wedge BG^{-V},$$

and this coaction induces the $H^*(BG)$-module structure on $H^*(BG^{-V})$. When G is finite, the skeletal filtration of EG gives rise to a spectral sequence converging from $H^*(G;H^*(S^{-V}))$ to $H^*(BG^{-V})$. Provided that G acts trivially on $H^*(S^{-V})$, for example if G is a p-group and we take cohomology with mod p coefficients, the spectral sequence collapses to an identification of $H^*(BG^{-V})$ with the free $H^*(BG)$-module generated by the fundamental class $\iota_V \in H^{-n}(S^{-V})$, $n = \dim V$. The diagram

$$\begin{array}{ccc}
BG^{-W} = EG \ltimes_G S^{-W} & \xrightarrow{\ \delta\ } & \Sigma^\infty BG_+ \wedge BG^{-W} \\
{\scriptstyle f=1\ltimes_G(e\wedge 1)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma^\infty(1\ltimes e)\wedge 1} \\
BG^{-V} \cong EG \ltimes_G(S^{W-V}\wedge S^{-W}) & \xrightarrow{\ \delta\ } & \Sigma^\infty(EG \ltimes_G S^{W-V})\wedge BG^{-W}
\end{array}$$

commutes by the naturality of δ. Here $EG \ltimes_G S^{W-V}$ is the Thom complex of the representation bundle $EG \ltimes_G(W-V) \to BG$ and, given an orientation, $(1 \ltimes_G e)^*\colon H^*(EG \ltimes_G s^{W-V}) \to H^*(BG)$ sends the Thom class to the Euler Class. With properly coherent choices of orientations, $\delta^*(\mu_{W-V} \otimes \iota_W) = \iota_V$, and the description of f^* in (1) of Theorem B follows. For general compact Lie groups, we can use the same argument to identify f^* after using the alternative Thom spectrum construction of BG^{-V} to calculate $H^*(BG^{-V})$.

In (2) of Theorem B, we start with a G-spectrum k_G indexed on U. We have the inclusion $i\colon \mathbb{R}^\infty = U^G \to U$, and the underlying nonequivariant spectrum k is just i^*k_G with G-action ignored. We have an inclusion $\iota\colon (i^*k_G)^G \to k$ and we say that k_G is split if there is a map $\zeta\colon k \to (i^*k_G)^G$ such that $\iota\zeta \simeq 1\colon k \to k$. This holds for such theories as cohomotopy, K-theory, and cobordism. When it holds, and not in general otherwise, we have isomorphisms

(*) $\quad k_G^*(i_*E) \cong k^*(E/G) \quad$ and $\quad k_*^G(\Sigma^{-A} i_*E) \cong k_*(E/G)$

for a free G-spectrum E indexed on $\mathbb{R}^\infty$ [13, II.8.4]. Here i_* is innocuous since free G-spectra live in the trivial universe. The presence of the adjoint representation A is essential, but of course $A = 0$ if G is finite. Now BG^{-V} is obtained by passage to orbits from $EG \ltimes S^{-V}$ and, as already explained, $i_*(EG \ltimes S^{-V})$ is equivalent to $EG_+ \wedge S^{-V}$. Thus (*) gives the isomorphism of (2) of Theorem B, and the diagram there is given by the naturality in E of the isomorphism (*) applied to the map $1 \ltimes (e \wedge 1)\colon EG \ltimes S^{-W} \to EG \ltimes S^{-V}$, from which f was obtained by passage to orbits over G.

We must still explain why $EG \ltimes_G S^{-V}$ is equivalent to the Thom spectrum of $-V\colon BG \to BO \times Z$. More generally, consider an arbitrary G-space Y and the natural map

$$\alpha\colon KO_G(Y) \longrightarrow KO_G(EG \times Y) \cong KO(EG \times_G Y)$$

induced by the projection $EG \times Y \to Y$ and the first isomorphism in (*). There are canonical Grassmannian classifying spaces $BO_G(U)$ and $BO(\mathbb{R}^\infty) \simeq BO \times Z$ which represent KO_G and KO, and the precise specification of these spaces [13, X§2] leads to an evaluation map

$$\epsilon\colon \mathcal{I}(U,\mathbb{R}^\infty) \times_G BO_G(U) \longrightarrow BO(\mathbb{R}^\infty).$$

Let $\chi\colon EG \to \mathcal{I}(U,\mathbb{R}^\infty)$ be a G-map. It turns out that, for a G-map $\xi\colon Y \to BO_G(U)$, $\alpha(\xi)$ is represented by the composite

$$EG \times_G Y \xrightarrow{\chi \times_G \xi} \mathcal{I}(U,\mathbb{R}^\infty) \times_G BO_G(U) \xrightarrow{\epsilon} BO(\mathbb{R}^\infty).$$

There is a Thom G-spectrum $M(\xi)$, and one sees by inspection of definitions [13, X.7.2] that $M(\epsilon\circ(\chi \ltimes_G \xi))$ is isomorphic to $\chi \ltimes_G M(\xi)$. We apply this fact with Y a point and $\xi = -V$, where the virtual representation $-V$ is viewed as an element of $RO(G) = KO_G(pt)$ and thus as a map from a point into $BO_G(U)$. Not surprisingly, $M(\xi) = S^{-V}$ in this situation. Since $\varepsilon\circ(\chi \times_G \xi)\colon BG \to BO \times Z$ is the map we called $-V$ before, we conclude that $M(-V)$ is indeed equivalent to $EG \times_G S^{-V}$. Moreover, as one would expect, the Thom diagonal $M(-V) \to \Sigma^{\infty}BG^{+}\wedge M(-V)$ corresponds under this equivalence to the coaction map δ described above.

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