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THE SEGAL CONJECTURE FOR ELEMENTARY ABELIAN p-GROUPS—II. p-ADIC COMPLETION IN EQUIVARIANT COHOMOLOGY

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LET G be a finite p-group, let EG be a free contractible G-space, and let π_G^* be equivariant stable cohomotopy theory. One version of the Segal conjecture reads as follows.

THEOREM. *The projection* $\mathrm{EG} \times \mathrm{X} \to \mathrm{X}$ *induces an isomorphism*

$$\pi_G^*(\mathrm{X})_p^\wedge \to \pi_G^*(\mathrm{EG} \times \mathrm{X})_p^\wedge \cong \pi^*(\mathrm{EG} \times_G \mathrm{X})_p^\wedge$$

for all finite G-CW *complexes* X.

This result implies an analogue for general finite groups, but we refer the reader to [25] and especially [5] for that. We shall give as efficient a proof of the theorem as present technology seems to allow, starting from the purely algebraic Ext calculation [4,1.1] of Adams, Gunawardena, and Miller as a given. When $G = (Z_p)^r$, the theorem is due to those authors. However, their original passage from [4,1.1] to this case of the theorem involved considerably more Ext calculation and other work than ours does and relied on [15] for the translation of a non-equivariant version of the theorem to the version stated. Their paper [4] therefore omitted the argument in favor of a reference to us, and we have borrowed their title with their permission.

Carlsson [9, 10] completed the proof of the theorem by reducing the general case to the case $G = (Z_p)^r$. The framework of our proof is the one he established, and it will be apparent that most of the main ideas are due to him. While our primary focus is on the elementary Abelian case, we include a complete proof since variants of Carlsson's arguments allow a number of simplifications. Along the way, we will generalize his framework and his reduction of the problem to elementary Abelian p-groups from cohomotopy to fairly general equivariant theories. This generalization is not digressive since our proof of the theorem when $G = (Z_p)^r$ is based on a naturality argument comparing cohomotopy to another theory for which the analogous result certainly holds.

We outline our work in §1 and then proceed to fill in the details. The arguments here were sketched in [22] and appeared originally in the preprints [12] and [26].

§1. STATEMENTS OF RESULTS

Let k_G^* be a represented cohomology theory on G-complexes ($= G$–CW complexes). We will give a precise definition in §2. Actually, we will specify a reduced bitheory $\tilde{k}_G^*(X; Y)$ on based G-complexes X and Y. This will be a cohomology theory in X (for fixed Y) and a homology theory in Y (for fixed X). We agree to write

(1.1) $$\tilde{k}_q^G(X;Y)=\tilde{k}_G^{-q}(X;Y),$$

(1.2) $$\tilde{k}_G^q(X)=\tilde{k}_G^q(X;S^0) \quad \text{and} \quad \tilde{k}_q^G(Y)=\tilde{k}_q^G(S^0;Y).$$

As usual, for unbased G-spaces X and Y, we set

(1.3) $$k_G^q(X)=\tilde{k}_G^q(X_+) \quad \text{and} \quad k_q^G(Y)=\tilde{k}_q^G(Y_+),$$

where X_+ denotes the union of X and a disjoint G-fixed basepoint.

We will also show how to construct from k_G^* a represented bitheory k_J^* for each subquotient $J=H/K$ of G. When $k_G^*=\pi_G^*$, k_J^* will be π_J^*. In general, k_J^* will depend on the extension $K\to H\to J$ and not just on the abstract group J. We sometimes use the notation $k_{H/K}^*$ to emphasize this fact. For $H\subset G$, we will have change of groups isomorphisms (e.g. [3, §5; 17, II])

(1.4) $$\tilde{k}_G^*(G_+\wedge_H X;Y)\cong\tilde{k}_H^*(X;Y) \quad \text{and} \quad \tilde{k}_G^*(X;G_+\wedge_H Y)\cong\tilde{k}_H^*(X;Y).$$

Here X in the first isomorphism and Y in the second need only be an H-space, and $G_+\wedge_H X$ is the based G-space generated by X. Explicitly, if H acts on $G_+\wedge X$ via $h(g,x)=(gh^{-1},hx)$, then $G_+\wedge_H X=(G_+\wedge X)/H$ with its evident left action by G.

While our theorems concern p-groups and p-adic completion, $\tilde{k}_G^*(X)_p^\wedge$ is not quite the right thing to study. Rather, we define

(1.5) $$\hat{k}_G^*(X;Y)=\lim(\tilde{k}_G^*(X_\alpha;Y)_p^\wedge),$$

where X_α runs over the finite subcomplexes of X. (The reader may prefer to think in terms of pro-groups.) These groups come with suspension isomorphisms in both variables. In order to have long exact sequences associated to cofibrations in both variables, we agree to restrict from now on to G-complexes Y with finite skeleta and to assume the following finite type hypothesis.

(1.6) Each group $\tilde{k}_q^G(Y)$ is finitely generated if Y has finite skeleta.

This assumption has the following consequence.

LEMMA 1.7. *For any subquotient* J *of* G, *each group* $\tilde{k}_J^q(X;Y)$ *is finitely generated if* X *is a finite* J-*complex and* Y *is a* J-*complex with finite skeleta.*

For $J\subset G$, this follows from (1.6) and both parts of (1.4) by an easy induction. The general case will follow from (3.2) below.

Use of inverse limits in (1.5) substitutes for the wedge axiom, but we note parenthetically that the groups $\hat{k}_G^*(X;Y)$ and $\tilde{k}_G^*(X;Y)_p^\wedge$ are usually isomorphic.

LEMMA 1.8. *If* X *and* Y *have finite skeleta and* $\lim^1\tilde{k}_G^*(X^n;Y)=0$, *then*

$$\tilde{k}_G^*(X;Y)_p^\wedge\cong\lim(\tilde{k}_G^*(X^n;Y)_p^\wedge).$$

The easy proof is given in §2. The conclusion applies to $\tilde{\pi}_G^*(X)$.

We wish to determine when the natural map

$$\hat{k}_G^*(X)\to\hat{k}_G^*(EG_+\wedge X)$$

is an isomorphism. Let $\tilde{E}G$ be the cofiber of the projection $EG_+\to S^0$, that is, the unreduced suspension of EG with one of its cone points as basepoint, and note that $(\tilde{E}G)^G=S^0$. It is equivalent to determine when $\hat{k}_G^*(\tilde{E}G\wedge X)=0$. As a non-equivariant space, $\tilde{E}G\wedge X$ is contractible, and it is natural to ask when $\hat{k}_G^*(X)=0$ for all contractible G-spaces X. By the

following lemma, whose proof is also given in §2, this question is no more general than the original one.

LEMMA 1.9. *Assume that* $\hat{k}^*_H$ *vanishes on contractible* H-*spaces for all proper subgroups* H *of* G. *If* $\hat{k}^*_G(X)=0$ *for any one contractible* G-*space* X *such that* $X^G=S^0$, *then* $\hat{k}^*_G$ *vanishes on contractible* G-*spaces.*

Following Carlsson, we use the cofibering $EG_+\to S^0\to\tilde{E}G$ in the second variable to obtain the fundamental exact sequence

$$(1.10)\qquad \ldots\to\hat{k}^q_G(X;EG_+)\to\hat{k}^q_G(X)\to\hat{k}^q_G(X;\tilde{E}G)\xrightarrow{\delta}\hat{k}^{q+1}_G(X;EG_+)\to\ldots.$$

The groups $\hat{k}^*_G(X;EG_+)$ carry the free part of the problem; the $\hat{k}^*_G(X;\tilde{E}G)$ carry the singular part. We shall prove the following result about the singular part in §§3–4. We have possibly different non-equivariant theories $k^*_{H/H}$ associated to subgroups H of G, and we let $j^*=k^*_{G/G}$ and $k^*=k^*_{e/e}$.

THEOREM A. *Suppose that* $\hat{k}^*_J$ *vanishes on contractible* J-*spaces for all proper subquotients* J *of* G. *Let* X *be a* G-*space such that* $X^G=S^0$ *and* X^H *is contractible for all proper subgroups* H.

(i) *If* G *is not elementary Abelian, then* $\hat{k}^*_G(X;\tilde{E}G)=0$.
(ii) *If* $G=(Z_p)^r$, *then* $\hat{k}^*_G(X;\tilde{E}G)$ *is the direct sum of* $p^{r(r-1)/2}$ *copies of* $\Sigma^{r-1}\hat{j}^*(S^0)$.

Up to G-homotopy type, there is only one X as specified in the theorem (e.g. [13]), and we shall display an explicit model in §8. By the theorem and (1.10), when G is not elementary Abelian we can only have $\hat{k}^*_G(X)=0$ if $\hat{k}^*_G(X;EG_+)=0$ and when G is elementary Abelian we can only have $\hat{k}^*_G(X)=0$ if the connecting homomorphism δ in (1.10) is an isomorphism.

To study $\hat{k}^*_G(X;EG_+)$, we need one hypothesis to allow reduction to a non-equivariant problem and another to ensure convergence of the relevant Adams spectral sequences. Both are clearly satisfied by π^*_G. We say that k^*_G is split if there is a natural map $\zeta: k^*(W)\to k^*_G(W)$ for spaces W regarded as G-trivial G-spaces such that the composite

$$k^*(W)\xrightarrow{\zeta}k^*_G(W)\xrightarrow{\pi^*}k^*_G(G\times W)\xrightarrow{\cong}k^*(W)$$

is an isomorphism, where π: $G\times W\to W$ is the projection. This condition implies isomorphisms (e.g. [25, §2; 3, §5; 17, II§8])

$$(1.11)\qquad \tilde{k}^*_G(X;Z)\cong\tilde{k}^*(X/G;Z)\quad\text{and}\quad\tilde{k}^*_G(W;Y)\cong\tilde{k}^*(W;Y/G),$$

where X and Y are G-free (away from their base points) and W and Z are G-trivial. We say that k_* is bounded below if $\tilde{k}_q(S^0)=0$ for all sufficiently small q. We shall prove the following result in §8, using a convergence result for inverse limits of Adams spectral sequences proven in §7. Let k be the spectrum representing k^*.

THEOREM B. *Let* k^*_G *be split and* k_* *be bounded below. Let* X *be a* G-*space such that* $X^G=S^0$ *and* X^H *is contractible for all proper subgroups* H.

(i) *If* G *is not elementary Abelian, then* $\hat{k}^*_G(X;EG_+)=0$.
(ii) *If* $G=(Z_p)^r$ *and* $H^q(k)=0$ *for all sufficiently large* q, *then* $\hat{k}^*_G(X;EG_+)$ *is the sum of* $p^{r(r-1)/2}$ *copies of* $\Sigma^r\hat{k}^*(S^0)$.

An immediate induction from (1.9) and Theorems A and B gives the following generalization of Carlsson's reduction theorem.

THEOREM C. *Let* G *be a finite* p-*group which is not elementary Abelian. Let* k_G^* *be a theory such that*

(i) k_J^* *vanishes on contractible* J-*spaces for all elementary Abelian subquotients* J *of* G;
(ii) k_J^* *is split and* $(k_{K/K})_*$ *is bounded below for all non-elementary Abelian subquotients* $J = H/K$ of G.

Then $\hat{k}_J^*$ *vanishes on contractible* J-*spaces for all subquotients* J *of* G, *including* G *itself.*

Thus we concentrate henceforward on the case $G = (Z_p)^r$. When (ii) of Theorems A and B both hold, we can only expect to have $\hat{k}_G^*(X) = 0$ if $\hat{j}^*(S^0) = \hat{k}^*(S^0)$. This shows the necessity of the restrictive cohomological hypothesis in (ii) of Theorem B since there are plenty of theories k_G^* for which $\hat{k}_G^*(X) = 0$ but $\hat{j}^*(S^0) \neq \hat{k}^*(S^0)$, for example equivariant K-theory and equivariant cohomotopy with coefficients in equivariant classifying spaces. In many cases, direct calculation of $\hat{k}_G^*(X; EG_+)$ seems prohibitively difficult, and determination of these groups falls out as an implication of a different proof that $\hat{k}_G^*(X) = 0$. See [22] and [24] for various examples and counter-examples.

In view of (1.9) and (1.10), the following result now completes the proof of the Segal conjecture.

THEOREM D. *Let* $G = (Z_p)^r$ *and let* X *be a* G-*space such that* $X^G = S^0$ *and* X^H *is contractible for all proper subgroups* H. *Assume that the Segal conjecture holds for* $G = (Z_p)^s$ *if* $s < r$. *Then*

$$\delta: \hat{\pi}_G^q(X; \tilde{E}G) \to \hat{\pi}_G^{q-1}(X; EG_-)$$

is an isomorphism for all q.

The possibility of such a proof was suggested in Carlsson's preprint [9]. Our argument was outlined in [22]. Carlsson's published paper contains a later sketch of a more calculational argument along roughly the same lines as ours [10, App. B].

By (ii) of Theorems A and B, δ in Theorem D is a morphism of free $\hat{\pi}^*(S^0)$-modules. Thus it suffices to show that δ is a bijection on generators, that is, that δ is an isomorphism when $q = r - 1$. In this degree, δ is a morphism of free $\hat{Z}_p$-modules on the same number of generators. It will thus be an isomorphism if it becomes an isomorphism when reduced mod p, and this will hold if it becomes a monomorphism when reduced mod p.

We use a naturality argument. In §5, we display a quite simple theory k_G^* with unit η: $\pi_G^* \to k_G^*$ such that $\hat{k}_J^*$ vanishes on contractible J-spaces for all subquotients J of G, including G itself. Since the sequence (1.10) is natural in theories, we obtain a commutative diagram

$$\begin{tikzcd} \hat{\pi}_G^{r-1}(X; \tilde{E}G) \arrow[r, "\delta"] \arrow[d, "\eta"'] & \hat{\pi}_G^r(X; EG_+) \arrow[d, "\eta"] \\ \hat{k}_G^{r-1}(X; \tilde{E}G) \arrow[r, "\delta"] & \hat{k}_G^r(X; EG_+). \end{tikzcd} \tag{1.11}$$

Since $\hat{k}_G^*(X) = 0$, the bottom map δ is an isomorphism. It therefore suffices to show that the left vertical arrow η becomes a monomorphism when reduced mod p. We shall verify this in §6 and so complete the proof of Theorem D.

§2. PRELIMINARY DEFINITIONS AND LEMMAS

We make our definitions precise and prove (1.8) and (1.9) here.

Let U be the sum of countably many copies of each of a set of representatives for the irreducible real representations of G. We assume given an inner product on U, and we define an indexing space V in U to be a finite dimensional sub G-inner product space. If $V \subset W$, we let $W-V$ be the orthogonal complement of V in W. A set of indexing spaces $\mathscr{A}$ is co-final if its union is all of U. For definiteness, the reader may want to take U to be the sum of countably many copies of the real regular representation Reg and $\mathscr{A}$ to be the sequence $\{n \text{ Reg} \mid n \geqslant 0\}$.

A G-prespectrum k_G indexed on a co-final set $\mathscr{A}$ consists of based G-spaces $k_G V$ for V in $\mathscr{A}$ and based G-maps $\sigma\colon \Sigma^{W-V} k_G V \to k_G W$ for $V \subset W$ in $\mathscr{A}$. Here $\Sigma^V X = X \wedge S^V$, where S^V is the 1-point compactification of V. We require σ to be the identity if $V = W$ and to satisfy the evident transitivity relation for $V \subset W \subset Z$. To avoid technical problems, we assume that k_G is a G–CW prespectrum in the sense of [17, I§8]. As explained there, this assumption results in no loss of generality. (The distinction between G-prespectra and G-spectra is also explained in [17]; we won't use G-spectra here.)

Now let $J = H/K$, where $K \lhd H \subset G$. Observe that the fixed point space U^K contains all irreducible representations of J infinitely often. We may restrict to a co-final set $\mathscr{A}$ in U such that $V^K = W^K$ implies $V = W$ for V and W in $\mathscr{A}$. For example, $\{n \text{ Reg}\}$ above satisfies this condition for any $K \subset G$. For a G-prespectrum k_G indexed on $\mathscr{A}$, we obtain a J-prespectrum k_J indexed on the indexing set $\mathscr{A}^K = \{V^K \mid V \in \mathscr{A}\}$ in U^K by letting $k_J(V^K) = (k_G V)^K$; the structural J-maps of k_J are obtained by passage to K-fixed points from the structural G-maps of k_G. When $K = e$, k_H is just k_G regarded as an H-spectrum.

A G-space X has a suspension G-prespectrum $\Sigma^\infty X$ with Vth space $\Sigma^V X$. The associated J-prespectrum is $\Sigma^\infty X^K$ if $J = H/K$. In particular, if k_G is the sphere G-prespectrum $\pi_G = \Sigma^\infty S^0$, then k_J is the sphere J-prespectrum π_J. In less elementary cases, the identification of k_J is less obvious. For example, when k_G represents K-theory or cobordism, it does not follow that k_J also represents K-theory or cobordism. See [17, II§9] for further discussion.

We may identify U^G with R^∞ and so embed R^q in U for all $q \geqslant 0$. For a finite based G-complex X and any based G-complex Y, define

$$\tilde{k}_G^q(X; Y) = \operatorname*{colim}_{V \supset R^q} [\Sigma^{V-R^q} X, Y \wedge k_G V]_G$$

$$\cong [X, \operatorname*{colim}_{V \supset R^q} \Omega^{V-R^q}(Y \wedge k_G V)]_G \quad \text{if} \quad q \geqslant 0$$

and $\tilde{k}_G^{-q}(X; Y) = \tilde{k}_G^0(\Sigma^q X; Y)$ if $q > 0$. The second form of the definition also applies to infinite G-complexes, but it will be essential to our work to think in terms of the first form. Here, if $q = 0$, the colimit is taken with respect to the composites

$$[\Sigma^V X, Y \wedge k_G V]_G \xrightarrow{\Sigma^{W-V}} [\Sigma^W X, Y \wedge \Sigma^{W-V} k_G V]_G \xrightarrow{(1 \wedge \sigma)_*} [\Sigma^W X, Y \wedge k_G W]_G,$$

and similarly for other values of q. There are suspension isomorphisms and exact sequences associated to cofiberings in both variables. For infinite dimensional X, there is also a $\lim^1$ exact sequence for the calculation of $\tilde{k}_G^*(X; Y)$ in terms of the $\tilde{k}_G^*(X^n; Y)$, and the following algebraic observation on the commutation of inverse limits with p-adic completion implies (1.8).

LEMMA 2.1. *Let $\{A_n\}$ be an inverse sequence of finitely generated Abelian groups such that $\lim^1 A_n = 0$. Then the natural homomorphism*

$$(\lim A_n)_p^\wedge \to \lim((A_n)_p^\wedge)$$

is an isomorphism.

Proof. For a fixed $q>0$, consider the two six term $\lim^1$ exact sequences obtained from the two short exact sequences

$$0\to {}_qA_n\to A_n\to p^qA_n\to 0 \quad \text{and} \quad 0\to p^qA_n\to A_n\to A_n/p^qA_n\to 0.$$

Here ${}_qA_n$ is the kernel of $p^q\colon A_n\to A_n$ and is thus finite. Both $\lim^1 {}_qA_n=0$ and $\lim^1 p^qA_n=0$, the latter because it is a quotient of $\lim^1 A_n=0$. We conclude that the displayed sequences remain exact on passage to inverse limits. Therefore

$$\lim A_n/p^q \lim A_n=\lim(A_n/p^qA_n)$$

for each q. The conclusion follows by interchange of limits over q and n.

We conclude this section with the following promised proof.

Proof of (1.9). Since $X^G=S^0$, we have a cofiber sequence $S^0\to X\to X/S^0$. Let X' be any other contractible G-space. Taking smash products, we obtain a cofiber sequence

$$X'\to X'\wedge X\to X'\wedge(X/S^0).$$

It suffices to prove that $\hat{k}^*_G(X'\wedge X)=0$ and $\hat{k}^*_G(X'\wedge(X/S^0))=0$. We claim first that $\hat{k}^*_G(W\wedge X)=0$ for any G-complex W. Since the zero skeleton W^0 and the skeletal subquotients W^n/W^{n-1} for $n>0$ are wedges of G-spaces of the form $G/H_+\wedge S^n$ and since we may as well assume that W is finite, we find by induction over skeleta and use of suspension that we need only verify the claim for $W=(G/H)_+$. If $H=G$, this holds by hypothesis. If $H\neq G$, then $\hat{k}^*_G((G/H)_+\wedge X)\cong\hat{k}^*_H(X)$, which is zero by the induction hypothesis. We claim next that $\hat{k}^*_G(X'\wedge Z)=0$ for any G-complex Z, such as X/S^0, such that Z^G is a point. Arguing as above, we need only verify this when $Z=(G/H)_+$ for a proper subgroup H, and here again the conclusion holds by the induction hypothesis.

§3. SUBQUOTIENT THEORIES, FAMILIES, AND S-FUNCTORS

This section gives several preliminaries needed for the proof of Theorem A, but we begin by saying a bit more about subquotient theories. Of course, the bitheories k^*_J for subquotients J are defined the same way as the bitheories k^*_G. However, there is an illuminating alternative description, due to Constenoble, which makes (1.7) clear. It is based on an elementary obstruction theoretic observation for which we shall have further use shortly.

A family $\mathscr{F}$ in G is a set of subgroups closed under subconjugacy. For a family $\mathscr{F}$ and G-complex X, we let $X_{\mathscr{F}}$ be the subcomplex consisting of those points of X whose isotropy groups are not in $\mathscr{F}$. There is a "universal $\mathscr{F}$-space" $E_{\mathscr{F}}$ characterized up to homotopy by $(E\mathscr{F})^H=\varnothing$ if $H\notin\mathscr{F}$ and $(E\mathscr{F})^H\simeq\{pt\}$ if $H\in\mathscr{F}$. Let $\tilde{E}\mathscr{F}$ be the cofiber of the projection $E\mathscr{F}_+\to S^0$, that is the unreduced suspension of $E\mathscr{F}$. Then $(\tilde{E}\mathscr{F})^H=S^0$ if $H\notin\mathscr{F}$ and $(\tilde{E}\mathscr{F})^H\simeq\{pt\}$ if $H\in\mathscr{F}$. Moreover, the G-homotopy type of $\tilde{E}\mathscr{F}$ is characterized by these properties [13]. For example, X in Theorems A, B and D is $\tilde{E}\mathscr{P}$ where $\mathscr{P}$ is the family of proper subgroups of G.

Both $E\mathscr{F}$ and $\tilde{E}\mathscr{F}$ can be taken as G-complexes with finite skeleta, and we shall only need to apply (1.6) to G-complexes Y of this form.

LEMMA 3.1. *For* G-*complexes* X *and* Y, *the inclusions* $X_{\mathscr{F}}\to X$ *and* $S^0\to\tilde{E}\mathscr{F}$ *induce bijections*

$$[X,\tilde{E}\mathscr{F}\wedge Y]_G\to[X_{\mathscr{F}},\tilde{E}\mathscr{F}\wedge Y]_G\leftarrow[X_{\mathscr{F}},Y]_G.$$

PROPOSITION 3.2. *Let* $J = H/K$, *where* $K < H \subset G$, *and let* $\mathscr{F}[K]$ *be the family of subgroups of* H *which do not contain* K. *Then, for finite* J-*complexes* X *and arbitrary* J-*complexes* Y,

$$\tilde{k}_J^*(X; Y) \cong \tilde{k}_H^*(X, \tilde{E}\mathscr{F}[K] \wedge Y).$$

Proof. For H-complexes W and Z, $W_{\mathscr{F}[K]} = W^K$ and

$$[W^K, Z^K]_J = [W^K, Z]_H \cong [W, \tilde{E}\mathscr{F}[K] \wedge Z]_H.$$

The conclusion follows upon letting W run through $\Sigma^{V-R^q}X$ (or $\Sigma^V\Sigma^{-q}X$ if $q<0$) as Z runs through $Y \wedge k_G V$.

The starting point of the proof of Theorem A is the case $\mathscr{F} = \{e\}$ of (3.1), which reads as follows. Let SX denote the singular set of a G-space X, namely the subspace of points with non-trivial isotropy subgroup.

LEMMA 3.3. *For* G-*complexes* X *and* Y, *the inclusions* $SX \to X$ *and* $S^0 \to \tilde{E}_G$ *induce natural bijections*

$$[X, \tilde{E}G \wedge Y]_G \to [SX, \tilde{E}G \wedge Y]_G \leftarrow [SX, Y]_G.$$

Therefore, for finite G-complexes X,

$$\tilde{k}_G^q(X, \tilde{E}G) = \begin{cases} \operatorname*{colim}_{V \supset R^q} [S(\Sigma^{V-R^q}X), k_G V]_G & \text{if } q \geqslant 0 \\ \operatorname*{colim}_{V} [S(\Sigma^V\Sigma^{-q}X), k_G V]_G & \text{if } q < 0. \end{cases} \tag{3.4}$$

We can replace the functor S on the right side of (3.4) by other suitable functors, and we shall prove Theorem A by approximating S by an equivalent filtered functor with explicitly calculable subquotients.

Carlsson codified the requisite conditions on functors in his notion of an "S-functor" [10, IV]. Briefly, an S-functor (T, τ) is a functor T from the category of based G-complexes to itself together with a natural map

$$\tau\colon T(X \wedge Y) \to (TX) \wedge Y$$

such that τ is the identity if $Y = S^0$, τ satisfies the evident transitivity relation on $T(X \wedge Y \wedge Z)$, and τ is a homeomorphism when G acts trivially on Y. (Actually, we will only apply T to finite complexes.) For the singular set functor S, the map $S(X \wedge Y) \to (SX) \wedge Y$ is just the inclusion. For any S-functor T, we define $\tilde{k}_G^*(X; T)$ by replacing S by T on the right side of (3.4). For $q = 0$, the colimit is taken with respect to the system of composites

$$[T(\Sigma^V X), k_G V]_G \xrightarrow{\Sigma^{W-V}} [\Sigma^{W-V} T(\Sigma^V X), \Sigma^{W-V} k_G V]_G \xrightarrow{[\tau, \sigma]_G} [T(\Sigma^W X), k_G W]_G$$

for $V \subset W$, and similarly for other values of q. The groups $\tilde{k}_G^q(X; T)$ are natural in X. If T preserves cofiber sequences, they give the terms of a cohomology theory on finite G-complexes, but we won't need this fact. In line with the definition of $\hat{k}_G^*(X; Y)$ in (1.5), we define

$$\hat{k}_G^*(X; T) = \lim \tilde{k}_G^*(X_\alpha; T)_p^\wedge \tag{3.5}$$

on infinite G-complexes X, where X_α runs over the finite sub-complexes. To ensure that lim preserves exact sequences here, we require $\tilde{k}_G^*(X; T)$ to be of finite type when X is finite. For the S-functors we shall use, this will follow easily from (1.7) and the calculational relationship of the theories $\tilde{k}_G^*(?; T)$ to the theories $\tilde{k}_J^*$.

A map ϕ: $(T, \tau) \to (T', \tau')$ of S-functors is a natural transformation ϕ: $T \to T'$ such that

$$\tau'\phi = (\phi \wedge 1)\tau\colon T(X \wedge Y) \to (T'X) \wedge Y$$

for all X and Y. We say that ϕ is an equivalence or a cofibration if each component map ϕ: $TX \to T'X$ is a G-homotopy equivalence or a G-cofibration. We extend the usual constructions of homotopy theory to S-functors spacewise. Thus wedges, smash products with spaces (such as cones and suspensions), pushouts, cofibers, and so on all exist in the category of S-functors. If ϕ: $T \to T'$ is a cofibration, we obtain a quotient S-functor T'/T with $(T'/T)(X) = T'X/TX$ and a canonical equivalence of S-functors $C_\phi \to T'/T$.

A map ϕ: $T \to T'$ induces a map ϕ^*: $\hat{k}^*_G(X; T') \to \hat{k}^*_G(X; T)$. If ϕ is an equivalence, then ϕ^* is an isomorphism. If ϕ is a cofibration, then ϕ^* fits into a long exact sequence

$$\ldots \to \hat{k}^q_G(X; T'/T) \to \hat{k}^q_G(X; T') \xrightarrow{\phi^*} \hat{k}^q_G(X; T) \to \hat{k}^{q+1}_G(X; T'/T) \to \ldots$$

The proof uses the fact, implied by the homeomorphism condition in the definition of an S-functor, that

$$\hat{k}^q_G(X; \Sigma T) \cong \hat{k}^q_G(\Sigma X; T) \cong \hat{k}^{q+1}_G(X; T).$$

Clearly $\hat{k}^*_G(X; T \vee T') \cong \hat{k}^*_G(X; T) \oplus \hat{k}^*_G(X; T')$ for any S functors T and T'.

The subquotients of our filtered approximation of S will be wedges of suspensions of S-functors of the following general form.

Definition 3.6. *Suppose given subgroups* $K \lhd H \subset G$. *Define an* S-*functor* $C(K, H)$ *by letting*

$$C(K, H)(X) = G_+ \wedge_H X^K.$$

The G-map τ: $G_+ \wedge_H (X^K \wedge Y^K) \to (G_+ \wedge_H X^K) \wedge Y$ is the extension to a G-map of the evident inclusion of H-spaces $X^K \wedge Y^K \to (G_+ \wedge_H X^K) \wedge Y$.

Subquotient theories enter into our work because of the following observation.

Lemma 3.7. *For* $K \lhd H \subset G$, $\hat{k}^*_G(X; C(K, H))$ *is isomorphic to* $\hat{k}^*_{H/K}(X^K)$.

Proof. It suffices to prove this for finite X, before passage to p-adic completion. For notational simplicity, we only give the verification in degree 0. Here the definitions and obvious isomorphisms give

$$\begin{aligned}\tilde{k}^0_G(X; C(K, H)) &= \operatorname{colim}[G_+ \wedge_H \Sigma^{V^K} X^K, k_G V]_G \\ &\cong \operatorname{colim}[\Sigma^{V^K} X^K, k_G V]_H \\ &\cong \operatorname{colim}[\Sigma^{V^K} X^K, (k_G V)^K]_{H/K} = \tilde{k}^0_{H/K}(X^K).\end{aligned}$$

§4. AN APPROXIMATION OF THE SINGULAR SET FUNCTOR

Our approximation of S is a variant of Carlsson's [10, V]. We show the somewhat surprising fact that, up to G-homotopy type, SX can be reconstructed functorially from its fixed point sets X^H for elementary Abelian subgroups H. (Carlsson uses all proper subgroups here; this choice introduces quite a bit of extra work, such as [10, IV 6–7 and V 1–3].) Precisely, we shall prove the following result.

THEOREM 4.1. *Let* G *be a finite* p-*group of* p-*rank* r. *There is an* S-*functor* A *and an equivalence* ϕ: A→S. A *has a filtration*

$$F_0A \subset F_1A \subset \ldots \subset F_{r-1}A = A$$

by successive cofibrations. If $B_0 = F_0A$ *and* $B_q = F_qA/F_{q-1}A$ *for* $0<q<r$, *there are isomorphisms of* S-*functors*

$$B_q \cong \bigvee_{[\omega]} \Sigma^q C(A(\omega), H(\omega)).$$

Here the ω *are strictly ascending chains* $(A_0, \ldots, A_q)$ *of non-trivial elementary Abelian subgroups of* G, $A(\omega) = A_q$, *and*

$$H(\omega) = \{g | gA_ig^{-1} = A_i \text{ for } 0 \leqslant i \leqslant q\}.$$

The wedge runs over one ω *in each orbit* $[\omega]$ *under the conjugation action of* G *on the set of such ascending chains. If* $G = (Z_p)^r$, *there is also an* S-*functor* $\tilde{A}$ *with a filtration*

$$F_0\tilde{A} \subset F_1\tilde{A} \subset \ldots \subset F_{r-2}\tilde{A} = \tilde{A}$$

by successive cofibrations. If $\tilde{B}_0 = F_0\tilde{A}$ and $\tilde{B}_q = F_q\tilde{A}/F_{q-1}\tilde{A}$ for $0<q<r-1$, *there are isomorphisms of* S-*functors*

$$\tilde{B}_q \cong \bigvee_{\omega} \Sigma^q C(A(\omega), G),$$

where the wedge runs over the strictly ascending chains $(A_0, \ldots, A_q)$ *of non-trivial proper subgroups of* G *and* $A(\omega) = A_q$. *Moreover, there is a cofibration* $\tilde{A} \to A$ *such that the quotient* $A/\tilde{A}$ *is equivalent to the wedge of* $p^{r(r-1)/2}$ *copies of the* S-*functor* $\Sigma^{r-1}C(G, G)$ *which sends* X *to* $\Sigma^{r-1}X^G$.

Before proving this, we show how it implies Theorem A. Recall that $X = \tilde{E}\mathscr{P}$ there; that is, $X^G = S^0$ and $X^H \simeq \{pt\}$ for $H \neq G$.

Proof of (i) of Theorem A. By (3.4) and the equivalence $A \to S$,

$$\hat{k}^*_G(X; \tilde{E}G) \cong \hat{k}^*_G(X; S) \cong \hat{k}^*_G(X; A).$$

By (3.7), the subquotient S-functors B_q of A satisfy

$$\hat{k}^n_G(X; B_q) \cong \sum_{[\omega]} \hat{k}^{n-q}_G(X; C(A(\omega), H(\omega))) \cong \sum_{[\omega]} \hat{k}^{n-q}_{J(\omega)}(X^{A(\omega)}),$$

where $J(\omega) = H(\omega)/A(\omega)$. If $A(\omega) \neq G$, then $X^{A(\omega)}$ is a contractible $J(\omega)$-space and, since $A(\omega) \neq e$, $\hat{k}^*_{J(\omega)}(X^{A(\omega)}) = 0$ by our hypothesis that $\hat{k}^*_J$ vanishes on contractible J-spaces for all proper subquotients J. If G is not elementary Abelian, then $A(\omega)$ can't be G and $\hat{k}^*_G(X; B_q) = 0$. Therefore $\hat{k}^*_G(X; A) = 0$ by induction up the filtration.

Proof of (ii) of Theorem A. Let $G = (Z_p)^r$ and recall that $j^* = k^*_{G/G}$. If $r = 1$, then $B_0 = A$ and $\hat{k}^*_G(X; B_0) \cong \hat{j}^*(S^0)$ since $\omega = (G)$ is the only possible chain. This proves the result in this case, so assume that $r \geqslant 2$. By the proof of part (i), $\hat{k}^*_G(X; \tilde{A}) = 0$. Therefore

$$\hat{k}^*_G(X; \tilde{E}G) \cong \hat{k}^*_G(X; A) \cong \hat{k}^*_G(X; A/\tilde{A}).$$

Since $\hat{k}^*_G(X; C(G, G)) = \hat{j}^*(S^0)$, $\hat{k}^*_G(X; \tilde{E}G)$ is the sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}\hat{j}^*(S^0)$.

The proof of (4.1) is based on ideas of Quillen [28,29]. Let $\mathscr{A} = \mathscr{A}(G)$ be the poset of non-trivial elementary Abelian subgroups of G. We regard $\mathscr{A}$ as a category in the opposite of the usual fashion, regarding an inclusion $A \subset B$ as a map $B \to A$. The group G acts on $\mathscr{A}$ by

conjugation of subgroups. If $G=(Z_p)^r$, we let $\tilde{\mathscr{A}}\subset\mathscr{A}$ be the poset of non-trivial proper subgroups of G. Here G acts trivially on $\mathscr{A}$ and $\tilde{\mathscr{A}}$ since G is Abelian.

Let B denote the classifying space functor on (topological) categories [32]. It is obtained by applying geometric realization (e.g. [20, §11]) to nerves, where the nerve of a category is the usual simplicial space with q-simplices the q-tuples of composable arrows; see [32] or [28, §1]. The functor B carries G-categories to G-spaces, and the following is a slight elaboration of [29, 2.2].

LEMMA 4.2. *If* $G\neq e$, *then* $B\mathscr{A}$ *is* G-*contractible. In particular,* $(B\mathscr{A})^H$ *is non-empty and contractible for every* $H\subset G$.

Proof. Choose a central subgroup B of order p. Since $A\subset AB\supset B$ and $AB\in\mathscr{A}$ for $A\in\mathscr{A}$, we have natural transformations from the functor $A\mapsto AB$ to the identity functor of $\mathscr{A}$ and to the constant functor at B. These functors and transformations are equivariant by the centrality of B, hence they induce G-homotopies connecting the identity map of $B\mathscr{A}$ to the constant map at the vertex B on passage to classifying spaces.

We construct our approximation of the singular set functor by parametrizing $\mathscr{A}$ by singular points of G-spaces X. We agree to regard X as a G-category with object and morphism spaces X and with structural maps (identity, source, target, composition) the identity map of X. Of course, the classifying space of this G-category is just X back again. We view SX similarly.

DEFINITION 4.3. *Define a topological* G-*category* $\mathscr{A}[X]$ *and a continu ous functor* ψ: $\mathscr{A}[X]\to SX$ *as follows. The objects of* $\mathscr{A}[X]$ *are pairs* (A,x), *where* $A\in\mathscr{A}$ *and* $x\in X^A$. *There is a morphism* $(A,x)\to(B,y)$ *whenever* $B\subset A$ *and* $y=x$. *The group* G *acts on objects by* $g(A,x)=(gAg^{-1},gx)$. *The set of objects is topologized as the disjoint union of the spaces* X^A. *The set of morphisms is topologized as the disjoint union over pairs* $B\subset A$ *of the spaces* X^A. *The functor* ψ *is given by the* X-*coordinate of objects and morphisms.*

PROPOSITION 4.4. For any X, $B\psi:B\mathscr{A}[X]\to SX$ *is a* G-*homotopy equivalence.*

Proof. By the G-Whitehead theorem, it suffices to prove that $(B\psi)^H:(B\mathscr{A}[X])^H\to(SX)^H$ is a homotopy equivalence for each $H\subset G$. We can pass to fixed points on the level of categories and functors, and the classifying space functor commutes with passage to fixed points. By Quillen's theorem A [28] (which holds for topological categories since the main input is the topological fact that geometric realization carries spacewise equivalences to equivalences [21, A.4]), it suffices to prove that $B(\psi^H/x)$ is contractible for each $x\in(SX)^H$. The comma category ψ^H/x has objects (A,x) with $x\in X^A$ and A fixed by H and morphisms $(A,x)\to(B,x)$ with $B\subset A$, $x\in X^A$, and A and B fixed by H. If G_x is the isotropy group of x, then $H\subset G_x$ and ψ^H/x is just a copy of $\mathscr{A}(G_x)^H$. Thus (4.2) gives the conclusion.

We are interested in based G-complexes X. The inclusion of the basepoint $*$ in X induces a G-cofibration $B\mathscr{A}[*]\to B\mathscr{A}[X]$. Clearly $B\psi$ factors through the quotient map $B\mathscr{A}[X]\to B\mathscr{A}[X]/B\mathscr{A}[*]$, and this quotient map is a G-homotopy equivalence since $B\mathscr{A}[*]$ is G-contractible. Define

$$AX=B\mathscr{A}[X]/B\mathscr{A}[*]$$

and let ϕ denote the induced G-homotopy equivalence $AX\to SX$. Given a second G-complex Y, define the G-category $\mathscr{A}[X]\wedge Y$ in the evident way and define a G-functor

$$\mathscr{A}[X\wedge Y]\to\mathscr{A}[X]\wedge Y$$

by sending $(A, x \wedge y)$ to $(A, x) \wedge y$. Upon passage to classifying spaces and then to quotients, we obtain a natural map

$$A(X \wedge Y) \to (AX) \wedge Y.$$

With these definitions, we have an S-functor A and an equivalence of S-functors $\phi: A \to S$.

Of course, realizations $|X_*|$ of simplicial G-spaces X_* come equipped with a natural filtration such that $F_0|X_*| = X_0$ and $F_q|X_*|/F_{q-1}|X_*|$ is G-homeomorphic to $\Sigma^q(X_q/sX_{q-1})$ for $q > 0$, where sX_{q-1} is the G-space of degenerate q-simplices. In particular, this applies to classifying spaces of topological G-categories.

By naturality, we have the induced filtration

$$F_q AX = F_q B\mathscr{A}[X]/F_q B\mathscr{A}[*]$$

of AX. The nondegenerate q-simplices of $\mathscr{A}$ are the strictly ascending chains $\omega = (A_0, \ldots, A_q)$ of non-trivial elementary Abelian subgroups of G. If G has p-rank r, $F_{r-1}AX = AX$ since there are no such chains when $q \geqslant r$. Let $A(\omega)$ and $H(\omega)$ be as in (4.1) and note that $A(\omega) \lhd H(\omega)$. Since we have collapsed out $B\mathscr{A}[*]$, we see immediately that, with the action of G ignored,

$$B_q X = \bigvee_{\omega} \Sigma^q X^{A(\omega)}.$$

It follows easily that the action of G on $B_q X$ induces a G-homeomorphism

$$\bigvee_{[\omega]} \Sigma^q (G_+ \wedge_{H(\omega)} X^{A(\omega)}) \cong \bigvee_{[\omega]} G_+ \wedge_{H(\omega)} \Sigma^q X^{A(\omega)} \to B_q X.$$

A simple comparison of definitions shows that

$$B_q \cong \bigvee_{[\omega]} \Sigma^q C(A(\omega), H(\omega))$$

as an S-functor.

Now let $G = (Z_p)^r$. We can perform all of the constructions above with $\mathscr{A}$ replaced by $\tilde{\mathscr{A}}$. The analysis of the resulting S-functors $\tilde{A}$ and $\tilde{B}_q$ is similar to that just given. Since the chains ω with $A(\omega) = G$ are all parametrized by X^G in AX, we find by inspection of definitions that

$$AX/\tilde{A}X \cong (B\mathscr{A}/B\tilde{\mathscr{A}}) \wedge X^G$$

for any X. Since $B\mathscr{A}$ is contractible, $B\mathscr{A}/B\tilde{\mathscr{A}}$ is equivalent to $\Sigma B\tilde{\mathscr{A}}$. By definition, $B\tilde{\mathscr{A}}$ is the Tits building of G, and a standard algebraic calculation (e.g. [35] or [29, p. 118]) shows that $B\tilde{\mathscr{A}}$ is equivalent to the wedge of $p^{r(r-1)/2}$ copies of the sphere S^{r-2}. This completes the proof of (4.1).

§5. THE CONSTRUCTION OF THE THEORY k_G^*

We here construct the test theory needed in our proof of Theorem D. We begin with perhaps the most obvious of all equivariant cohomology theories, letting $h_G^*(X)$ be the ordinary mod p cohomology of the orbit space X/G. Equivalently (by the dimension axiom), h_G^* is Bredon cohomology with constant coefficients Z_p [8]. It is characterized by

$$h_G^q(G/H) = \begin{cases} Z_p & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

and the requirement that a G-map between orbits induces the identity map on cohomology.

We define $k_G^*(X) = h_G^*(EG \times X)$; this is just Borel cohomology, the mod p cohomology of the orbit space $EG \times_G X$. Clearly $\tilde{k}_G^*(X) = 0$ if X is a contractible based G-space since $EG_+ \wedge X$ is then G-contractible.

Obviously h_G^* is a ring theory, and we let η: $\pi_G^* \to h_G^*$ be its unit; that is, $\eta(f) = f^*(1) \in h_G^*(X)$ for $f \in \pi_G^*(X)$, where $1 \in h_G^0(pt)$ and f is regarded as a stable map $X_+ \to S^0$ of some degree. We define $\eta: \pi_G^* \to k_G^*$ to be the composite $\varepsilon^* \eta$, where ε^*: $h_G^* \to k_G^*$ is induced by the projection ε: $EG \times X \to X$.

We need to know that h_G^* is representable or, equivalently, that the Z-graded theory h_G^* just specified extends to an $RO(G)$-graded theory. This has been proven by several authors; it is the simplest example of the general principle that Bredon cohomology with coefficients in a Mackey functor is $RO(G)$-gradable [16; 17, V§9]. Exactly as non-equivariantly, any representable G-cohomology theory can be represented by an ΩG-prespectrum [17, I§6]. Thus let h_G^* be represented by an ΩG-prespectrum h_G, so that the adjoint structure maps $\tilde{\sigma}$: $h_G V \to \Omega^{W-V} h_G W$ are G-homotopy equivalences for indexing G-spaces $V \subset W$. Then, for $q \geqslant 0$,

$$\tilde{h}_G^q(X) = [X, h_G R^q]_G \quad \text{and} \quad \tilde{h}_G^{-q}(X) = [\Sigma^q X, h_G(0)]_G.$$

With $X = G/H_+$, it follows that $h_G(0)^H$ is a $K(Z_p, 0)$ for all $H \subset G$. An equivariant generalization [17, I.7.13] of a standard non-equivariant argument shows that $(h_G V)^H$ is $(n_H\text{-}1)$-connected, where $n_H = \dim V^H$. Since $\Omega^V h_G V \simeq h_G(0)$, $h_G V$ is a non-equivariant $K(Z_p, n)$, $n = \dim V$. It can be shown further that $(h_G V)^H$ is a $K(Z_p, n_H)$ for all $H \subset G$, but we won't need this much. We will need that $[X, h_G V]$ is a Z_p-vector space for all X and V, and this holds because h_G is a ring G-prespectrum with unit of order p. These vector spaces are finite dimensional when X is a finite G-complex because they are so when X is an orbit. This last fact can easily be verified by a standard inductive procedure as, for example, in the proof of [17, I.7.12].

Let $\pi_G = \{S^V\}$ be the sphere G-prespectrum. The unit of h_G^* is represented by a map of G-prespectra $\eta: \pi_G \to h_G$. In fact, η is determined by η: $S^0 \to h_G(0)$ and compatibility with the structural maps. Non-equivariantly, $\eta: S^V \to h_G V$ is just the fundamental class of $\tilde{H}^n(S^V)$, $n = \dim V$.

While we have now stated all that we shall use about h_G, it is easy to describe an explicit construction. Let N_p denote the mod p infinite symmetric product functor. For a based space X, $N_p X$ is the quotient of the infinite symmetric product NX obtained by identifying all p-fold sums $y + \ldots + y$ to zero. Remember that NX is the free topological monoid generated by X, with the basepoint of X set equal to zero. There is an evident natural inclusion ι: $X \to N_p X$, and addition induces a natural pairing ϕ: $N_p X \wedge N_p Y \to N_p(X \wedge Y)$.

If G acts on X, then G also acts on $N_p X$. An explicit model for h_G is obtained by setting $h_G V = N_p S^V$ and letting σ: $\Sigma^{W-V} h_G V \to h_G W$ be the composite

$$(N_p S^V) \wedge S^{W-V} \xrightarrow{1 \wedge \iota} (N_p S^V) \wedge (N_p S^{W-V}) \xrightarrow{\phi} N_p S^W.$$

We restrict to Vs such that $V^G \neq 0$ unless $V = 0$. The point is that an elaboration of an unpublished argument of Segal [33] shows that the adjoints $\tilde{\sigma}$: $N_p S^V \to \Omega^{W-V} N_p S^W$ are then G-homotopy equivalences (and $N_p S^0 = Z_p$). With this model, η: $\pi_G \to h_G$ is given by the inclusions ι: $S^V \to N_p S^V$.

We represent k_G^* in terms of h_G. For based G-spaces X and Y, let $F(X, Y)$ be the function space of based maps $X \to Y$ with G acting by conjugation. In particular, $\Omega^V X = F(S^V, X)$. Define k_G to be the function ΩG-prespectrum $F(EG_+, h_G)$. Its Vth space is $F(EG_+, h_G V)$ and its adjoint structural equivalences are the evident maps

$$F(EG_+, h_G V) \xrightarrow{F(1, \tilde{\sigma})} F(EG_+, \Omega^{W-V} h_G W) \cong \Omega^{W-V} F(EG_+, h_G W).$$

The natural bijection $[X, F(Y, Z)]_G \cong [X \wedge Y, Z]_G$ makes clear that k_G represents k_G^* since h_G represents h_G^*. The unit η: $\pi_G \to k_G$ is the composite

$$\pi_G \xrightarrow{\eta} h_G = F(S^0, h_G) \xrightarrow{\varepsilon^*} F(EG_+, h_G) = k_G.$$

§6. PROPERTIES OF THE THEORY k_G^*

We need a few lemmas about k_G^* and the k_J^* to complete the proof of Theorem D outlined in §1. For the first, recall that $j = k_{G/G}$ and note that $\pi_{G/G} = \{S^{V^G}\}$ is a copy of the non-equivariant sphere prespectrum π.

LEMMA 6.1. *The unit* η: $\pi \to j$ *maps* $1 \in \tilde{\pi}^0(S^0)$ *to an element of* $\tilde{j}^0(S^0)$ *which is non-zero mod* p.

Proof. Note that

$$\tilde{j}^0(S^0) = \operatorname{colim}[S^{V^G}, F(EG_+, h_G V)^G] = \operatorname{colim}[S^{V^G} \wedge EG_+, h_G V]_G.$$

Let e: $S^0 \to S^{V - V^G}$ be the inclusion of $\{0, \infty\}$, so that $1 \wedge e$: $S^{V^G} \to S^V$ is the inclusion of the fixed point set. Clearly η maps $1 \in \tilde{\pi}^0(S^0) = \operatorname{colim}[S^{V^G}, S^{V^G}]$ to the element represented by the composite G-map (for V suitably large)

$$\alpha_V: S^{V^G} \wedge EG_+ \xrightarrow{1 \wedge e \wedge 1} S^V \wedge EG_+ \xrightarrow{\eta \wedge 1} h_G V \wedge EG_+ \xrightarrow{1 \wedge \varepsilon} h_G.$$

It will prove more convenient to rewrite this as the composite

$$\alpha_V: S^{V^G} \wedge EG_+ \xrightarrow{1 \wedge \varepsilon} S^{V^G} \xrightarrow{1 \wedge e} S^V \xrightarrow{\eta} h_G V. \tag{6.2}$$

To show that $\eta(1)$ is non-zero mod p, it suffices to show that, if $V \neq V^G$ and V is complex, then α_V induces a non-trivial homomorphism on passage to mod p Borel cohomology. Thus consider

$$1 \wedge_G \alpha_V: EG_+ \wedge_G (S^{V^G} \wedge EG_+) \to EG_+ \wedge_G h_G V.$$

Since $1 \times \varepsilon$: $EG \times EG \to EG$ is a G-homotopy equivalence, ε: $EG \to \{*\}$,

$$1 \wedge_G (1 \wedge \varepsilon): EG_+ \wedge_G (S^{V^G} \wedge EG_+) \to EG_+ \wedge_G S^{V^G} = \Sigma^n BG_+, \quad n = \dim V^G,$$

is a homotopy equivalence and may be ignored. Now consider the following composite, where $W = V - V^G$.

$$\Sigma^n BG_+ = \Sigma^n (EG_+ \wedge_G S^0) \xrightarrow{\Sigma^n (1 \wedge_G e)} \Sigma^n (EG_+ \wedge_G S^W) = EG_+ \wedge_G S^V \xrightarrow{1 \wedge_G \eta} EG_+ \wedge_G h_G V.$$

With G action ignored, η: $S^V \to h_G V$ is the fundamental class. Consideration of the standard spectral sequence for the calculation of Borel cohomology makes clear that the Thom class $\mu_V \in \tilde{H}^*(EG_+ \wedge_G S^V)$ of the representation bundle $EG \times_G V \to BG$ is $(1 \wedge_G \eta)^*(v)$ for some class $v \in \tilde{H}^*(EG_+ \wedge_G h_G V)$. Of course, $\mu_V = \Sigma^n \mu_W$, where $\mu_W \in \tilde{H}^*(EG_+ \wedge_G S^W)$ is the Thom class. By definition or inspection, $(1 \wedge_G e)^*(\mu_W)$ is the Euler class $\chi(W)$. Since G is elementary Abelian and $W^G = 0$, $\chi(W)$ is non-zero. Therefore $\alpha_V^*(v) = \Sigma^n \chi(W)$ is non-zero.

LEMMA 6.3. *For every subquotient* J *of* G, *including* G *itself,* $\hat{k}_J^*$ *vanishes on contractible spaces.*

Proof. Let $J=H/K$. For a finite J-complex X and for $q\geqslant 0$,

$$\begin{aligned}\tilde{k}_J^q(X) &= \operatorname*{colim}_{V\supset R^q}[\Sigma^{V^K-R^q}X,(k_G V)^K]_J\\ &= \operatorname*{colim}_{V\supset R^q}[\Sigma^{V^K-R^q}X,F(EG_+,h_G V)]_H\\ &= \operatorname*{colim}_{V\supset R^q}[\Sigma^{V^K-R^q}X\wedge EG_+,h_G V]_H\\ &= \operatorname*{colim}_{V\supset R^q}\lim_n[\Sigma^{V^K-R^q}X\wedge EG_+^n,h_G V]_H.\end{aligned}$$

Here the last equality holds since the relevant $\lim^1$ terms clearly vanish. Of course, $\tilde{k}_J^{-q}(X)=\tilde{k}_J^0(\Sigma^q X)$ for $q>0$. For a general J-complex X, $\hat{k}_J^q(X)=\lim_\alpha \tilde{k}_J^q(X_\alpha)$, where X_α runs over the finite subcomplexes of X (p-adic completion not being needed since the $\tilde{k}_J^q(X_\alpha)$ are Z_p-vector spaces). Now suppose that X is contractible. Then $(X\wedge EG_+)^L$ is contractible for all $L\subset H$ and $X\wedge EG_+$ is H-contractible by the equivariant Whitehead theorem. Using compactness or a cellular contracting homotopy, we see that, for each pair (α, m), there exists a pair (β,n) such that $X_\alpha\subset X_\beta$, $m\leqslant n$, and the inclusion $X_\alpha\wedge EG_+^m\to X_\beta\wedge EG_+^n$ is null H-homotopic. Thus, for $q\geqslant 0$,

$$[\Sigma^{V^K-R^q}X_\beta\wedge EG_+^n,h_G V]_H\to[\Sigma^{V^K-R^q}X_\alpha\wedge EG_+^m,h_G V]_H$$

is zero for every V (and similarly with q replaced by zero and X by $\Sigma^q X$). It follows formally, by inspection from the definitions of limits and colimits, that $\hat{k}_J^*(X)=0$.

At this point, the proof of Theorem D would be complete were it not for a minor technical catch. Let $X=\tilde{E}\mathscr{P}$. It would appear that (6.3) allows us to quote Theorem A to conclude that $\hat{k}_G^*(X;\tilde{E}G)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}\hat{j}^*(S^0)$. By naturality, it would appear that the map

$$\eta\colon \hat{\pi}_G^{r-1}(X;\tilde{E}G)\to\hat{k}_G^{r-1}(X;\tilde{E}G) \tag{6.4}$$

is the sum of $p^{r(r-1)/2}$ copies of the $(r-1)$st suspension of the unit map $\eta\colon \tilde{\pi}^0(S^0)\to\tilde{j}^0(S^0)$ and is therefore a monomorphism mod p by (6.1).

The catch is that the exact sequences used in §4 to prove Theorem A were derived under the assumption (1.6), that $\tilde{k}_*^G(Y)$ is of finite type for all G-complexes Y with finite skeleta. It is not hard to see that this assumption fails for our theory k_*^G. We need to check that we have enough finiteness to justify the fundamental exact sequence (1.10) for k_*^G and enough information about $\hat{k}_G^{r-1}(X;\tilde{E}G)$ to conclude that η in (6.4) is a monomorphism mod p.

Clearly $\tilde{k}_*^G(Y)$ is of finite type when Y is finite by an inductive argument starting from the fact that $\tilde{k}_*^G(G/H_+)=\tilde{k}_*^H(S^0)$ is just $H^*(BH)$. In general, we have long exact sequences

$$\ldots\to\tilde{k}_{q+1}^G(Y^n/Y^{n-1})\to\tilde{k}_q^G(Y^{n-1})\to\tilde{k}_q^G(Y^n)\to\tilde{k}_q^G(Y^n/Y^{n-1})\to\ldots.$$

Here Y^n/Y^{n-1} is a wedge of suspensions $\Sigma^n G/H_+$, and $\tilde{k}_q^G(\Sigma^n G/H_+)=H^{n-q}(BH)$. For fixed q, this is non-zero for infinitely many n, so we cannot expect the colimit $\tilde{k}_q^G(Y)=\operatorname{colim}\tilde{k}_q^G(Y^n)$ to be attained for fixed n. However, if $Y=Y'_+$ for a free G-complex Y', then only orbits with $H=e$ and thus $\tilde{H}^*(BH)=0$ appear and there is no problem.

In particular, $\tilde{k}_q^G(EG_+)$ is of finite type, hence $\tilde{k}_*^G(X;EG_+)$ is of finite type if X is a finite G-complex. By the long exact sequence associated to the cofiber sequence $EG_+\to S^0\to\tilde{E}G$, the same is true of $\tilde{k}_*^G(X;\tilde{E}G)$. On passage to limits over the finite subcomplexes of X, this justifies the fundamental exact sequence (1.10).

We now turn to the injectivity of η in (6.4). By (3.4) and (4.1), we have isomorphisms

$$\hat{k}_G^*(X;\tilde{E}G)\cong\tilde{k}_G^*(X;S)\cong\tilde{k}_G^*(X;A).$$

By (4.1), we also have a cofiber sequence of S-functors $\tilde{A}\to A\to A/\tilde{A}$ such that $\hat{k}_G^*(X;A/\tilde{A})$ is the sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}\hat{j}^*(S^0)$. Finite type hypotheses are irrelevant for this much. These statements are natural in k_G, and there results a commutative diagram

$$\begin{array}{ccccccc}
\oplus\Sigma^{r-1}\hat{\pi}^0(S^0) & \cong & \hat{\pi}_G^{r-1}(X;A/\tilde{A}) & \xrightarrow{\cong} & \hat{\pi}_G^{r-1}(X;A) & \cong & \hat{\pi}_G^{r-1}(X;\tilde{E}G) \\
\oplus\Sigma^{r-1}\eta\downarrow & & \downarrow\eta & & & & \downarrow\eta \\
\oplus\Sigma^{r-1}\hat{j}^0(S^0) & \cong & \hat{k}_G^{r-1}(X;A/\tilde{A}) & \to & \hat{k}_G^{r-1}(X;A) & \cong & \hat{k}_G^{r-1}(X;\tilde{E}G).
\end{array}$$

Clearly (6.1) implies that η on the left is a monomorphism mod p. If we could argue with exact sequences as in the proof of Theorem A, we could conclude that the bottom arrow is an isomorphism and we would be done. The groups on the bottom are Z_p vector spaces, and the following lemma shows that enough of the cited proof goes through to allow us to conclude that η on the right is a monomorphism mod p. This will complete the proof of Theorem D.

LEMMA 6.5. *The map* $\hat{k}_G^{r-1}(X;A/\tilde{A})\to\hat{k}_G^{r-1}(X;A)$ *restricts to a monomorphism on the image of* η.

Proof. Consider the cofiber sequences $F_{q-1}\tilde{A}\to F_q\tilde{A}\to\tilde{B}_q$ of S-functors of (4.1). Each $\tilde{B}_q$ is a wedge of suspensions of S-functors $C(K,G)$, where

$$\hat{k}_G^*(X;C(K,G))=\hat{k}_{G/K}^*(X^K).$$

Thus $\hat{k}_G^*(X;\tilde{B}_q)=0$ for $0\leqslant q\leqslant r-2$ by (6.3). We claim first that $\hat{k}_G^*(X;\tilde{A})=0$. This isn't immediate since, in the absence of the finite type assumption, we don't know that the inverse limits used in passing from $\hat{k}_G^*(X_\alpha;?)$ to $\hat{k}_G^*(X;?)$ preserve exact sequences derived from cofiberings of S-functors. The situation is saved by the fact that the proof of (6.3) gives a particularly strong reason for the vanishing of the $\hat{k}_G^*(X;\tilde{B}_q)$, and this reason is inherited by all $\hat{k}_G^*(X;F_q\tilde{A})$.

To explain this, we look back at the definitions in §3. For an S-functor T, define

$$T^j_{\alpha,v,m}=\begin{cases}[T(\Sigma^{V-RJ}X_\alpha)\wedge EG^m_+,h_GV]_G & \text{if } j\geqslant 0\\ [T(\Sigma^V\Sigma^{-j}X_\alpha)\wedge EG^m_+,h_GV]_G & \text{if } j<0.\end{cases}$$

By co-finality, the definition of k_G in terms of h_G, and the $\lim^1$ exact sequence, we find that

$$k_G^j(X;T)=\lim_\alpha\operatorname{colim}_v\lim_m T^j_{\alpha,v,m}.$$

When $T=C(K,H)$, we see from (3.7) and the proof of (6.3) that, for each fixed pair (α,m), there exists a pair (β,n) such that $X_\alpha\subset X_\beta$, $m\leqslant n$, and $T^*_{\beta,v,n}\to T^*_{\alpha,v,m}$ is zero for every V. This condition is obviously inherited by wedges of S-functors, and it is easily checked that if this condition holds for T' and T'', where $T'\to T\to T''$ is a cofiber sequence of S-functors, then it also holds for T. It therefore holds for all $\tilde{B}_q$ and, inductively, for all $F_q\tilde{A}$.

We use this condition $\tilde{A}$ to complete the proof of the lemma. Consider the system of exact sequences

$$\tilde{A}^{r-2}_{\alpha,v,m}\to(A/\tilde{A})^{r-1}_{\alpha,v,m}\to A^{r-1}_{\alpha,v,m}$$

derived from the cofiber sequence $A \to A/\tilde{A} \to \Sigma\tilde{A}$. We have $(A/\tilde{A})(X) = (B\mathscr{A}/B\tilde{\mathscr{A}}) \wedge X^G$, where $B\mathscr{A}/B\tilde{\mathscr{A}}$ is equivalent to a wedge of $p^{r(r-1)/2}$ copies of the $(r-1)$-sphere. Since $X^G = S^0$, we may set $X_0 = S^0$ and restrict to $X_\alpha \supset X_0$, so that $X_\alpha^G = S^0$ for all α. Then the system $\{(A/\tilde{A})^{r-1}_{\alpha,v,m}\}$ is constant in α, with $(A/\tilde{A})^{r-1}_{\alpha,v,m}$ being the sum of $p^{r(r-1)/2}$ copies of $[S^{V^G} \wedge EG^m_+, h_G V]_G$. Let x be a non-zero element of

$$Im\,\eta \subset \tilde{k}_G^{r-1}(X; A/\tilde{A}) = \operatorname*{colim}_v \lim_m (A/\tilde{A})^{r-1}_{0,v,m}.$$

For some fixed V sufficiently large, we may choose a representative $\{x_{v,m}\}$ for x with $x_{v,m} \in (A/\tilde{A})^{r-1}_{0,v,m}$. Since the map $\alpha_v: S^{V^G} \wedge EG_+ \to h_G V$ of (6.2) factors through $S^{V^G} \wedge EG^m_+$ for all m and is non-trivial for all complex representations $V \neq V^G$, we see that $x_{v,m} \neq 0$ if $V \neq V^G$ and that $x_{v,m}$ cannot then map to zero in $(A/\tilde{A})^{r-1}_{0,w,m}$ for any $W \supset V$. Now suppose for a contradiction that x maps to zero in $\tilde{k}_G^{r-1}(X; A)$. This means that, if we write $x_{\alpha,v,m}$ for $x_{v,m}$ regarded as an element of $(A/\tilde{A})^{r-1}_{\alpha,v,m}$, then, for each α, there exists $V_\alpha \supset V$ such that $x_{\alpha,v,m}$ maps to zero in $A^{r-1}_{\alpha,v_\alpha,m}$ for all m. For each m, we can choose $\alpha \geqslant 0$ and $n \geqslant m$ such that the map $\tilde{A}^{r-2}_{\alpha,v,n} \to \tilde{A}^{r-2}_{0,v,m}$ is zero for all V. A chase of the following diagram, in which $W = V_\alpha$, will convince the reader that $x_{v,m}$ maps to zero in $(A/\tilde{A})^{r-1}_{w,m}$. This contradicts the assumption that x is a non-zero element of Im η and so proves the lemma.

$$\begin{tikzcd}
 & (A/\tilde{A})^{r-1}_{\alpha,v,n} \arrow[d] \arrow[dr] & \\
 & & (A/\tilde{A})^{r-1}_{\alpha,v,m} \arrow[dd] \\
\tilde{A}^{r-2}_{\alpha,w,n} \arrow[r] \arrow[d, "0"'] & (A/\tilde{A})^{r-1}_{\alpha,w,n} \arrow[r] \arrow[dr] & A^{r-1}_{\alpha,w,n} \\
\tilde{A}^{r-2}_{0,w,m} \arrow[r] & (A/\tilde{A})^{r-1}_{0,w,m} = & (A/\tilde{A})^{r-1}_{\alpha,w,m}
\end{tikzcd}$$

§7. INVERSE LIMITS OF ADAMS SPECTRAL SEQUENCES

The only novelty in our proof of Theorem B is the recognition that inverse limits of Adams spectral sequences converge. This is not wholly obvious. Indeed, Adams [1, p. 7], who was considering the case $G = Z_2$ before the Segal conjecture was conjectured, expressed considerable skepticism on this point: "Even if there is an Adams spectral sequence starting from $\mathrm{Ext}_A^{**}(Z_2[x, x^{-1}], Z_2)$, it is likely to be very hard to prove anything useful about its convergence". Accordingly, we shall go into a fair amount of detail. Unfortunately, most of the work is in an unpublished preprint of Boardman [7]. We are told that an argument also appears in Wegmann's thesis [36].

Write $\{E_r X\}$ for the classical mod p Adams spectral sequence associated to a spectrum X. If X is p-complete, bounded below, and of finite type over $\hat{Z}_p$, then

$$E_2 X = \mathrm{Ext}_A(H^*(X), Z_p)$$

and $\{E_r X\}$ converges strongly to $\pi_*(X)$; see Adams [2, §15]. Of course, $E_r X$ is naturally a differential $E_r S^0$-module.

PROPOSITION 7.1. *Assume given an inverse sequence*

$$\ldots \to X_{n+1} \to X_n \to \ldots \to X_0$$

of spectra such that each X_n *is* p*-complete, bounded below, and of finite type over* $\hat{Z}_p$. *Let* $\{E_r\}$ *be the inverse limit of the spectral sequences* $\{E_r X_n\}$.

(i) E_2 *is isomorphic to* Ext_A (colim $H^*(X_n)$, Z_p).
(ii) E_r *is a differential* $E_r S^0$*-module.*
(iii) $\{E_r\}$ *converges strongly to* $\lim \pi_*(X_n)$.

Proof. The essential, obvious, fact is that lim is an exact functor on the category of finitely generated $\hat{Z}_p$-modules (or Z_p-modules). For (i), this fact and [11, V.9.1* and V.9.5*] imply that Ext converts colimits to limits in the present context. Since (ii) is obvious, it remains to prove (iii). We closely follow Boardman's careful study of convergence [7]. We can construct Adams resolutions

$$\ldots \to Y_{n,s+1} \to Y_{n,s} \to \ldots \to Y_{n,0} = X_n$$

such that each $Y_{n,s}$ is p-complete, bounded below, and of finite type over $\hat{Z}_p$. By the exactness of lim, the limit of the standard exact couples used to construct the spectral sequences $\{E_r X_n\}$ is an exact couple whose associated spectral sequence is $\{E_r\}$. The filtration of $\lim \pi_*(X_n)$ associated to the exact couple is given by

$$F^s \lim \pi_*(X_n) = \operatorname{Im}(\lim F^s \pi_*(X_n)).$$

Strong convergence is the assertion that $E_\infty^{s,*} = F^s/F^{s+1}$ and the filtration is complete and Hausdorff; see [11, XV§2] and [7, §5] for discussion. We have

(a) $\lim\limits_s \pi_*(Y_{n,s}) = 0$ and $\lim\limits_s{}^1 \pi_*(Y_{n,s}) = 0$;

(b) $\lim\limits_r{}^1 E_r^{s,*} X_n = 0$, hence $\lim\limits_r{}^1 Z_r^{s,*} X_n = 0$.

In the language of Boardman [7; 5.5, 5.6 and 8.1], (a) asserts that $\{E_r X_n\}$ converges conditionally to $\pi_*(X_n)$ and (b) is then equivalent to strong convergence; see [7; 2.2, 18.4 and 18.5] for the validity of (a) and (b). We claim that

(c) $\lim\limits_s \lim\limits_n \pi_*(Y_{n,s}) = 0$ and $\lim\limits_s{}^1 \lim\limits_n \pi_*(Y_{n,s}) = 0$;

(d) $\lim\limits_r{}^1 \lim\limits_n E_r^{s,*} X_n = 0$, hence $\lim\limits_r{}^1 \lim\limits_n Z_r^{s,*} X_n = 0$.

By another quotation of Boardman [7, 8.1], this will prove the result. The first half of (c) holds by the first half of (a) and the standard interchange of limits isomorphism

$$\lim_s \lim_n \pi_*(Y_{n,s}) \cong \lim_n \lim_s \pi_*(Y_{n,s}).$$

Since $\lim^i = 0$ for $i > 1$ in the context of sequential limits, spectral sequences of Roos [31, Thm. 3] for the computation of $\lim^i$ of product inverse systems collapse to give extensions

$$\begin{array}{c} 0 \to \lim\limits_s{}^1 \lim\limits_n \pi_*(Y_{n,s}) \to \lim\limits_{s,n}{}^1 \pi_*(Y_{n,s}) \to \lim\limits_s \lim\limits_n{}^1 \pi_*(Y_{n,s}) \to 0 \\ \| \\ 0 \to \lim\limits_n{}^1 \lim\limits_s \pi_*(Y_{n,s}) \to \lim\limits_{n,s}{}^1 \pi_*(Y_{n,s}) \to \lim\limits_n \lim\limits_s{}^1 \pi_*(Y_{n,s}) \to 0. \end{array}$$

By (a), this implies the second half of (c). Similarly, we have extensions

$$\begin{array}{c} 0\to\lim_r{}^1\lim_n E_r^{s,*}X_n\to\lim_{r,n}{}^1 E_r^{s,*}X_n\to\lim_r\lim_n{}^1 E_r^{s,*}X_n\to 0 \\ \| \\ 0\to\lim_n{}^1\lim_r E_r^{s,*}X_n\to\lim_{n,r}{}^1 E_r^{s,*}X_n\to\lim_n\lim_r{}^1 E_r^{s,*}X_n\to 0. \end{array}$$

Our finiteness hypotheses imply that each $\lim_r E_r^{s,t}X_n$ is a finite dimensional Z_p vector space. Therefore $\lim_n{}^1\lim_r E_r^{s,*}X_n=0$. Via the extensions, this fact and (b) imply (d).

§8. THE CALCULATION OF $\hat{k}_G^*(X;EG_+)$

To prove Theorem B, we introduce a particularly convenient model for $X=\tilde{E}\mathscr{P}$, following Carlsson. Thus let V be the reduced regular complex representation of G and let X be the union of the spheres S^{nV}. The inclusion of S^{nV} in $S^{(n+1)V}$ is obtained by smashing the identity map of S^{nV} with the inclusion $e:S^0\to S^V$. Since $V^G=0$, $X^G=S^0$. If $H\neq G$, then V contains a trivial H-summand, hence e^H is null homotopic and X^H is contractible. The groups $\tilde{k}_G^*(S^{nV};EG_+)$ admit non-equivariant interpretations in terms of the Thom spectra BG^{-nV} of the virtual representations $-nV$. Such spectra were first introduced by Boardman [6]. Carlsson [10, App. A] gave an *ad hoc* construction adequate for the present purposes. A systematic account of generalized Thom spectra is given in [17, IX–X], and the derivation of the following theorem from that work is explained in [23]. We use complex representations in order to have the orientations needed to prove part (i), in which mod p cohomology is understood. We write $\chi(V)$ for the Euler class of V.

THEOREM 8.1. *There are Thom spectra* $\mathrm{BG}^{-\mathrm{V}}$ *for complex representations* V *of* G *and maps* $\mathrm{f}:\mathrm{BG}^{-\mathrm{W}}\to\mathrm{BG}^{-\mathrm{V}}$ *for inclusions* $\mathrm{V}\subset\mathrm{W}$ *which satisfy the following properties.*

(i) $\mathrm{H}^*(\mathrm{BG}^{-\mathrm{V}})$ *is a free* $\mathrm{H}^*(\mathrm{BG})$*-module on one generator* ι_{V} *of degree* $-\dim_{\mathbf{R}}\mathrm{V}$, *and* $\mathrm{f}^*:\mathrm{H}^*(\mathrm{BG}^{-\mathrm{V}})\to\mathrm{H}^*(\mathrm{BG}^{-\mathrm{W}})$ *is the morphism of* $\mathrm{H}^*(\mathrm{BG})$*-modules specified by* $\mathrm{f}^*(\iota_{\mathrm{V}})=\chi(\mathrm{W}-\mathrm{V})\iota_{\mathrm{W}}$.

(ii) *If* $\mathrm{k}_{\mathrm{G}}^*$ *is split with underlying non-equivariant theory* k^*, *then* $\mathrm{k}_*(\mathrm{BG}^{-\mathrm{V}})$ *is isomorphic to* $\tilde{\mathrm{k}}_*^{\mathrm{G}}(\mathrm{S}^{\mathrm{V}};\mathrm{EG}_+)$ *and the diagram*

$$\begin{array}{ccc} \mathrm{k}_*(\mathrm{BG}^{-\mathrm{W}}) & \xrightarrow{\mathrm{f}_*} & \mathrm{k}_*(\mathrm{BG}^{-\mathrm{V}}) \\ \wr\| & & \wr\| \\ \tilde{\mathrm{k}}_*^{\mathrm{G}}(\mathrm{S}^{\mathrm{W}};\mathrm{EG}_+) & \xrightarrow{\mathrm{e}^*} & \tilde{\mathrm{k}}_*^{\mathrm{G}}(\mathrm{S}^{\mathrm{V}};\mathrm{EG}_+) \end{array}$$

commutes, where $\mathrm{e}:\mathrm{S}^{\mathrm{V}}\to\mathrm{S}^{\mathrm{W}}$ *is the inclusion.*

Recalling the notation $k_q^G=k_G^{-q}$ and our definition of $\hat{k}_G^*(X;EG_+)$ as an inverse limit, we obtain the following consequence of (ii). We again take V to be the reduced regular representation.

COROLLARY 8.2. *If k_G^* is split and $X = \cup S^{nV}$, then*

$$\hat{k}_G^{-q}(X; EG_+) = \lim k_q(BG^{-nV})_p^\wedge .$$

Here we have passed to the stable category (replacing the prespectrum k by its associated spectrum in the context of [17, 23]), so that $k_*(BG^{-nV}) = \pi_*(k \wedge BG^{-nV})$. Henceforward, we agree to complete all spectra at p without change of notation. We have enough finiteness that this only entails smashing with the Moore spectrum $M\hat{Z}_p$. By (7.1) and (8.2), we have a spectral sequence

$$\{E_r\} = \lim \{E_r(k \wedge BG^{-nV})\}$$

converging from

$$E_2 = \mathrm{Ext}_A(H^*(k) \otimes \mathrm{colim}\, H^*(BG^{-nV}), Z_p)$$

to $\hat{k}_G^*(X; EG_+)$ under the hypotheses of Theorem B. By (8.1.i), the co-limit is taken with respect to the homomorphisms

$$\chi(V)\colon H^*(BG^{-nV}) \to H^*(BG^{-(n+1)V}).$$

Here $\chi(V) \in H^*(BG)$ restricts to zero in $H^*(BH)$ for any proper subgroup H since $V^H \neq 0$.

Proof of (i) of Theorem B. If G is not elementary Abelian, a theorem of Quillen [30] implies that $\chi(V)$ is nilpotent. Therefore $\mathrm{colim} H^*(BG^{-nV}) = 0$.

Henceforward, let $G = (Z_p)^r$ and let $L = \chi(V) \in H^{2(p^r-1)}(BG)$. If $p > 2$,

$$H^*(BG) = Z_p[y_1, \ldots, y_r] \otimes E[x_1, \ldots, x_r],$$

$\deg x_i = 1$ and $\deg y_i = 2$. If $p = 2$, we can write $H^*(BG)$ in the same way additively, but with $y_i = x_i^2$. In either case, L is the product of the non-zero elements of the vector space spanned by the y_i, hence $\mathrm{colim} H^*(BG^{-nV})$ is the localization $H^*(BG)\,[L^{-1}]$.

The following algebraic calculation is due to Lin [19] when $p = 2$ and $r = 1$, to Gunawardena [14] when $p > 2$ and $r = 1$, and to Adams, Gunawardena and Miller [4,1.1] when $r > 1$. It is part of the last three authors' proof of the Segal conjecture for $G = (Z_p)^r$ but, according to Adams [private communication], "it lies on the direct line of proof and only one-third of the way through" and the other two-thirds "include the trickiest part". A second approach to part of the calculation has been given by Priddy and Wilkerson [27]. The bulk of the calculation relies heavily on the pioneering work of Singer [18, 34].

THEOREM 8.3. *Let* St *denote the* Z_p*-module* $H^*(BG)\,[L^{-1}] \otimes_A Z_p$ *of* A*-indecomposable elements of* $H^*(BG)\,[L^{-1}]$ *and regard* St *as a trivial* A*-module. Then* St *is concentrated in degree* $-r$ *and has dimension* $p^{r(r-1)/2}$, *and the quotient homomorphism* $\varepsilon\colon H^*(BG)\,[L^{-1}] \to St$ *induces an isomorphism*

$$\mathrm{Ext}_A(K \otimes St, Z_p) \to \mathrm{Ext}_A(K \otimes H^*(BG)\,[L^{-1}], Z_p)$$

for any finite dimensional A*-module* K.

Since K clearly contains a non-zero A-trivial submodule, the result for general K follows inductively from the result for $K = Z_p$ given in [4, 1.1].

Remark 8.4. It is possible to put an action of the general linear group $GL(r, Z_p)$ on all objects in sight, algebraic or topological, and the fundamental exact sequence (1.10) is $GL(r, Z_p)$-equivariant. As proven in [4, 1.1], *St* in the previous theorem is the Steinberg module, a fact which lurks behind but need not be invoked in the proof of the Segal conjecture.

The equivariant context is irrelevant to the completion of the proof of (ii) of Theorem B, which is contained in the following result. Remember that we are implicitly completing spectra at p.

THEOREM 8.5. *Let* k *be a* p-*complete spectrum which is bounded below, of finite type over* $\hat{Z}_p$, *and cohomologically bounded above. Let* Y *be the wedge of* $p^{r(r-1)/2}$ *copies of* S^{-r}. *Then there is a compatible system of maps* α: $Y \to BG^{-nV}$ *which induces an isomorphism* $\pi_*(k \wedge Y) \to \lim \pi_*(k \wedge BG^{-nV})$.

Proof. First consider $k = S^0$. By (8.3) and standard facts about $\mathrm{Ext}_A(Z_p, Z_p)$, we have $E_2^{s,t} = 0$ if $t - s < -r$, $E_2^{0,-r} = St$, $E_2^{s,s-r} = a_0^s E_2^{0,-r}$, and $E_2^{s,s-r+1} = 0$ unless $p = 2$ and $s = 1$, when a_0 annihilates this group. Here $a_0 \in E_2^{1,1} S^0$ is the canonical generator. Therefore the elements of $E_2^{s,s-r}$ are all non-bounding permanent cycles. Since multiplication by a_0 detects multiplication by p, we see by convergence that $\lim \pi_{-r}(BG^{-nV})$ is a free $\hat{Z}_p$-module on $p^{r(r-1)/2}$ generators. Choose generators α_i. Each α_i may be viewed as a compatible system of maps $S^{-r} \to BG^{-nV}$, and their wedge sum is the required system α. The systems α_i are detected by the mod p Hurewicz homomorphism

$$\lim \pi_{-r} BG^{-nV} \to \lim E_2^{0,-r}(BG^{-nV}) \to \lim H_{-r}(BG^{-nV}),$$

and α induces ε: $H^*(BG)[L^{-1}] \to St$ on passage to cohomology. Now return to the general case. The system $1 \wedge \alpha$: $k \wedge Y \to k \wedge BG^{-nV}$ induces a map of spectral sequences

$$\{E_r(1 \wedge \alpha)\}: \{E_r(k \wedge Y)\} \to \lim \{E_r(k \wedge BG^{-nV})\} = \{E_r\},$$

and $E_2(1 \wedge \alpha)$ is the isomorphism of (8.3). The conclusion follows from the convergence of $\{E_r\}$ and the comparison theorem (in the form given by Boardman [7, 5.27]).

Remark 8.6. Let $BG^{-\infty V}$ be the homotopy inverse limit of the spectra BG^{-nV}. The natural map $\pi_*(BG^{-\infty V}) \to \lim \pi_*(BG^{-nV})$ is an isomorphism since the relevant $\lim^1$ terms vanish. We conclude that the system α induces an equivalence $Y \to BG^{-\infty V}$.

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