Topology Vol. 27, No. 1, pp. 7–21, 1988
Printed in Great Britain.

0040 9383/88 $3.00 + .00
Pergamon Press plc

A GENERALIZATION OF THE SEGAL CONJECTURE

J. F. ADAMS, J.-P. HAEBERLY, S. JACKOWSKI and J. P. MAY

(*Received* 13 *May* 1986)

§1. INTRODUCTION

THEOREM 1.4 below is a generalization of the Segal conjecture about equivariant cohomotopy. It asserts an invariance property of the G-cohomology-theory $S^{-1}\pi_G^*(—)_I^\wedge$ obtained from equivariant cohomotopy π_G^* by first localizing with respect to a general multiplicatively-closed subset S in the Burnside ring $A(G)$, and then completing with respect to a general ideal $I \subset A(G)$. We first explain how we place previous "localization theorems" and "completion theorems" in one setting by formulating suitable invariance statements.

Let G be a finite group; all our G-spaces will be G-CW complexes [23]. Let $\mathscr{H}$ be some class of subgroups and let $f: X \to Y$ be a G-map. We will say that f is an "$\mathscr{H}$-equivalence" if the induced map of fixed-point-sets $f^H: X^H \to Y^H$ is an ordinary homotopy equivalence for each $H \in \mathscr{H}$. (Thus we may assume without loss of generality that $\mathscr{H}$ is closed under passing to conjugate subgroups.) Let h be a functor defined on G-spaces and G-maps; we will say that h is "$\mathscr{H}$-invariant" if it carries each $\mathscr{H}$-equivalence to an isomorphism in the target category of h. The same property was previously introduced in [34] and studied further in [35].

In particular, let $\mathscr{H}$ be the class of all subgroups $H \subset G$; then an $\mathscr{H}$-equivalence is just a G-homotopy-equivalence, and every G-cohomology-theory is $\mathscr{H}$-invariant.

To place "localization theorems" in this setting, we assume that $\mathscr{H}$ is closed under passing to conjugate subgroups and larger subgroups. Then for any X we have an $\mathscr{H}$-fixed-point subcomplex

$$X^{\mathscr{H}} = \cup\{X^H : H \in \mathscr{H}\},$$

and the inclusion $i: X^{\mathscr{H}} \to X$ is an $\mathscr{H}$-equivalence.

Remark 1.1. In this case, h is $\mathscr{H}$-invariant iff $h(i): h(X) \to h(X^{\mathscr{H}})$ is iso for each X. ("Only if" is clear; and we will explain the converse in §7.)

"Localization theorems" usually state that $h(i)$ is iso when h is a functor obtained by localization, $h = S^{-1}k$, and $S, \mathscr{H}$ are suitably related. Such theorems go back to Segal [29, Prop. 4.1].

We place "completion theorems" in this setting. A class $\mathscr{H}$ which is closed under passing to conjugate subgroups and smaller subgroups is called a "family". A G-space Y qualifies as a universal space $E\mathscr{F}$ for the family $\mathscr{F}$ if Y^H is contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$. For background on spaces $E\mathscr{F}$, see [27, 10, 11 p. 175, 13]. For any X the projection

$$p: E\mathscr{F} \times X \to X$$

is an $\mathscr{F}$-equivalence.

Remark 1.2. In this case, h is $\mathscr{F}$-invariant iff $h(p){:}h(X)\to h(E\mathscr{F}\times X)$ is iso for each X. ("Only if" is clear; and if $f{:}X\to Y$ is an $\mathscr{F}$-equivalence, then $1\times f{:}\ E\mathscr{F}\times X\to E\mathscr{F}\times Y$ is a G-homotopy-equivalence.)

"Completion theorems" usually state that $h(p)$ is iso when h is a functor obtained by completion, $h=k(-)_I^{\wedge}$, and I, $\mathscr{F}$ are suitably related. Such theorems go back to Atiyah and Segal [6].

We will show that it makes sense to look for a "best possible" invariance result.

THEOREM 1.3. *For each G-cohomology-theory h^* satisfying the axioms given in §7, there is a unique minimal class $\mathscr{H}$ such that h^* is $\mathscr{H}$-invariant.*

[Note that as $\mathscr{H}$ decreases, the $\mathscr{H}$-invariance property gets stronger, because less data on f suffice to prove $h(f)$ iso.]

We seek specific invariance results (preferably best possible) for particular functors. The functors we consider are progroup-valued. The role of progroups in this subject has been recognized ever since the work of Atiyah and Segal [6]. Let h be a functor from finite G-CW complexes to R-modules. Then h yields a progroup-valued functor $\mathbf{h}$ defined on all G-CW complexes X; we define $\mathbf{h}(X)$ to be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite G-CW subcomplexes of X. Localization of promodules over R (with respect to a multiplicative set $S\subset R$) is done termwise: $S^{-1}\{M_\alpha\}=\{S^{-1}M_\alpha\}$. To complete promodules (with respect to an ideal $I\subset R$) we define $\{M_\alpha\}_I^{\wedge}$ be the inverse system $\{M_\alpha/I^rM_\alpha\}$, where α runs as before and r runs over the non-negative integers. In particular, even if X is a finite complex, the completion $\mathbf{h}(X)_I^{\wedge}$ is a progroup.

We take h to be equivariant cohomotopy—see [1] or [30].

THEOREM 1.4. *The theory $S^{-1}\pi_G^*(-)_I^{\wedge}$ (progroup-valued equivariant cohomotopy localized at $S\subset A(G)$ and completed at $I\subset A(G)$) is $\mathscr{H}$-invariant, where*

$$\mathscr{H}=\cup\{Supp(P){:}\,P\cap S=\varnothing\,\&\,P\supset I\}.$$

Here P runs over prime ideals of $A(G)$, and Supp(P) is the support of P, which we define following Dress [12]. [$H\in$Supp(P) if P comes from H via the restriction map $A(G)\to A(H)$ and P does not come from any $K<H$. Dress shows that Supp(P) is a single conjugacy class of subgroups H.]

Our companion paper on K-theory [2] shows that a theorem precisely analogous to (1.4) holds for equivariant K-theory; one just replaces the Burnside ring $A(G)$ by the representation ring $R(G)$, and "supports" in the sense of Dress [12] by "supports" in the sense of Segal [28].

Originally we sought the special case $S=\{1\}$ of (1.4); this goes as follows.

THEOREM 1.5. *For any family $\mathscr{F}$ the theory $\pi_G^*(-)_{I(\mathscr{F})}^{\wedge}$, equivariant cohomotopy completed at*

$$I(\mathscr{F})=\bigcap_{H\in\mathscr{F}}\mathrm{Ker}(A(G)\to A(H)),$$

is $\mathscr{F}$- invariant.

COROLLARY 1.6. *There is a pro-isomorphism*

$$\pi_G^*(X)^{\wedge}_{I(\mathscr{F})} \leftrightarrow \pi_G^*(E\mathscr{F} \times X)$$

natural in the G-space X.

On the right of (1.6) we can omit the completion at $I(\mathscr{F})$, because $\pi_G^*(E\mathscr{F} \times X)$ is already complete (see §6). Given this, the result follows from (1.5) and (1.2).

We refer to our companion paper [2] for the application of (1.6) to calculate the equivariant cohomotopy of equivariant classifying spaces.

We may pass from the inverse systems in (1.6) to their inverse limits. We assume that X is a finite G-CW complex; then the inverse system $\pi_G^*(X)^{\wedge}_{I(\mathscr{F})}$ is Mittag–Leffler; therefore the pro-isomorphic inverse system $\pi_G^*(E\mathscr{F} \times X)$ is Mittag–Leffler; therefore its inverse limit is the representable G-cohomotopy of $E\mathscr{F} \times X$. All this goes back to [6].

The classical case is that in which $\mathscr{F} = \{1\}$, $E\mathscr{F}$ becomes EG and the completion is done using the augmentation ideal $\mathrm{Ker}(\varepsilon: A(G) \rightarrow Z)$. In this case (1.6) becomes the Segal conjecture, which has been proved by the combined efforts of a number of mathematicians, by far the greatest contribution being due to Carlsson [8].

Compared with the special case $\mathscr{F} = \{1\}$, the general case (1.5), (1.6) has more flexibility, and (1.4) has more flexibility still. By adjusting S and I, we can obtain results about functors closer to cohomotopy, at the price of using stronger hypotheses on our spaces and maps. Conversely, (1.4) shows what price (in terms of S and I) will pay for a given level of invariance (every class $\mathscr{H}$ arises for suitable S and I, usually for many).

One of us [25] has obtained a further generalization of (1.4). In this he replaces the representing spectrum for cohomotopy, that is the sphere spectrum, by the suspension spectrum of a suitable classifying space. (See appendix.)

As for history: completion theorems of the general form of (1.6) were proposed by one of us [17, 18]. For equivariant K-theory (over a compact Lie group G), such a theorem was proved independently, using different approaches, by two of us [16, 19]. The analogy between K-theory and cohomotopy led to the starting-point of the present work, an attempt to prove (1.6). The statement (1.4) grew out of our attempts to explain our proof of (1.6); in order to prove completion theorems in cohomotopy, we were driven to use intermediate results which involved localization as well as completion, and involved classes $\mathscr{H}$ which were not families.

The rest of this paper is organised as follows. Necessary preliminaries about progroups come in §2, and necessary preliminaries about the Burnside ring come in §3. §4 and §5 go to proving (1.4); §6 deduces (1.5) and (1.6); and finally, §7 covers (1.1) and (1.3).

The proof of (1.4) may be summarized as follows. We assemble the result from information "over the rationals", which is easy to come by, and p-adic information, which we derive ultimately from Carlsson [8]. The assembly job is done by (2.3), which is our main algebraic weapon. Carlsson proceeds from his p-adic result to the I-adic statement of the Segal conjecture by quoting the work of May and McClure [26]; our main proof, in §5, subsumes and generalises that part of the proof of the Segal conjecture. (Note that even for p-groups (1.4) gives some new information, because its proof builds in "rational" information.) The steps of our main argument prove special cases of (1.4) which grow successively more general.

In the course of upgrading our information in §5, we need a relation between equivariant cohomotopy over a group G and equivariant cohomotopy over a quotient group G/H. We prepare this result in §4. The difference between the proof of (1.4) and that in [2] is explained by the fact that this relation works much better in cohomotopy than in K-theory, while the

Euler class is much more accessible in K-theory than in cohomotopy. Otherwise the only topological ingredient worth mentioning in §5 is the use of "transfer" in (5.4).

§2. PROGROUPS

In this section we will summarize what we need about progroups. The language of progroups is due to Grothendieck [15] and may be found in [4, 6] and later references.

Inverse systems of Abelian groups, indexed on directed sets, qualify as progroups. The progroups which arise in the examples given in §1 are of this form. However, at the end of §7 we assume that h^* carries any direct limit of G-spaces to an inverse limit in the category of progroups. To construct an inverse limit in the category of progroups, you take all the data contained in your inverse system of progroups, and interpret it as a single progroup [4]. To make this idea work as stated, one generalizes the allowable indexing systems to "filtering categories".

If $\{M_\alpha\}$ and $\{N_\beta\}$ are progroups, one defines

$$\operatorname{Prohom}(\{M_\alpha\},\{N_\beta\})=\varprojlim_{\beta}\varinjlim_{\alpha}\operatorname{Hom}(M_\alpha,N_\beta),$$

where both limits are taken in the category of groups. There is a unique sensible definition for the composite of prohomomorphisms. The progroups and prohomomorphisms make up a category. A prohomomorphism $\{M_\alpha\}\to\{N_\beta\}$ is a pro-isomorphism if it is an isomorphism in this category.

In §1 we introduced a progroup-valued functor $\mathbf{h}$, giving the definition on objects as $\mathbf{h}(X)=\{h(X_\alpha)\}$. It is easy to supply the definition of h on maps.

The main use of the language of progroups is to make statements about inverse systems which cannot be expressed as statements about their limits. These are mostly statements about exactness. In fact, the category of progroups is an Abelian category, in which one can conduct exactness arguments.

LEMMA 2.1. *The functor $S^{-1}\pi_G^*(-)_I^\wedge$ of* (1.4) *carries pairs and cofiberings to pro-exact sequences.*

Of course, the assertion about "cofiberings" assumes that one introduces the reduced theory $\tilde{\pi}_G^*$ and uses it in the usual way.

It may be reassuring, and help in checking lemmas and details, if we make the definition of "pro-exact" utterly explicit. Let

$$\mathbf{L}\xrightarrow{\mathbf{f}}\mathbf{M}\xrightarrow{\mathbf{g}}\mathbf{N}$$

be a sequence of two prohomomorphisms whose composite is the zero prohomomorphism. By definition, the element

$$\mathbf{f}\in\varprojlim_{\beta}\varinjlim_{\alpha}\operatorname{Hom}(L_\alpha,M_\beta)$$

is a system of compatible elements

$$f_\beta\in\varinjlim_{\alpha}\operatorname{Hom}(L_\alpha,M_\beta),$$

and each f_β is an equivalence class of representatives

$$f_{\alpha\beta}\in\operatorname{Hom}(L_\alpha,M_\beta).$$

The sequence is pro-exact at $\mathbf{M}$ if for each such representative

$$L_\alpha \xrightarrow{f_{\alpha\beta}} M_\beta$$

there is a diagram

$$\begin{array}{ccc} & & M_\gamma \xrightarrow{g_{\gamma\delta}} N_\delta \\ & & \downarrow m \\ L_\alpha & \xrightarrow{f_{\alpha\beta}} & M_\beta \end{array}$$

in which m is a map of $\mathbf{M}$, $g_{\gamma\delta}$ is a representative for some component g_δ of g, and

$$m(\operatorname{Ker} g_{\gamma\delta}) \subset \operatorname{Im} f_{\alpha\beta}.$$

Cultural aside: inverse limits in the category of progroups preserve pro-exactness.

Proof of (2.1). If X_α is a finite G-CW complex, then $\pi_G^n(X_\alpha)$ is a finitely generated Z-module [1] and therefore finitely generated over $A(G)$. Thus $S^{-1}\pi_G^n(X_\alpha)$ is finitely generated over the Noetherian ring $S^{-1}A(G)$. The Artin–Rees lemma [5] may now be used to show that if $X_\alpha \subset Y_\beta$ is a finite pair, the sequence

$$\cdots \to \left\{\frac{S^{-1}\pi_G^n(Y_\beta, X_\alpha)}{(S^{-1}I)^r S^{-1}\pi_G^n(Y_\beta, X_\alpha)}\right\} \to \left\{\frac{S^{-1}\pi_G^n(Y_\beta)}{(S^{-1}I)^r S^{-1}\pi_G^n(Y_\beta)}\right\} \to \left\{\frac{S^{-1}\pi_G^n(X_\alpha)}{(S^{-1}I)^r S^{-1}\pi_G^n(X_\alpha)}\right\} \to \cdots$$

is proexact. Varying the finite pair, we get enough to prove the required proexactness statement for a general pair $X \subset Y$. Similarly for cofiberings.

LEMMA 2.2. *In order to prove that a G-cohomology theory h^* is $\mathscr{H}$-invariant, it is sufficient to verify the following special case: if Z is a pointed* G-*space such that Z^H is contractible for $H \in \mathscr{H}$, then $\tilde{h}^*(Z) = 0$.*

The proof of (2.2) would be clear if h^* were group-valued. We would assume given an $\mathscr{H}$-equivalence $f: X \to Y$, and apply the assumed property of h^* to the mapping-cone $Z = Y \cup_f CX$. We would then use the exact cohomology sequence of a cofibering (which is the only significant assumption on h^* we need) to show that $h^*(f)$ is an isomorphism.

Of course, this proof carries over to progroup-valued functors, and it is for this purpose that we have stated (2.1) explicitly. The equation "$\tilde{h}^*(Z) = 0$" should now be read "$\tilde{h}^*(Z)$ is prozero". Here a progroup $\{M_\alpha\}$ is prozero if it is a zero object in the category of progroups, and this is equivalent to the following explicit condition: for each of its objects M_α, the progroup has a zero map

$$M_\beta \xrightarrow{0} M_\alpha.$$

Now we need a result for proving that progroups are prozero, and what follows is our main algebraic weapon. Let $\mathbf{M} = \{M_\alpha\}$ be a pro-object of finitely generated modules over a Noetherian ring R; let S be a multiplicative subset of R, and let I be an ideal in R.

LEMMA 2.3. $S^{-1}\mathbf{M}_I^\wedge$ *is prozero iff* $S_P^{-1}\mathbf{M}_P^\wedge$ *is prozero for each prime ideal* $P \subset R$ *such that* $P \cap S = \varnothing$ *and* $P \supset I$.

Here S_P^{-1} means "localization at P"; that is, the multiplicative set S_P is the complement of P.

Proof. It is immediate that if $S^{-1}\mathbf{M}_I^{\wedge}$ is prozero then so are all the other $S_P^{-1}\mathbf{M}_P^{\wedge}$; we have to argue in the other direction.

First we note that it is enough to prove the special case $S=\{1\}$, in which data are given for all $P\supset I$ and the conclusion is $\mathbf{M}_I^{\wedge}=0$. For then to prove (2.3) in the generality given, we apply the special case to the promodule $S^{-1}\mathbf{M}$ over $S^{-1}R$; the primes Q of $S^{-1}R$ for which we require data correspond to the primes P of R for which we have data.

Assuming $S=\{1\}$, we take a typical term in $\mathbf{M}_I^{\wedge}$, say $T=M_\alpha/(I^rM_\alpha)$. We will find a finite number of prime ideals $P_1, P_2, \ldots, P_n$ containing I and integers $s(i)$ such that the map

$$T=M_\alpha/(I^rM_\alpha)\to\bigoplus_i S_{P_i}^{-1}(M_\alpha/(I^r+P_i^{s(i)})M_\alpha)$$

is mono.

In fact, we take $P_1, P_2, \ldots, P_n$ to be the associated prime ideals of T, which are finite in number by a standard result [22]. These prime ideals contain I^r, and therefore contain I. Let $L_i\subset T$ be the submodule annihilated by P_i. By the Artin–Rees lemma [5] there exists $s(i)$ such that

$$L_i\cap P_i^{s(i)}T\subset P_iL_i=0.$$

We will show that the kernel K_i of the map

$$T\to S_P^{-1}(T/P_i^{s(i)}T)$$

does not have P_i as an associated prime.

For suppose it did, and for convenience write P, L, s, K instead of $P_i, L_i, s(i), K_i$. Then we would have a monomorphism $R/P\to K$, which must map into L. Since $L\to T/P^sT$ is mono by the choice of s, we would get a monomorphism $R/P\to T/P^sT$. Since localization preserves exactness, we would get the following commutative diagram.

$$\begin{tikzcd}
R/P \arrow[r, "\text{mono}"] \arrow[d, "\text{mono}"'] & T/P^sT \arrow[d] \\
S_P^{-1}(R/P) \arrow[r, "\text{mono}"] & S_P^{-1}(T/P^sT)
\end{tikzcd}$$

But the diagonal is zero because we assumed R/P mapped into K. This contradiction shows that K_i does not have P_i as an associated prime.

But then the kernel of

$$T\to\bigoplus_i S_{P_i}^{-1}(T/P_i^{s(i)}T)$$

has no associated primes, and must be zero as claimed.

Given $T=M/(I^rM_\alpha)$, we now have the following commutative diagram for any map $m\colon M_\beta\to M_\alpha$ in $\mathbf{M}$.

$$\begin{tikzcd}
M_\beta/(I^rM_\beta) \arrow[r] \arrow[d] & M_\alpha/(I^rM_\alpha)=T \arrow[d, "\text{mono}"] \\
\bigoplus_i S_{P_i}^{-1}(M_\beta/(I^r+P_i^{s(i)})M_\beta) \arrow[r] & \bigoplus_i S_{P_i}^{-1}(M_\alpha/(I^r+P_i^{s(i)})M_\alpha)
\end{tikzcd}$$

For each P_i our hypotheses allow us to choose m so that $S_{P_i}^{-1}M_\beta$ maps to zero in $S_{P_i}^{-1}(M_\alpha/P_i^{s(i)}M_\alpha)$. We can do this for a finite number of i, and so ensure that the lower horizontal arrow is zero; then m must be zero. This proves (2.3).

§3. THE BURNSIDE RING

In this section we will say what we need about the Burnside ring.

The Burnside ring $A(G)$ is the Grothendieck group constructed from (finite) G-sets [12]. For each subgroup $H \subset G$ there is a homomorphism of rings

$$\phi_H: A(G) \to Z$$

which carries a G-set W to $|W^H|$; ϕ_H depends only on the conjugacy class of H. With these maps as components, we obtain a map

$$\Phi: A(G) \to \prod_{(H)} Z,$$

where the product runs over all conjugacy classes (H); Φ is mono. By the going-up theorem [5], each prime ideal P of $A(G)$ is the restriction of a prime ideal of $\prod_{(H)} Z$; that is, it may be written in the form

$$q(H,p) = \phi_H^{-1}(p),$$

for some H and some prime ideal (p) in Z. Here (p) is clearly determined by P; however, we may still get the same ideal $q(H,p)$ for different choices of H. Fix a prime $p > 0$; for each subgroup $H \subset G$, let H_p be the smallest normal subgroup of H such that H/H_p is a p-group. Then $(H_p)_p$ is a characteristic subgroup of H_p, and hence normal in H, so $(H_p)_p = H_p$; thus H_p is "p-perfect", meaning that any quotient of it which is a p-group is trivial. Dress [12] says that H and K are p-equivalent, and writes $H \sim_p K$, if H_p is conjugate to K_p; he shows that $q(H,p) = q(K,p)$ iff $H \sim_p K$. The "support" of $q(H,p)$ is then the conjugacy class of H_p. For $p = 0$ we can interpret this discussion in the same way as for any other prime which does not divide $|G|$; H_0 becomes H, and 0-equivalence becomes conjugacy.

In the rest of this paper we shall make free use of localization with respect to prime ideals in $A(G)$. Integer denominators are sometimes more convenient than general elements of $A(G)$, and we can reduce to that case. Let P be a prime in $A(G)$, and let (p) be its counter-image under $Z \to A(G)$; we write $S_{(p)}^{-1}$ for localization over Z at (p).

LEMMA 3.1. *The map $S_{(p)}^{-1}A(G) \to S_P^{-1}A(G)$ is epi.*

To prove this conveniently, we discuss the idempotents in $S_{(p)}^{-1}A(G)$. Such idempotents have been used by several authors [11 p8, 14, 3, 31]. We continue to write ϕ_H after localizing at (p). If $H \sim_p K$ then $\phi_H(x) \equiv \phi_K(x) \bmod p$ for any x; in particular, if e is idempotent then $\phi_H(e)$ must be constant at 0 or 1 as H runs over a p-equivalence class. By a standard result of commutative algebra [7] the Boolean algebra of idempotents in $S_{(p)}^{-1}A(G)$ is canonically isomorphic to the Boolean algebra of open-and-closed sets in spec $S_{(p)}^{-1}A(G)$. This spectrum has been explicitly described by Dress [12]; it is the disjoint union of finitely many open sets, each containing just one of the ideals $q(H,p)$. There is therefore just one primitive idempotent e_H in $S_{(p)}^{-1}A(G)$ for each conjugacy class of p-perfect subgroups H, given by

$$\phi_K(e_H) = \begin{cases} 1 \text{ if } K \sim_p H \\ 0 \text{ if } K \not\sim_p H. \end{cases}$$

These idempotents can also be obtained by more elementary methods. For $(p)=0$ we can interpret this discussion in the obvious way.

Proof of (3.1). Let $H \in \operatorname{Supp}(P)$ and let $e=e_H$ be the corresponding idempotent. Consider the map

$$A(G) \to S_{(p)}^{-1}A(G) \to eS_{(p)}^{-1}A(G).$$

This is a map of rings which carries every element of S_P to an invertible element. (The target $eS_{(p)}^{-1}A(G)$ is a local ring because it has only one maximal ideal, and the counter-image of that maximal ideal is P.) It is also universal among such maps. [Any such map carries e to an invertible element and $(1-e)$ to zero.] This characterizes the target as $S_P^{-1}A(G)$. But clearly the map

$$S_{(p)}^{-1}A(G) \to eS_{(p)}^{-1}A(G)$$

is epi.

It may be helpful to know that the localized cohomology theory $S_P^{-1}h^*$ is the same as that obtained by first localizing over Z to get $S_{(p)}^{-1}h^*$, and then taking the summand $e_H S_{(p)}^{-1}h^*$; compare [20].

§4. TOPOLOGICAL PRELIMINARIES

In this section we prove a topological result needed for the main proof. Let G be a finite group and (p) a given prime; let $H=G_p$ and let P be the corresponding prime ideal $q(G,p)=q(H,p)$ in $A(G)$, as in §3.

PROPOSITION 4.1. *Then there is an isomorphism*

$$S_P^{-1}\tilde{\pi}_G^n(X) \cong S_{(p)}^{-1}\tilde{\pi}_{G/H}^n(X^H)$$

natural as X runs over finite pointed G-spaces.

Results of this sort were known to Araki [3] McClure [21] and probably others. We separate off the first part of the proof.

LEMMA 4.2. *Restriction gives a natural isomorphism*

$$S_P^{-1}\{X,Y\}^G \xrightarrow{\cong} S_P^{-1}\{X^H,Y\}^G.$$

Here X runs over finite pointed G-spaces; Y runs over pointed G-spaces which may be infinite; and $\{X,Y\}^G$ means stable G-homotopy-classes of stable G-maps.

Sketch proof of (4.2). $S_P^{-1}\{X,Y\}^G$ is one group of a G-cohomology-theory which is zero on all the G-cells

$$(G/K)\times E^m, (G/K)\times S^{m-1}$$

of X which are not in X^H. See [20], Theorem 4.8.

Proof of (4.1). By suspending X if necessary we can assume $n \geqslant 0$.

First we construct the natural transformation. Restriction on H-fixed-point-sets gives a natural map

$$\tilde{\pi}_G^n(X) \to \tilde{\pi}_{G/H}^n(X^H).$$

This is a map of $A(G)$-modules, provided we make $A(G)$ act on $\tilde{\pi}^n_{G/H}(X^H)$ via the homomorphism $\theta\colon A(G)\to A(G/H)$ which carries a G-set W to W^H. Notice now that G/H is a p-group, $S_{(p)}^{-1}A(G/H)$ is a local ring, and the counter-image of its unique maximal ideal in $A(G)$ is P. Thus θ carries an element of $A(G)$ not in P to an element invertible in $S_{(p)}^{-1}A(G/H)$. So we get an induced map

$$\phi\colon S_P^{-1}\tilde{\pi}^n_G(X)\to S_{(p)}^{-1}\tilde{\pi}^n_{G/H}(X^H).$$

We show that ϕ is epi. The map

$$S_{(p)}^{-1}\tilde{\pi}^n_G(X^H)\to S_{(p)}^{-1}\tilde{\pi}^n_{G/H}(X^H)$$

is split epi because any representative (G/H)-map is also a G-map. *A fortiori*,

$$S_P^{-1}\tilde{\pi}^n_G(X^H)\to S_{(p)}^{-1}\tilde{\pi}^n_{G/H}(X^H)$$

is epi. The map

$$S_P^{-1}\tilde{\pi}^n_G(X)\to S_P^{-1}\tilde{\pi}^n_G(X^H)$$

is epi by (4.2) applied to $Y=S^n$.

We show that ϕ is mono. Take an element of $S_P^{-1}\tilde{\pi}^n_G(X)$; using (3.1), we may write it $[f]/d$, where d is an integer prime to p and f is a representative G-map

$$S^V\wedge X\xrightarrow{f}S^V\wedge S^n$$

for a suitable representation V of G. Now assume $[f]/d\in\operatorname{Ker}\phi$. Then after increasing both d and V, we may assume that the restriction of f to $S^{V^H}\wedge X^H$ is G -nullhomotopic. Thus $[f]$ maps to zero in

$$S_P^{-1}\{S^{V^H}\wedge X^H, S^V\wedge S^n\}^G.$$

But then $[f]$ maps to zero in

$$S_P^{-1}\{S^V\wedge X, S^V\wedge S^n\}^G$$

by (4.2) applied to $Y=S^V\wedge S^n$. Thus $[f]/d=0$.

§5. THE MAIN PROOF

In this section we will prove Theorem 1.4. By (2.3) it is sufficient to consider $S_P^{-1}\pi^*_G(X)^\wedge_P$; in this case the only relevant assumption is the contractibility of X^H for one conjugacy class of H.

LEMMA 5.1. *Let G be a p-group. If X^1, the underlying space of X, is contractible, then $\tilde{\pi}^*_G(X)^\wedge_{(p)}$ is prozero.*

The result remains true in a trivial way if we take (p) to be the prime ideal (0), for we have to interpret it so that $G=1$ is the only group which qualifies.

Proof of (5.1). Carlsson [8] proves that the inverse limit of the inverse system

$$\left\{\frac{\tilde{\pi}^n_G(X_\alpha)}{p^r\tilde{\pi}^n_G(X_\alpha)}\right\}$$

is zero. Since the groups of this inverse system are finite groups, it follows that the inverse system is prozero.

Now let G be a finite group and (p) a given prime; let $H=G_p$ and $P=q(G,p)=q(H,p)$ be as in §4.

LEMMA 5.2. *If X^H is contractible, then $S_P^{-1}\tilde{\pi}_G^*(X)_P^\wedge$ is prozero.*

Proof. By (4.1) we have an isomorphism of progroups

$$\left\{\frac{S_P^{-1}\tilde{\pi}_G^n(X_\alpha)}{p^r S_P^{-1}\tilde{\pi}_G^n(X_\alpha)}\right\} \overset{\cong}{\longleftrightarrow} \left\{\frac{S_{(p)}^{-1}\tilde{\pi}_{G/H}^n(X_\alpha{}^H)}{p^r S_{(p)}^{-1}\tilde{\pi}_{G/H}^n(X_\alpha{}^H)}\right\}.$$

Since G/H is a p-group and X^H is contractible, the right-hand side is prozero by (5.1). That is, $S_P^{-1}\tilde{\pi}_G^*(X)_{(p)}^\wedge$ is prozero. Since completion at P is more drastic than completion at (p), the result follows.

Now let P be a general prime ideal $q(H,p)$ in $A(G)$, where H is p-perfect.

PROPOSITION 5.3. *If X^H is contractible, then $S_P^{-1}\tilde{\pi}_G(X)_P^\wedge$ is prozero.*

The proof involves a construction. Let N be the normalizer of H in G, and let F/H be a Sylow p-subgroup of N/H, where F/H is interpreted as H/H if $(p)=0$.

LEMMA 5.4. *$S_P^{-1}\tilde{\pi}_G^n(X)_P^\wedge$ is a direct summand in*

$$\{S_P^{-1}\pi_G^n((G/F)\times X_\alpha, G/F)\}_P^\wedge.$$

Here $\{S_P^{-1}\pi_G^n((G/F)\times X_\alpha, G/F)\}$ is a progroup in which X_α runs over the finite subcomplexes of X.

LEMMA 5.5. *If X^H is contractible, then $\{S_P^{-1}\pi_G^n((G/F)\times X_\alpha, G/F)\}_P^\wedge$ is prozero.*

Proof of 5.4. For each finite G-space X_α we have a G-covering-map

$$(G/F)\times X_\alpha, G/F \xrightarrow{\varpi} X_\alpha, pt$$

natural in X_α. This gives the following commutative diagram, in which "Tr" means transfer—see, for example, [1] or [21].

$$\begin{tikzcd}
 & \pi_G^n((G/F)\times X_\alpha, G/F) \arrow[dr, "\mathrm{Tr}"] & \\
\pi_G^n(X_\alpha, pt) \arrow[ur, "\varpi^*"] \arrow[rr, "{[G/F]}"] & & \pi_G^n(X_\alpha, pt)
\end{tikzcd}$$

The horizontal arrow is multiplication by the class of G/F in $A(G)$. Using the fact that H is p-perfect and F/H is a p-group, we find

$$\phi_H(G/F)=|N/F|$$

which is prime to p by the choice of F. Thus $[G/F]$ does not lie in P, and on localizing at P we get the following commutative diagram, which is natural for maps of X_α.

$$\begin{tikzcd}
 & S_P^{-1}\pi_G^n((G/F)\times X_\alpha, G/F) \arrow[dr, "\mathrm{Tr}"] & \\
S_P^{-1}\pi_G^n(X_\alpha, pt) \arrow[ur, "\varpi^*"] \arrow[rr, "\cong"] & & S_P^{-1}\pi_G^n(X_\alpha, pt)
\end{tikzcd}$$

The conclusion follows.

Proof of (5.5). Of course we use the natural isomorphism

$$\pi_G^n((G/F)\times X_\alpha, G/F)\cong \pi_F^n(X_\alpha, pt).$$

This isomorphism is a map of $A(G)$-modules, if we make $A(G)$ act on $\pi_F^n(X_\alpha, pt)$ via the restriction map i^*: $A(G)\to A(F)$. It follows that

$$\{S_P^{-1}\pi_G^n((G/F)\times X_\alpha, G/F)\}\hat{_P} = \{S^{-1}\pi_G^n(X_\alpha, pt)\}\hat{_I},$$

where on the right-hand side localization and completion are done over $A(F)$, taking

$$S = i^* S_P, \qquad I = (i^*P)A(F).$$

We now wish to prove that $S^{-1}\tilde{\pi}_F^n(X)\hat{_I}$ is prozero. By (2.3), it is sufficient to prove that $S_Q^{-1}\tilde{\pi}_F^n(X)\hat{_Q}$ is prozero for each prime ideal Q of $A(F)$ such that $Q\cap S=\varnothing$ and $Q\supset I$. Equivalently, we have to consider prime ideals Q whose counter-image in $A(G)$ is P. We will show there is only one such ideal, namely the ideal $q(H,p)$ of $A(F)$.

Any such Q has to be an ideal $q(K,p)$ of $A(F)$ for the same p and some $K\subset F$ which is p-perfect and conjugate to H in G; it follows that $K=H$.

Thus (5.2) applies and shows $S_Q^{-1}\tilde{\pi}_F^n(X)\hat{_Q}=0$. This proves (5.5).

Proposition 5.3 follows from (5.4) and (5.5). Theorem 1.4 follows immediately by assembling (2.2), (2.3) and (5.3).

§6.

In this section we deduce (1.5) and (1.6).

Proof of (1.5). We deduce (1.5) from (1.4) by taking the ideal I in (1.4) to be the ideal $I(\mathscr{F})$ in (1.5). For this it is enough to show that if $\mathscr{F}$ is a family, then the class

$$\mathscr{H} = \{\mathrm{Supp}(P): P\supset I(\mathscr{F})\}$$

in (1.4) is contained in $\mathscr{F}$.

Since $\mathscr{F}$ is a family, the ideal

$$I(\mathscr{F}) = \bigcap_{H\in\mathscr{F}} \mathrm{Ker}(A(G)\to A(H))$$

is the intersection of the prime ideals $q(H,0)$ over $H\in\mathscr{F}$. If $P\supset I(\mathscr{F})$, then P must contain one of these ideals $q(H,0)$. According to Dress [12], this means that $P=q(H,p)$ for some H and some p. Since $H_p\subset H\in\mathscr{F}$ and $\mathscr{F}$ is a family, we have $\mathrm{Supp}(P)\subset\mathscr{F}$. This holds for each $P\supset I(\mathscr{F})$, so $\mathscr{H}\subset\mathscr{F}$.

[Of course, $\mathscr{H}=\mathscr{F}$ since every $H\in\mathscr{F}$ is the support of an ideal $q(H,0)$.]

We turn to (1.6). Let $\mathscr{F}$ be a family.

LEMMA 6.1. *Let Y be a finite G-CW complex such that Y^H is empty for $H\notin\mathscr{F}$. Then $\pi_G^*(Y)$ is annihilated by some power of $I(\mathscr{F})$.*

LEMMA 6.2. *Let Y be a G-space such that Y^H is empty for $H\notin\mathscr{F}$. Then the canonical pro-map*

$$\pi_G^*(Y)\to\pi_G^*(Y)\hat{_{I(\mathscr{F})}}$$

is a pro-isomorphism.

We omit the proof of (6.1) and the proof of (6.2) from (6.1); both are sufficiently well known, and the ideas go back to [6].

Lemma 6.2 applies to $Y=E\mathscr{F}\times X$ and shows that in (1.6), the right-hand side $\pi_G^*(E\mathscr{F}\times X)$ is already complete.

§7

In this section we explain (1.1) and prove (1.3).

We say that a G-CW complex X is an "$\mathscr{H}$–complex" if its G-cells are all of the form $G/H\times E^m$ with $H\in\mathscr{H}$. It is easy to prove the appropriate generalization of the "theorem of J. H. C. Whitehead"; this says that if X is an $\mathscr{H}$–complex and $f\colon Y\to Z$ is an $\mathscr{H}$-equivalence, then the induced map

$$f_*\colon[X,Y]^G\to[X,Z]^G$$

is a bijection. In particular, an $\mathscr{H}$-equivalence between $\mathscr{H}$–complexes is a G-homotopy-equivalence; this was certainly known to previous authors [34, §1].

Assume, as in (1.1), that $\mathscr{H}$ is closed under passing to larger subgroups. Then an $\mathscr{H}$-equivalence $f\colon X\to Y$ induces a map $f^{\mathscr{H}}\colon X^{\mathscr{H}}\to Y^{\mathscr{H}}$ which is an $\mathscr{H}$-equivalence between $\mathscr{H}$–complexes, and therefore a G-homotopy-equivalence by the remarks above. Now (1.1) follows.

One may also deduce the result from [21, II, 9.3].

We turn to (1.3). Our assumptions on h^* are as follows. It is Z-graded and satisfies Eilenberg–Steenrod Axioms 1–6, with the words "exact" and "isomorphism" interpreted as "pro-exact" and "pro-isomorphism" if h^* is progroup-valued. No axiom of "suspension with respect to arbitrary representations" is required.

Proof of (1.3). First we define the required class $\mathscr{H}$. For any subgroup K, let $\mathscr{C}(K)$ be the complement of the conjugacy class (K), i.e., the class of subgroups not conjugate to K. We lay down that K is not in $\mathscr{H}$ if and only if h^* is invariant with respect to $\mathscr{C}(K)$. It follows that if h^* if $\mathscr{L}$-invariant, then $\mathscr{L}\supset\mathscr{H}$; for if $K\notin\mathscr{L}$, then $\mathscr{C}(K)\supset\mathscr{L}$, so the $\mathscr{L}$-invariance of h^* implies the $\mathscr{C}(K)$-invariance, and $K\notin\mathscr{H}$. This justifies the words "unique minimal" in (1.3). It remains to prove that h^* is $\mathscr{H}$-invariant.

Let $\mathscr{F}$ be a family, and let $\mathscr{F}'=\mathscr{F}\cup(H)$ be the "adjacent" family obtained by adjoining the conjugacy class of a subgroup H all of whose proper subgroups lie in $\mathscr{F}$ Consider the map

$$i\colon X\wedge(E\mathscr{F}\sqcup P)\to X\wedge(E\mathscr{F}'\sqcup P).$$

Then i^K is an equivalence for $K\not\approx H$, so if $H\notin\mathscr{H}$ it follows that $h^*(i)$ is iso..If $H\in\mathscr{H}$, the same conclusion follows trivially if we assume X^H contractible. Suppose then that X^H is contractible for all $H\in\mathscr{H}$. We can get from the empty family to the family of all subgroups by a finite number of the steps considered above; so $h^*(i)$ is iso for the map

$$i\colon X\wedge P\to X\wedge S^0.$$

That is, $\tilde{h}^*(X)=0$. Now (2.2) shows that h^* is $\mathscr{H}$-invariant.

We remark that this proof carries over when G becomes a compact Lie group. We need one more assumption on h^*: it carries any direct limit of G-spaces to an inverse limit in the category of progroups (see §2). The finite induction implicit in the proof above is replaced by an appeal to Zorn's Lemma, using the class $\mathscr{C}$ of families $\mathscr{F}$ such that

$$\tilde{h}^*(X\wedge(E\mathscr{F}\sqcup P))=0.$$

The induction starts because this class contains the empty family. We have to show that for any totally ordered subset $\{\mathscr{F}_\alpha\}$ of families in $\mathscr{C}$, the union $\mathscr{F} = \bigcup_\alpha \mathscr{F}_\alpha$ will serve as an upper bound in $\mathscr{C}$. For this we construct the homotopy-limit $\operatorname{Holim}_{\vec{\alpha}} E\mathscr{F}_\alpha$. The construction of this limit involves extending G-maps $(\partial\sigma^n) \times E\mathscr{F}_\alpha \to E\mathscr{F}_\beta$ over $\sigma^n \times E\mathscr{F}_\alpha$ by induction over n, but this is certainly possible in view of the properties of $E\mathscr{F}_\alpha$, $E\mathscr{F}_\beta$ and the fact that we always have $\mathscr{F}_\alpha \subset \mathscr{F}_\beta$. We observe that $\operatorname{Holim}_{\vec{\alpha}} E\mathscr{F}_\alpha$ qualifies as $E\mathscr{F}$, so

$$X \wedge (E\mathscr{F} \sqcup P) = \operatorname{Holim}_{\vec{\alpha}}(X \wedge (E\mathscr{F}_\alpha \sqcup P))$$

and the assumed property of h^* gives

$$\tilde{h}^*(X \wedge (E\mathscr{F} \sqcup P)) = 0.$$

Zorn's lemma now shows that $\mathscr{C}$ has a maximal element $\mathscr{F}$. If $\mathscr{F}$ were not the family of all subgroups, then there would be a subgroup H minimal among subgroups not in $\mathscr{F}$, and the argument above applied to $\mathscr{F}' = \mathscr{F} \cup (H)$ would yield a contradiction. Thus we conclude $\tilde{h}^*(X) = 0$, as before.

Acknowledgements—J.F.A. and S.J. thank the University of Chicago and Northwestern University, and S.J. similarly thanks also the Forschungsinstitut für Mathematik ETH Zurich, for hospitality during discussions on the successive versions of this paper.

REFERENCES

1. J. F. Adams: Prerequisites for Carlsson's lecture, *Lecture Notes in Mathematics*, Springer, Berlin **1051** (1984), 483–532.
2. J. F. Adams, J.-P. Haeberly, S. Jackowski and J. P. May: A generalization of the Atiyah–Segal completion theorem, *Topology* **27** (1988), 1–6.
3. S. Araki: Equivariant stable homotopy theory and idempotents of Burnside rings, preprint (1983).
4. M. Artin and B. Mazur: Etale homotopy, *Lecture Notes in Mathematics*, Springer, Berlin **100** (1969), 154–166.
5. M. F. Atiyah and I. G. Macdonald: *Introduction to commutative algebra*, Addison–Wesley U.K. (1969), 62–107.
6. M. F. Atiyah and G. B. Segal: Equivariant K-theory and completion, *J. Diff. Geom.* **3** (1969), 1–18.
7. H. Bass: Algebraic K-theory, W. A. Benjamin U.S.A. (1968), p. 103.
8. G. Carlsson: Equivariant stable homotopy and Segal's Burnside ring conjecture, *Ann. of Math.* **120** (1984), 189–224.
9. J. Caruso and J. P. May: Completions in equivariant cohomology theory. To appear.
10. T. tom Dieck: Orbitypen und aquivariante Homologie I, *Arch. Math.* **23** (1972), 307–317.
11. T. tom Dieck: Transformation groups and representation theory, *Lecture Notes in Mathematics*, Springer, Berlin **766** (1979).
12. A. Dress: A characterisation of solvable groups, *Math. Zeit.* **110** (1969), 213–217.
13. A. D. Elmendorff: Systems of fixed point sets, *Trans. Am. Math. Soc.* **277** (1983), 275–284.
14. D. Gluck: Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex, *Illinois J. Math.* **25** (1981), 63–67.
15. A. Grothendieck: Technique de descente et théorems d'existence en géometrie algébrique II, Seminaire Bourbaki 12ième année, (1959–60), exp. 195.
16. J.-P. Haeberly: Completions in equivariant K-theory, Thesis, University of Chicago (1983).
17. S. Jackowski: Localisation and completion theorems in equivariant cohomology theories, Thesis, University of Warsaw (1976).
18. S. Jackowski: Equivariant K-theory and cyclic subgroups, *Lecture Notes Lond. Math. Soc. C.U.P.* **26** (1977), 76–92.

19. S. Jackowski: Families of subgroups and completion, *J. Pure and Appl. Algebra* **37** (1985), 167–179.
20. C. Kosniowski: Equivariant cohomology and stable cohomotopy, *Math. Ann.* **210** (1974), 83–104.
21. L. G. Lewis, Jr, J. P. May and M. Steinberger (with contributions by J. E. McClure): Equivariant stable homotopy theory, *Lecture Notes in Mathematics*, Springer, Berlin, **1213** (1986).
22. H. Matsumara: Commutative Algebra, W. A. Benjamin, U.S.A. (1970), p. 52.
23. T. Matumoto: on G-CW-complexes and a theorem of J. H. C. Whitehead, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* **18** (1971/72), 363–374.
24. J. P. May: The completion conjecture in equivariant cohomology, *Lecture Notes in Mathematics*, Springer, Berlin **1051** (1984), 620–637.
25. J. P. May: A further generalization of the Segal conjecture, in preparation.
26. J. P. May and J. E. McClure: A reduction of the Segal conjecture, *Can. Math. Soc. Conference Proceedings*, Vol. 2 (1982), 209–222.
27. R. S. Palais: The classification of G-spaces, *Memoirs Am. Math. Soc.* **36** (1960).
28. G. B. Segal: The representation ring of a compact Lie group, *Inst. Hautes Etudes Sci. Publ. Math.* **34** (1968), 113–128.
29. G. B. Segal: Equivariant K-theory, *Inst. Hautes Etudes Sci. Publ. Math.* **34** (1968), 129–151.
30. G. B. Segal: Equivariant stable homotopy theory, in *Proceedings of the International Congress of Mathematicians* 1970, Gauthier–Villars, Paris, **2** (1971), 59–63.
31. T. Yoshida: Idempotents in Burnside rings and Dress induction theorem, *J. Algebra* **80** (1983), 90–105.
32. L. G. Lewis, Jr., J. P. May and J. E. McClure: Classifying G-spaces and the Segal conjecture, *Can. Math. Soc. Conference Proceedings*, **2** (1982), 165–179.
33. J. P. May: Stable maps between classifying spaces, *Contemp. Math.* **37** (1985), 121–129.
34. R. M. Seymour: On G-cohomology theories and Kunneth formulae, *Can. Math. Soc. Conference Proceedings*, **2** (1982), 251–271.
35. R. M. Seymour: A Kunneth formula for $RO(G)$-graded equivariant cohomology theories, preprint.

APPENDIX: BY J. P. MAY

In [32] the Segal conjecture was generalized to the assertion that equivariant cohomotopy with coefficients in equivariant classifying spaces is $\{1\}$-invariant. This generalization specializes to give a calculation of the stable maps between classifying spaces in the non-equivariant world [33]. In a later paper [25], I will use the theorem below to prove that equivariant cohomotopy with coefficients in equivariant classifying spaces satisfies the $\mathscr{H}$-invariance property analogous to (1.4) above. This generalization of (1.4) specializes to give a calculation of the equivariant stable maps between equivariant classifying spaces.

The proof of (1.4) takes given information about p-groups and p-adic completion, namely that supplied by Carlsson and quoted as (5.1), and derives from it the strongest implications. This idea works in considerable generality.

Let h_G^* be a Z-graded cohomology theory defined on G-CW complexes. We want h_G^* to take values in modules over $A(G)$, and the natural way to ensure this is to require h_G^* to be $RO(G)$-gradable, or equivalently, representable. We also require h_G^* to be of finite type, in the sense that each $h_G^q(G/H)$ is finitely generated, and this ensures that each $h_G^q(X)$ is finitely generated when X is a finite G-CW complex. We obtain a Z-graded progroup-valued cohomology theory on general G-CW complexes by setting $\mathbf{h}_G^*(X)=\{h_G^*(X_\alpha)\}$ as in §1.

We need a relation like (4.1), and this requires us to construct representable theories $h_{H/K}^*$ on H/K-CW complexes for subquotient groups H/K of G. There is a sensible way to do this [24, p. 626; 9;3; 21, II, §9], the evident analog of (4.1) holds [3; 21, V, §6] and so does the analog of (5.4). When h_G^* is stable G-cohomotopy with coefficients in a G-space Y, the theory associated to H/K is just stable H/K-cohomotopy with coefficients in the H/K-space Y^K. In this case, the proofs above of (4.1) and (5.4) apply with only notational changes.

From here, one can argue exactly as in §5 to reach the following conditional conclusion.

Theorem. *Suppose that $h_{H/K}^*$ is of finite type for each subquotient H/K of G and that $\tilde{\mathbf{h}}_{H/K}^*(X)_{(p)}^{\wedge}=0$ whenever H/K is a p-group and X is a nonequivariantly contractible H/K-space. Then the theory $S^{-1}\mathbf{h}_G^*(-)_I^{\wedge}$ is $\mathscr{H}$-invariant, where*

$$\mathscr{H}=\cup\{\mathrm{Supp}(P)|P\cap S=\varnothing \,\&\, P\supset I\}.$$

A comparable reduction from general p-groups to elementary Abelian p-groups can be obtained by Carlsson's methods [24, 9].

Department of Pure Mathematics,
University of Cambridge,
16 Mill Lane,
Cambridge CB2 1SB, U.K.

Dept. of Mathematics
University of Washington
Seattle, WA 98195
U.S.A.

Mathematical Institute
University of Warsaw
Palac Kultury i Nauki IXp
00–901 *Warszawa*
Poland

Dept. of Mathematics
University of Chicago
Chicago, IL 60637
U.S.A.