Topology Vol. 27, No. 1, pp. 1-6, 1988
Printed in Great Britain.

0040 9383/88 $3.00+.00
Pergamon Press plc

A GENERALIZATION OF THE ATIYAH–SEGAL COMPLETION THEOREM

J. F. ADAMS, J.-P. HAEBERLY, S. JACKOWSKI and J. P. MAY

(Received 9 June 1986)

§1. INTRODUCTION

WE SHALL give a geodesic path from the equivariant Bott periodicity theorem to the generalized completion theorem in equivariant K-theory.

Let G be a compact Lie group. Our G-spaces are understood to be G-CW complexes and we let K_G^* denote the progroup valued G-cohomology theory specified by

$$K_G^n(X) = \{K_G^n(X_\alpha)\},$$

where X_α runs over the finite subcomplexes of X. For a subgroup H of G, we have a restriction homomorphism r_H^G: $R(G) \to R(H)$ and we let I_H^G be its kernel. (Subgroups are understood to be closed.) A set $\mathscr{F}$ of subgroups of G closed under subconjugacy is called a family. We let $(K_G^*)_{\mathscr{F}}^{\wedge}$, the $\mathscr{F}$-adic completion of K_G^*, denote the progroup valued G-cohomology theory specified by

$$K_G^n(X)_{\mathscr{F}}^{\wedge} = \{K_G^n(X_\alpha)/JK_G^n(X_\alpha)\},$$

where J runs over the finite products of ideals I_H^G with $H \in \mathscr{F}$. (The relevant information about progroups is summarized in [2, §2].)

THEOREM 1.1. *If a G-map $f: X \to Y$ restricts to a homotopy equivalence $f^H: X^H \to Y^H$ for each $H \in \mathscr{F}$, then $(f^*)_{\mathscr{F}}^{\wedge}$: $(K_G^* Y)_{\mathscr{F}}^{\wedge} \to (K_G^* X)_{\mathscr{F}}^{\wedge}$ is an isomorphism.*

The same assertion holds with K_G and $R(G)$ replaced by KO_G and $RO(G)$.

Theorem 1.1 was first conjectured in 1976 [8] and was first proven, independently, by two of us in 1983 [7, 9]. The case $\mathscr{F} = \{1\}$ is the Atiyah–Segal completion theorem of [4], and the proof in [7] follows [4] in outline. The proof in [9] contained the key idea of proceeding by direct induction rather than giving unitary groups and tori a privileged role. Our variant of this idea exploits an argument due to Carlsson [5] in cohomotopy to obtain an immediate reduction to quotation of Bott periodicity for the equivariant K-theory of G-spheres. It is to be emphasized that our argument, like that of [9], includes a new proof of the original Atiyah–Segal theorem.

We use (1.1) to compute equivariant K-theory characteristic classes in §2. We prove (1.1) in §3 and make a few remarks on it in §5. In §4, we use (1.1) to prove the following mixed localization and completion theorem. Its cohomotopy analog was the main result of our paper [2], and more discussion of such invariance theorems may be found there. Pro-$R(G)$-modules are localized termwise, $S^{-1}\{M_\alpha\} = \{S^{-1}M_\alpha\}$.

THEOREM 1.2. *Let $S \subset R(G)$ be a multiplicative set, let $I \subset R(G)$ be an ideal, and define*

$$\mathscr{H} = \cup\{\mathrm{Supp}(P) | P \cap S = \phi \text{ and } P \supset I\}.$$

If a G-map $f\colon X \to Y$ restricts to a homotopy equivalence $f^H\colon X^H \to Y^H$ for all $H \in \mathscr{H}$, then $S^{-1}(f^*)^{\wedge}_I\colon S^{-1}K^*_G(Y)^{\wedge}_I \to S^{-1}K^*_G(X)^{\wedge}_I$ is an isomorphism. The same assertion holds with K_G and $R(G)$ replaced by KO_G and $RO(G)$.

Here P runs over prime ideals of $R(G)$ and $\mathrm{Supp}(P)$ is the support of P as defined by Segal [13]: $H \in \mathrm{Supp}(P)$ if P comes from H via the restriction map $R(G) \to R(H)$ and P does not come from any $K \subset H$. Segal shows that $\mathrm{Supp}(P)$ is a single conjugacy class of (topologically) cyclic subgroups H. The theorem has content even when $S = \{1\}$ and $I = 0$.

COROLLARY 1.3. *If a G-map $f\colon X \to Y$ restricts to a homotopy equivalence $f^H\colon X^H \to Y^H$ for all cyclic subgroups H, then $f^*\colon K^*_G(Y) \to K^*_G(X)$ is an isomorphism.*

For finite groups G, this result goes back to [8].

§2. EQUIVARIANT K-THEORY OF CLASSIFYING SPACES

The main motivation for Theorem 1.1 comes from the following consequence (which is actually equivalent to the theorem).

Let $E\mathscr{F}$ be a universal $\mathscr{F}$-free G-space, so that $(E\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and is empty if $H \notin \mathscr{F}$. For any G-space X, the projection $E\mathscr{F} \times X \to X$ restricts to a homotopy equivalence $(E\mathscr{F} \times X)^H \to X^H$ for each $H \in \mathscr{F}$, so (1.1) gives an isomorphism $K^*_G(X)^{\wedge}_{\mathscr{F}} \to K^*_G(E\mathscr{F} \times X)^{\wedge}_{\mathscr{F}}$. For a G-space Y, such as $E\mathscr{F} \times X$, all of whose isotropy groups are in $\mathscr{F}$, the groups of the inverse system $K^*_G(Y)$ are $\mathscr{F}$-adically complete. For a finite G-CW complex X, the inverse system $K^*_G(X)^{\wedge}_{\mathscr{F}}$ satisfies the Mittag–Leffler condition. These facts imply that the algebraic completion $K^*_G(X)^{\wedge}_{\mathscr{F}}$ is isomorphic to the topological completion $K^*_G(E\mathscr{F} \times X)$.

COROLLARY 2.1. *If X is a finite G-CW complex, then the projection $E\mathscr{F} \times X \to X$ induces an isomorphism $K^*_G(X)^{\wedge}_{\mathscr{F}} \to K^*_G(E\mathscr{F} \times X)$.*

McClure has obtained interesting applications [12]. For example, he has shown that $K^*_G(X)$ is detected by the family of finite subgroups of G, so that a G-vector bundle is stably trivial if it is stably trivial when regarded as an H-vector bundle for each finite subgroup H of G.

With X a point, the original Atiyah–Segal completion theorem specializes to a calculation of the K-theory of classifying spaces in terms of completions of representation rings. There is an analogous specialization of (2.1) to the calculation of the K_G-theory of classifying G-spaces. To see this, let Π be a normal subgroup of a compact Lie group Γ with quotient group G. The orbit projection $q\colon Y \to Y/\Pi$ of a Π-free Γ-space is a kind of equivariant bundle, and there is a universal bundle $E(\Pi;\Gamma) \to B(\Pi;\Gamma)$ of this sort. Classically, $\Gamma = G \times \Pi$, and q is then called a principal (G, Π)-bundle. For example, a smooth G-n-plane bundle has an associated principal $(G, O(n))$-bundle. The universal Π-free Γ-space $E(\Pi;\Gamma)$ is just $E\mathscr{F}$, where $\mathscr{F} = \mathscr{F}(\Pi;\Gamma)$ is the family of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$, and we have the following calculation of $K_G(B(\Pi;\Gamma))$.

COROLLARY 2.2. *The projection $E(\Pi;\Gamma) \to \{pt\}$ induces an isomorphism*

$$R(\Gamma)^{\wedge}_{\mathscr{F}(\Pi;\Gamma)} \xrightarrow{\cong} K_\Gamma(E(\Pi;\Gamma)) \cong K_G(B(\Pi;\Gamma)).$$

Parenthetically, we insert the analogous specialization of [2, (1.6)]. The case $\Pi = 1$ is Segal's original version of the Segal conjecture.

COROLLARY 2.3. *Let* Γ *be finite. The projection* $E(\Pi;\Gamma)\to\{pt\}$ *induces an isomorphism*

$$A(\Gamma)\hat{}_{\mathscr{J}(\Pi;\Gamma)}\xrightarrow{\cong}\pi^0_\Gamma(E(\Pi;\Gamma))\cong\pi^0_G(B(\Pi;\Gamma)).$$

The right-hand change of groups isomorphisms in (2.2) and (2.3) are standard; see [14, 2.1] for K-theory, [1, 5.3] for cohomotopy, and [11, II§8] for general theories.

§3. PROOF OF THEOREM 1.1

Since $(K^*_G)\hat{}_{\mathscr{J}}$ is a progroup valued cohomology theory (as explained in [2]), exact sequences derived from cofibre sequences imply that (1.1) is equivalent to the following vanishing theorem.

THEOREM 3.1. $\tilde{K}^*_G(X)\hat{}_{\mathscr{J}}$ *is pro-zero for every based G-space such that* X^H *is contractible for each* $H\in\mathscr{J}$, *and similarly for* $(KO^*_G)\hat{}_{\mathscr{J}}$.

We deduce this from a special case. Let U be the sum of countably many copies of each of a countable set of non-trivial representations V_i of G such that each $V_i^G=0$ and some $V_i^H\neq 0$ if H is a proper subgroup of G. For K^*_G, we restrict attention to complex representations. For KO^*_G, we restrict attention to Spin representations with dimension divisible by eight. Since the arguments are otherwise identical, we concentrate on the complex case from now on. Let Y be the colimit of the one-point compactifications S^V of the finite dimensional subrepresentations V of U. Since each $V_i^G=0$, $Y^G=S^0$. If $V\subset W$ and $(W-V)^H\neq 0$, where $W-V$ is the complement of V in W, then the inclusion $S^V\to S^W$ is null H-homotopic. It follows that Y^H is contractible and Y is H-contractible for $H\neq G$.

LEMMA 3.2. *If* $\mathscr{J}$ *is proper* ($G\notin\mathscr{J}$), *then* $\tilde{K}^*_G(Y)\hat{}_{\mathscr{J}}$ *is pro-zero.*

To deduce (3.1) from (3.2), we need to know the behavior of $(\tilde{K}^*_G)\hat{}_{\mathscr{J}}$ with respect to restriction to subgroups. For $H\subset G$, let $\mathscr{J}|H$ be the family of subgroups of H which are in $\mathscr{J}$.

LEMMA 3.3. *For any based G-space X*,

$$\tilde{K}^*_G(G/H)_+\wedge X)\hat{}_{\mathscr{J}}\cong\tilde{K}^*_H(X)\hat{}_{\mathscr{J}|H}.$$

Proof of (3.1). Since $\mathscr{J}$ is a family, the equivariant Whitehead theorem shows that X is H-contractible and thus $\tilde{K}^*_H(X)$ is pro-zero for $H\in\mathscr{J}$. We must show that $\tilde{K}^*_G(X)\hat{}_{\mathscr{J}}$ is pro-zero. To avoid triviality, we assume that $G\notin\mathscr{J}$. Since the descending chain condition on subgroups allows induction, we may assume that $\tilde{K}^*_H(X)\hat{}_{\mathscr{J}|H}$ is pro-zero for all proper subgroups $H\subset G$. Since $Y^G=S^0$, we have a cofibre sequence

$$S^0\to Y\to Y/S^0.$$

Taking smash products with X, we obtain a cofibre sequence

$$X\to X\wedge Y\to X\wedge(Y/S^0).$$

It suffices to prove that $\tilde{K}^*_G(X\wedge Y)\hat{}_{\mathscr{J}}$ and $\tilde{K}^*_G(X\wedge(Y/S^0))\hat{}_{\mathscr{J}}$ are both pro-zero. We claim first that $\tilde{K}^*_G(W\wedge Y)\hat{}_{\mathscr{J}}$ is pro-zero for any G-CW complex W. Since the zero skeleton W^0 and the skeletal quotients W^n/W^{n-1} for $n>0$ are wedges of G-spaces of the form $(G/H)_+\wedge S^n$ and since we may as well assume that W is finite, we need only verify this for $W=(G/H)_+ \wedge S^n$ and thus, by suspension, for $W=(G/H)_+$. Here (3.2) gives the conclusion if $H=G$ and (3.3) and the H-contractibility of Y give the conclusion of $H \neq G$. We claim next that $K^*_G(X \wedge Z)^{\wedge}_{\mathscr{J}}$ is pro-zero for any G-CW complex Z, such as Y/S^0, such that Z^G is a point. Arguing as above, we need only verify this when $Z=(G/H)_+$ for a proper subgroup H, and here the conclusion holds by (3.3) and the induction hypothesis.

The argument just given is an adaptation of the preliminary steps in Carlsson's proof of the Segal conjecture [5].

Proof of (3.2). By Bott periodicity [3, 14], $\tilde{K}^*_G(S^V)$ is the free $\tilde{K}^*_G(S^0)$-module generated by the Bott class $\lambda_V \in \tilde{K}^0_G(S^V)$. Moreover, λ_V restricts to the Bott class in $\tilde{K}^0_H(S^V)$ for each $H \subset G$. The Euler class $\chi_V \in R(G)=\tilde{K}^0_G(S^0)$ is $e^*(\lambda_V)$, where e: $S^0 \to S^V$ is the evident inclusion. If $H \neq G$ and $V^H \neq 0$, then e is null H-homotopic and $X_V \in I^G_H$. If $V \subset W$, then the inclusion i: $S^V \to S^W$ is $1 \wedge e$, e: $S^0 \to S^{W-V}$. Since $\lambda_W = \lambda_{W-V}\lambda_V$, the homomorphism i^*: $\tilde{K}^*_G(S^W) \to \tilde{K}^*_G(S^V)$ is given by the formula

$$i^*(x\lambda_W) = x\chi_{W-V}\lambda_V,$$

for $x \in \tilde{K}^*_G(S^0)$; that is, i^* is multiplication by χ_{W-V}.

We may view $\tilde{K}^*_G(Y)^{\wedge}_{\mathscr{J}}$ as the inverse limit in the category of progroups of the $\tilde{K}^*_G(Y)/J\tilde{K}^*_G(Y)$, where J runs over the finite products of ideals I^G_H with $H \in \mathscr{J}$ (see [2, §2]). So it suffices to prove that $\tilde{K}^*_G(Y)/J\tilde{K}^*_G(Y)$ is pro-zero for each such J. This means that, for each V, there exists $W \supset V$ such that

$$i^*\colon \tilde{K}^*_G(S^W)/J\tilde{K}^*_G(S^W) \to \tilde{K}^*_G(S^V)/J\tilde{K}^*_G(S^V)$$

is zero. If $J=I^G_{H_1} \ldots I^G_{H_n}$ and we choose $W-V$ to be the sum of representations W_i such that $W_i^{H_i} \neq 0$, then i^* is zero since it is multiplication by $\chi_{W_1} \ldots \chi_{W_n} \in J$.

Since $\tilde{K}^*_G((G/H)_+ \wedge X) \cong \tilde{K}^*_H(X)$ as pro-$R(G)$-modules, where $R(G)$ acts on $\tilde{K}^*_H(X)$ through r^G_H: $R(G) \to R(H)$, the following algebraic fact implies (3.3).

LEMMA 3.4. *The $\mathscr{J}$-adic and $(\mathscr{J}|H)$-adic topologies coincide on $R(H)$.*

This follows from Segal's results on $R(G)$ [13, §3]. The key point is the following observation about supports of prime ideals, which can be derived from [13, 3.5 or 3.7].

LEMMA 3.5. *If $S \subset H$ is a support of a prime ideal $Q \subset R(H)$ and if $P=(r^G_H)^{-1}(Q) \subset R(G)$, then S is a support of P.*

Proof of (3.4). If $L \in \mathscr{J}|H$, then $r^G_H(I^G_L)\,R(H) \subset I^H_L$ since $r^H_L r^G_H = r^G_L$. Conversely, if $K \in \mathscr{J}$ and if $I=r^G_H(I^G_K)R(H)$, then I contains some product of ideals I^H_L with $L \in \mathscr{J}|H$. To see this, note that some product of prime ideals $Q \supset I$ is contained in I and that any prime ideal $Q \subset R(H)$ contains I^H_S, where $S \subset H$ is a support of Q. So it suffices to check that S is in $\mathscr{J}$ when Q contains I. If $P=(r^G_H)^{-1}(Q) \subset R(G)$, then S is a support of P and P contains I^G_K. Since $R(K)$ is finitely generated and thus integral over $R(G)/I^G_K$ [13, 3.2], $P=(r^G_K)^{-1}(P')$ for some prime ideal $P' \subset R(K)$. Therefore P has a support $S' \subset K$. Since any two supports of a given prime ideal are conjugate [13, 3.7] and S' is in $\mathscr{J}$, S is in $\mathscr{J}$.

Remark 3.6. The previous two lemmas remain valid for $RO(G)$. The essential points are that any prime ideal Q of $RO(G)$ is the restriction of a prime ideal P of $R(G)$ and that if P is also the restriction of $P' \neq P$, then P' is the complex conjugate of P.

§4. PROOF OF THEOREM 1.2

Again, (1.2) is equivalent to the following vanishing theorem.

THEOREM 4.1. *$S^{-1}\tilde{K}_G^*(X)_I^\wedge$ is pro-zero for every based G-space such that X^H is contractible for each $H \in \mathscr{H}$, and similarly for $S^{-1}(KO_G^*)_I^\wedge$.*

As a matter of algebra [2, 2.3], it suffices to prove that $S_P^{-1}\tilde{K}_G^*(X)_P^\wedge$ is pro-zero for each prime ideal $P \subset R(G)$ such that $P \cap S = \phi$ and $P \supset I$. Here S_P^{-1} means "localization at P"; that is, the multiplicative set S_P is the complement of P. Let $H \in \mathrm{Supp}(P)$ and let $\mathscr{J}$ be the family of subgroups of G subconjugate to H. By (3.1), $\tilde{K}_G^*(Y)_{\mathscr{J}}^\wedge$ is pro-zero if Y^K is contractible for all $K \in \mathscr{J}$. Since P contains I_H^G, it follows that $\tilde{K}_G^*(Y)_P^\wedge$ is pro-zero, and a fortiori $S_P^{-1}\tilde{K}_G^*(Y)_P^\wedge$ is pro-zero. For X as in (4.1), X^H is contractible but X^K need not be contractible for $K \subset H$. However, we can embed X as a subcomplex of a G-CW complex Y such that $Y^K = X^K$ for all K which contain a conjugate of H and Y^K is contractible for all other K. For example, we can take $Y = X \wedge \tilde{E}\mathscr{G}$, where $\mathscr{G}$ is the family of subgroups of G which do not contain a conjugate of H and $\tilde{E}\mathscr{G}$ is the unreduced suspension of $E\mathscr{G}$ with one of the cone points as basepoint; the inclusion of S^0 in $\tilde{E}\mathscr{G}$ induces the inclusion of X in Y. The classical localization theorem [14, 4.1] implies that $S_P^{-1}\tilde{K}_G^*(Y) \to S_P^{-1}\tilde{K}_G^*(X)$ is a pro-isomorphism; *a fortiori* $S_P^{-1}\tilde{K}_G^*(Y)_P^\wedge \to S_P^{-1}\tilde{K}_G^*(X)_P^\wedge$ is a pro-isomorphism and $S_P^{-1}\tilde{K}_G^*(X)_P^\wedge$ is pro-zero. In more detail, let $\{Y_\alpha\}$ run over the finite subcomplexes of Y and let $X_\alpha = X \cap Y_\alpha$. Then $S_P^{-1}\tilde{K}_G^*(Y_\alpha) \to S_P^{-1}\tilde{K}_G^*(X_\alpha)$ is an isomorphism for each α by induction up the finitely many cells of Y_α not in X_α since these cells are of orbit type G/K with $K \in \mathscr{G}$ and since $S_P^{-1}R(K) = 0$ for such K by [13.3.7].

Remark 4.2. Every collection $\mathscr{H}$ of cyclic subgroups of G can be realized, generally in several ways, as

$$\mathscr{H} = \cup\{\mathrm{Supp}(P) | P \cap S = \phi \text{ and } P \supset I\}$$

for some multiplicative set S and ideal I.

§5. REMARKS

We conclude with three unrelated comments.

Remark 5.1. Clearly (2.1) remains valid when X is finitely dominated but not necessarily finite. The extra generality is significant because locally linear compact topological G-manifolds are compact G-ENR's and are therefore finitely dominated, but they need not have the homotopy types of finite G-CW complexes (even stably). It is also possible to rework our proofs for general compact G-spaces such that each $K_G^n(X)$ is finitely $R(G)$-generated.

Remark 5.2. Let $\mathscr{J}$ be a family in G. As one of us observed [9], (3.4) and Segal's results in [13, p. 121] imply that $\mathscr{J}$ contains all topologically cyclic subgroups of G iff the $\mathscr{J}$-adic topology on $R(G)$ is complete. McClure [12] has proven that $\mathscr{J}$ contains all finite cyclic subgroups of G iff the $\mathscr{J}$-adic topology is Hausdorff on all finitely generated $R(G)$-modules. (This fact is the key to the proof of his result cited in §2.)

Remark 5.3. One might expect the Real K-theory case to work equally well. However, in the presence of a non-trivial involution on G, use of families other than $\{1\}$ leads to difficulty. A Real G-space is the same thing as a $\tilde{G}$-space, where $\tilde{G}$ is the semi-direct product of G and Z_2

determined by the involution, and $KR_{\tilde{G}}^*$ is a cohomology theory on $\tilde{G}$-spaces. For a general subgroup L of G, we do not have a good description of $KR_{\tilde{G}}^*(G/L)_+ \wedge X)$; if $L = \tilde{H}$ for a Real subgroup H of G, then this is $KR_{\tilde{H}}^*(X)$. This suggests that we should restrict attention to Real families in G, but some of our arguments require use of actual families in $\tilde{G}$.

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D.P.M.M.S., 16 *Mill Lane*,
*Cambridge CB*2 1*SB*, *England*.

Dept. of Mathematics, University of Washington,
Seattle, *WA* 98195, *U.S.A.*

Dept. of Mathematics, University of Warsaw,
Palac Kultury i Nauki IXp,
00–901 *Warsaw*, *Poland*.

Dept. of Mathematics, University of Chicago,
Chicago, *IL* 60637, *U.S.A.*