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MAXIMAL IDEALS IN THE BURNSIDE RING OF A COMPACT LIE GROUP

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ABSTRACT. A new criterion is found for deciding whether or not two maximal ideals in the Burnside ring of a compact Lie group coincide. One consequence is that certain algebraic and topological localizations in equivariant stable homotopy theory are naturally isomorphic.

Let G be a compact Lie group and H a (closed) subgroup. For a prime p, let $q(H,p)$ be the kernel of the homomorphism $A(G) \to Z_p$ obtained by sending the equivalence class of a finite G-CW complex X to the mod p reduction of the Euler characteristic $\chi(X^H)$. All maximal ideals of the Burnside ring $A(G)$ are of this general form. Our main goal is to obtain a satisfactory criterion for determining when $q(J,p) = q(H,p)$. The reader is referred to [**1**, 5.7 or **2**, V, §3] for background and earlier results. We shall obtain the following result as a consequence of our criterion.

THEOREM 1. *If $H \subset J \subset K$ and $q(H,p) = q(K,p)$, then $q(J,p) = q(K,p)$.*

This gives a positive answer to a question raised in [**2**, V.3.8]. As explained in detail in [**2**, V.8.9], the theorem has the following consequence in equivariant stable homotopy theory. Let $WH = NH/H$.

THEOREM 2. *Let $q = q(K,p)$, where $|WK|$ is finite and prime to p, and let*

$$\mathcal{E} = \{H \mid q(J,p) \neq q \text{ for all } J \subset H\},$$
$$\mathcal{F}' = \{H \mid (H) \leq (K)\} \quad \text{and} \quad \mathcal{F} = \mathcal{E} \cap \mathcal{F}'.$$

Then, for finite G-CW spectra X and general G-spectra Y, the algebraic localization $[X,Y]_q^G$ is naturally isomorphic to the topological localization $[X, E(\mathcal{F}',\mathcal{F}) \wedge Y]_{(p)}^G$.

Here $E(\mathcal{F}',\mathcal{F})$ is a based G-CW complex characterized up to G-equivalence by the requirement that $E(\mathcal{F}',\mathcal{F})^H$ be equivalent to S^0 if $H \in \mathcal{F}' - \mathcal{F}$ and be contractible otherwise (see e.g. [**2**, V, §7]). The importance of Theorem 2 is that $[X,Y]_{(p)}^G$ can be computed in terms of the various algebraic localizations $[X,Y]_q^G$ (by [**2**, V.5.5]). The relevance of Theorem 1 is that it implies

$$\mathcal{F}' - \mathcal{F} = \{H \mid q(H,p) = q(K,p)\}.$$

To see this, recall that, for any maximal ideal q of residual characteristic p, there is one and, up to conjugacy, only one K with $|WK|$ finite and prime to p such that

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$q = q(K,p)$; moreover, this K is maximal in $\{H|q(H,p) = q\}$ (see [**1**, 5.7.2 or **2**, V.3.1]). For a given H, we let H^p denote a subgroup of G such that $q(H,p) = q(H^p,p)$ and $|WH^p|$ is finite and prime to p.

If G is finite and H_p is the smallest normal subgroup of H such that H/H_p is a p-group, then $q(H,p) = q(J,p)$ if and only if H_p is conjugate to J_p. We wish to generalize this criterion to the compact Lie case. Say that a group is p-perfect if it admits no nontrivial quotient p-groups. Then H_p above can also be characterized as the maximal p-perfect subgroup of H. In the compact Lie case, we let H'_p denote the maximal p-perfect subgroup of H. It can be constructed explicitly as the inverse image in H of $(H/H_0)_p \subset H/H_0$, where H_0 is the component of the identity element. We then define $H_p \subset NH'_p$ to be the inverse image of a maximal torus in WH'_p. H_p is still p-perfect, but now WH_p is finite [**1**, 5.7.5 or **2**, V.3.3]. Note that $q(H,p) = q(H'_p,p) = q(H_p,p)$ since H/H'_p is a p-group and H_p/H'_p is a torus [**1**, 5.7.1 or **2**, V.3.6]. The following is our main result. We are indebted to the referee for its proper formulation.

THEOREM 3. *Let $q = q(K,p)$, where $|WK|$ is finite and prime to p, and let $L = K_p$.*

(i) *$L = K'_p$; that is, L is the maximal p-perfect subgroup of K; moreover, L is the unique normal p-perfect subgroup of K whose quotient is a finite p-group.*

(ii) *L is maximal in $\{H \mid q(H,p) = q$ and H is p-perfect$\}$; up to conjugacy, this property uniquely characterizes L.*

(iii) *If $H \subset G$, then $q(H,p) = q$ if and only if H_p is conjugate to L.*

(iv) *If $H \subset L$ is p-perfect and $q(H,p) = q$, then $HT = L$, where T is the component of the identity element of the center of L.*

Part (iii) can be restated as follows.

COROLLARY 4. *$q(H,p) = q(J,p)$ if and only if H_p is conjugate to J_p.*

Parts (i) and (iv) imply Theorem 1.

PROOF OF THEOREM 1. We are given $H \subset J \subset K$ and $q(H,p) = q(K,p)$. Expanding K if necessary, we may assume that $|WK|$ is finite and prime to p. We have $H'_p \subset J'_p \subset K'_p$ since passage to maximal p-perfect subgroups preserves inclusions. By (i), $K'_p = L$. By (iv), $H'_pT = L$. Therefore $J'_pT = L$. Since J'_pT/J'_p is a torus,

$$q(J,p) = q(J'_p,p) = q(J'_pT,p) = q(K,p).$$

The proof of Theorem 3 depends on the following observation.

PROPOSITION 5. *If H is p-perfect and WH is finite, then, up to conjugation, H is a normal subgroup of H^p with quotient a finite p-group.*

PROOF. The argument is identical to the proof of [**1**, 5.7.8 or **2**, V.3.6(iii)], which give the same conclusion under the stronger hypothesis that $|H/H_0| \not\equiv 0 \bmod p$ rather than that H is p-perfect.

PROOF OF THEOREM 3. (i) By the proposition, L is subconjugate to K. Since K/K'_p is finite and L/K'_p is a torus, this implies $L = K'_p$. By the maximality of K'_p, if $H \subset K$ is p-perfect, then $H \subset L$. If, further, $H \triangleleft K$ and K/H is a p-group, then $H = L$ since L is p-perfect.

(ii) Since K is maximal in $\{H \mid q(H,p) = q\}$ and is characterized up to conjugacy by this property, (ii) follows immediately from (i).

(iii) Since $q(H,p) = q(H_p,p)$, sufficiency is clear. Applying the proposition to H_p and then applying part (i) to H^p, we see that H_p is conjugate to $(H^p)_p$. This implies necessity.

(iv) Since H is p-perfect and $H \lhd HT$ with quotient a torus, the construction of H_p shows that we may assume $HT \subset H_p$. Since H_p/H is a torus, $HT \lhd H_p$ and H_p/HT is a torus. As a compact subgroup of the discrete group of automorphisms modulo inner automorphisms of HT, H_p/SHT is finite and thus trivial, where S is the centralizer of HT in H_p. Clearly S contains T and is the center of H_p. Since $q(H,p) = q$, H_p is conjugate to L by part (iii). Therefore the identity component of S is isomorphic to, and thus equal to, T. It follows that $H_p = HT \subset L$ and hence $H_p = HT = L$.

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