Weak equivalences and quasifibrations *

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Quasifibrations are essentially fibrations up to weak homotopy. They play a fundamental role in homotopy theory since a variety of important constructions give rise to quasifibrations which fail to be fibrations. Quasifibrations were introduced in a basic 1958 paper by Dold and Thom [2], and some refinements of their work were added by Hardie in 1970 [4]. The importance of quasifibrations to the study of classifying spaces and fibrations was first established in a 1959 paper of Dold and Lashof [1], and a systematic account was given in [5]. Quasifibrations played an essential role in Quillen's 1973 paper [6] in which he introduced the higher algebraic K-groups of rings. They have been applied in quite a large number of more recent papers.

Despite their importance, quasifibrations have not been treated in any textbook, and I know of no better published reference than the original paper (in German) of Dold and Thom. Around 1972, I proved a new theorem about weak homotopy equivalences of pairs of spaces and observed that the basic facts about quasifibrations are very easy consequences of that result. I've never published this material, which was intended as part of a still projected volume on the homotopical foundations of algebraic topology. In view of its close connection to the theme of the Montreal conference, I thought that I would seize the occasion to give an exposition.

We give some preliminaries and state our theorem about weak equivalences in section 1. We explain the application to the theory of quasifibrations in section 2. We prove the theorem about weak equivalences in section 3.

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* This paper is in final form and no version of it will be submitted for publication elsewhere.

§1. Weak equivalences of pairs

A map $f: X \to Y$ of spaces is said to be an n-equivalence if, for all $x \in X$, $f_*: \pi_q(X,x) \to \pi_q(Y,f(x))$ is a bijection for $0 \leq q < n$ and a surjection for $q = n$. A map $f: (X,A) \to (Y,B)$ of pairs of spaces is said to be an n-equivalence if $(f_*)^{-1}\mathrm{Im}(\pi_0 B \to \pi_0 Y) = \mathrm{Im}(\pi_0 A \to \pi_0 X)$ and, for all $a \in A$, $f_*: \pi_q(X,A,a) \to \pi_q(Y,B,f(a))$ is a bijection for $1 \leq q < n$ and a surjection for $q = n$. The condition on components means that if $f(x)$ can be connected to a point of B, then x can be connected to a point of A; it is automatically satisfied when X and Y are path connected. In both the absolute and the relative cases, f is said to be a weak equivalence if it is an n-equivalence for all n.

By the evident long exact sequences and the five lemma, plus some tedious extra details to handle fundamental groups, we have the following relationship between weak equivalences of pairs and of their constituent spaces.

Lemma 1.1. Let $f: (X,A) \to (Y,B)$ be a map of pairs such that both $f_*: \pi_0 A \to \pi_0 B$ and $f_*: \pi_0 X \to \pi_0 Y$ are bijections. If any two of the three maps $f: A \to B$, $f: X \to Y$, and $f: (X,A) \to (Y,B)$ are weak equivalences, then so is the third.

Our new theorem on weak equivalences of pairs is a kind of analog in the context of excisive triads. Recall that a triad $(X; A, B)$ is said to be excisive if X is the union of the interiors of A and B.

Theorem 1.2. Let $f: (X; X_1, X_2) \to (Y; Y_1, Y_2)$ be a map of excisive triads such that $f: (X_i, X_1 \cap X_2) \to (Y_i, Y_1 \cap Y_2)$ is an n-equivalence for $i = 1$ and $i = 2$. Then $f: (X, X_i) \to (Y, Y_i)$ is an n-equivalence for $i = 1$ and $i = 2$.

No useful conclusion could be derived with an assumption on only one of the pairs $(X_i, X_1 \cap X_2)$. While this result really does seem to be new, the following immediate consequence of the lemma and theorem is folklore; a proof appears in Gray [3, 16.24].

Corollary 1.3. Let $f: (X; X_1, X_2) \to (Y; Y_1, Y_2)$ be a map of excisive triads such that $f: X_1 \cap X_2 \to Y_1 \cap Y_2$, $f: X_1 \to Y_1$, and $f: X_2 \to Y_2$ are weak equivalences. Then $f: X \to Y$ is a weak equivalence.

In turn, this implies a local criterion for a map to be a weak equivalence.

Corollary 1.4. Let $f\colon X \to Y$ be a map and let $\mathcal{O}$ be an open cover of Y which is closed under finite intersections. If $f\colon f^{-1}U \to U$ is a weak equivalence for all $U \in \mathcal{O}$, then $f\colon X \to Y$ is a weak equivalence.

Proof. Let $\mathcal{C}$ be the collection of subspaces V of Y such that V is a union of spaces in $\mathcal{O}$, $f\colon f^{-1}V \to V$ is a weak equivalence, and $f\colon f^{-1}(U \cap V) \to U \cap V$ is a weak equivalence for all $U \in \mathcal{O}$. Order $\mathcal{C}$ by inclusion. The union of a chain in $\mathcal{C}$ is in $\mathcal{C}$ by an obvious colimit argument, and $\mathcal{C}$ is nonempty since it contains $\mathcal{O}$. Thus $\mathcal{C}$ has a maximal element V. Suppose $V \neq Y$. Then there is a $U \in \mathcal{O}$ which is not contained in V. The previous corollary implies that $U \cup V$ is in $\mathcal{C}$, contradicting the maximality of V.

§2. Quasifibrations

If $p\colon E \to B$ is a fibration, then $p\colon (E, p^{-1}A) \to (B, A)$ is a weak equivalence for all nonempty subspaces A of B; in particular, $p\colon (E, p^{-1}b) \to (B, b)$ is a weak equivalence for all $b \in B$ (e.g. [9, p.187]). The notion of a quasifibration turns this desirable property into a definition.

Definition 2.1. A surjective map $p\colon E \to B$ is a quasifibration if $p\colon (E, p^{-1}b) \to (B, b)$ is a weak equivalence for all $b \in B$.

It is to be emphasized that this notion does not properly belong to fibration theory since the pullback of a quasifibration need not be a quasifibration.

Assume given a fixed surjective map $p\colon E \to B$. We shall derive various criteria for p to be a quasifibration.

Clearly $p\colon E \to B$ is a quasifibration if and only if its restriction $p^{-1}C \to C$ is a quasifibration for each path component C of B. Thus we may as well restrict attention to path connected base spaces B.
Of course, if p is a quasifibration, then, for $b \in B$ and $x \in p^{-1}b$, the exact sequence of homotopy groups of the pair $(E, p^{-1}b)$ yields an exact sequence

$$\cdots \to \pi_{n+1}(B,b) \to \pi_n(p^{-1}b,x) \to \pi_n(E,x) \to \pi_n(B,b) \to \cdots \to \pi_0(B,b).$$

Let $Np = \{(x,\omega) \mid \omega: I \to B, \omega(1) = p(x)\} \subset E \times B^I$ and let $q: Np \to B$ be the fibration specified by $q(x,\omega) = \omega(0)$; thus $q^{-1}b$ is the usual homotopy theoretic fiber of p over b. If $\lambda: E \to Np$ is the natural equivalence, $\lambda(x) = (x, c_{p(x)})$, then $q \circ \lambda = p$ and λ restricts to a map $p^{-1}b \to q^{-1}b$ for each $b \in B$. Clearly p is a quasifibration if and only if $\lambda: (E, p^{-1}b) \to (Np, q^{-1}b)$ is a weak equivalence for all $b \in B$. By Lemma 1.1, this holds if and only $\lambda: p^{-1}b \to q^{-1}b$ is a weak equivalence for all $b \in B$. With B connected, the fibers $q^{-1}b$ all have the same homotopy type, hence the fibers $p^{-1}b$ all have the same weak homotopy type if p is a quasifibration.

Say that a subspace A of B is distinguished if the restriction $p: p^{-1}A \to A$ is a quasifibration. Since $p: (E, p^{-1}A, p^{-1}a) \to (B, A, a)$ induces a map of long exact sequences of homotopy groups of triples, the five lemma and some tedious verifications on the π_1 level give the following observation.

Lemma 2.2. Let A be a distinguished subspace of B. Then the maps $p: (E, p^{-1}a) \to (B, a)$ are weak equivalences for all $a \in A$ if and only if the map $p: (E, p^{-1}A) \to (B, A)$ is a weak equivalence.

The following analog of Corollary 1.3, which is the heart of the Dold-Thom theory of quasifibrations, is now a direct consequence of Theorem 1.2. This observation is perhaps the main point of our work.

Corollary 2.3. Let $(B; B_1, B_2)$ be an excisive triad. If $B_1 \cap B_2$, B_1, and B_2 are distinguished, then $p: E \to B$ is a quasifibration.

Proof. With (B, A) replaced by $(B_i, B_1 \cap B_2)$, Lemma 2.2 gives that

$$p: (p^{-1}B_i, p^{-1}B_1 \cap p^{-1}B_2) \to (B_i, B_1 \cap B_2)$$

is a weak equivalence for $i = 1$ and $i = 2$. By Theorem 1.2,

$$p: (E, p^{-1}B_i) \to (B, B_i)$$

is a weak equivalence for $i = 1$ and $i = 2$. By Lemma 2.2 applied with $A = B_i$, $p: (E, p^{-1}b) \to (B, b)$ is a weak equivalence for all $b \in B_i$, $i = 1$ and $i = 2$, and thus for all $b \in B$.

The proof of Corollary 1.4 applies to give the quasifibration analog of that result.

Corollary 2.4. Let $\mathcal{O}$ be an open cover of B which is closed under finite intersections. If each $U \in \mathcal{O}$ is distinguished, then $p: E \to B$ is a quasifibration.

These results are usually used in conjunction with the following observation. Recall that a homotopy $h_t: B \to B$ is a deformation of B onto A if $h_0 = \mathrm{Id}$, $h_t(a) = a$ for $a \in A$, and $h_1(B) \subset A$.

Lemma 2.5. Let A be a distinguished subspace of B. Suppose there exist deformations h of B onto A and H of E onto $p^{-1}A$ such that $p \circ H_1 = h_1 \circ p$ and $H_1: p^{-1}b \to p^{-1}h_1(b)$ is a weak equivalence for all $b \in B$. Then $p: E \to B$ is a quasifibration.

Proof. By Lemma 1.1, $H_1: (E, p^{-1}b) \to (p^{-1}A, p^{-1}h_1(b))$ is a weak equivalence for all $b \in B$. Passage to homotopy groups from the commutative diagram

$$\begin{array}{ccc} (E, p^{-1}b) & \xrightarrow{H_1} & (p^{-1}A, p^{-1}h_1(b)) \\ p\downarrow & & \downarrow p \\ (B, b) & \xrightarrow[h_1]{} & (A, h_1(b)) \end{array}$$

gives the conclusion.

Say that B is filtered if it is given as the union of an increasing sequence of subspaces F_nB such that each inclusion $F_nB \to F_{n+1}B$ is a cofibration. By an evident colimit argument, a map $p: E \to B$ is a quasifibration if each F_nB is distinguished. The following immediate inductive consequence of Corollary 2.3 and Lemma 2.5 is probably the most generally useful criterion for p to be a quasifibration.

Theorem 2.6. Let $p: E \to B$ be a map onto a filtered space B and suppose that the following conditions hold.

(i) F_0B and each open subset of each $F_nB - F_{n-1}B$ are distinguished.

(ii) For each $n \geq 1$, there is an open neighborhood U_n of $F_{n-1}B$ in F_nB and there are deformations h of U_n onto $F_{n-1}B$ and H of $p^{-1}U_n$ onto $p^{-1}F_{n-1}B$ such that $p \circ H_1 = h_1 \circ p$ and $H_1: p^{-1}b \to p^{-1}h_1(b)$ is a weak equivalence for each $b \in U_n$.

Then each F_nB is distinguished and $p: E \to B$ is a quasifibration.

There is an alternative criterion that often applies when E and B are built up from successive compatible pushout diagrams.

Theorem 2.7. Let $p: E \to B$ be a map of filtered spaces such that $F_nE = p^{-1}F_nB$ for $n \geq 0$ and, for $n \geq 1$, $p: F_nE \to F_nB$ is obtained by passage to pushouts from a commutative diagram of the form

$$\begin{array}{ccccc} F_{n-1}E & \xleftarrow{g_n} & D_n & \xrightarrow{j_n} & E_n \\ p\downarrow & & \downarrow q_n & & \downarrow p_n \\ F_{n-1}B & \xleftarrow[f_n]{} & A_n & \xrightarrow[i_n]{} & B_n. \end{array}$$

Suppose that the following conditions hold.

(i) F_0B is distinguished.

(ii) Each map $p_n: E_n \to B_n$ is a fibration.

(iii) Each map $i_n: A_n \to B_n$ is a cofibration.

(iv) Each right square is a pullback.

(v) $g_n: (q_n)^{-1}(a) \to p^{-1}f_n(a)$ is a weak equivalence for all $a \in A_n$.

Then each F_nB is distinguished and $p: E \to B$ is a quasifibration.

The inductive step here is a consequence of the second of the following two lemmas, which are due to Hardie [4]. Both refer to a commutative diagram

$$(*) \qquad \begin{array}{ccccc} E & \xleftarrow{g} & D & \xrightarrow{j} & E' \\ p\downarrow & & \downarrow q & & \downarrow p' \\ B & \xleftarrow[f]{} & A & \xrightarrow[i]{} & B'. \end{array}$$

Lemma 2.8. If, in (*), p, q, and p' are quasifibrations and the maps $g: q^{-1}(a) \to p^{-1}f(a)$ and $j: q^{-1}(a) \to (p')^{-1}i(a)$ are weak equivalences for all $a \in A$, then the induced map $s: M(j,g) \to M(i,f)$ of double mapping cyclinders is a quasifibration.

Proof. $M(i,f) = B \cup_f (A\times I) \cup_i B'$ is the union of $B \cup_f (A\times[0,2/3])$ and $(A\times[1/3,1]) \cup_i B'$, and similarly for $M(j,g)$. The conclusion follows easily from Lemma 2.5 and Corollary 2.3.

Lemma 2.9. Suppose that (*) satisfies the following conditions.

(i) p is a quasifibration.

(ii) p' is a fibration.

(iii) i is a cofibration.

(iv) The right square is a pullback.

(v) $g: q^{-1}(a) \to p^{-1}f(a)$ is a weak equivalence for all $a \in A$.

Then the map $r: E \cup_g E' \to B \cup_f B'$ induced by p and p' is a quasifibration.

Proof. We have the commutative diagram

$$\begin{array}{ccc} M(j,g) & \xrightarrow{\beta} & E \cup_g E' \\ s\downarrow & & \downarrow r \\ M(i,f) & \xrightarrow[\alpha]{} & B \cup_f B'. \end{array}$$

Since i and j are cofibrations (the latter by [9, I.7.14]), the quotient maps α and β are homotopy equivalences by a standard result on pushouts of equivalences. The map s is a quasifibration by the previous lemma. By Lemma 1.1 and a chase of the diagram, it suffices to show that $\beta: s^{-1}(x) \to r^{-1}(x)$ is a weak equivalence for each $x \in M(i,f)$. If $x \in B$ or $x \in B' - i(A)$, β is a homeomorphism. If $x = (a,s)$, where $a \in A$ and $0 < s \leq 1$, then it is easy to see that β can be identified with the weak equivalence $g: q^{-1}(a) \to p^{-1}f(a)$.

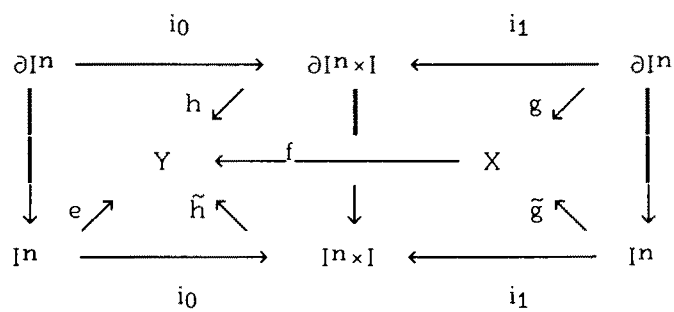
§3. The proof of Theorem 1.2

We begin with an analysis of the notion of an n-equivalence. In the absolute case, we have the following result. We omit the proof since a generalized version of the based analog is given in [6, Lemma 1] and we shall shortly be proving the more difficult relative analog.

Lemma 3.1. For each $n \geq 1$, the following statements about a map $f: X \to Y$ are equivalent.

(i) For each $x \in X$, $f_*: \pi_q(X,x) \to \pi_q(Y,fx)$ is an injection for $q = n-1$ and a surjection for $q = n$.

(ii) If $h: e \simeq fg$ on ∂I^n in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.

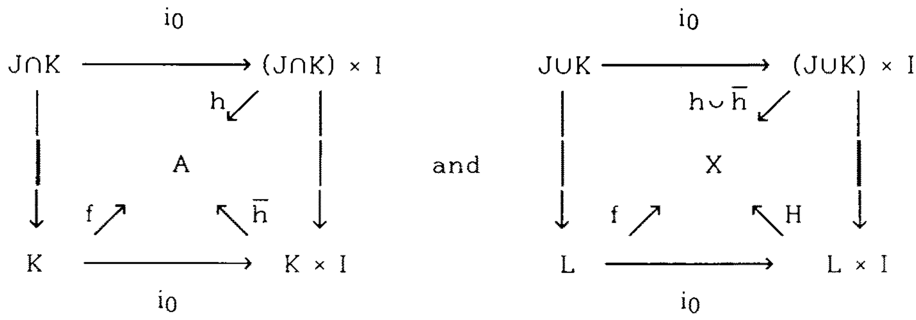


(iii) The conclusion of (ii) holds when $e = fg$ on ∂I^n and h is the constant homotopy.

In order to prove the relative analog, we will need the following relative homotopy extension property.

Limma 3.2 (relative HEP). Let $(L; J, K)$ be a triad such that the inclusions $J \cap K \to K$ and $J \cup K \to L$ are cofibrations. Then any homotopy $h: (J, J \cap K) \times I \to (X,A)$ of the restriction of a map $f: (L,K) \to (X,A)$ extends to a homotopy $H: (L,K) \times I \to (X,A)$ of f.

Proof. This holds by two applications of the usual HEP:



Before proceeding, we must fix some notations. Let

$$J^0 = \{0\} \text{ and } J^n = (\partial I^n \times I) \cup (I^n \times \{0\}) \subset I^{n+1} \text{ for } n \geq 1.$$

For a pair (X,A) with basepoint $a \in A$, we take

$$\pi_n(X, A, a) = [(I^n, \partial I^n, J^{n-1}), (X, A, a)] \text{ for } n \geq 1.$$

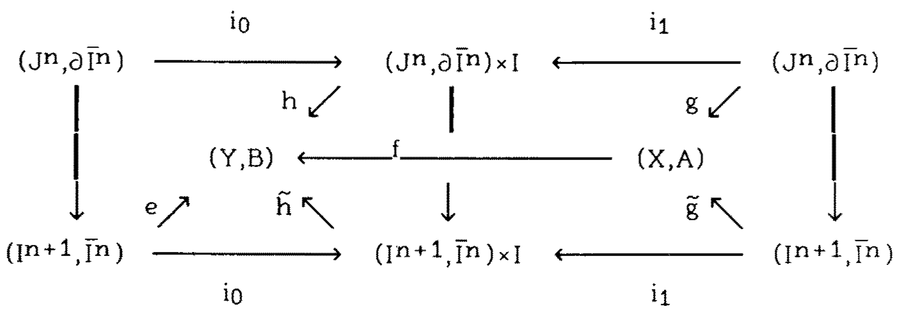
Let $\bar{I}^n = I^n \times \{1\}$ and $\partial \bar{I}^n = \partial I^n \times \{1\} = J^n \cap \bar{I}^n$. Define the negative of a homotopy h to be h traversed from 1 to 0 and define the sum $h_1 + \cdots + h_j$ of homotopies $h_i: f_{i-1} \simeq f_i$ to be the homotopy obtained by traversing h_i on the interval $[(i-1)/j, i/j]$.

The following lemma and its proof are due to Sugawara [8].

Lemma 3.3. For each $n \geq 0$, the following statements about a map $f: (X,A) \to (Y,B)$ are equivalent.

(i) For each $a \in A$, $f_*: \pi_q(X, A, a) \to \pi_q(Y, B, fa)$ is an injection for $q = n$ and a surjection for $q = n+1$. (When $n = 0$, replace the injectivity statement by $(f_*)^{-1}\mathrm{Im}(\pi_0 B \to \pi_0 Y) = \mathrm{Im}(\pi_0 A \to \pi_0 X)$.)

(ii) If $h: e \simeq fg$ on J^n in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.



(iii) The conclusion of (ii) holds when $e = fg$ on J^n and h is the constant homotopy.

<u>Proof</u>. We shall leave to the reader the minor modifications of proofs needed when $n = 0$. Of course, (ii) implies (iii) trivially, and (iii) implies (i) by appropriate specializations. A direct proof that (i) implies (ii) is possible, but it is simpler to prove that (iii) implies (ii) and (i) implies (iii).

<u>(iii) implies (ii)</u>. Assume given $h: e \simeq fg$ on J^n in the diagram of (ii). By application of relative HEP to the triad $(I^{n+1}; J^n, \bar{I}^n)$, there is a homotopy $j: (I^{n+1},\bar{I}^n)\times I \to (Y,B)$ of e which extends h. Since $j_1 = fg$ on J^n, (iii) gives a map $\tilde{g}: (I^{n+1},\bar{I}^n) \to (X,A)$ such that $\tilde{g} = g$ on J^n and a homotopy $k: j_1 \simeq f\tilde{g}$ such that k extends the constant homotopy h' at fg on J^n. Choose a homotopy $L: (J^n \times I, \partial\bar{I}^n \times I)\times I \to (Y,B)$ from $h+h'$ to h which is constant at fg on both $J^n\times\{0\}$ and $J^n\times\{1\}$. By application of relative HEP to the triad $(I^{n+2}; J^n\times I \cup I^{n+1}\times\partial I, I^n\times I)$, there is a homotopy $\tilde{L}: (I^{n+2}, \bar{I}^n\times I)\times I \to (Y,B)$ of $j+k$ which extends the union of L and the constant homotopies at e and f on $I^n\times\{0\}$ and $I^n\times\{1\}$. Let $\tilde{h} = \tilde{L}_1: e \simeq f\tilde{g}$. Then $\tilde{h}$ extends h, as required.

<u>(i) implies (iii)</u>. Assume that $e = fg$ on J^n and that h is the constant homotopy in the diagram of (ii). Let $* = (0, \cdots, 0, 1) \in I^{n+1}$ and let $a = g(*)$ and $b = f(a)$. Since $(J^n, \partial\bar{I}^n, *)$ is equivalent to $(I^n, \partial I^n, J^{n-1})$, $g: (J^n, \partial\bar{I}^n, *) \to (X, A, a)$ may be regarded as representing an element of $\pi_n(X, A, a)$. Since e is defined on I^{n+1} with $e(\bar{I}^n) \subset B$, fg represents the trivial element of $\pi_n(Y, B, b)$. Since f is injective on π_n, there is a homotopy $j: (J^n, \partial\bar{I}^n, *)\times I \to (X, A, a)$ from g to the trivial map $\bar{a}$ at a. Relative HEP gives a homotopy $K: (I^{n+1},\bar{I}^n)\times I \to (Y,B)$ of e which extends fj. Since $fj_1 = \bar{b}$, $K_1: (I^{n+1}, \partial I^{n+1}, J^n) \to (Y, B, b)$ represents an element of $\pi_{n+1}(Y, B, b)$. Since f is surjective on π_{n+1}, there is a map $J_1: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, a)$ and a homotopy $L: K_1 \simeq fJ_1$ of maps of triples. Another application of relative HEP (with unit interval reversed) gives a homotopy $J: (I^{n+1},\bar{I}^n)\times I \to (X,A)$ which ends at J_1 and extends j. Let $\tilde{g} = J_0$.

Certainly $\tilde{g}$ extends $j_0 = g$, and we have the homotopy $K+L-fJ$: $fg \simeq fg$ on $J^n \times I$. Choose any homotopy $M: (J^n \times I, \partial \bar{I}^n \times I) \times I \to (Y,B)$ from $fj+\bar{b}-fj$ to the constant homotopy at fg such that M is constant at fg on both $J^n \times \{0\}$ and $J^n \times \{1\}$. Relative HEP gives a homotopy $\tilde{M}: (I^{n+2}, \bar{I}^n \times I) \times I \to (Y,B)$ of $K+L-fJ$ which extends the union of M and the constant homotopies at e and $f\tilde{g}$ on $I^{n+1} \times \{0\}$ and $I^{n+1} \times \{1\}$. Let $\tilde{h} = \tilde{M}_1: e \simeq f\tilde{g}$; $\tilde{h}$ is constant at fg on J^n, as required.

Proof of Theorem 1.2. Replacing X by the mapping cylinder of f with its evident induced decomposition as an excisive triad, we may assume without loss of generality that f is an inclusion. Suppose given maps $g: (J^q, \partial \bar{I}^q) \to (X, X_i)$ and $e: (I^{q+1}, \bar{I}^q) \to (Y, Y_i)$ such that $fg = e$ on J^q, where $0 \le q \le n$ and $i = 1$ or $i = 2$. By the previous lemma, it suffices to construct an extension $\tilde{g}: (I^{q+1}, \bar{I}^q) \to (X, X_i)$ of g and a homotopy $\tilde{h}: e \simeq f\tilde{g}$ of maps $(I^{q+1}, \bar{I}^q) \to (Y, Y_i)$ such that $\tilde{h}$ restricts on J^q to the constant homotopy h at fg. Cubically subdivide I^{q+1} so finely that the image under e of each closed subcube lies entirely in the interior of Y_j, $j = 1$ or $j = 2$. Since f is an inclusion, the image under g of the intersection of each subcube with J^q lies entirely in the interior of X_j for the same j. Regard I^{q+1} as $I^q \times I$. The subdivision of I^{q+1} gives a cubical subdivision of I^q and a partition of I into subintervals $I_r = [v_{r-1}, v_r]$, where $0 = v_0 < v_1 < \cdots < v_s = 1$. We shall construct $\tilde{g}$ and $\tilde{h}_t$ on the spaces $K \times I_r$, where K runs through the cubical cells of I^q and $1 \le r \le s$, proceeding by induction on r and, for fixed r, by induction on the dimension of K. We shall so arrange things that

(a) $\tilde{g}(K \times I_r) \subset X_j$ and $\tilde{h}_t(K \times I_r) \subset Y_j$ if $e(K \times I_r) \subset \mathrm{Int}(Y_j)$;
(b) $\tilde{g}(K \times \{v_r\}) \subset X_1 \cap X_2$ and $\tilde{h}_t(K \times \{v_r\}) \subset Y_1 \cap Y_2$ if $e(K \times \{v_r\}) \subset Y_1 \cap Y_2$.

Since $e(\bar{I}^q) \subset X_i$, (a) and the case $r = s$ of (b) ensure that $\tilde{g}(\bar{I}^q) \subset X_i$ and $\tilde{h}_t(\bar{I}^q) \subset Y_i$. At each stage, the given maps g and $h_t = fg$ on J^q and the induction hypothesis specify maps $\tilde{g}$ and $\tilde{h}_t$ on $\partial K \times I_r \cup K \times \{v_{r-1}\}$, where ∂K is empty if K is a vertex. If either $e(K \times \{v_r\})$ is not contained in $Y_1 \cap Y_2$ or $e(K \times I_r)$ is contained in the intersection of the interiors of Y_1 and Y_2, we simply choose a representation (d,u) of $(K \times I_r, \partial K \times I_r \cup K \times \{v_{r-1}\})$ as a DR-pair and specify $\tilde{g}$ and $\tilde{h}_t$ on $K \times I_r$ by

$$\tilde{g}(x) = \tilde{g}d_1(x) \quad \text{and} \quad \tilde{h}_t(x) = \begin{cases} ed_{2t}(x) & \text{if } 0 \le t \le 1/2 \\ \tilde{h}_{2t-1}d_1(x) & \text{if } 1/2 \le t \le 1. \end{cases}$$

If $e(K\times\{v_r\})$ is contained in $Y_1 \cap Y_2$ and $e(K\times I_r)$ is contained in the interior of just one of the Y_j, the induction hypothesis gives

$$\tilde{g}: (\partial K\times I_r \cup I_r\times\{v_{r-1}\}, \partial K\times\{v_r\}) \to (X_j, X_1 \cap X_2)$$

and a homotopy $\tilde{h}: e \simeq f\tilde{g}$ of maps

$$(\partial K\times I_r \cup I_r\times\{v_{r-1}\}, \partial K\times\{v_r\}) \to (Y_j, Y_1 \cap Y_2).$$

Application of (ii) of Lemma 3.3 to the n-equivalence

$$f: (X_j, X_1 \cap X_2) \to (Y_j, Y_1 \cap Y_2)$$

gives the required extensions of $\tilde{g}$ and $\tilde{h}_t$ to $K\times I_r$.

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