The Work of J. F. Adams[1]

I first met Frank here in Manchester in 1964, when this building was being planned. I remember from the first feeling that he was a far more impressive man than the anecdotes of his exploits had led me to expect, and a far nicer one. I also felt humbled by the sheer amount of mathematics that he knew and perhaps more so by the amount that he somehow assumed I knew. I feel a little the same way now, faced with this audience and this topic. Still, I don't want to spend much time in reminiscence.[2] I want rather to give a quick guided tour through Frank's work, largely letting it speak for itself.

I should say that Frank's collected works are to be published in the near future by the Cambridge University Press. Like this talk, the collected works are organized by subject matter rather than by strict chronology. However, I will begin not quite at the beginning of his work with a sequence of four papers submitted between 1955 and 1958. All dates cited are dates of submission, not necessarily of appearance.

A. The cobar construction, the Adams spectral sequence, higher order cohomology operations, and the Hopf invariant one problem

1. *On the chain algebra of a loop space* (1955, with Peter Hilton) **[5]**[3]
2. *On the cobar construction* (1956) **[6]**

Let K be a CW-complex with trivial 1-skeleton. In the first paper, a DGA-algebra $A(K)$ is constructed whose homology is the Pontryagin algebra $H_*(\Omega K)$; as an algebra, $A(K)$ is free on generators in bijective correspondence with the cells of K (other than the vertex). As Kathryn Hess explained in her talk a few hours ago, this Adams-Hilton model is small enough to be of concrete value for computations and is still being used and studied today. In the second paper, a larger, but functorial, DGA is given whose homology is $H_*(\Omega K)$, namely the cobar construction $F(C_*(K))$. This construction was discussed in John McCleary's talk on Hochschild homology. Nowadays, an obvious and trivial next step after the introduction of

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[1] Reconstruction and expansion of the talk given at the conference, most of which was not written out beforehand

[2] A more personal tribute has been published in The Mathematical Intelligencer, Vol. 12, No. 1, 1990, 40–48.

[3] Details of publication of Adams' works discussed here can be found in the complete bibliography which follows this paper.

the cobar construction would be to filter it and so arrive at what is called the Eilenberg-Moore spectral sequence for the computation of $H_*(\Omega K)$. In fact, Moore and Adams were already in contact before this paper was written, and it was cited by Eilenberg and Moore as an important precursor to their work.

3. *On the structure and applications of the Steenrod algebra* (1957) [9]

Adams viewed this paper as a step towards the solution of the Hopf invariant one problem. The main theorem states that if $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ both contain elements of Hopf invariant one, then $n \leq 4$. It is now chiefly celebrated for the introduction of the Adams spectral sequence converging from $\mathrm{Ext}_A^{s,t}(H^*(X), Z_p)$ to ${}_p\pi_*^s(X)$. Products are defined in the spectral sequence when $X = S^0$, and the sub-Hopf algebras A_r are used to compute products of the elements h_i inductively, where h_i corresponds to Sq^{2^i}. The basic argument runs as follows. Let $n = 2^m$, $m \geq 3$. Assuming that h_m is a permanent cycle, $h_0(h_m)^2$ would survive to E_∞ if $d_2 h_{m+1} = 0$. This would contradict the fact that, in $\pi_*^s(S^0)$, $2x^2 = 0$ if $\deg(x)$ is odd. This seems straightforward enough today, but it was revolutionary at the time. The idea of reducing such a fundamental topological problem as Hopf invariant one to the non-triviality of a particular differential in a spectral sequence was quite new and unexpected.

Adams was curiously modest about the Adams spectral sequence. He always referred to it as a formalization of the Cartan-Serre method of killing homotopy groups. I think we all see it as something very much more than that. Its introduction was a watershed, and it substantially raised the level of algebraic sophistication of our subject.

4. *On the non-existence of elements of Hopf invariant one* (1958) [14]

If $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one, then $n = 1, 2, 4,$ or 8. The proof is based on showing that Sq^{2^m} decomposes in terms of secondary cohomology operations if $m > 3$. The paper contains definitive homological algebra for the study of Ext_A, including minimal resolutions and the cobar construction with its $\smile$ and $\smile_1$ products. It uses Milnor's description of A^* to redo the calculations in the previous paper. It gives a detailed study of stable secondary cohomology operations via universal examples, which are generalized two-stage Postnikov systems. The results include axioms for the operations, existence and uniqueness theorems, the relationship between the operations and Tor_A^2, and a Cartan formula. Particular operations are studied via homological algebra, and a key computation in CP^∞ is used to start the induction which shows that the undetermined constants in the decompositions of the Sq^{2^m} are non-zero.

Adams was a problem solver. He introduced exactly the tools he needed to solve the problems he studied, and he had relatively little interest in Bourbaki style analysis of the foundations or in systematic calculations. He had an extraordinary talent for proving important and easily formulated conceptual theorems through a mix of new ideas, new foundational constructions, and adroit calculations. The solution of the Hopf invariant one problem was the first of many such successes.

B. Applications of K-theory

1. *Vector fields on spheres* (1961) [**23**]

Having so spectacularly solved the Hopf invariant one problem, Adams turned next to the vector fields problem. It was natural for him to try cohomology operations here too. A 1960 note [**20**] gave a partial result, and he was still working in cohomology in July, 1961, when he gave a series of lectures in Berkeley. When the solution came, however, it used K-theory and Adams wrote of his cohomological efforts: "The author's work on this topic may be left in decent obscurity, like the bottom nine-tenths of an iceberg." Write $n = (2a+1)2^b$ and $b = c + 4d$ and let $\rho(n) = 2^c + 8d$. Hurwicz-Radon and Eckman had shown that there exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}. Adams proved that there do not exist $\rho(n)$ such fields. It suffices to show that the truncated projective space $\mathbb{R}P^{m+\rho(m)}/\mathbb{R}P^{m-1}$ is not coreducible (the bottom cell is not a retract up to homotopy) for any m. He introduced what are now called the Adams operation ψ^k into real and complex K-theory, he calculated the K-theory of truncated projective spaces, with their Adams operations, and he showed that there is no splitting of their real K-theory which is compatible with the operations. All of Adams' papers are well written, but the exposition in this classic paper is especially lovely.

For background, James, in part, and Atiyah had shown that the bundle $O(n)/O(n-k) \to S^{n-1}$ admits a cross-section if and only if n is a multiple of the order of the image of the canonical line bundle in $\tilde{J}(\mathbb{R}P^{k-1})$, and analogously in the complex and quaternionic cases. Curiously, it was left to Atiyah and Bott to observe that Adams' calculations actually imply that $\widetilde{KO}(\mathbb{R}P^k) \cong \tilde{J}(\mathbb{R}P^k)$. This group is cyclic of order $2^{\varphi(k)}$, where $\varphi(k)$ is the number of j such that $0 < j \leq k$ and $j \equiv 0, 1, 2,$ or $4 \bmod 8$. Mark Mahowald discussed the significance of this calculation in his talk.

2. *On complex Stiefel manifolds* (1964, with Grant Walker) [**29**]

It is shown that $U(n)/U(n-k) \to S^{2n-1}$ admits a cross-section if and only if M_k divides n; here $\nu_p(M_k) = \sup\{r+\nu_p(r) \,|\, 1 \le r \le (k-1)/(p-1)\}$ if $p \le k$ and $\nu_p(M_k) = 0$ if $p > k$. Atiyah and Todd had shown that the condition is necessary, and they had conjectured that it is sufficient. As already noted, Atiyah had reduced the problem to a calculation in $\widetilde{J}(\mathbb{C}P^{k-1})$, and this paper analyzes $\widetilde{J}(\mathbb{C}P^n)$ by the methods of $J(X)$-I,II. It gives a worked example of the general study in those papers.

3. *On the groups* $J(X)$*-I* (1963), *II* (1963), *III* (1963), *IV* (1965) ([**25**], [**28**], [**31**], [**35**])

The program in this fundamentally important cycle of papers is to give effective means for computing the group $J(X) = \widetilde{J}(X) \oplus Z$ of fiber homotopy equivalence classes of stable vector bundles over a finite CW-complex X. The basic idea is to give computable upper and lower bounds $J''(X)$ and $J'(X)$ for $J(X)$ and to show that the two bounds coincide. Thus $J(X)$ would be captured in the diagram of epimorphisms

$$\begin{array}{ccc} & \nearrow & J''(X) \\ & & \downarrow \\ KO(X) & \longrightarrow & J(X) \\ & & \downarrow \\ & \searrow & J'(X). \end{array}$$

That $J''(X)$ really is an upper bound depends on the celebrated Adams conjecture: "If k is an integer, X is a finite CW-complex and $y \in KO(X)$, then there exists a non-negative integer $e = e(k,y)$ such that $k^e(\psi^k - 1)y$ maps to zero in $J(X)$."

As Michael Crabb explained in his talk, it is now possible to give a fairly elementary proof of the Adams conjecture. It is fortunate that such an argument was not discovered early on. The proofs of the Adams conjecture by Sullivan and Quillen led to a veritable cornucopia of new mathematics, including localizations and completions of spaces and the higher algebraic K-groups of rings.

J(X)-I. The Adams conjecture is proven if y is a linear combination of $O(1)$ and $O(2)$ bundles or if $X = S^{2n}$ and y is a complex bundle. The proof is based on the Dold theorem mod k: if there is a fiberwise map $E_\xi \to E_\eta$ of degree $\pm k$ on each fiber, then $k^e\xi$ and $k^e\eta$ are fiber homotopy equivalent for some $e > 0$.

J(X)-II. The group $J''(X)$ is specified as $KO(X)/W(X)$, where $W(X)$ is the subgroup generated by all elements $k^{e(k)}(\psi^k - 1)y$ for a suitable function e (independent of y). The cannibalistic classes ρ^k are defined by the formula $\rho^k(\xi) = \varphi^{-1}\psi^k\varphi(1)$ on Spin(8n)-bundles ξ, and it is shown that they can be defined more generally after localization. If ξ and η are fiber homotopy equivalent, then $\rho^k(\xi) = \rho^k(\eta)[\psi^k(1+y)/(1+y)]$ for some $y \in \widetilde{K}(X)$ (independent of k). $J'(X)$ is specified as $KO(X)/V(X)$, where $V(X)$ is the subgroup of those x such that $\rho^k(x) = \psi^k(1+y)/(1+y)$ in $KO(X) \otimes Z[1/k]$ for all $k \neq 0$ and some $y \in \widetilde{K}O(X)$. Explicit computations give the groups $KO(\mathbb{R}P^n) = J''(\mathbb{R}P^n) = J'(\mathbb{R}P^n)$ and $J''(S^n) = J'(S^n)$. The latter calculations imply that $J(\pi_{8n+i}(SO)) = Z_2$ if $i = 0$ or 1 and that $J(\pi_{4n-1}(SO))$ is cyclic of order $m(2n)$, where $m(2n)$ is the denominator of $B_n/4n$, although Adams was left with an ambiguity when n is even because he only had the complex and not the real Adams conjecture for bundles over spheres. Of course, these basic calculations are essential to the understanding of the stable homotopy groups of spheres.

J(X)-III. The main theorem of the series is proven: $J'(X) = J''(X)$. This is based on the fundamental commutative diagram

$$\begin{array}{ccc} \sum_k \widetilde{K}SO(X) & \xrightarrow{\sum k^{e(k)}(\psi^k - 1)} & \widetilde{K}SO(X) \\ \downarrow{\scriptstyle \sum \vartheta^k} & & \downarrow{\scriptstyle \prod \rho^\ell} \\ 1 + \widetilde{K}SO(X) & \xrightarrow{\prod \psi^\ell/1} & \prod_\ell 1 + \widetilde{K}SO(X) \otimes Z[1/\ell] \end{array}$$

The diagram is obtained by summing individual diagrams for pairs (k, ℓ), and the ϑ^k are constructed in the course of the character theoretic proof. The main theorem follows from the fact that this diagram is a weak pullback. The paper also explains and exploits the modular periodicity of the Adams operations.

The paper has a tantalizing last section. It asks for a theory Sph(X) of stable spherical fibrations in which $J(X)$ is a direct summand mapped to by $KO(X)$; Sph(X) should be represented by $BF \times Z$, where F is the monoid of homotopy equivalences of spheres. It also asks for a theory Sph($X; kO$) of kO-oriented stable spherical fibrations and gives a number of probable consequences. With characteristic honesty, Adams wrote of this discussion "I will not call the results "theorems", since the underlying assumptions have not been stated precisely enough." This section makes vividly clear just how prescient this whole series of papers was. Many relevant and now standard tools were unavailable to Adams, but he foresaw much that would later be formulated and proven with them.

For example, an alternative version of the diagram above can be constructed conceptually by exploiting localized classifying spaces rather than representation theory. At $p = 2$, the relevant diagram is:

$$
\begin{array}{ccccc}
BO & & & & \\
\downarrow{\scriptstyle \gamma^3} & \overset{\psi^3-1}{\searrow} & & & \\
SF/\,\mathrm{Spin} & \longrightarrow & B\,\mathrm{Spin} & \xrightarrow[J]{} & BSF \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} & & \| \\
BO_{\otimes} & \longrightarrow & B(SF;kO) & \longrightarrow & BSF \\
& \underset{\psi^3/1}{\searrow} & \downarrow{\scriptstyle c(\psi^3)} & & \\
& & B\,\mathrm{Spin}_{\otimes} & &
\end{array}
$$

Here $B(SF;kO)$ classifies $\mathrm{Sph}(F;kO)$, $c(\psi^3)$ is the universal cannibalistic class determined by ψ^3, and μ is given by the Atiyah-Bott-Shapiro orientation. The rows are fibration sequences, so μ determines ν and the Adams conjecture determines γ^3. The composite $c(\psi^3)\circ\mu$ is ρ^3, the composite $\nu\circ\gamma^3$ can be taken as ϑ^3, and these two maps are 2-local equivalences.[4] I discussed this approach with Frank, who had envisioned something of the sort. He very much liked it, but he rightly emphasized that you can't proceed this way before you have the Adams conjecture.

J(X)-IV. The results of I–III are applied to computations in the stable homotopy groups of spheres. The starting point is an abstract analysis of the "d and e invariants" of a half exact functor k from the homotopy category to an abelian category $\mathcal{A}$. For $f : X \to Y$, $d(f) = f^* \in \mathrm{Hom}(k(Y), k(X))$. If $d(f) = 0$ and $d(\Sigma f) = 0$, then $e(f)$ is the class of

$$0 \to k(\Sigma X) \to k(Cf) \to k(Y) \to 0$$

in $\mathrm{Ext}^1(k(Y), k(\Sigma X))$. In the applications, $\mathcal{A}$ consists of finitely generated Abelian groups with Adams operations, k is taken to be $\widetilde{K}$ or $\widetilde{KO}$, and X and Y are taken to be spheres or Moore spaces, for which the Hom and Ext target groups are readily computed.

[4]For details, see Chapter V of [J. P. May (with contributions by Nigel Ray, Frank Quinn, and J. Tornehave) E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Vol 577, 1977].

Calculations of these invariants on the groups π_r^s for $r > 1$ are related to $J : \pi_r(SO) \to \pi_r^s$. Here $d = 0$ except in the real case with $r \equiv 1$ or 2 mod 8, when d detects a direct summand $Z_2 \not\subset \mathbf{Im}\, J$ generated by elements μ_r. For any r, the real e invariant detects $\mathbf{Im}\, J$ as a direct summand, the case $r \equiv 7$ mod 8 being incomplete in the paper since the full Adams conjecture was not yet available. Various composition products and Toda brackets are detected by means of Yoneda and Massey products in the target groups. This seems a little like a magical boot strap operation since the Hom and Ext calculations involved are fairly elementary. The conception is a marvelous example of algebraic modeling of topological phenomena.

The complex e-invariant is determined by the Chern character, and, via Adams' paper on the Chern character, to be discussed shortly, this leads to a proof by K-theory of the Hopf invariant one problem for any prime p. Finally, the e-invariant is used to prove that if Y is the mod p^f Moore space for an odd prime p (with bottom cell in a suitable odd dimension) and if $r = 2(p-1)p^{f-1}$, then there is a map $A : \Sigma^r Y \to Y$ which induces an isomorphism on $\widetilde{K}$, so that all of the iterates of A are essential. You have heard about these vitally important periodicity maps in several talks, for example those of Katsumi Shimomura, Doug Ravenel, and especially that of Pete Bousfield.

4. *K-theory and the Hopf Invariant* (1964, with Michael Atiyah) [**34**]

This paper gives the beautiful and definitive K-theoretic proof of the Hopf invariant one result for all primes p. For $p = 2$, it is based on the relation $\psi^2\psi^3 = \psi^3\psi^2$ applied in the obvious 2-cell complex. Alain Jeanneret showed us that this trick can still be used to good effect to obtain new results.

5. *Geometric dimension of bundles over* $\mathbb{R}P^n$ (1974) [**51**]

Since Kee Y. Lam's talk gave a rather complete summary of the results in this nice paper, I will not discuss its main thrust. However, in view of the current interest in periodicity maps, I want to mention an addendum that it gives to the discussion of Moore spaces in J(X)-IV: for $n \geq 5$, there is a map $\Sigma^8 Y_n \to Y_n$ which induces an isomorphism on $\widetilde{KSp}$, where Y_n is the mod 2 Moore space with bottom cell in dimension n.

C. Characteristic classes and calculations in K-theory and cobordism

1. *On formulae of Thom and Wu* (1961) [**19**]

In this beautiful early example of modern algebraic modeling, Adams shows that any Poincaré duality algebra over the Steenrod algebra has "Wu classes" and thus Stiefel Whitney classes which satisfy all of the same formulas which relate these classes in the cohomology of differential manifolds. The proof is based on the construction and analysis of a suitable universal left and right A-algebra.

2. *On Chern characters and the structure of the unitary group* (1960) **[18]**

Using Bott periodicity to study the Postnikov system of $BU[2q,\ldots,\infty)$, Adams defines characteristic classes $ch_{q,r}(\xi) \in H^{2q+2r}(X;\mathbb{Z})$ for stable bundles ξ over (2q-1)-connected spaces. If ch_r is the r^{th} component of the Chern character, then $ch_{q,r}(\xi)$ rationalizes to $m(r)ch_{q+r}(\xi)$, and the $ch_{q,r}(\xi)$ relate appropriately to Steenrod operations when reduced mod p. Yuli Rudjak noted that some of the ideas Adams introduced here are relevant to the study of the orientability of various kinds of bundles.

3. *Chern characters revisited* (1971) **[47]**

This gives a more modern and sophisticated approach to the Chern character. The image of $H_*(bu;\mathbb{Z})$ in $H_*(bu;\mathbb{Q}) = \mathbb{Q}[u]$, $\deg u = 2$, is shown to be the subgroup generated by $\{u^r/m(r) \mid r \geq 0\}$. The elegant one prime at a time proof is based on the simple A-module structure of $H^*(bu;\mathbb{Z}_p)$. Viewing ch_r as a map $bu \to K(\mathbb{Q}, 2r)$, it follows that the image of $m(r)ch_r : bu_n(X) \to H_{n-2r}(X;\mathbb{Q})$ is integral for any X.

4. *The Hurewicz homomorphism for MU and BP* (1970, with Arunas Liulevicius) **[45]**

This paper gives a nice proof via the Adams spectral sequence of an interpretation of the Hattori-Stong theorem: $\pi_*(BP) \to \pi_*(k \wedge BP)$ is a split monomorphism, and similarly for MU, where k represents connective K-theory.

This is one of the very few papers in which Adams allowed his coauthor to do the actual writing. Adams preferred to hold pen in hand himself, although he paid careful attention to the suggestions of his collaborators.

The following five papers can be viewed as a series in which Adams applied to K-theory the algebraic foundations that he established for the calculational study of generalized homology and cohomology theories.

5. *Hopf algebras of cooperations for real and complex K-theory* (1970, with Albert Harris and Robert Switzer) **[42]**

Using the stable Adams operations in localized K-theory to obtain integrality conditions, $K_*(K)$ is computed as a subring of $K_*(K) \otimes \mathbb{Q}$, which is a ring of finite Laurent series on two variables; $KO_*(KO)$ is also determined. Francis Clarke showed us how to relate this to the study of the ring of cooperations in elliptic homology.

6. *Operations of the Nth kind in K-theory* (1972) **[48]**

In a report on work with David Baird, Adams indicates that, for K-theory localized at an odd prime p, $\mathrm{Ext}^{s,t}_{K_*(K)}(\widetilde{K}_*(X), \widetilde{K}_*(Y)) = 0$ for all finite X and Y and all t when $s \geq 3$. He then speculates about stable homotopy theory "seen through the spectacles of K-theory". Pete Bousfield's beautiful talk on the structure of stable homotopy theory localized at K showed us how these speculations have come to fruition.

7. *Operations on K-theory of torsion-free spaces* (1975, with Peter Hoffman) **[54]**

For integers $n < m$, this paper computes the ring of those operations $K(X) \to K(X)$ which are defined and natural for CW-spectra X such that $\pi_r(X) = 0$ for $r < 2n$, $H_r(X)$ is free for all r, and $H_r(X) = 0$ for $r > 2m$.

8. *Stable operations on complex K-theory* (1976, with Francis Clarke) **[59]**

It is observed that although linear combinations of ψ^1 and ψ^{-1} are the only obvious stable operations, $K^0(K)$ is actually uncountable.

9. *Primitive elements in the K-theory of BSU* (1975) **[56]**

The kernel and cokernel of $PK^0(BU; k) \to PK^0(BSU; k)$ are computed for any commutative ring k. In particular, somewhat surprisingly, it is found that $PK^0(BU; \widehat{Z}_p) \to PK^0(BSU; \widehat{Z}_p)$ is an isomorphism.

D. Stable homotopy and generalized homology

Especially during his last few years at Manchester, Adams made frequent extended trips to the United States. His usual destination was Chicago, a place where he always felt very comfortable and at home. Some of his most influential writing is in notes prepared for delivery in lecture series at Chicago (1967, 1970, and 1971) and at a conference in Seattle (1968). According to Nigel Ray, he tried out some of these lectures on people at Manchester.

1967 *S. P. Novikov's work on operations on complex cobordism* * [49][5]

Novikov's work in question was only available in Russian at the time, and it was quite difficult reading even for those who knew Russian. Adams' clear exposition allowed the quick assimilation of this material into the main stream of algebraic topology in the West.

1968 *Lectures on generalized homology* (Seattle) **[39]**

1. *The universal coefficient theorem and the Künneth theorem*

[5]The three lecture notes denoted * are in the "Chicago blue book" (titled after the 1971 lectures). The University of Chicago Press will keep it in print, and I would like to be told if anybody has trouble obtaining a copy.

This classic account shows that the four "UCT's" imply the four "KT's" by specialization, that two of the UCT's imply the other two by duality, and that one UCT can be viewed as a special case of the ASS (Adams' preferred abbreviation for the Adams spectral sequence). It gives a treatment of the remaining UCT, by Atiyah's method in K-theory, that still seems to represent the state of the art. The account is applied to the Conner-Floyd theorem and to other relations between K and MU.

2. *The Adams spectral sequence*

This account of the generalized ASS shows much progress beyond earlier tries at generalization. The now generally accepted preference for homology over cohomology is expounded. Convergence is not studied here.

3. *Hopf algebra and comodule structure*

Definitive foundations are given for the algebra used to describe E_2 of the generalized ASS in terms of homology. The material here was taken for granted in quite a few talks at this conference, for example those of Doug Ravenel, Katsumi Shimomura, and Vladimir Vershinin.

4. *Splitting generalized cohomology theories with coefficients*

This gives a splitting of KU and a parallel splitting of MU via idempotents; the former is still the standard reference, but the latter was soon superseded by Quillen's approach via formal group laws.

5. *Finiteness theorems*

A systematic generalized treatment of coherent rings is given. One application gives that, for finite CW-complexes X, $MU^*(X)$ admits a finite MU^*-resolution by finitely generated free modules (as was also shown by Conner and Smith). Another application (due to J. Cohen) shows that a space Y with non-trivial reduced mod p cohomology has infinitely many non p-trivial stable homotopy groups.

1970 *Quillen's work on formal groups and complex cobordism* * [49]

Just as the 1967 lectures allowed the rapid assimilation of Novikov's work, so these lectures allowed the rapid assimilation of Quillen's work. I remember these lectures as great fun. The first eight strictly alternated algebra and topology, giving a connected development of the theory of formal groups on the one hand and a clear exposition of their role in topology on the other. The calculations of $MU^*(MU)$ and $BP^*(BP)$ of Novikov and Quillen were reworked as explicit calculations of $MU_*(MU)$ and $BP_*(BP)$. These calculations have been cited in several talks here.

An interesting survey, *Algebraic topology in the last decade* [43], based on a lecture given at a conference at the University of Wisconsin, also dates from 1970.

1971 *Stable homotopy and generalized homology* * [**49**]

This classic lecture series is still the best introduction to its topics, although many of us prefer more idealistic approaches to the construction of the stable homotopy category. Its treatment of convergence of the ASS marked a major improvement over earlier work. The important idea of studying the stable homotopy category by means of localizations at spectra was first introduced here.

E. Lectures on Lie groups (1969) [38]

This excellent exposition of the basics of Lie theory, with emphasis on the representation theory of compact Lie groups, is based on lectures given at Manchester.[6] Working tools for many of the papers in the next two groups can be found here.

F. Finite H-spaces and compact Lie groups

1. *The sphere considered as an H-space mod p* (1960) [**16**]

The very first appearance of the localization of a space appeared here, presaging the study of finite H-spaces via localization. Adams gave the slogan "an odd sphere is an H-space mod p" as early as 1956.

2. *H-spaces with few cells* (1960) [**21**]

It is shown that if $H^*(G; Z_2) \cong E(x_q, x_n)$, $q \leq n$, then q and n are both one of 1, 3, 7, or else $(q,n) = (1,2), (3,5), (7,11)$, or $(7,15)$. Adams left open the question of realizability of the last two, but that was settled in the negative in John Hubbuck's thesis.

3. *Finite H-spaces and algebras over the Steenrod algebra* (1978, with Clarence Wilkerson) [**63**]

Explicit necessary and sufficient algebraic conditions are given on an unstable A-algebra P which ensure that $P \cong H^*(BT)^W$ for a torus T and finite group W. These conditions hold if P is a polynomial algebra on generators of even degree prime to p, and W is then a p-adic generalized reflection group; moreover, there is a space X such that $P \cong H^*(X)$. The program carried out in this paper is due to Wilkerson, and it is one that Frank particularly liked. The paper contains lovely algebra. Here is a nice quote; it follows a description of a proof based on appeal to Bezout's theorem: "We hope that the reader finds something appealing in this strategy; we shall

[6] It was reprinted by the University of Chicago Press in 1982. Again, I would like to be told if anybody has trouble obtaining it.

not risk spoiling this impression by giving the details. The proof we shall present is our second."

4. *Finite H-spaces and Lie groups* (1980) [**66**]

This "unsatisfactory report" explains the difficulties involved in trying to obtain Adams-Wilkerson type results in K-theory. The requisite algebraic models would require a filtration; the natural choice is not the obvious choice but the rational filtration, defined in terms of the Chern character, and one must relate the Adams operations to it. This leads to a notion of good integrality which is satisfied in the absence of torsion and in some but not all classical cases with torsion. An amusing "letter from E_8" explains that he needs so much torsion in his integral cohomology in order to ensure the absence of torsion in his K-theory.

This report was given in a conference in honor of Saunders MacLane in Aspen, Colorado. Frank and I went there together, and we had agreed to entertain the category theorists with silly lectures—my own started with a warning that it would contain a deliberate categorical blunder that would destroy the validity of all of the results. We spent the afternoons climbing in the Rockies. Going up from Independence Pass, at 11,000 feet, to the nearest peak, at over 13,000 feet, entailed quite a bit of crawling on hands and knees over snow and ice, but the view was well worth the effort. Frank and I seemed to do some such daft thing at all of the conferences we attended together.

Frank had great affection for the low-dimensional and exceptional Lie groups. Their study was a lifetime hobby, and the next three papers are "Snippets of a book I have in preparation." It is hoped that his unpublished notes will be edited and published in the not too distant future; appropriate people at this conference have agreed to do the work.

5. *Spin(8), triality, F_4 and all that* (1981) [**67**]

This paper gives a nice conceptual explanation of the exceptional symmetry of $D_4 = \mathrm{Spin}(8)$ given by the fact that $\mathrm{Out}(\mathrm{Spin}(8)) \cong \Sigma_3$. The exceptional 27-dimensional Jordan algebra J is constructed as the sum of $\mathbb{R}^3$ and the three irreducible representations of Spin(8), together with a linear, a bilinear, and a trilinear form. Then F_4 is the group of $\mathbb{R}$-linear maps $J \to J$ which preserve all three forms, and Spin(8) is the subgroup which fixes the elements of $\mathbb{R}^3$. The subgroup of F_4 which preserves the preferred basis elements of $\mathbb{R}^3$ maps onto Σ_3 with kernel Spin(8), giving a clear and attractive proof that the outer automorphisms of Spin(8) are specified by conjugation by elements of F_4.

6. *The fundamental representations of* E_8 (1985) [**73**]

The object here is to describe explicit polynomial generators for $R(E_8)$. Let α be the adjoint representation. Then $R(E_8)$ is generated by $\lambda^i\alpha$ for $1 \leq i \leq 5$, β, $\lambda^1\beta$, and γ, and explicit concrete descriptions of β and γ are given. In this area, Frank expected a lot from his readers; he thought that everybody should understand Lie groups as well as he did. Here is the argument for one fairly obscure lemma: "Sketch proof. Take the obvious steps yourself—it's quicker than going to the library."

7. *2-tori in* E_8 (1986) [**77**]

The phenomena uncovered here are interesting and intricate. Maximal 2-tori in E_8 fall into two conjugacy classes, one of rank 9 and one of rank 8, and explicit constructions are given. Any 2-torus in E_8 is conjugate to one in $Ss^+(16)$, and maximal 2-tori in $Ss^+(16)$ remain maximal in E_8. However, maximal 2-tori in $Ss^+(16)$ fall into four conjugacy classes, two of rank 9 and two of rank 8, while maximal 2-tori of Spin(16) fall into two conjugacy classes, both of rank 9. For n even, the maximal 2-tori in $PO(n)$ are determined. For $n \equiv 0 \mod 8$ and $n > 8$, the maximal 2-tori of $PSO(n)$ and $Ss^+(n)$ are determined .

This kind of explicit information is obviously relevant to Quillen's work on the mod 2 cohomology of compact Lie groups, and it will be necessary to the program Mimura described to us of understanding the mod 2 cohomology of exceptional Lie groups in terms of 2-tori.

G. Maps between classifying spaces of compact Lie groups

1. *Maps between classifying spaces* (1975, with Zafer Mahmud) [**55**]

Stefan Jackowski gave us a summary of this important paper (abbreviated MBCS below) in his talk on the same topic, so I shall say relatively little. Let G and G' be compact connected Lie groups with maximal tori T and T' and Weyl groups W and W'. Let $\vartheta : H^*(BG';\mathbb{Q}) \to H^*(BG;\mathbb{Q})$ be a homomorphism of $\mathbb{Q}$-algebras which "commutes with Steenrod operations for all sufficiently large primes p". There is an "admissible map" $\varphi : \pi_1(T)\otimes \mathbb{Q} \to \pi_1(T') \otimes \mathbb{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc} H^*(BG';\mathbb{Q}) & \xrightarrow{\vartheta} & H^*(BG;\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*(BT';\mathbb{Q}) & \xrightarrow{\varphi^*} & H^*(BT;\mathbb{Q}) \end{array}$$

Two choices of φ differ by composition by an element of W'. Moreover, any

ϑ is induced by a map f defined after finite localization. Conversely, given an admissible map φ, there is a unique homomorphism ϑ such that the diagram commutes, and ϑ can be induced by a map f defined after finite localization. Thus there is a bijective correspondence between homomorphisms ϑ induced by maps f defined after finite localization and W'-equivalence classes of admissible maps φ. For any $f : BG \to BG'$, $f^* : K(BG') \to K(BG)$ carries $R(G')$ into $R(G)$. There is a detailed Lie theoretic analysis of admissible maps, and there are lots of concrete calculational examples and case by case calculations.

This paper initiated serious work on this topic, long before others were interested. The talks of Jackowski, Stewart Priddy, and others made clear that this is now a thriving area of algebraic topology.

2. *Maps between classifying spaces II* (1978) [**62**]

In this fascinating and relatively neglected paper, G and G' are compact, but not necessarily connected, Lie groups. The group $FF(X)$ of "formally finite elements of $K(X)$" is defined as the group of differences of elements annihilated by all but finitely many λ^i. Clearly $f^* : K(X') \to K(X)$ carries $FF(X')$ into $FF(X)$ for any $f : X \to X'$. Let $\alpha : R(G) \to K(BG)$ be the natural map. With $X = BG$, $\mathbf{Im}(\alpha) \subset FF(BG)$. Adams proves that equality holds if G is finite or if $\pi_0(G)$ is the union of its Sylow subgroups. In these cases, $f^* : K(BG') \to K(BG)$ carries $R(G')$ into $R(G)$ for any map $f : BG \to BG'$, but this is not true in general. However, for any G there is an $n > 0$ such that $nFF(BG) \subset \mathbf{Im}(\alpha)$. If G is monogenic and $x \in FF(BG)$, then $x = \alpha(\rho)$ for an honest representation ρ of G, but there is a finite p-group G and an element $x \in FF(BG)$ such that x is not of the form $\alpha(\rho)$ for any representation ρ. For finite G, the crux is that $x \in \mathbf{Im}(\alpha)$ if the total exterior power $\lambda_t(x)$ is a rational function of t. However, there is a G such that $\pi_0(G)$ is a p-group and an x such that $\lambda_t(x)$ is a rational function of x and yet $nx \notin \mathbf{Im}(\alpha)$ for any $n > 0$. The elements of formal dimension 2 in $K(BSL(2,5))$ are analyzed in detail.

This paper is independent of MBCS, and I think that its ideas can be exploited further. The basic method is to approximate G by its finite subgroups. At the time, this was a novel idea. In particular, finite approximations of toral extensions over finite groups first appear here. There is also an attractive theory of characters $\widehat{\chi}_g$ defined on elements of $K(BG, \widehat{Z}_p)$, and it is observed that $K(BG)$ injects into the product of the $K(BG, \widehat{Z}_p)$. If $\widehat{\chi}_g(x) = 0$ for all g of prime power order, then $x = 0$. These characters take values in $\widehat{Z}_p \otimes \mathbb{C}$, but they lie in $\mathbb{C}$ if $\lambda_t(x)$ is a rational function of

t. Another pleasant result, which should have been known before, is that a continuous class function $G \to \mathbb{C}$ which restricts to a virtual character for every finite subgroup of G is a virtual character.

3. *Maps between classifying spaces III* (1983, with Mahmud) [**71**]

As in MBCS, G and G' are compact connected Lie groups. For any $f : BG \to BG'$, $f^* : R(G') \to R(G)$ carries $RO(G')$ to $RO(G)$ and $RSp(G')$ to $RSp(G)$. There is a canonical, easily determined, "Dynkin element" $\delta \in Z(G)$, defined up to multiplication by squares of elements of order 4 in $Z(G)$, whose behavior on self-conjugate representations distinguishes real from symplectic representations. Under suitable hypotheses, necessary to rule out counterexamples, it is shown that the admissible map associated to f^* in MBCS preserves Dynkin elements (up to indeterminacy).

4. *Maps between p-completed classifying spaces* (1988, with Zdzislaw Wojtkowiak) [**81**]

Working in the context of MBCS, complete all spaces at p. It is observed that a result of Dwyer and Zabrodsky on maps $B\pi \to BG'$ for finite p-groups π implies that, given $f : BG \to BG'$, there is a compatible map $BT \to BT'$, unique up to the action of W'. It follows that all of the results about admissible maps in MBCS carry over to "admissible maps" $\pi_1(T) \otimes \hat{Z}_p \to \pi_1(T') \otimes \hat{Z}_p$, since these maps are obtained by extension of scalars from p-local admissible maps.

H. Modules over the Steenrod algebra and their Ext groups

1. *A finiteness theorem in homological algebra* (1960) [**17**]

This gives a general vanishing result of the form $\mathrm{Ext}_A^{s,t}(k,k) = 0$ in a range $s\varepsilon < t < sq$. When A is the mod p Steenrod algebra, $\varepsilon = 1$ and $q = 2(p-1)$, which is not best possible.

2. *A periodicity theorem in homological algebra* (1961, 1965) [**26**], [**36**]

The 1965 paper gives details of results sketched in Adams' 1961 Berkeley notes. Consider $H^{s,t}(A) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$, where A is the mod 2 Steenrod algebra. It is shown that $H^{s,t}(A) = 0$ for $0 < s < t < U(s)$, where $U(s)$ is approximately $3s$ and is best possible. It is also shown that, in small regions near the vanishing line, there are periodicity isomorphisms $\pi_r : H^{s,t}(A) \cong H^{s+2^r, t+3\cdot 2^r}(A)$ given by $\pi_r(x) = \langle h_{r+1}, (h_0)^{2^r}, x\rangle$.

From his earliest work in stable homotopy theory, Adams was very concerned with the possibility of discerning systematic periodicity phenomena in stable homotopy groups. Pertaining to the algebraic periodicity just mentioned, he asked in his 1961 Berkeley lecture notes: "What geometric phenomena can one find which show a periodicity and which on passing to algebra give the sort of periodicity encountered in the last lecture?". His persistent interest in this kind of question has perhaps helped shape the present intense focus on periodicity phenomena in stable homotopy theory that was evidenced, for example, in the talks of Mark Mahowald, Jack Morava, Ethan Devinatz, Mike Hopkins, Doug Ravenel, Don Davis, and Martin Bendersky.

3. *Modules over the Steenrod algebra* (1970, with Harvey Margolis) [**41**]

This gives Adams' clarification of interesting results of Margolis. Let B be a sub Hopf algebra of the mod 2 Steenrod algebra. A bounded below B-module M is free if and only if $H(M; P_t^s) = 0$ for all $P_t^s \in B$ with $s < t$. If M is free, then M is injective. If $f : M \to N$ induces an isomorphism on homology for all P_t^s, then M and N are stably equivalent, in the sense that they become isomorphic after adding suitable free modules to each.

4. *Sub-Hopf-algebras of the Steenrod algebra* (1973, with Margolis) [**50**]

All sub Hopf algebras of the mod p Steenrod algebra are constructed.

5. *What we don't know about* $\mathbb{R}P^\infty$ (1972) [**48**]

Consideration of truncated projective spaces led Adams to several conjectures which, in retrospect, can be viewed as versions of the Segal conjecture for the group $\mathbb{Z}_2$. One can form an A-algebra $P = \mathbb{Z}_2[x, x^{-1}]$ which, as Adams observes, is certainly not the cohomology of any spectrum. He conjectures that $\mathrm{Ext}_A(P, \mathbb{Z}_2) \cong \mathrm{Ext}_A(\Sigma^{-1}\mathbb{Z}_2, \mathbb{Z}_2)$, and he explains how this might come about in terms of the sub Hopf algebras A_r.

6. *Calculation of Lin's Ext groups* (1979, with W. H. Lin, Don Davis, and Mark Mahowald) [**64**]

This paper gives Adams' simplification of Davis and Mahowald's simplification of Lin's proof of Adams' Ext conjecture just stated. Exactly as Adams envisioned, the proof is by inductive use of the A_r. This implies the Segal conjecture for $\mathbb{Z}_2$, although the topology is not presented here.

7. *The Segal conjecture for elementary abelian p-groups* (1984, with Jeremy Gunawardena and Haynes Miller) [**76**]

This gives the Ext calculation which was used in the original proof of the Segal conjecture for elementary Abelian p-groups, although, again, the topology is not presented here. Let V be an elementary p-group of rank n. Let $H^*(V)_{\mathrm{loc}}$ be the localization obtained by inverting βx for all non-zero elements $x \in H^1(V)$. The quotient map $H^*(V)_{\mathrm{loc}} \to \mathbb{Z}_p \otimes_A H^*(V)_{\mathrm{loc}}$ induces an isomorphism on $\mathrm{Tor}_A(\mathbb{Z}_p, ?)$ and $\mathbb{Z}_p \otimes_A H^*(V)_{\mathrm{loc}}$ is 0 except in degree $-n$, where it has rank $p^{n(n-1)/2}$ and can be identified with the Steinberg representation of $\mathrm{Aut}(V) = \mathrm{GL}(n, \mathbb{Z}_p)$. For $p = 2$ and $n = 1$, this is the result of the paper above; for $p > 2$ and $n = 1$, it is Gunawardena's thesis. The proof is based on use of the Singer construction. Various other Ext calculations are also given, including as a very special case the isomorphism $\mathbb{Z}_p\{\mathrm{Hom}\,\mathbb{Z}_p(U,V)\} \cong \mathrm{Hom}_A(H^*(V), H^*(U))$ that Nick Kuhn took as the starting point of his talk.

The three authors of this paper originally planned a sequel giving the topology, but it was superseded by a sequel by three other authors that gave an efficient use of their Ext calculations to prove the Segal conjecture for elementary Abelian p-groups, together with a version of Carlsson's reduction of the conjecture for all finite p-groups to this case.[7]

Several people at the conference asked me how this proof of the Segal conjecture for p-groups compares to the beautiful new proof that Jean Lannes presented to us. Elementary Abelian p-groups play no special role in Lannes' direct induction, and this is a line of attack that Adams himself pursued without success. Lannes uses his magic functor T (about which more later) as a substitute for the Singer construction. However, his proof gives the nonequivariant form of the Segal conjecture, and, to generalize from p-groups to all finite groups, one requires the original equivariant form. Both arguments should be of lasting interest .

I. Miscellaneous papers in homotopy and cohomology theory

1 *An example in homotopy theory* (1957) [**7**]

2. *An example in homotopy theory* (1963, with Grant Walker) [**27**]

The first of these notes displays two CW-complexes which are of the same n-type for all n but are not homotopy equivalent. The second displays a phantom map from a CW-complex to a finite-dimensional CW-complex. Each example was the first of its kind.

3. *A variant of E. H. Brown's representability theorem* (1970) [**40**]

A group-valued contravariant functor on the homotopy category of finite CW-complexes is shown to be representable if it satisfies the wedge and Mayer-Vietoris axioms. Brown showed this for functors taking countable sets as values; thus the result eliminates the annoying countability assumption at the price of insisting on group values. A consequence is that any homology theory which satisfies the direct limit axiom is representable.

4. *Idempotent functors in homotopy theory* (1973) [**52**]

The notion of an idempotent functor is advertised in connection with localization and completion theory. Axioms are given on a set S of maps

[7] J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups II. p-adic completions in equivariant cohomology. Topology 26(1987), 413–433.

in the homotopy category of connected CW-complexes which ensure, by Brown's theorem, that S is the set of equivalences for an idempotent functor. The key axiom, which is hard to verify in practice, is an analog of the solution set condition in the adjoint functor theorem.

In fact, this paper results from a failed swindle presented in 1973 lectures at Chicago. Adams presented the theory without the solution set analog, thereby constructing localizations and completions without doing a shred of work. Pete Bousfield, who is not generally an excitable chap, was sitting behind me, and I vividly remember him tapping me on the shoulder and asking "How does he know it is a set?" However, even Frank's rare mistakes had good effects. Pete was inspired by these lectures to develop the theory of localizations of spaces and spectra at generalized homology theories. As the talks of Devinatz, Hopkins, Bousfield, Greenlees, and others made clear, such localizations are of fundamental importance in homotopy theory.

5. *The Kahn-Priddy theorem* (1972) [**46**]

In this semi-expository paper, it is shown, among other things, that the 2-local stable map $\mathbb{R}P^\infty \to S^0$ which induces an epimorphism on homotopy groups is essentially unique, and similarly for odd primes.

6. *Uniqueness of BSO* (1975, with Stewart Priddy) [**58**]

With all spaces and spectra localized or completed at a prime p, if X is a connective spectrum whose zero$^{\text{th}}$ space is equivalent to BSO, then X is equivalent to bso. That is, the space BSO is the zero$^{\text{th}}$ space of only one connective spectrum. The same is true for BSU, and a variant is true for BO and BU. The proof of this remarkable result is well worth summarizing. First, using ideas from Adams' work on the Chern character, it is shown that the Postnikov system argument for the calculation of the mod p cohomology of bso works equally well for X, so that there is an isomorphism of A-modules $H^*(X) \cong H^*(\text{bso})$. Second, the theory of stable equivalence of modules over A_1 (for $p = 2$) and $E[x, y]$ (for $p > 2$) introduced by Adams and Margolis allows sufficient calculation of the groups $\text{Ext}_A(H^*(Y), H^*(X))$ to see that it looks to the eyes of E_∞ of the ASS as if there ought to be an equivalence $X \to \text{bso}$. Finally, an approximation of X in the form $X \simeq W \wedge M$, where W has finite skeleta and M is the Moore spectrum for $Z_{(p)}$ or $\widehat{Z}_{(p)}$, gets around the convergence problem for the ASS. As important motivating examples, it follows that $BSO_\otimes$ and, when $p > 2$, F/PL are equivalent to BSO as infinite loop spaces.

Stewart Priddy told us something about his collaboration with Frank on this project. Actually, I proposed the problem to both authors, but my own ideas on the subject led nowhere.

7. *A generalization of the Segal conjecture* (1986, with Jean-Pierre Haeberly, Stefan Jackowski, and J. P. May) [78]

For a finite group G, a multiplicative subset S of the Burnside ring $A(G)$, and an ideal I in $A(G)$, the cohomology theory $S^{-1}(\pi_G^*(?))_I^\wedge$ is $\mathcal{H}$-invariant, where $\mathcal{H} = \bigcup\{\text{Supp}(P) \mid P \cap S = \varphi, P \supset I\}$. That is, a G-map $f : X \to Y$ which restricts to an equivalence $f^H : X^X \to Y^H$ for $H \in \mathcal{H}$ induces an isomorphism $S^{-1}(\pi_G^*(f))_I^\wedge$. The proof reduces the general case to Carlsson's equivariant version of the original Segal conjecture, which is the special case in which G is a p-group, S is empty, and $I = (p)$.

The paper developed in a typically competitive manner, with successive generalizations leading to the present elegant, but perhaps hard to assimilate, conclusion; needless to say, the ultimate formulation was due to Adams.

8. *A generalization of the Atiyah-Segal completion theorem* (1986, with Haeberly, Jackowski, and May) [79]

For a compact Lie group G, a multiplicative subset S of the representation ring $R(G)$, and an ideal I in $R(G)$, the cohomology theory $S^{-1}(K_G^*(?))_I^\wedge$ is $\mathcal{H}$-invariant, where $\mathcal{H} = \bigcup\{\text{Supp}(P) \mid P \cap S = \varphi, P \supset I\}$, and similarly for real K-theory. The proof is by a surprisingly easy direct reduction to quotation of equivariant Bott periodicity. The case in which S is empty and I is the augmentation ideal is the original Atiyah-Segal completion theorem, which is thus given a new proof. The case $I = 0$ is the Segal localization theorem. With S empty, the theorem is due to the second and third authors; the proof given is a simplification of Jackowski's.

I got to hold pen in hand on this one, Frank on the previous one.

9. *Atomic spaces and spectra* (1988, with Nick Kuhn) [80]

This last work is a quintessential example of Adams' characteristic algebraic modeling of topological phenomena. If X is a p-complete space or spectrum of finite type, then $[X, X]$ is a profinite monoid with zero; in the spectrum case, it is a profinite ring. As a matter of algebra, if M is a profinite monoid with zero, then either M contains a non-trivial idempotent or M is"good", in the sense that every element of M is either invertible or topologically nilpotent; if M is a profinite ring and is good, it is local with radical as maximal ideal and $M/\,\text{rad}(M)$ is a finite field. If X is indecomposable, then $[X, X]$ is good. If $[X, X]$ is good, then X is atomic and prime. There is a p-local spectrum of finite type which is indecomposable and for which $[X, X]$ is not good. Every finite field of characteristic p arises as $[X, X]/\,\text{rad}[X, X]$ for an indecomposable stable summand X of the classifying space of some finite p-group; every finite field of characteristic 2 so arises from some finite spectrum X. John Hubbuck told us more about this topological realizability of finite fields.

J. Infinite loop spaces(1975 IAS lecture notes) [61]

If you want to get graduate students interested in algebraic topology, this is a very good book to have them read. It contains capsule introductions to a variety of topics, none of which have yet made it to any text book. The exposition is delightful, despite the elephantine jokes. "I am very grateful to J. P. May, B. J. Sanderson and S. B. Priddy for reading the first draft of this book $\cdots$; I have benefited greatly from their comments. It goes without saying that I accept the responsibility for any jokes which remain." It was deliberately thin on details, except for its discussion of the transfer and its quite full treatment of the Madsen, Snaith, Tornehave theorem that passage to maps of zero$^{\text{th}}$ spaces gives an injection $[bso, bso] \to [BSO, BSO]$.

K. Two unpublished expository papers

There are two expository papers of Adams that were distributed in hand-written form but not published during his lifetime; they will appear in his collected works.

1. *Two theorem of J. Lannes* (1986) [83]

Lannes introduced a functor T which is characterized by an adjunction $\mathrm{Hom}_{\mathcal{U}}(T(M), P) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*(B\mathbb{Z}_p) \otimes P)$, where $\mathcal{U}$ is the category of unstable A-modules. He proved that T is exact and that it preserves tensor products. In the case $p = 2$, Adams gives an alternative approach to these two theorems "which aims to make $T(M)$ more accessible to direct calculation". Nick Kuhn told us about some of the useful formulas to be found in this paper.

2. *The work of M. J. Hopkins* (1988) [84]

This gives a concise and clear discussion of the theorems of Ethan Devinatz, Mike Hopkins, and Jeff Smith. It explains the three versions of their nilpotence theorem and also their theorem about the existence and essential uniqueness of v_n-self maps, and it raises interesting questions about "generators" and "Euler characteristics" for the classes of finite spectra that arise in the proof of the latter theorem. This capsule summary was the culmination of a series of lectures at Cambridge which gave a rather complete exposition of the nilpotence theorems.

Perhaps the first paragraph of the paper provides a fitting close to this discussion of Frank's work.

"The work I shall report has the following significance. At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate."

In fact, this happy state of affairs is a testament to Adams' own contributions, which led the way and paved the ground. The real tribute to Frank at this conference was the mathematical content of the talks. Our subject is thriving, and nowhere more so than in the directions that he himself pioneered.

Beyond his published work, Adams contributed to the development of our subject in many other ways. He was a good friend and mentor to many of us. His correspondence with mathematicians all over the world was extraordinary. He wrote me well over 1,000 handwritten pages, and I was only one of many regular correspondents. He was especially generous and helpful to young mathematicians just starting out. There are many proofs attributed to Adams in the work of others, and there are many proofs and some essentially complete rewrites that come from his "anonymous" referee's reports. He wrote many perspicacious comments on papers for Mathematical Reviews. He was a force to be reckoned with at any occasion such at this, and his guiding presence and astute questions enlivened many a conference.

As the two unpublished papers illustrate, Frank throughout his career made it his business to learn and assimilate fully all of the most important new developments in algebraic topology. Moreover, he worked hard to ease the assimilation of these developments by others. When Frank started out in 1955, it was perfectly possible for one man to understand thoroughly all that there was to know in algebraic topology. This was still more or less possible when I met him in 1964. But our field has by now expanded and developed in so many directions that no one person can expect to master all of it, as Frank tried so hard to do. Many of its most vital directions were set in course by Frank Adams, and his influence will permeate the subject for the foreseeable future. His untimely death is a great loss to our subject, and a great personal loss to all of us who knew him well. He will long be missed, and long be remembered. His place in the history of mathematics is assured.

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