Completions of G-spectra at ideals of the Burnside ring

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On May 3, 1988, Frank Adams wrote us a letter suggesting the topic of the present paper. Referring to the first author's paper **[10]**, he wrote as follows:

> You interest the reader in a problem which makes sense for all finite groups G; but you only give the answer when G is a p-group. Perhaps one should try to give the answer in a form valid for all finite groups G? I hope the answer is
>
> $$D(EG_+) \simeq (S^0)^{\wedge}_I.$$
>
> Here EG_+ is the universal G-space and $D(EG_+)$ is its equivariant functional dual, just as you have it; and the operation $(\cdot)^{\wedge}_I$ takes the equivariant completion of a G-spectrum at an ideal of the Burnside ring $A(G)$—in this case the augmentation ideal. This would be pleasingly reminiscent of Graeme's original statement.
>
> The first point to settle is whether the necessary completion already exists in the literature. I don't know off hand. [He then points out that references **[17]**, **[19]**, and **[20]** fail to construct such completions.] Did someone pass up a chance to set up a useful general theory?

We took up Frank's challenge, and we offer the results in his memory. We think that he would have enjoyed the ideas.

Expanding on Frank's letter, let G be a finite group, let EG_+ be the union of EG and a disjoint basepoint and let $\pi : EG_+ \to S^0$ be the projection. Then the Segal conjecture asserts that π induces an isomorphism from the I-adic completion of the stable G-cohomotopy groups of S^0 to the stable G-cohomotopy groups of EG_+, where I is the augmentation ideal of $A(G)$. This is a statement about the behavior on homotopy groups of the map of dual G-spectra

$$\pi^* : S = D(S^0) \to D(EG_+),$$

where S is the sphere G-spectrum and $D(X) = F(X, S)$.

Generalizations of the Segal conjecture, such as those in [**3, 7, 23**], admit similar descriptions in terms of more general G-spectra than S_G and more general ideals in $A(G)$. It is natural to ask whether such results can be interpreted as saying that the relevant maps of G-spectra are completions at the appropriate ideals. If we define the notion of completion of a G-spectrum in terms of behavior on homotopy groups, then of course the answer is obviously yes. We shall give a more interesting and powerful notion of completion, one specified in terms of an appropriate universal property, and we shall prove that, on suitably restricted kinds of G-spectra, our completions of G-spectra do indeed induce completions of homotopy groups and are characterized by that property.

In fact, we shall construct $X_I^\wedge$ when G is an arbitrary compact Lie group, I is a finitely generated ideal in $A(G)$, and X is an arbitrary G-spectrum. The G-spectrum $X_I^\wedge$ will be a suitable Bousfield localization of X, and the homotopy groups of $X_I^\wedge$ will be determined algebraically by the homotopy groups of X via the left derived functors $L_i^I(M)$ of the I-adic completion functor $M_I^\wedge$ on $A(G)$-modules M. These derived functors are identically zero for $i \geq 2$. When $A(G)$ is Noetherian and M is finitely generated, $L_0^I(M) \cong M_I^\wedge$ and $L_1^I(M) = 0$.

For finite groups G, the new construction leads to generalizations of both the Segal conjecture and the Atiyah-Segal completion theorem. Previous versions of these results start with the restrictions of the appropriate cohomology theory to, say, finite G-CW complexes or to pro-group valued theories defined in terms of such restricted theories. Our constructions allow us to extend these results to the represented theories defined on arbitrary G-spaces or G spectra. In a later paper [**13**], which gives a number of applications of the present theory, we shall prove the very surprising result that, for the periodic K-theory spectrum K_G, the projection $K_G \wedge EG_+ \to K_G$ induces an equivalence upon completion at the augmentation ideal I of $A(G)$; equivalently, the Tate spectrum associated to K_G, which turns out to be a rational G-spectrum, has trivial completion.

In section 1, we give an elementary cohomological construction of completions that applies only when $A(G)$ is Noetherian and is restricted to G-spectra that are bounded below and of finite type. However, this version of completion has the important virtue of being characterized by its behavior on homotopy groups or on certain ordinary homology or cohomology groups. We give our general construction of completions in section 2, and we describe its behavior on homotopy groups in section 3. We give the promised application to the Segal conjecture and Atiyah-Segal completion theorem in section 4. This requires both constructions. The homotopical characterization of our cohomological construction allows us to use existing completion theorems to identify $Y_I^\wedge$ for certain G-spectra Y and ideals I. Our general construction allows us to deduce an identification of $F(X,Y)_I^\wedge$ for arbitrary G-spectra X.

The proofs of our calculational results (as opposed to the construction and characterization of completions) involve an algebraic study of the relationship between derived functors of completions and Grothendieck's local cohomology groups. In deference to Adams' dictum about the writing of algebraic topology "... one writes algebra only as required" [**22**, p.48], we have separated out the general algebraic study for publication elsewhere [**12**]. However, the key piece of algebra is the proof that $L_i^I(M) = 0$ for $i \geq 2$. This follows from a general theorem of Grothendieck when $A(G)$ is Noetherian, but we give a self-contained proof which works for all G in section 6; several other algebraic proofs are also deferred to that final section. Topological proofs, in particular the determination of the homotopy groups of completed G-spectra, are given in section 5. For completeness, we quickly run through the simpler theory of localization of G-spectra in an appendix.

At the referee's request, we have added a brief introduction that explains our parallel definitions of homological and cohomological completions. In his words, "the connection between corresponding completions is interesting even in the nonequivariant case". The material of section 1 originally appeared in the short preprint [14], with Mike Hopkins as a joint author. Also at the referee's request, we have incorporated that elementary part of the theory into the more sophisticated general theory, at the price of jettisoning Hopkins' very pretty proof of Theorem 1.4 below. It is a pleasure to thank Hopkins for his contribution to this project. We also thank Bill Dwyer, Dick Swan, and Gennady Lyubeznik for enjoyable and helpful conversations.

§0. General definitions of localizations and completions.

We begin by describing a general conceptual approach to the theory of localizations and completions that has been part of the second author's way of thinking about the subject since the mid 1970's. Like Bousfield's much more substantive study [4], these ideas were inspired by Frank Adams' marvelous 1973 lectures at the University of Chicago.

Our general definitions make sense in any category with suitable properties, but we shall use the language of spectra for specificity. The reader is warned that the words "localization" and "completion" are used with a certain whimsical interchangeability, both here and in the literature. The following notion is all inclusive.

DEFINITION 0.1. *Let $\mathcal{W}$ be any class of spectra, to be thought of as a class of acyclic spectra.*

(i) *A map $f : X \to Y$ is a $\mathcal{W}$-equivalence if its cofiber is in $\mathcal{W}$.*

(ii) *A spectrum Z is $\mathcal{W}$-complete if $[W, Z] = 0$ for all $W \in \mathcal{W}$.*

(iii) *A $\mathcal{W}$-completion of a spectrum X is a $\mathcal{W}$-equivalence from X to a $\mathcal{W}$-complete spectrum.*

There are Eckmann-Hilton dual ways of giving content to these general definitions, one homological and the other cohomological.

DEFINITION 0.2. *Let $\mathcal{T}$ be any class of spectra, to be thought of as a class of test spectra.*

(i) *A spectrum W is $\mathcal{T}$-acyclic if $W \wedge T$ is contractible for all $T \in \mathcal{T}$.*

(ii) *A spectrum W is $\mathcal{T}^*$-acyclic if $F(W, T)$ is contractible for all $T \in \mathcal{T}$.*

When $\mathcal{W}$ is the class of $\mathcal{T}$-acyclic spectra, we refer to the concepts of Definition 0.1 as "$\mathcal{T}$-equivalence", "$\mathcal{T}$-local", and "$\mathcal{T}$-localization". When $\mathcal{W}$ is the class of $\mathcal{T}^*$-acyclic spectra, we refer to these concepts as "$\mathcal{T}^*$-equivalence", "$\mathcal{T}^*$-complete", and "$\mathcal{T}^*$-completion".

We think of (i) as specifying homological acyclicity and (ii) as specifying cohomological acyclicity. If $\mathcal{T}$ consists of a single spectrum E, then $\mathcal{T}$-localization is exactly Bousfield's notion of E-localization [4], and he proved that E-localizations always exist. His proof generalizes readily to the equivariant setting. (He works in Adams' version [**1**] of the stable category, but the argument is similar and a bit simpler in the equivariant stable category constructed in [**17**].) We omit the proof since we shall give explicit constructions of the localizations that we need.

Various elementary properties of E-localizations are listed in [4, §1], and they remain valid in the equivariant context. In particular, two spectra E and E' are said to be Bousfield equivalent if they determine the same class of homologically acyclic spectra. They then also determine the same class of equivalences, the same local spectra, and the same localization functor. It is an important observation that the function spectrum $F(E', X)$ is E-local for any spectrum E' that is Bousfield equivalent to E, by an immediate application of the adjunction

$$[W, F(E', X)] \cong [W \wedge E', X].$$

In fact, our completions will turn out to be just such function spectra.

More generally, we say that two classes $\mathcal{T}$ and $\mathcal{T}'$ are Bousfield equivalent if they determine the same homologically acyclic spectra. Again, they then determine the same equivalences, the same local spectra, and the same localizations. In practice, $\mathcal{T}$ is usually Bousfield equivalent to $\{E\}$ for some spectrum E.

Dually, we say that two classes $\mathcal{T}$ and $\mathcal{T}'$ are cohomologically equivalent if they determine the same class of cohomologically acyclic spectra. Of course, this is a different notion than Bousfield equivalence. The relationship between acyclic and complete spectra has a different flavor in the cohomological and homological contexts. In the former, we have the following tautological observation, which shows that taking the complete spectra is a kind of closure operation.

LEMMA 0.3. *Let $C\mathcal{T}$ be the class of $\mathcal{T}^*$-complete spectra. Then $\mathcal{T}$ and $C\mathcal{T}$ are cohomologically equivalent, and $C\mathcal{T}$ is the largest class of spectra that is cohomologically equivalent to $\mathcal{T}$.*

We will first define I-completions cohomologically, using a certain class $\mathcal{K}$ of I-torsion Eilenberg-MacLane spectra as test spectra. That class will be cohomologically equivalent to a larger class $\mathcal{T}$ of I-torsion spectra, and we will take $\mathcal{T}$-localization as our primary definition of I-completion. In turn, $\mathcal{T}$ will be Bousfield equivalent to $\{E\}$ for a certain spectrum E. We do not claim that $\mathcal{K}$ and $\{E\}$ are either cohomologically or Bousfield equivalent. However, when restricted to bounded below spectra, which seems to be the limit of utility of the cohomological notion, $\mathcal{K}^*$-completion and E-localization will automatically agree because the class of bounded below $\mathcal{K}^*$-acyclic spectra will turn out to be the same as the class of bounded below $\mathcal{T}$-acyclic spectra.

§1. A cohomological construction of completions at I.

Let G be a compact Lie group. We work in the stable category $\overline{h}G\mathcal{S}$ of G-spectra constructed in [**17**], and spectra and maps are understood to be G-spectra and G-maps throughout. Let $\mathcal{O}$ be the full subcategory of $\overline{h}G\mathcal{S}$ whose objects are the suspension spectra of orbits G/H_+ (where H is a closed subgroup of G and the plus denotes addition of a disjoint basepoint). A Mackey functor M is an additive contravariant functor $\mathcal{O} \to \mathcal{A}b$, written $M(G/H)$ on objects. When G is finite, this is equivalent to the standard definition of Dress [**9**], by [**17**, V.9.9]. Since $A(G) = [S,S]_G$ and S is the suspension spectrum of G/G_+, it follows formally from the definition that each $M(G/H)$ is an $A(G)$-module. For any spectrum X and integer n, we have the n^{th} homotopy group Mackey functor $\underline{\pi}_n(X)$. Its value on G/H is

$$\pi_n(X^H) = [G/H_+ \wedge S^n, X]_G,$$

and its contravariant functoriality on $\mathcal{O}$ is obvious.

Let I be an ideal of $A(G)$. For an $A(G)$-module M, we define $M_I^\wedge = \lim M/I^rM$. We define the I-adic completion of Mackey functors termwise, so that

$$M_I^\wedge(G/H) = M(G/H)_I^\wedge.$$

We extend any other functor on $A(G)$-modules to Mackey functors in the same termwise fashion. There results a map of Mackey functors $\gamma : M \to M_I^\wedge$. When M is of finite type, in the sense that it takes values in finitely generated Abelian groups and thus in finitely generated $A(G)$-modules, we call γ the I-adic completion of M. We say that a Mackey functor is "I-adic" if it takes values in finitely generated $A(G)_I^\wedge$-modules. Observe that $\gamma : M \to M_I^\wedge$ is an isomorphism if M is I-adic. Experience with the case $G = e$ and $I = (p)$ teaches us that, due to lack of exactness, I-adic completion cannot be the appropriate algebraic completion functor on general Mackey functors M. We shall define the appropriate general functor in section 3.

A Mackey functor M determines an Eilenberg-MacLane spectrum $HM = K(M,0)$ which is uniquely characterized up to isomorphism in $\overline{h}G\mathcal{S}$ by $\underline{\pi}_0(HM) = M$ and $\underline{\pi}_n(HM) = 0$ if $n \neq 0$. Maps $M \to M'$ of Mackey functors determine and are determined by maps $HM \to HM'$ of Eilenberg-MacLane spectra; we shall denote corresponding maps by the same letter. See [**16**] and [**17**, V§9] for background. We let $H_G^*(X; M)$ denote the cohomology theory on G-spectra X which is represented by HM. All of our cohomology theories will be taken to be Z-graded. However, since our theories are represented by G-spectra, they are $RO(G)$-gradable. When X is a based G-space, $H_G^*(X;M)$ is just the classical reduced Bredon cohomology of X [6].

Now return to the context of Definitions 0.1 and 0.2.

DEFINITION 1.1. *Let $\mathcal{K}_I$ be the class of Eilenberg-MacLane spectra $K(N,q)$, where N runs over the Mackey functors N such that $IN = 0$ and q runs over the integers. Note that a spectrum W is $\mathcal{K}_I^*$-acyclic if and only if $H_G^*(W;N) = 0$ for all such Mackey functors N.*

LEMMA 1.2. *Let W be $\mathcal{K}_I^*$-acyclic. Then $H_G^*(W;N) = 0$ if $I^rN = 0$ for some $r \geq 1$ or if $N = M_I^\wedge$ for some Mackey functor M. Therefore $H(M_I^\wedge)$ is $\mathcal{K}_I^*$-complete for any Mackey functor M.*

PROOF: Inductive use of Bockstein exact sequences gives the result when $I^rN = 0$. Since cohomology commutes with products in the coefficient variable, the conclusion for $M_I^\wedge$ follows from the Bockstein exact sequence associated to the evident short exact sequence

$$0 \to M_I^\wedge \to \prod M/I^rM \to \prod M/I^rM \to 0.$$

A spectrum X is said to be bounded below if the groups $\pi_n(X^H)$ are zero for all H and all n less than some fixed n_0.

PROPOSITION 1.3. *If Z is bounded below and if each $H\underline{\pi}_n(Z)$ is $\mathcal{K}_I^*$-complete, then Z is $\mathcal{K}_I^*$-complete.*

PROOF: Since Z is bounded below, it has a Postnikov decomposition. The conclusion follows by induction up the tower and passage to limits. In fact, the dualization of the proof of the Whitehead theorem for CW-complexes that is explained on the space level in [18, Theorems 5*, 5#] works equally well on the spectrum level. The essential technical point is that, in the good category of spectra of [17], we can work with fibration sequences of spectra exactly as we work with fibration sequences of spaces. (Even nonequivariantly, such an argument would not work quite so simply in traditional categories of CW-spectra, which bear essentially the same relationship to our category of spectra as the category of CW-complexes and cellular maps bears to the category of general spaces.)

Henceforward in this section, we assume that $A(G)$ is Noetherian. This holds if and only if $A(G)$ is a finitely generated Abelian group. More concretely, it holds if and only if the Weyl group of G acts trivially on the maximal torus or, equivalently, if G is a central toral extension over a finite group; see [**8**, §5.10]. By the latter description, $A(H)$ is Noetherian for all (closed) subgroups H of G. These are precisely the compact Lie groups G for which the most naive version of the Segal conjecture is valid [**24**]. With this assumption, the Artin-Rees lemma implies that I-adic completion is an exact functor on the category of Mackey functors of finite type.

Of course, provided that it exists, the $\mathcal{K}_I^*$-completion of a spectrum X is unique up to canonical equivalence since it is specified by a universal property. We shall construct $\mathcal{K}_I^*$-completions by induction up Postnikov towers, and the following cohomological result will allow us to start the induction. It will be given a best possible generalization in Theorem 3.2, so we omit the short direct proof that was found by Hopkins.

THEOREM 1.4. *If M is a Mackey functor of finite type, then the map $\gamma : HM \to H(M_I^\wedge)$ is a $\mathcal{K}_I^*$-equivalence and therefore a $\mathcal{K}_I^*$-completion.*

The homology theory represented by the Burnside ring Mackey functor $\underline{A}$, $\underline{A}(G/H) = A(H)$, plays the same role equivariantly that integral homology plays nonequivariantly, and we have a Mackey functor version of this theory with $\underline{H}_n(X;\underline{A}) = \underline{\pi}_n(X \wedge H\underline{A})$. Since $H\underline{A}$ can be constructed by killing the higher homotopy groups of S, we see that there is a Hurewicz isomorphism $\underline{\pi}_n(X) \cong \underline{H}_n(X;\underline{A})$ when X is (n-1)-connected.

We can now state and prove the main results of this section.

PROPOSITION 1.5. *Let Z be bounded below. The following conditions on Z are equivalent, and they imply that Z is $\mathcal{K}_I^*$-complete.*

(i) *Each Mackey functor $\underline{\pi}_n(Z)$ is I-adic.*

(ii) *$[K, Z]_G$ is I-adic for all finite G-CW spectra K.*

If G is finite, the following condition is also equivalent to (i).

(iii) *Each Mackey functor $\underline{H}_n(Z;\underline{A})$ is I-adic.*

PROOF: By Lemma 1.2 and Proposition 1.3, (i) implies that Z is I-complete.

(i) if and only if (ii). Since I-adic completion is exact, (i) implies (ii) by induction on the number of cells of K; (ii) implies (i) by taking $K = (G/H)_+ \wedge S^n$ for any subgroup H and integer n.

(ii) if and only if (iii). Given (ii), Spanier-Whitehead duality implies that $\underline{\pi}_*(Z \wedge K)$ is I-adic for any finite G-spectrum K. The skeleta of $H\underline{A}$ are finite if G is finite, and (iii) follows. Conversely, let Z_n be the (n-1)-connected cover of Z and consider the fibration $Z_{n+1} \to Z_n \to K(\underline{\pi}_n(Z), n)$. Assume inductively that $\underline{H}_*(Z_n; \underline{A})$ is I-adic and that $\underline{\pi}_q(Z)$ is I-adic for $q \le n$. By the implication (ii) implies (iii), $\underline{H}_*(K(\underline{\pi}_n(Z), n); \underline{A})$ is I-adic, and it follows that $\underline{H}_*(Z_{n+1}; \underline{A})$ is I-adic. By the Hurewicz theorem, $\underline{\pi}_{n+1}(Z) \cong \underline{\pi}_{n+1}(Z_{n+1})$ is isomorphic to $\underline{H}_{n+1}(Z_{n+1}; \underline{A})$ and is therefore I-adic.

A spectrum X is said to be of finite type if each $\pi_n(X^H)$ is finitely generated.

THEOREM 1.6. *Let X be bounded below and of finite type. Then the following conditions on a map $\gamma : X \to X_I^\wedge$ from X to a $\mathcal{K}_I^*$-complete spectrum are equivalent. Moreover, there exists one and, up to homotopy, only one such map γ.*

(i) *γ is a $\mathcal{K}_I^*$-equivalence; that is, γ is a $\mathcal{K}_I^*$-completion of X.*

(ii) *Each $\gamma_* : \underline{\pi}_n(X) \to \underline{\pi}_n(X_I^\wedge)$ is I-adic completion.*

(iii) *$\gamma_* : [K, X]_G \to [K, X_I^\wedge]_G$ is I-adic completion for all finite K.*

If G is finite, the following condition is also equivalent to (i)–(iii).

(iv) *$X_I^\wedge$ is bounded below and $\gamma_* : \underline{H}_*(X; \underline{A}) \to \underline{H}_*(X_I^\wedge; \underline{A})$ is I-adic completion.*

PROOF: Conditions (ii), (iii), and (iv) are equivalent by arguments exactly like those in the previous proof. We shall construct γ satisfying (i) through (iv). Since (i) implies the uniqueness of γ, it will follow that (i) implies (ii). We first give the promised construction and then prove that (ii) implies (i). We may replace X by a Postnikov tower $\lim X_n$, and we construct $(X_n)_I^\wedge$ inductively; X_n is the trivial G-spectrum for sufficiently small n, and we take $(X_n)_I^\wedge$ to be trivial for such n. Assume given a map $\gamma_n : X_n \to (X_n)_I^\wedge$ which satisfies (i) and (ii) and consider the following diagram:

$$\begin{array}{ccccccc}
K(\underline{\pi}_{n+1}(X), n+1) & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \xrightarrow{k} & K(\underline{\pi}_{n+1}(X), n+2) \\
\gamma\downarrow & & \gamma_{n+1}\downarrow & & \gamma_n\downarrow & & \downarrow\gamma \\
K(\underline{\pi}_{n+1}(X)_I^\wedge, n+1) & \longrightarrow & (X_{n+1})_I^\wedge & \longrightarrow & (X_n)_I^\wedge & \xrightarrow{k_I^\wedge} & K(\underline{\pi}_{n+1}(X)_I^\wedge, n+2).
\end{array}$$

The top row is the fiber sequence induced by the k-invariant $k = k^{n+2}$. Since γ_n is an I-completion and $K(\underline{\pi}_{n+1}(X)_I^\wedge, n+2)$ is I-complete, there is a map $k_I^\wedge : (X_n)_I^\wedge \to K(\underline{\pi}_{n+1}(X)_I^\wedge, n+2)$, unique up to homotopy, such that the right square commutes up to homotopy. The bottom row is defined to be the fiber sequence induced by $k_I^\wedge$. A standard fiber sequence argument gives a map γ_{n+1} which makes the left square commute up to homotopy and the middle square commute on the nose. By the five lemma, γ_{n+1} is a $\mathcal{K}_I^*$-equivalence, and γ_{n+1} clearly satisfies (ii). By the previous proposition, $(X_{n+1})_I^\wedge$ is $\mathcal{K}_I^*$-complete. The commutativity of the middle squares for all n allows us to define $X_I^\wedge = \lim(X_n)_I^\wedge$ and obtain a well-defined inverse limit map $\gamma : X \to X_I^\wedge$. Clearly $H_G^*(X;M)$ is the colimit of the $H_G^*(X_n;M)$, the colimit being achieved in each degree, and similarly for $X_I^\wedge$. Therefore γ is a $\mathcal{K}_I^*$-equivalence, and γ clearly satisfies (ii). Again by the previous proposition, $X_I^\wedge$ is $\mathcal{K}_I^*$-complete. Returning to the remaining implication, assume (ii). By "cocellular approximation", we may assume that X and $X_I^\wedge$ are given as Postnikov towers and that γ is cocellular, that is, the inverse limit of a tower of compatible maps $X_n \to (X_I^\wedge)_n$. We then have diagrams like those of the construction and can deduce (i).

REMARK 1.7: Even for nonequivariant p-adic completion, the integral homological conditions in the above two results do not appear in the literature. It is critical to observe that these characterizations only work stably, because of our use of duality in their proofs. Bousfield and Kan [**5**, 5.7] point out that the integral homology of the (unstable) p-adic completion of an n-sphere, n odd, contains large $\mathbb{Q}$-modules in dimensions kn for $k \geq 2$.

§2. The general construction of completions at I.

Return again to the context of Definitions 0.1 and 0.2. We must specify the appropriate class $\mathcal{T}_I$ of test spectra.

DEFINITION 2.1. *Let I be an ideal in $A(G)$. A spectrum T is an I-torsion spectrum if, for each $\alpha \in I$, there exists a positive integer k such that $\alpha^k : T \to T$ is null homotopic. Observe that, if I is finitely generated, we can choose a k such that $\alpha^k : T \to T$ is null homotopic for all $\alpha \in I$. let $\mathcal{T}_I$ be the collection of all I-torsion spectra. Define the I-completion of X (or the completion of X at I), denoted $X_I^\wedge$, to be its $\mathcal{T}_I$-localization.*

When I is finitely generated, we shall prove shortly that $X_I^\wedge$ exists for any X. In general, an ideal I may be countably generated. We are confident that I-completions still exist but, as explained at the end of the section, we do not have a construction starting from the given definitions (which may not be quite the right ones for infinitely generated ideals). We assume that I is finitely generated except where otherwise specified.

For $\alpha \in A(G)$, let S/α be the cofiber of $\alpha : S \to S$, let $S[\alpha^{-1}]$ be the telescope of countably many instances of $\alpha : S \to S$, and let $M(\alpha)$ be the fiber of the evident inclusion $S \to S[\alpha^{-1}]$. For $r \geq 1$, there are maps of cofiber sequences

$$\begin{array}{ccccccc} S & \xrightarrow{\alpha^r} & S & \longrightarrow & S/\alpha^r & \longrightarrow & S^1 \\ \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overline{\alpha}} & & \| \\ S & \xrightarrow{\alpha^{r+1}} & S & \longrightarrow & S/\alpha^{r+1} & \longrightarrow & S^1. \end{array}$$

Passing to telescopes, we obtain the cofiber sequence

$$S \to S[\alpha^{-1}] \to S/\alpha^\infty \to S^1.$$

Thus $M(\alpha) \simeq \Sigma^{-1} S/\alpha^\infty$. For a finite set $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ of elements of $A(G)$, let $S/\underline{\alpha} = S/\alpha_1 \wedge \cdots \wedge S/\alpha_n$ and $M(\underline{\alpha}) = M(\alpha_1) \wedge \cdots \wedge M(\alpha_n)$. By Lemma 5.5 below, $M(\underline{\alpha})$ is equivalent to a G-CW spectrum whose cellular chain complex admits a convenient algebraic description.

The following result implies that I-completion, alias $\mathcal{T}_I$-localization, is the same thing as Bousfield localization at either $S/\underline{\alpha}$ or $M(\underline{\alpha})$.

THEOREM 2.2. *Let $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of generators for the ideal I. Then the following properties of a spectrum W are equivalent.*
(i) W is I-acyclic: $W \wedge T$ is contractible for all I-torsion spectra T.
(ii) W is $S/\underline{\alpha}$-acyclic: $W \wedge S/\underline{\alpha}$ is contractible.
(iii) W is $M(\underline{\alpha})$-acyclic: $W \wedge M(\underline{\alpha})$ is contractible.
That is, $\mathcal{T}_I$ is Bousfield equivalent to $\{S/\underline{\alpha}\}$ and $S/\underline{\alpha}$ is Bousfield equivalent to $M(\underline{\alpha})$.

PROOF: Let $\underline{\beta} = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $J = (\alpha_1, \ldots, \alpha_{n-1}) \subset I$, and $\alpha = \alpha_n$. We proceed by induction on n.

Step 1: the case $n = 1$: Clearly $\alpha^2 : S/\alpha \to S/\alpha$ is null homotopic, hence so is $(b\alpha)^2$ for any $b \in A(G)$. Thus S/α is an I-torsion spectrum and (i) implies (ii). Assume (ii). By Verdier's axiom in the stable category, there are cofibrations $S/\alpha \to S/\alpha^r \to S/\alpha^{r-1}$. Inductively, $W \wedge S/\alpha^r$ is contractible for all r. Since telescopes commute with smash products, $W \wedge M(\alpha)$ is contractible. Now assume (iii). Then $S \wedge W \to S[\alpha^{-1}] \wedge W$ is an equivalence and therefore so is $\alpha : W \to W$. If T is an I-torsion spectrum, then $W \wedge T$ is contractible since $\alpha : W \wedge T \to W \wedge T$ is a nilpotent equivalence. Therefore (i)–(iii) are equivalent when $I = (\alpha)$.

Step 2: the inductive step. Assume that (i)–(iii) are equivalent for $\underline{\beta}$ and observe that $S/\underline{\alpha} = S/\underline{\beta} \wedge S/\alpha$ and $M(\underline{\alpha}) = M(\underline{\beta}) \wedge M(\alpha)$. If $\lambda = \sum b_i \alpha_i \in I$, then $\lambda^{2n} : S/\underline{\alpha} \to S/\underline{\alpha}$ is null homotopic. Thus $S/\underline{\alpha}$ is an I-torsion spectrum and (i) implies (ii). Assume (ii). Then $W \wedge S/\alpha \wedge M(\underline{\beta})$ is contractible by the induction hypothesis and $W \wedge M(\underline{\alpha})$ is contractible by Step 1. Finally, assume (iii). Since an I-torsion spectrum T is also a J-torsion spectrum, $W \wedge M(\alpha) \wedge T$ is contractible by the induction hypothesis. As in Step 1, $W \wedge T$ is then contractible since the map $\alpha : W \wedge T \to W \wedge T$ is a nilpotent equivalence.

We now change notation and write $M(\underline{\alpha}) = M(I)$. Up to equivalence, $M(I)$ is in fact independent of the choice of generating set $\underline{\alpha}$ (as would be false for $S/\underline{\alpha}$). Observe that there is a canonical map $M(I) \to S$ and thus a natural map $X = F(S, X) \to F(M(I), X)$. We have already noted (in §0) that its target is $M(I)$-local and thus I-complete.

THEOREM 2.3. *The natural map $X \to F(M(I), X)$ is an I-completion.*

PROOF: We use the notations of the previous theorem and its proof. By that result, it suffices to show that $X \to F(M(\underline{\alpha}), X)$ is an $S/\underline{\alpha}$-equivalence. We proceed by induction on n.

Step 1: the case $n = 1$. We must show that $X \wedge S/\alpha \to F(M(\alpha), X) \wedge S/\alpha$ is an equivalence or, equivalently, that $F(S[\alpha^{-1}], X) \wedge S/\alpha$ is contractible. Since S/α is finite and is the dual of S^{-1}/α, standard equivalences give

$$F(S[\alpha^{-1}], X) \wedge S/\alpha \simeq F(S[\alpha^{-1}], X \wedge S/\alpha) \simeq F(S[\alpha^{-1}] \wedge S^{-1}/\alpha, X).$$

Since $\alpha : S[\alpha^{-1}] \to S[\alpha^{-1}]$ is an equivalence and $\alpha^2 : S^{-1}/\alpha \to S^{-1}/\alpha$ is null homotopic, $S[\alpha^{-1}] \wedge S^{-1}/\alpha$ is contractible.

Step 2: the inductive step. The map $X \to F(M(\underline{\alpha}), X)$ can be written as the composite

$$X \to F(M(\underline{\beta}), X) \to F(M(\alpha), F(M(\underline{\beta}), X)) \simeq F(M(\underline{\alpha}), X).$$

Smashed with $S/\underline{\alpha} = S/\underline{\beta} \wedge S/\alpha$, the first map becomes an equivalence by the induction hypothesis and the second map becomes an equivalence by the case $n = 1$ applied with X replaced by $F(M(\underline{\beta}), X)$.

This explicit construction of I-completions has the following important immediate consequence.

COROLLARY 2.4. *For any spectra X and Y,*

$$F(X, Y_I^\wedge) = F(X, F(M(I), Y)) \cong F(M(I), F(X, Y)) = F(X, Y)_I^\wedge.$$

Therefore, if $\gamma : Y \to Y_I^\wedge$ is an I completion, then so is

$$\gamma_* : F(X, Y) \to F(X, Y_I^\wedge).$$

We must still connect up our two constructions of completions. The following parenthetical observation was suggested by the referee.

PROPOSITION 2.5. *Let $\mathcal{T}_I^!$ be the class of bounded below I-torsion spectra.*
(i) *$\mathcal{T}_I$ is Bousfield equivalent to $\mathcal{T}_I^!$.*
(ii) *$\mathcal{T}_I^!$ is cohomologically equivalent to $\mathcal{K}_I$.*

PROOF: Part (i) is immediate from Theorem 2.2. Part (ii) is implied by Lemma 1.2 and Proposition 1.3 since, if T is an I-torsion spectrum, then some power of I annihilates each $\underline{\pi}_n(T)$.

Although we do not know how to make mathematical use of this fact, it may lend plausibility to the following comparison of our two notions of completion at I. For clarity, we use the language of Definition 0.2.

THEOREM 2.6. *Let W, X, Y, and Z be spectra and let $f : X \to Y$ be a map.*
(i) *If W is $\mathcal{T}_I$-acyclic, then W is $\mathcal{K}_I^*$-acyclic.*
(ii) *If W is bounded below and $\mathcal{K}_I^*$-acyclic, then W is $\mathcal{T}_I$-acyclic.*
(iii) *If Z is $\mathcal{K}_I^*$-complete, then Z is $\mathcal{T}_I$-local.*
(iv) *If Z is bounded below and $\mathcal{T}_I$-local, then Z is $\mathcal{K}_I^*$-complete.*
(v) *If X is bounded below and $\gamma : X \to X_I^\wedge$ is a $\mathcal{T}_I$-localization, then γ is a $\mathcal{K}_I^*$-completion.*
(vi) *If X is bounded below and $\gamma : X \to X_I^\wedge$ is a $\mathcal{K}_I^*$-completion, then γ is a $\mathcal{T}_I$-localization.*

Obviously (i) implies (iii), and (i) and (iv) imply (v). Moreover, (vi) implies (ii) since, if W is $\mathcal{K}_I^*$-acyclic, then $W \to *$ is a $\mathcal{K}_I^*$-completion. Thus it suffices to prove (i), (iv), and (vi).

PROOF OF (i). Let $I = (J, \alpha)$. The result is trivial if $I = 0$ and we assume inductively that it holds for J. Assume that $IN = 0$ and thus $JN = 0$ and $\alpha N = 0$. Suppose that W is $\mathcal{T}_I$-acyclic and note that $W \wedge S^{-1}/\alpha$ is then

$\mathcal{T}_J$-acyclic. Since $S^{-1}/\alpha \simeq D(S/\alpha)$, we have isomorphisms

$$[W,(HN)/\alpha]_G^* \cong [W, HN\wedge S/\alpha]_G^* \cong [W\wedge S^{-1}/\alpha, HN]_G^* = H_G^*(W\wedge S^{-1}/\alpha; N).$$

These groups are zero because $W\wedge S^{-1}/\alpha \simeq *$ if $J=0$, and by the induction hypothesis if $J \neq 0$. Since $\alpha N = 0$, the evident cofibration $HN \to HN \to (HN)/\alpha$ gives rise to short exact sequences

$$0 \to H_G^q(W;N) \to [W,(HN)/\alpha]_G^q \to H_G^{q+1}(W;N) \to 0.$$

Therefore $H_G^*(W;N) = 0$.

Proof of (iv). In the next section, we will give a new algebraic definition of an "I-complete Mackey functor". If Z is $\mathcal{T}_I$-local, the homotopy group Mackey functors $\underline{\pi}_*(Z)$ are I-complete by Theorems 3.3 and 3.5 and the spectra $H\underline{\pi}_*(Z)$ are therefore $\mathcal{K}_I^*$-complete by Theorem 3.1. If, further, Z is bounded below, then Z is $\mathcal{K}_I^*$-complete by Proposition 1.3.

Proof of (vi). By (v), the $\mathcal{T}_I$-localization $\gamma : HM \to (HM)_I^\wedge$ is a $\mathcal{K}_I^*$-completion for any M. If X is bounded below, then its $\mathcal{K}_I^*$-completion $\gamma : X \to X_I^\wedge$ can and, up to equivalence, must be constructed by induction up the Postnikov tower, exactly as in the proof of Theorem 1.6 but with $K(M_I^\wedge, n)$'s replaced by $K(M,N)_I^\wedge$'s. Since the functor $F(M(I), ?)$ preserves fibrations and limits, the $\mathcal{T}_I$-localizations $F(M(I), X_n)$ and $F(M(I), X)$ give a precise realization of this inductive construction of the $\mathcal{K}_I^*$-completion of X.

We complete this section by considering the behavior of completions with respect to inclusions of ideals. The following observations are easily verified from the definitions.

Proposition 2.7. *Let $J \subset I$, where I is any ideal, not necessarily finitely generated. Then the following conclusions hold.*

(i) *If T is an I-torsion spectrum, then T is a J-torsion spectrum.*

(ii) *T is an I-torsion spectrum if and only if T is a J-torsion spectrum for all finitely generated ideals $J \subset I$.*

(iii) *If W is a J-acyclic spectrum, then W is an I-acyclic spectrum.*

(iv) *If $f : X \to Y$ is a J-equivalence, then f is an I-equivalence.*

(v) *If Z is I-complete, then Z is J-complete.*

(vi) *If it exists, the completion* $X \to X_I^\wedge$ *of* X *at* I *factors uniquely through the completion* $X \to X_J^\wedge$ *of* X *at* J, *and the resulting map* $X_J^\wedge \to X_I^\wedge$ *is the completion of* $X_J^\wedge$ *at* I.

(vii) *Let* $Y = \operatorname{hocolim} X_J^\wedge$, *where the homotopy colimit runs over the set of finitely generated subideals* $J \subset I$. *Then any map from* X *to an* I*-complete spectrum factors uniquely through the canonical map* $X \to Y$, *and* $\underline{\pi}_*(Y) \cong \operatorname{colim} \underline{\pi}_*(X_J^\wedge)$.

PROOF OF (vii): If Z is I-complete, then Z is J-complete for all J and, for $J \subset J'$, the map $[X_{J'}^\wedge, Z]_G \to [X_J^\wedge, Z]_G$ induced by $X_J^\wedge \to X_{J'}^\wedge$ is an isomorphism. Therefore the relevant $\lim^1$ term vanishes and the canonical map $[X_I^\wedge, Z]_G \to \lim[X_J^\wedge, Z]_G$ is an isomorphism. The rest follows.

Assuming that I-completions exist, we conclude that $X_I^\wedge \cong Y_I^\wedge$. Of course, Y itself would be $X_I^\wedge$ if it were I-complete, but presumably this fails.

§3. Statements of results about the homotopy groups of completions at I.

We wish to describe the homotopy groups $X_I^\wedge$ in terms of the homotopy groups of X. In general, the I-adic completion functor is neither left nor even right exact. Its left derived functors L_i^I are studied in [**12**]. They are constructed on an $A(G)$-module M simply by taking the homology of the complex obtained by applying I-adic completion to a free resolution of M. There is a natural epimorphism $\varepsilon : L_0^I M \to M_I^\wedge$, which is an isomorphism when $A(G)$ is Noetherian and M is finitely generated. In general, the natural map $\gamma : M \to M_I^\wedge$ factors as $\varepsilon \circ \eta$, $\eta : M \to L_0^I M$. Setting aside any preconceived algebraic ideas, let us say that a Mackey functor M is I-complete if the natural map $\eta : M \to L_0^I M$ is an isomorphism. The following definitive generalization of Lemma 1.2 indicates the relevance of this notion to cohomology. It will be proven in section 5.

THEOREM 3.1. *The following conditions on a Mackey functor* M *are equivalent.*

(i) HM *is* $\mathcal{K}_I^*$*-complete: if* W *is* $\mathcal{K}_I^*$*-acyclic, then* $H_G^*(W; M) = 0$.

(ii) HM *is* I*-complete: if* W *is* $\mathcal{T}_I$*-acyclic, then* $[W, HM]_G = 0$.

(iii) M *is* I*-complete:* $\eta : M \to L_0^I M$ *is an isomorphism.*

Similarly, the following result is the definitive generalization of Theorem 1.4 (compare Theorem 3.11 below). Parts (ii) and (iii) are immediate consequences of part (i) and Theorem 2.6.

THEOREM 3.2. *Let M be a Mackey functor.*

(i) *The I-completion of HM is a two-stage Postnikov system with*

$$\underline{\pi}_0((HM)_I^\wedge) = L_0^I(M) \quad \textit{and} \quad \underline{\pi}_1((HM)_I^\wedge) = L_1^I(M).$$

(ii) *If $L_I^1 M = 0$, then $\gamma : HM \to (HM)_I^\wedge = H(L_0^I M)$ is a $\mathcal{K}_I^*$-equivalence.*

(iii) *If $L_I^1 M \neq 0$, then there does not exist a $\mathcal{K}_I^*$-equivalence $HM \to HM'$ where HM' is $\mathcal{K}_I^*$-complete.*

Theorem 3.2(i) will also be proven in section 5, where it will be shown to imply the first part of the following theorem. The second part follows from the first by Corollary 2.4. Let Y_G^* denote the cohomology theory represented by a spectrum Y.

THEOREM 3.3. *Let X and Y be any spectra.*

(i) *There are natural short exact sequences*

$$0 \to L_1^I(\underline{\pi}_{q-1}(X)) \to \underline{\pi}_q(X_I^\wedge) \to L_0^I(\underline{\pi}_q(X)) \to 0.$$

(ii) *There are natural short exact sequences*

$$0 \to L_1^I(Y_G^{q+1}(X)) \to (Y_I^\wedge)_G^q(X) \to L_0^I(Y_G^q(X)) \to 0.$$

As a matter of algebra, the $L_i^I(M)$ admit the following descriptions. We abbreviate $A = A(G)$ in the rest of this section.

THEOREM 3.4. *Let M be either an A-module or a Mackey functor.*

(i) *There is a natural short exact sequence*

$$0 \to \lim{}^1 \operatorname{Tor}_1^A(A/I^r, M) \to L_0^I(M) \to M_I^\wedge \to 0.$$

(ii) *There is a natural short exact sequence*

$$0 \to \lim{}^1 \operatorname{Tor}_2^A(A/I^r, M) \to L_1^I(M) \to \lim \operatorname{Tor}_1^A(A/I^r, M) \to 0;$$

if I is a principal ideal, then the $\lim^1$ kernel term is zero.

(iii) *$L_i^I(M) = 0$ for $i \geq 2$.*

Except for (iii), this is proven in [**12**, 1.1 and 1.5]. The notion of an I-complete Mackey functor is clarified by the following result, which is proven in [**12**, 4.1].

THEOREM 3.5. *For any Mackey functor M, the Mackey functors $M_I^\wedge$, $L_0^I M$, and $L_1^I M$ are I-complete. If M is I-complete, then $L_1^I M = 0$.*

So far we have stated our results in a language that should be reasonably familiar to topologists. However, Grothendieck's local cohomology groups $H_I^n(M)$ and certain analogous local homology groups $H_n^I(M)$ of A-modules M play an essential intermediate role in the passage from the topology to the algebra. We shall give the relevant definitions in section 5. As the remarks after the next theorem make clear, local cohomology occurred implicitly in the Bousfield-Kan treatment [**5**] of p-adic completion. The following result is proven in [**12**, 2.5 and 3.5].

THEOREM 3.6. *There are natural isomorphisms $L_i^I(M) \cong H_i^I(M)$, and*
(i) *there is a natural short exact sequence*

$$0 \to \mathrm{Ext}_A^1(H_I^1(A), M) \to L_0^I(M) \to \mathrm{Hom}_A(H_I^0(A), M) \\ \to \mathrm{Ext}_A^2(H_I^1(A), M) \to 0;$$

(ii) *there is a natural isomorphism*

$$L_1^I(M) \cong \mathrm{Hom}_A(H_I^1(A), M).$$

REMARK 3.7: Working nonequivariantly, with $A = Z$, suppose that $I = (p)$ for a prime p. Then $H_I^0(A) = 0$ and $H_I^1(A) = Z/p^\infty$. Therefore

$$L_0^{(p)}(M) = \mathrm{Ext}(Z/p^\infty, M) \quad \text{and} \quad L_1^{(p)}(M) = \mathrm{Hom}(Z/p^\infty, M),$$

as was first observed by Bousfield and Kan [**5**, VI.2.1]. Since $\mathrm{Tor}(Z/qZ, M)$ is the kernel $\Gamma(q; M)$ of $q : M \to M$, Theorem 3.4 gives a short exact sequence

$$0 \to \lim{}^1 \Gamma(p^r; M) \to L_0^{(p)}(M) \to M_I^\wedge \to 0$$

and reinterprets $L_1^{(p)}(M)$ as $\lim \Gamma(p^r; M)$. Therefore, $L_0^{(p)}(M) = M_p^\wedge$ and $L_1^{(p)}(M) = 0$ if the p-power torsion of M is of bounded order.

There is an analog of the last statement in our general context.

DEFINITION 3.8. *For an A-module or Mackey functor M and an element $\alpha \in A$, let $\Gamma(\alpha; M)$ be the kernel of $\alpha : M \to M$. Observe that $\Gamma(\alpha^r; M)$ is contained in $\Gamma(\alpha^{r+1}; M)$ and say that M has bounded α-torsion if this* *ascending chain stabilizes. For A itself, $\Gamma(\alpha; A) = \Gamma(\alpha^2; M)$ since A is a subring of a product of copies of Z. Say that a sequence $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ of elements of A is pro-regular for M if $M/(\alpha_1, \dots, \alpha_{i-1})^r M$ has bounded α_i-torsion for $r \geq 0$ and $1 \leq i \leq n$.*

Clearly any $\underline{\alpha}$ is pro-regular for M if A is Noetherian and M is finitely generated. The following result, which will be proven in section 6, implies that any $\underline{\alpha}$ is pro-regular for M when M is A itself and therefore when M is any free A-module.

THEOREM 3.9. *For any finitely generated ideal J and any element $\alpha \in A$, the A-module A/J has bounded α-torsion.*

Given Theorem 3.9, the following result is a special case of [**12**, 1.9]. Actually, its first conclusion holds under a somewhat weaker notion of a "pro-regular sequence" than that specified in Definition 3.8; see [**12**, 1.8]

THEOREM 3.10. *Let $I = (\alpha_1, \dots, \alpha_n)$ and write $J = (\alpha_1, \dots, \alpha_{n-1})$ and $\alpha = \alpha_n$. If $\underline{\alpha}$ is a pro-regular sequence for M, then $L_0^I(M) \cong M_I^\wedge$ and $L_i^I(M) = 0$ for $i > 0$. Moreover, the following conclusions hold for any M.*
(i) $L_0^I(M) \cong L_0^\alpha(L_0^J(M))$.
(ii) *There is a short exact sequence*

$$0 \to L_0^\alpha(L_1^J(M)) \to L_1^I(M) \to L_1^\alpha(L_0^J(M)) \to 0.$$

Theorems 3.3 and 3.10 have the following implication.

THEOREM 3.11. *If $\underline{\alpha}$ is a pro-regular sequence for each $\underline{\pi}_q(X)$, then $\gamma_* : \underline{\pi}_*(X) \to \underline{\pi}_*(X_I^\wedge)$ is completion at I. In particular, if $\underline{\alpha}$ is a pro-regular sequence for M, then $(HM)_I^\wedge = H(M_I^\wedge)$.*

§4. The Segal conjecture and the Atiyah-Segal completion theorem.

Let G be finite in this section. Recall that a family $\mathcal{F}$ in G is a collection of subgroups of G which is closed under subconjugacy. A G-space X is an $\mathcal{F}$-space if X^H is empty for $H \notin \mathcal{F}$. There is a universal $\mathcal{F}$-space $E\mathcal{F}$ characterized by the property that $(E\mathcal{F})^H$ is non-empty and contractible for $H \in \mathcal{F}$. A family determines an ideal $I\mathcal{F}$ in $A(G)$, namely the intersection over $H \in \mathcal{F}$ of the kernels I_H^G of the restriction homomorphisms $A(G) \to A(H)$.

The cohomology groups of a spectrum X with coefficients in a spectrum Y are the homotopy groups of $F(X,Y)^G$. In particular, the stable cohomotopy groups of X are the homotopy groups of $D(X)^G$. If X is a finite G-CW spectrum and Y is a G-CW spectrum with finite skeleta, then $F(X,Y)$ is bounded below and of finite type, so that the elementary cohomological construction of completions of Theorem 1.6 applies and we can recognize completions by their behavior on homotopy groups. In any case, it is natural to ask if the $I\mathcal{F}$-completion of $F(X,Y)$ is equivalent to $F(E\mathcal{F}_+ \wedge X, Y)$. The latter kind of geometric completion has long played an important role in the study of equivariant cohomology theories; see for example [**8**, Ch.7; **17**, Ch.V].

Our definitions imply that, for any ideal $I \subset A(G)$ and any $H \subset G$, $M(I)$ regarded as an H-spectrum is $M(r^G_H(I)A(H))$, where r^G_H is the restriction $A(G) \to A(H)$; moreover, $M(0) \simeq S$. For a family $\mathcal{F}$, the canonical map $M(I\mathcal{F}) \to S$ is an $\mathcal{F}$-equivalence, in the sense that it is an H-equivalence for all $H \in \mathcal{F}$. Therefore, by the $\mathcal{F}$-Whitehead theorem [**17**, II.2.2], there is a unique map $\xi : \Sigma^\infty E\mathcal{F}_+ \to M(I\mathcal{F})$ over S. Thus, for any spectra X and Y, we have a commutative diagram of canonical maps

$$\begin{array}{ccc} & \nearrow^{\gamma} & F(M(I\mathcal{F}) \wedge X, Y) \cong F(X, F(M(I\mathcal{F}), Y)) \\ F(X,Y) & & \downarrow \xi^* \qquad\qquad \downarrow \xi^* \\ & \searrow_{\pi^*} & F(E\mathcal{F}_+ \wedge X, Y) \cong F(X, F(E\mathcal{F}_+, Y)). \end{array}$$

Via Corollary 22.4, this implies a striking reinterpretation of completion theorems in terms of the cohomological behavior of the map ξ; via Theorem 3.3(ii), it also implies a calculational generalization of such theorems.

THEOREM 4.1. *The map $\xi^* : F(M(I\mathcal{F}), Y) \to F(E\mathcal{F}_+, Y)$ is an equivalence if and only if the map $\pi^* : Y \to F(E\mathcal{F}_+, Y)$ is an $I\mathcal{F}$-completion. When this holds,*

$$\pi^* : F(X,Y) \to F(E\mathcal{F}_+ \wedge X, Y)$$

is an $I\mathcal{F}$-completion for any X, and there are short exact sequences

$$0 \to L_1^{I\mathcal{F}}(Y_G^{q+1}(X)) \to Y_G^q(E\mathcal{F}_+ \wedge X) \to L_0^{I\mathcal{F}}(Y_G^q(X)) \to 0.$$

In particular, if Y has finite skeleta and $\pi^* : Y \to F(E\mathcal{F}_+, Y)$ induces I-adic completion on homotopy groups or satisfies one of the other equivalent conditions of Theorem 1.6, then we can deduce the conclusion of the theorem for arbitrary X. This allows us to generalize all of the existing completion theorems to the corresponding represented theories without doing any further work. For example, the generalized Segal conjecture proved in [**3**, 1.6] asserts that $S \to D(E\mathcal{F}_+)$ is an $I\mathcal{F}$-completion for any family $\mathcal{F}$. The arguments of [**10**] [**11**] interpret the conclusion in terms of ordinary p-adic completion.

There is no general criterion for determining whether or not the conclusion of Theorem 4.1 holds for a given spectrum Y. It fails if Y represents connective equivariant K-theory, and it also fails for the suspension spectra of general G-spaces. The following results, which are immediate consequences of [**23**, 1.1] and [**7**, 3.1], respectively, give the most general positive conclusions presently available.

THEOREM 4.2. *Let $G = \Gamma/\Pi$, where Γ is a compact Lie group and Π is a normal subgroup, let $E(\Pi,\Gamma)$ be the universal Π-free Γ-space, and let $Y = E(\Pi,\Gamma)_+ \wedge_\Pi Z$ for a finite Γ-CW spectrum Z. Then $\pi^* : Y \to F(E\mathcal{F}_+, Y)$ is an $I\mathcal{F}$-completion for any family $\mathcal{F}$.*

THEOREM 4.3. *Let Y be the suspension G-spectrum of the function G-space $F(K,L)$, where K is a G-space of finite type and L is a finite G-CW complex such that, for $H \subset G$, L^H is a simple space and $\pi_i(L^H) = 0$ if $i \leq \dim(K^H)$. Then $\pi^* : Y \to F(EG_+, Y)$ is an IG-completion, where IG is the augmentation ideal of $A(G)$.*

Both of these results generalize the original Segal conjecture, which is the case $Y = S$ and $I = IG$. In this case, we have

$$\pi_G^q(EG_+ \wedge X) \cong \pi^q(EG_+ \wedge_G X),$$

by [**17**, II.8.4], and we can now compute the nonequivariant stable cohomotopy groups of the Borel construction $EG_+ \wedge_G X$ from the equivariant stable cohomotopy groups of X for any X.

COROLLARY 4.4. *For any spectrum X, there are short exact sequences*

$$0 \to L_1^{IG}(\pi_G^{q+1}(X)) \to \pi^q(EG_+ \wedge_G X) \to L_0^{IG}(\pi_G^q(X)) \to 0.$$

By the generalization of the Atiyah-Segal completion theorem given in [**2**] together with the following algebraic result, which will be proven at the end of section 6, Theorem 4.1 also applies when Y represents real or complex periodic equivariant K-theory. Recall that passage from finite G-sets to their permutation representations induces a ring homomorphism $\rho : A(G) \to R(G)$ which factors through the real representation ring $RO(G)$. Regard $R(G)$-modules and $RO(G)$-modules as $A(G)$-modules by pullback along ρ, and remember that G is finite here.

THEOREM 4.5. *Let $J\mathcal{F} \subset R(G)$ be the intersection over $H \in \mathcal{F}$ of the kernels J_H^G of the restrictions $R(G) \to R(H)$. Then, for any $R(G)$-module M, the $A(G)$-modules $M_{J\mathcal{F}}^{\wedge}$ and $M_{I\mathcal{F}}^{\wedge}$ are naturally isomorphic. The analogous result holds for $RO(G)$.*

Just as for cohomotopy, we have natural isomorphisms

$$K_G^*(EG_+ \wedge X) \cong K^*(EG_+ \wedge_G X) \text{ and } KO_G^*(EG_+ \wedge X) \cong KO^*(EG_+ \wedge_G X),$$

by [**17**, II.8.4], and we can now compute the nonequivariant K-groups of $EG_+ \wedge_G X$ from the equivariant K-groups of X.

COROLLARY 4.6. *For any spectrum X, there are short exact sequences*

$$0 \to L_1^{IG}(K_G^{q+1}(X)) \to K^q(EG_+ \wedge_G X) \to L_0^{IG}(K_G^q(X)) \to 0,$$

and similarly for real K-theory.

As a bizarre application, we have the following purely algebraic conclusion about completions of representation rings of finite groups.

COROLLARY 4.7. *Let Π be a normal subgroup of a finite group Γ with quotient group G and let $I(\Pi;\Gamma) = I\mathcal{F}$, where $\mathcal{F}$ is the family of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$. Then*

$$R(\Gamma)_{I\Gamma}^{\wedge} \cong L_0^{IG} R(\Gamma)_{I(\Pi;\Gamma)}^{\wedge} \ ,$$

where $A(G)$ acts on $R(\Gamma)_{I(\Pi;\Gamma)}^{\wedge}$ through $A(G) \to R(G) \to R(\Gamma)$. The same conclusion holds for real representation rings.

PROOF: Let $X = B(\Pi;\Gamma) = E(\Pi;\Gamma)/\Pi$, where $E(\Pi;\Gamma)$ is the universal Π-free Γ-space. Then $EG \times E(\Pi;\Gamma)$ is a free contractible Γ-space, so that $B\Gamma \simeq (EG \times E(\Pi;\Gamma))/\Gamma = EG \times_G X$. By [**2**, 2.1 and 2.2], $K_G^1(X) = 0$ and $K_G^0(X) \cong R(\Gamma)_{I(\Pi;\Gamma)}^{\wedge}$. The previous corollary gives the conclusion.

The reader may object that it is unnatural to appeal to the Burnside ring when studying K-theory. Ideally, we should instead study completions of K-module spectra at ideals of $R(G)$. By doing so, we could expect to generalize the K-theory case of Theorem 4.1 to arbitrary compact Lie groups. We believe that this can be done, but the arguments would entail a discussion of highly structured module spectra over highly structured ring spectra, which would take us far afield.

As a final observation, the following parenthetical remark on duality generalizes [**21**], to which we refer for more discussion. Note that, since completions are given by Bousfield localizations, $(X \wedge Y)^{\wedge}_{I} \simeq (X^{\wedge}_{I} \wedge Y)^{\wedge}_{I}$.

REMARK 4.8: Let $\nu : D(X) \wedge Y \to F(X, Y)$ be the canonical duality map. If X is finite and Y satisfies the conclusion of Theorem 4.1, then both horizontal arrows and the left vertical arrow, hence also the right vertical arrow, are equivalences in the commutative diagram

$$\begin{array}{ccc} (DX \wedge Y)^{\wedge}_{I\mathcal{F}} & \longrightarrow & (D(E\mathcal{F}_{+} \wedge X) \wedge Y)^{\wedge}_{I\mathcal{F}} \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\nu}\downarrow \\ F(X, Y)^{\wedge}_{I\mathcal{F}} & \longrightarrow & F(E\mathcal{F}_{+} \wedge X, Y). \end{array}$$

Thus, up to completion, duality holds for the infinite spectrum $E\mathcal{F}_{+} \wedge X$ in the theory represented by Y. The intuitive explanation is that we can replace $E\mathcal{F}_{+}$ by the finite dimensional spectrum $M(I\mathcal{F})$.

§5. Algebraic definitions and topological proofs.

We begin by summarizing some basic definitions from [**12**]. For the moment, A can be any commutative ring.

Define the cofiber Ck of a map $k : X \to Y$ of chain complexes by $(Ck)_i = Y_i \oplus X_{i-1}$, with differential $d_i(y, x) = (d_i(y) + k_{i-1}(x), -d_{i-1}(x))$. Define the suspension ΣX by $(\Sigma X)_i = X_{i-1}$, with differential $-d$. We have a short exact sequence $0 \to Y \to Ck \to \Sigma X \to 0$, and the connecting homomorphism of the derived long exact sequence in homology is k_*. Given a sequence of chain maps $f^r : X^r \to X^{r+1}$, $r \geq 0$, define a map $\iota : \oplus X^r \to \oplus X^r$ by $\iota(x) = x - f^r(x)$ for $x \in X^r$. Define the homotopy colimit, or telescope, of the sequence $\{f^r\}$ to be $C\iota$ and denote it $\mathrm{Tel}(X^r)$. Then $H_i(\mathrm{Tel}(X^r)) = \mathrm{Colim}\, H_i(X^r)$. The composite of the projection from $C\iota$ to its second variable and the canonical map $\oplus X^r \to \mathrm{Colim}(X^r)$ is a homology isomorphism $\mathrm{Tel}(X^r) \to \mathrm{Colim}(X^r)$.

DEFINITIONS 5.1. *For $\alpha \in A$, let $K_*(\alpha)$ denote the map $\alpha : A \to A$ regarded as a chain complex $d_0 : K_0(\alpha) \to K_{-1}(\alpha)$. The identity map in degree 0 and multiplication by α in degree -1 specify a chain map $K_*(\alpha^r) \to K_*(\alpha^{r+1})$. Define $K_*(\alpha^\infty) = \operatorname{Colim} K_*(\alpha^r)$, so that $K_*(\alpha^\infty)$ is the cochain complex $A \to A[1/\alpha]$. For a sequence $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$, let $K_*(\underline{\alpha}) = K_*(\alpha_1) \otimes \cdots \otimes K_*(\alpha_n)$. Taking tensor products, we obtain a chain map $K_*(\underline{\alpha}^r) \to K_*(\underline{\alpha}^{r+1})$. Define $K_*(\underline{\alpha}^\infty) = \operatorname{Colim} K_*(\underline{\alpha}^r)$. Observe that the homology isomorphism $\operatorname{Tel} K_*(\underline{\alpha}^r) \to K_*(\underline{\alpha}^\infty)$ gives a projective approximation of the flat cochain complex $K_*(\underline{\alpha}^\infty)$. For any of these complexes $K_*(?)$, let $K^\cdot(?)$ denote the complex regraded as a cochain complex, $K^i(?) = K_{-i}(?)$.*

DEFINITIONS 5.2. *Let M be an A-module and define the local cohomology groups of M at $I = (\alpha_1, \dots, \alpha_n)$ to be*

$$H_I^*(M) \cong H^*(K^\cdot(\underline{\alpha}^\infty) \otimes M) \cong H^*(\operatorname{Tel} K^\cdot(\underline{\alpha}^r) \otimes M).$$

Define the local homology groups of M at I by

$$H_*^I(M) \cong H_*(\operatorname{Hom}(\operatorname{Tel} K^\cdot(\underline{\alpha}^r), M)).$$

Note that both $H_I^*(M)$ and $H_*^I(M)$ are covariant functors of M. It is not obvious from the definitions just given that these functors depend only on I and not on the choice of its generators. We refer the reader to [**15**] and [**12**, §2] for discussion of this point.

Now let $A = A(G)$. We must prove Theorems 3.1, 3.2(i), and 3.3(i). We begin with the last two. By Theorem 3.6 and the algebraic fact that $H_q^I(M) = 0$ for $q \geq 2$, as will be proven in the next section, these results can be restated as follows in terms of local homology.

THEOREM 5.3. *For any Mackey functor M, $(HM)_I^\wedge$ is (-1)-connected and*

$$\underline{\pi}_q((HM)_I^\wedge) = H_q^I(M) \text{ for } q \geq 0.$$

THEOREM 5.4. *For any spectrum X, there is a natural short exact sequence*

$$0 \to H_1^I(\underline{\pi}_{q-1}(X)) \to \underline{\pi}_q(X_I^\wedge) \to H_0^I(\underline{\pi}_q(x)) \to 0.$$

The proof of Theorem 5.3 depends on a chain level understanding of the cohomology theory represented by HM, as set out briefly in [**16**] where these $RO(G)$ gradable cohomology theories were introduced. Details on G-CW spectra are given in [**17**, II§5].

If X is a G-CW spectrum, we have a cellular chain complex $\underline{C}_*(X)$ in the Abelian category of Mackey functors with $\underline{C}_q(X) = \underline{\pi}_q(X^q/X^{q-1})$. For a Mackey functor M, we obtain a cochain complex of Abelian groups by taking homomorphisms of Mackey functors:

$$C^*(X; M) = \mathrm{Hom}(\underline{C}_*(X), M).$$

Its cohomology groups $H^q_G(X; M)$ agree with the cohomology groups represented by HM, namely

$$[X, HM]^q_G = \pi^G_{-q} F(X, HM).$$

Let us say that a G-CW spectrum X is special if all of its cells have trivial orbit type, so that X^q/X^{q-1} is a wedge of copies of S^q. We then write $C_*(X) = \underline{C}_*(X)(G/G)$. This is a chain complex of $A(G)$-modules, and we have an isomorphism

$$(*) \qquad C^*(X; M) = \mathrm{Hom}(\underline{C}_*(X), M) \cong \mathrm{Hom}_{A(G)}(C_*(X), M(G/G))$$

since a map of Mackey functors with represented domain functor $[?, S]_G$ is determined by the image of the identity map of S.

The cellular theory of special G-CW spectra is exactly like the cellular theory of nonequivariant spectra, as set out in [**17**, VIII§2]. The chain complex $C_*(X \wedge Y)$ of a smash product of special G-CW spectra is isomorphic to $C_*(X) \otimes C_*(Y)$. If $f : X \to Y$ is a cellular map, then the chain complex $C_*(Cf)$ of its cofiber is isomorphic to the algebraic cofiber $C(C_*(f))$ of its map of cellular chains. If $f_r : X_r \to X_{r+1}$ is a sequence of cellular maps, we may take the telescope of the X_r to be the cofiber of the map $\bigvee X_r \to \bigvee X_r$ whose restriction to X_r is $1 - f_r$. With this definition, the chain complex $C_*(\mathrm{Tel}\, X_r)$ is isomorphic to the algebraic telescope $\mathrm{Tel}(C_*(X_r))$. Moreover, if we regard a special G-CW spectrum as an H-spectrum for $H \subset G$, then it inherits a structure of special H-CW spectrum with the same cells.

Returning to the matter at hand, recall that $X^\wedge_I = F(M(\underline{\alpha}), X)$. We have the following application of the discussion just given.

LEMMA 5.5. *The spectrum $M(\underline{\alpha})$ is equivalent to a special G-CW spectrum whose cellular chain complex is isomorphic to* $\mathrm{Tel}\, K_*(\underline{\alpha}^r)$.

PROOF: Up to equivalence, telescopes commute with smash products, hence $M(\underline{\alpha})$ is equivalent to the telescope of the spectra $S^{-1}/\alpha_1^r \wedge \cdots \wedge S^{-1}/\alpha_n^r$. This telescope is clearly a special G-CW spectrum. In view of the remarks above and the description of $K_*(\underline{\alpha}^r)$ given in Definitions 5.1, its cellular chain complex is isomorphic to $\mathrm{Tel}\, K_*(\underline{\alpha}^r)$.

PROOF OF THEOREM 5.3: We have

$$\pi_q^G(HM)_I^\wedge = \pi_q^G F(M(\underline{\alpha}), HM) = H_G^{-q}(M(\underline{\alpha}); M).$$

The regrading in the last equality matches that in the last sentence of Definitions 5.1, and $(*)$ together with the previous lemma imply that this group is $H_q^I(M(G/G))$. The argument passes to subgroups H of G via the chain of isomorphisms

$$\pi_q^H(HM)_I^\wedge = \pi_q^H(HN)_J^\wedge \cong H_q^J(N(H/H)) = H_q^I(M(G/H)).$$

Here $J = r(I)A(H)$, $r : A(G) \to A(H)$, and N is the Mackey functor $M|\,H$, so that $N(H/K) = M(G/K)$. The first equality holds since, when regarded as H-spectra, $M(\underline{\alpha})$ and HM are $M(r(\underline{\alpha}))$ and HN. The last equality holds since the chain complexes from which the two groups are computed are the same.

PROOF OF THEOREM 5.4: The proof is essentially the same as the folklore argument for the special case of nonequivariant p-adic completion. Like any other Bousfield localization, I-completion preserves cofiber sequences. If X is bounded below, it has a Postnikov tower $\{X_i\}$ with cofiber sequences $K(\underline{\pi}_i(X), i) \to X_i \to X_{i-1}$. By Theorem 5.3 and inductive use of the long exact homotopy sequences of the completions of these cofiber sequences, Theorem 5.4 holds for each X_i and thus for X. If X is bounded above, say $\underline{\pi}_r(X) = 0$ for $r > m$, then $X_I^\wedge$ is also bounded above. In fact, $\pi_r^H(X_I^\wedge) \cong [G/H_+ \wedge S^r \wedge M(\underline{\alpha}), X]_G$, and it follows from the dimensions of the cells occurring in the CW decomposition of $M(\underline{\alpha})$ that $\underline{\pi}_r(X_I^\wedge) = 0$ for $r > m+n$, where n is the number of generators of I. To determine $\underline{\pi}_q(X_I^\wedge)$ for a general X, we form the cofibration

$$X[q-n,\infty) \to X \to X(-\infty, q-n),$$

where the second arrow is obtained by killing the homotopy groups $\underline{\pi}_r(X)$ for $r \geq q-n$. When we pass to completions, the first arrow still induces an isomorphism on $\underline{\pi}_q$, and the calculation of $\underline{\pi}_q(X)$ follows.

PROOF OF THEOREM 3.1: This can be proven by a detailed study of the algebraic structure of I-complete Mackey functors, starting from Theorem 3.4(i), but it is easier to use topology. Statements (ii) and (iii) are equivalent by Theorems 3.2(i) and 3.5, which imply that each says that the identity map $HM \to HM$ is an I-completion. Theorem 2.6(iii) gives that (i) implies (ii). It remains to prove that (ii) implies (i). Thus assume that HM is I-complete and W is $\mathcal{K}_I^*$-acyclic. We must show that $H_G^*(W;M) = 0$. Since HM is equivalent to $F(M(I), HM)$ and $M(I)$ is the telescope of a sequence of finite I-torsion spectra T, HM is a homotopy inverse limit of spectra $F(T, HM) \simeq HM \wedge D(T)$. Since $D(T)$ is a finite I-torsion spectrum, the homotopy group systems of $HM \wedge D(T)$ are I-torsion. This implies that $[W, HM \wedge D(T)]_G^* = 0$, by Lemma 1.2 and Proposition 1.3, and therefore $H_G^*(W;M) = 0$ by the $\lim^1$ exact sequence.

§6. Algebraic proofs.

Let $A = A(G)$. We proved Theorems 3.4, 3.5, 3.6, and 3.10 in [**12**], modulo Theorem 3.9 and the assertion that $L_i^I(M) = 0$ for $i \geq 2$. We prove these facts here, starting with the latter. Since $L_i^I(M) \cong H_i^I(M)$, it suffices to show that $H_i^I(M) = 0$ for $i \geq 2$. By [**12**, **3.1**], there is a spectral sequence which converges to $H_*^I(M)$ and has E_2-term $\mathrm{Ext}^*(H_I^*(A), M)$. It is immediate from the form of the spectral sequence that the following result implies the desired vanishing.

THEOREM 6.1. *Let $I = (\alpha_1, \dots, \alpha_n)$ be a finitely generated ideal in the Burnside ring $A = A(G)$. Then $H_I^i(A) = 0$ for $i \geq 2$.*

PROOF: Let ΦG be the set of conjugacy classes of those subgroups H of G which have finite index in their normalizers. Additively, A is the free abelian group on generators $[G/H]$ with $(H) \in \Phi G$. Let NH be the normalizer of H in G and $WH = NH/H$. The set of orders of the groups WH has a least common multiple w. Give ΦG the Hausdorff metric. Then ΦG is a totally disconnected compact Hausdorff space. Let C be the ring of continuous functions $\Phi G \to \mathbb{Z}$. There is an embedding $\varphi : A \to C$ and we let $Q = C/A$. Then $wQ = 0$. See [8] or [**17**, Ch.V] for the proofs of the given statements. We have a long exact sequence

$$\cdots \to H_I^{i-1}(Q) \to H_I^i(A) \to H_I^i(C) \to H_I^i(Q) \to \cdots.$$

It suffices to prove that $H_I^i(C) = 0$ for $i \geq 2$ and $H_I^i(Q) = 0$ for $i \geq 1$.

LEMMA 6.2. *Let D be the product of finitely many copies of $\mathbb{Z}$. If $i \geq 2$, then $H_I^i(M) = 0$ for any ideal I in D and any D-module M.*

PROOF: If $D = \mathbb{Z}$, then I is principal and the conclusion is immediate from the definitions. The constructions all commute with finite direct products, and the conclusion follows.

LEMMA 6.3. *Under the hypotheses of Theorem 6.1, $H_I^i(C) \doteq 0$ for $i \geq 2$.*

PROOF: Let $J = IC \subset C$. Inspection of definitions shows that the groups $H_I^i(C)$ and $H_J^i(C)$ are isomorphic since they are computable from isomorphic cochain complexes. In C, the $\varphi(\alpha_i)$ take finitely many values. We can choose finitely many disjoint open and closed subsets Γ_k of ΦG with characteristic functions δ_k, $\delta_k(H) = 1$ for $(H) \in \Gamma_k$ and $\delta_k(H) = 0$ for $(H) \notin \Gamma_k$, such that each $\varphi(\alpha_i)$ is a linear combination of the δ_k. If D is the subring of C generated by the δ_k, then D is isomorphic to a finite product of copies of $\mathbb{Z}$. If K is the ideal in D generated by the $\varphi(\alpha_i)$, then $J = KC$ and the groups $H_J^i(C)$ and $H_K^i(C)$ are isomorphic since they are computable from isomorphic cochain complexes. Now the previous lemma gives the conclusion.

LEMMA 6.4. *Under the hypotheses of Theorem 6.1, if R is any sub A-module of Q, then $H_I^i(R) = 0$ for $i \geq 1$.*

PROOF: Observe that $H_I^0(R) = \{r \mid I^k \cdot r = 0 \text{ for some } k\} \subset R$. Write $I = (J, \alpha)$, where $J = (\alpha_1, \ldots, \alpha_{n-1})$ and $\alpha = \alpha_n$. The result is trivial when $I = 0$, and we assume it for J. If $\underline{\beta} = \{\alpha_1, \ldots, \alpha_{n-1}\}$, then $K^{\cdot}(\underline{\alpha}^\infty) = K^{\cdot}(\underline{\beta}^\infty) \otimes K^{\cdot}(\alpha^\infty)$ and a standard double complex argument gives a spectral sequence which converges from $E_2^{p,q} = H_\alpha^p(H_J^q(R))$ to $H_I^{p+q}(R)$. We have $E_2^{p,q} = 0$ for $q \geq 1$ by the induction hypothesis, and it is obvious that $E_2^{p,q} = 0$ for $p \geq 2$. Since $H_J^0(R) \subset R$, it only remains to prove that $H_\alpha^1(R) = 0$ for $R \subset Q$. Since $H_\alpha^1(R)$ is the cokernel of the natural map $\iota : R \to R[1/\alpha]$, it suffices to prove that ι is an epimorphism. The kernel of ι is the group $H_\alpha^0(R)$ of α-power torsion elements in R. Let $\overline{R} = R/H_\alpha^0(R)$. It suffices to prove that $\alpha : \overline{R} \to \overline{R}$ is an isomorphism, and it is clearly a monomorphism. We shall prove that, for any $r \in R$, the subgroup $\langle \alpha, r \rangle$ of R generated by $\{\alpha^k r \mid k \geq 0\}$ is finite. The same will then hold for the image $\langle \alpha, \overline{r} \rangle$ of this group in $\overline{R}$. Therefore $\alpha : \langle \alpha, \overline{r} \rangle \to \langle \alpha, \overline{r} \rangle$ will be a bijection and the conclusion will follow. Let $\rho \in C$ map to $r \in Q$. We can choose finitely many open and closed subsets Γ_j of ΦG such that $\varphi(\alpha)$ and ρ are both linear combinations of the characteristic functions δ_j of the Γ_j.

Then each $\alpha^k\rho = \varphi(\alpha)^k\rho$ is also a linear combination of the δ_j. If d_j is the image of δ_j in Q, then $wd_j = 0$. Thus the d_j generate a finite subgroup of Q which contains $\langle\alpha, r\rangle$.

We must still prove Theorem 3.9, which can be restated as follows.

THEOREM 6.5. *Let $J = (\beta_1, \ldots, \beta_q)$ be a finitely generated ideal in the Burnside ring $A = A(G)$ and let α be any element of A. There is an $n > 0$ such that if $\gamma \in A$ and $\alpha^s\gamma \in J$ for some $s \geq n$, then $\alpha^n\gamma \in J$. That is, α^n annihilates any element of A/J that α^s annihilates.*

We shall derive this from the following result.

THEOREM 6.6. *Let B' be a finitely generated subring of A that contains the β_i among its generators. Then there is a Noetherian ring B such that $B' \subset B \subset A$ and $P^n \subset Q$ for each P-primary ideal Q in a well chosen reduced primary decomposition of the ideal K of B generated by $\{\beta_1, \ldots, \beta_q\}$, where $n > 0$ is an integer depending only on the β_i.*

PROOF OF THEOREM 6.5, ASSUMING THEOREM 6.6: Assume that $\alpha^s\gamma \in J$. Write $\alpha^s\gamma = \sum a_i\beta_i$ and let B' be generated by α, γ, the β_i and the a_i. Let B and K be as in Theorem 6.6. Then $\alpha^s\gamma \in K$, hence $\alpha^s\gamma$ is in each primary ideal Q of any reduced primary decomposition of K. Either $\gamma \in Q$ or $\alpha \in P$, where Q is P-primary. In the latter case, Theorem 6.6 gives that $\alpha^n \in Q$ if our reduced primary decomposition is well chosen. Then $\alpha^n\gamma \in Q$, hence $\alpha^n\gamma \in K$ in B and thus $\alpha^n\gamma \in J$ in A.

We can specify n as follows. Let ΦG, C, φ, and w be as in the proof of Theorem 6.1 and let ν be the maximum over all primes p of the exponents $\nu_p(w)$. Each $\varphi(\beta_i)$ takes finitely many values $m_{i,r}$. For each prime p and each sequence $m_R = \{m_{1,r_1}, \ldots, m_{q,r_q}\}$, let $n(p,R)$ be maximal such that $p^{n(p,R)}$ divides each m_{i,r_i}. Then let n' be the maximum over p and R of the $n(p,R)$. Define $n = n' + \nu$.

PROOF OF THEOREM 6.6: Let $\{\xi_j\} \supset \{\beta_j\}$ be a finite set of generators for B'. Each $\varphi(\xi_j)$ takes finitely many values. For each sequence $S = \{s_j\}$ of such values, the subset Γ_S of ΦG consisting of those (H) such that $\varphi(\xi_j)(H) = s_j$ for each j is open and closed. Let $\Pi = \{\Gamma_S\}$. Then Π is a finite partition of ΦG into a disjoint union of open and closed subsets. Let $D \subset C$ be the ring of those continuous functions on ΦG which are constant on each $\Gamma \in \Pi$. Then D is the product of copies of $\mathbb{Z}$ indexed on the $\Gamma \in \Pi$. Let $B = A \cap D$ in C. Since $wC \subset A$, $wD \subset B$. Moreover, for any ideal L

of B, $wLD \subset L(wD) \subset LB = L$, so that w annihilates LD/L. Clearly D is Noetherian, and it follows that B is Noetherian.

The analysis of prime ideals and localizations in B is precisely parallel to the analysis in A that is given in [8] and [**17**, V§§3,5]. Of course, $\mathrm{Spec}(D) = \Pi \times \mathrm{Spec}(\mathbb{Z})$; the prime ideals of D are of the form

$$\overline{q}(\Gamma, p) = \{d \mid d(\Gamma) \equiv 0 \mod p\},$$

where $\Gamma \in \Pi$ and p is a prime or zero. The prime ideals of B are of the form $q(\Gamma, p) = \overline{q}(\Gamma, p) \cap B$. Here $q(\Gamma, 0) = q(\Gamma', 0)$ if and only if $\Gamma = \Gamma'$, and $B \to B/q(\Gamma, 0) \otimes \mathbb{Q}$ is the localization of B at $q(\Gamma, 0)$. Let $\Pi(\Gamma, p)$ be the set of Γ' such that $q(\Gamma', p) = q(\Gamma, p)$ and let $I(\Gamma, p)$ be the intersection over $\Gamma' \in \Pi(\Gamma, p)$ of the prime ideals $q(\Gamma', 0)$. Then $B \to B/I(\Gamma, p) \otimes \mathbb{Z}_{(p)}$ is the localization of B at $q(\Gamma, p)$. Moreover, the localization of D at $q(\Gamma, p)$ is the product of copies of $\mathbb{Z}_{(p)}$ indexed on the $\Gamma' \in \Pi(\Gamma, p)$, and the map $B_{q(\Gamma,p)} \to D_{q(\Gamma,p)}$ is a monomorphism.

Let K be the ideal of B generated by $\{\beta_1, \ldots, \beta_q\}$. Let $K = \bigcap Q_i$ be a reduced primary decomposition, let Q be any of the Q_i, and let $P = q(\Gamma, p)$ be the radical of Q. We want to show that $P^n \subset Q$. Let $\lambda : B \to B_P$ be the localization of B at P and let M be the maximal ideal of B_P. Recall (e.g. [**26**, pp.223–228]) that Q is P-primary if and only if $Q = \lambda^{-1}(R)$ where R is M-primary and that R is M-primary if and only if $M \supset R \supset M^t$ for some t. If $p = 0$, then P is the only P-primary ideal and there is nothing to prove. Thus assume that p is a non-zero prime. With $Q = \lambda^{-1}(R)$, let $R' = R + M^n$ (with n defined as above) and $Q' = \lambda^{-1}(R')$. We claim that if we replace Q by Q', then we still have a reduced primary decomposition of K. Since $P^n \subset Q'$, this claim will immediately imply Theorem 6.6.

To prove the claim, let L be the intersection of the prime ideals $q(\Gamma', 0)_P$ in B_P, where Γ' runs over those elements of $\Pi(\Gamma, p)$ such that $K \subset q(\Gamma', 0)$. Obviously these $q(\Gamma', 0)$ are minimal prime ideals of K; since they are themselves the only $q(\Gamma', 0)$-primary ideals, they must appear in any reduced primary decomposition of K [**26**, p.211]. Thus it suffices to show that $R \cap L = R' \cap L$. This means that $R' \cap L \subset R$, and this will hold if $M^n \cap (R + L) \subset R$. Regarded as a B-module via φ, the localization D_P of D is the sum of a copy of $Z_{(p)}$ for each $\Gamma' \in \Pi(\Gamma, p)$. Its submodule $K_P D_P$ is the sum of (0) for $\Gamma' \in \Pi(\Gamma, p)$ such that $K \subset q(\Gamma', 0)$ and $(p^{m(\Gamma')})$ for $\Gamma' \in \Pi(\Gamma, p)$ such that $K \not\subset q(\Gamma', 0)$, where $m(\Gamma')$ is maximal such that $p^{m(\Gamma')}$ divides each $\beta_i(\Gamma')$. By the definition of n', $n' \geq m(\Gamma')$ for each Γ'.

Let RD_P be the sum of $(p^{r(\Gamma')})$. Since $K_P \subset R$, $m(\Gamma') \geq r(\Gamma')$. Clearly LD_P is the sum of (0) for Γ' such that $K \subset q(\Gamma', 0)$ and $Z_{(p)}$ for Γ' such that $K \not\subset q(\Gamma', 0)$. It follows that $(R+L)D_P$ is the sum of $(p^{r(\Gamma')})$ for Γ' such that $K \subset q(\Gamma', 0)$ and $Z_{(p)}$ for Γ' such that $K \not\subset q(\Gamma', 0)$. Therefore

$$M^{n'} D_P \cap (R+L)D_P \subset RD_P.$$

Also, $M^{\nu_p(w)} \subset (p^{\nu_p(w)})$ and $p^{\nu_p(w)} D_P \subset B_P$. Since $n \geq n' + \nu_p(w)$,

$$\begin{aligned} M^n \cap (R+L) &\subset (M^{\nu_p(w)} M^{n'}) D_P \cap (R+L) D_P \\ &\subset (p^{\nu_p(w)} M^{n'}) D_P \cap (R+L) D_P \\ &= p^{\nu_p(w)} (M^{n'} D_P \cap (R+L) D_P) \\ &\subset p^{\nu_p(w)} RD_P \subset RB_P = R. \end{aligned}$$

The middle equality follows from the inequalities $n' \geq r(\Gamma')$ and a check of submodules in the summands $\mathbb{Z}_{(p)}$ of D_P.

We left the following unfinished piece of business in section 4.

Proof of Theorem 4.5: Remember that G is finite in this result. Clearly $M/(I\mathcal{F} \cdot M)^r = M/(I'\mathcal{F} \cdot M)^r$, where $I'\mathcal{F}$ is the ideal of $R(G)$ generated by $\rho(I\mathcal{F})$, and $I'\mathcal{F} \subset J\mathcal{F}$ since ρ commutes with restriction. It suffices to show that some power of $J\mathcal{F}$ is contained in $I'\mathcal{F}$ or, equivalently, that $J\mathcal{F}$ is nilpotent in $R(G)/I'\mathcal{F}$. This will hold if any prime ideal P of $R(G)$ which contains $I'\mathcal{F}$ also contains $J\mathcal{F}$. Clearly $Q = \rho^* P$ contains $I\mathcal{F}$, hence Q contains I^G_H for some $H \in \mathcal{F}$. It suffices to prove that P contains J^G_H, and this will certainly hold if the support C of P is contained in H (up to conjugacy). Recall that the support of P is a subgroup C which is minimal such that P comes from a prime ideal of $R(C)$. Since the support of Q is contained in H (see e.g. [**3**, p.17]), it suffices to prove that C is also the support of Q.

Let $\mathcal{H}$ and $\mathcal{E}$ denote the sets of conjugacy classes of subgroups of G and of elements of G, respectively. Passage from elements g to cyclic groups $\langle g \rangle$ gives a function $\mathcal{E} \to \mathcal{H}$. Let $\mathbb{Z}^{\mathcal{H}}$ and $\mathbb{C}^{\mathcal{E}}$ be the rings of functions $\mathcal{H} \to \mathbb{Z}$ and $\mathcal{E} \to \mathbb{C}$, and let $\pi : \mathbb{Z}^{\mathcal{H}} \to \mathbb{C}^{\mathcal{E}}$ be the evident map of rings. Then the following diagram commutes, where $\varphi(S)(H)$ is the cardinality of S^H for a finite G-set S and $\chi(\alpha)(g)$ is the trace of $\alpha(g)$ for a representation α:

$$\begin{array}{ccc} A(G) & \xrightarrow{\rho} & R(G) \\ \varphi \downarrow & & \chi \downarrow \\ \mathbb{Z}^{\mathcal{H}} & \xrightarrow{\pi} & \mathbb{C}^{\mathcal{E}}. \end{array}$$

Incidentally, this implies that ρ is injective if and only if G is cyclic. According to Segal [**25**], the support C is a cyclic group $\langle g \rangle$. If the prime P is minimal, then P consists of those α such that $\chi(\alpha)(g) = 0$, and the diagram implies that $Q = q(C, 0)$, whose support is also C. If P is maximal (the only other case), then P has residual characteristic a prime p, and Segal shows [**25**, p.122] that p does not divide $|C|$. Since ρ commutes with the restrictions $A(G) \to A(C)$ and $R(G) \to R(C)$, it is clear that the support of Q is a subgroup D of C, so that $Q = q(D, p)$. Since P contains the minimal prime with support C, Q contains $q(C, 0)$ and therefore Q also equals $q(C, p)$. This implies that C/D is a p-group and thus that $C = D \subset H$.

Appendix: localizations of G-spectra

Let C be a multiplicatively closed subset of $A(G)$. Since we are interested in localizations, we may as well assume that $0 \notin C$ and that C is saturated, in the sense that $xy \in C$ implies $x \in C$ and $y \in C$. Equivalently, C is the complement of the (set theoretical) union of a nonempty set of prime ideals. An $A(G)$-module N is C-local if $\gamma : N \to N$ is an isomorphism for all $\gamma \in C$. A map $\lambda : M \to C^{-1}M$ is the C-localization of an $A(G)$-module M if $C^{-1}M$ is C-local and any $A(G)$-map from M to another C-local $A(G)$-module N factors uniquely through $C^{-1}M$. The kernel of λ is the set of elements annihilated by some $\gamma \in C$. When $C = A(G) - P$, it is usual to write $C^{-1}M = M_P$ and to call it the localization of M at P.

We show how to mimic this algebraic construction on spectra.

DEFINITIONS A.1. *A spectrum Y is said to be C-local if $\gamma : Y \to Y$ is an equivalence for all $\gamma \in C$. A map $\lambda : X \to C^{-1}X$ is said to be a C-localization if $C^{-1}X$ is C-local and if any map from X to a C-local spectrum Y factors uniquely (up to homotopy) through $C^{-1}X$.*

Say that T is a C-torsion spectrum if $\gamma : T \to T$ is null homotopic for some $\gamma \in C$.

LEMMA A.2. *The following conditions on a spectrum Y are equivalent.*

(i) *Y is C-local.*

(ii) *$Y \wedge T$ is contractible for all C-torsion spectra T.*

(iii) *$Y \wedge T$ is contractible for all finite C-torsion spectra T.*

(iv) *Each Mackey functor $\underline{\pi}_n(Y)$ is C-local.*

PROOF: If Y is C-local and $\gamma : T \to T$ is null homotopic, then the map $\gamma : Y \wedge T \to Y \wedge T$ is a null homotopic equivalence and $Y \wedge T$ is contractible. If $Y \wedge T$ is contractible for all finite C-torsion spectra T, then $\gamma : Y \to Y$ is an equivalence for $\gamma \in C$ since $Y \wedge S/\gamma$ is contractible. This shows the equivalence of (i)–(iii). Since $\gamma_* : \underline{\pi}_*(X) \to \underline{\pi}_*(X)$ is multiplication by γ, (i) and (iv) are equivalent by the Whitehead theorem in the stable category.

We can choose a sequence $\{\gamma_i\}$ of elements of C such that, for any $\gamma \in C$, there exist $i \geq 1$ and $\delta \in C$ such that $\gamma_1 \cdots \gamma_i = \gamma\delta$.

PROPOSITION A.3. *Let $C^{-1}X$ be the telescope of the sequence of maps $\gamma_i : X \to X$ and let $\lambda : X \to C^{-1}X$ be the canonical map. Then λ is a C-localization of X. Moreover, $\lambda_* : \underline{\pi}_*(X) \to \underline{\pi}_*(C^{-1}X)$ is C-localization, and λ is characterized by this property.*

PROOF: For $\gamma \in C$, $\gamma : C^{-1}X \to C^{-1}X$ is the telescope of the maps $\gamma : X \to X$ on the terms of the telescope. Its cofiber is the telescope of the sequence of maps $\gamma_i : X/\gamma \to X/\gamma$. Applying the cofinality assumed of the sequence $\{\gamma_i\}$ to the powers of γ, we see that γ divides a cofinal sequence of the γ_i. Since $\gamma^2 : X/\gamma \to X/\gamma$ is null homotopic, the cofiber is contractible and $\gamma : C^{-1}X \to C^{-1}X$ is an equivalence. Thus $C^{-1}X$ is C-local. If Y is C-local, the maps $(\gamma_i)^* : [X, Y]_G \to [X, Y]_G$ are isomorphisms for any X. Applying the $\lim^1$ exact sequence, we find immediately that $\lambda^* : [C^{-1}X, Y]_G \to [X, Y]_G$ is an isomorphism. The first clause of the last statement holds since our construction mimics a standard algebraic construction of C-localization, and the second clause follows from the universal property and the Whitehead theorem.

Observe that $C^{-1}X$ is naturally equivalent to $X \wedge C^{-1}S$ and that $C^{-1}(X \wedge Y)$ is naturally equivalent to $C^{-1}X \wedge C^{-1}Y$.

PROPOSITION A.4. *Y is C-local if and only if it is $C^{-1}S$-local, and C-localization coincides with Bousfield localization at $C^{-1}S$.*

PROOF: Write $E = C^{-1}S$ and let F be the fiber of the canonical map $S \to E$. Observe that F is the telescope of the sequence $S^{-1}/\gamma_1 \cdots \gamma_n$. If Y is C-local, then $[W \wedge S^{-1}/\gamma, Y]^G_* = [W, Y \wedge S/\gamma]^G_* = 0$ for all $\gamma \in C$ since

S^{-1}/γ is dual to S/γ and $Y \wedge S/\gamma$ is contractible. If W is E-acyclic, then $W \simeq W \wedge F$ and $[W,Y]^G_* = 0$ by the $\lim^1$ exact sequence. Conversely, let Y be E-local. If T is a finite C-torsion spectrum then so is $D(T)$, hence $X \wedge D(T)$ is E-acyclic for any X. Thus $[X, Y \wedge T]^G_* = [X \wedge D(T), Y]^G_* = 0$ and $Y \wedge T$ is contractible. It is clear from the construction that the C-localization $X \to C^{-1}X$ is an E-equivalence.

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