IDEMPOTENTS AND LANDWEBER EXACTNESS IN BRAVE NEW ALGEBRA

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Abstract. We explain how idempotents in homotopy groups give rise to splittings of homotopy categories of modules over commutative S-algebras, and we observe that there are naturally occurring equivariant examples involving idempotents in Burnside rings. We then give a version of the Landweber exact functor theorem that applies to MU-modules.

In 1997, not long after [6] was written, I gave an April Fool's talk on how to prove that BP is an E_∞ ring spectrum or equivalently, in the language of [6], a commutative S-algebra. Unfortunately, the problem of whether or not BP is an E_∞ ring spectrum remains open. However, two interesting remarks emerged and will be presented here. One concerns splittings along idempotents and the other concerns the Landweber exact functor theorem.

One of the nicest things in [6] is its one line proof that KO and KU are commutative S-algebras. This is an application of the following theorem [6, VIII.2.2], or rather the special case that follows.

Theorem 1. *Let R be a cell commutative S-algebra, A be a cell commutative R-algebra, and M be a cell R-module. Then the Bousfield localization $\lambda : A \longrightarrow A_M$ of A at M can be constructed as the inclusion of a subcomplex in a cell commutative R-algebra. In particular, the commutative R-algebra A_M is a commutative S-algebra by neglect of structure.*

The cell assumptions can always be arranged by use of the cofibrant replacement constructions in [6], so they result in no loss of generality. The theorem specializes as follows to algebraic localizations at elements of $R_* = \pi_*(R)$ [6, VIII.4.2].

Theorem 2. *Let R be a cell commutative S-algebra and X a set of elements of R_*. The localization $\lambda : R \longrightarrow R[X^{-1}]$ that induces the algebraic localization $R_* \longrightarrow R_*[X^{-1}]$ can be constructed as the unit of a cell commutative R-algebra.*

The connective real K-theory spectrum ko is a commutative S-algebra by multiplicative infinite loop space theory [11], and KO is the localization $ko[\beta^{-1}]$ obtained by inverting the Bott class. Therefore KO is a commutative ko-algebra and thus a commutative S-algebra. That's the one line. Complex K-theory works similarly.

As a matter of algebra, idempotents give localizations. Since MU arises in nature as an E_∞ ring spectrum, that being the paradigmatic example that led to the definition [10], one might try to prove that BP is a Bousfield localization of MU and thus a commutative MU-algebra. That is April Fool's nonsense, but the basic idea has a correct version with other applications, as we shall explain. Essentially

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the same idea occurred independently to Schwänzl, Vogt, and Waldhausen, who gave quite different applications [13, 14].

Definition 3. Let R be a cell commutative S-algebra and let $e \in R_0$ be an idempotent element. As a matter of algebra, $R_*[e^{-1}] = eR_*$. Define eR to be the cell commutative R-algebra $R[e^{-1}]$ of Theorem 2.

Theorem 4. *Let $1 = e_1 + \cdots + e_n$ where the e_i are orthogonal idempotents in R_*. Then the canonical map*

$$\varepsilon : R \longrightarrow e_1R \times \cdots \times e_nR$$

of commutative R-algebras is a weak equivalence. Therefore the category of R-modules is Quillen equivalent to the product of the categories of e_iR-modules.

Proof. The first statement is obvious. The second statement follows from the next two results. The first is implicit in [6, III.4.2 and VII.4.8] and explicit in [9, I.3.6] and the second is proven by an easy formal argument. □

Theorem 5. *If $f : R \longrightarrow Q$ is a weak equivalence of commutative S-algebras, then the extension of scalars functor $f_* : \mathscr{M}_R \longrightarrow \mathscr{M}_Q$ and the pullback of structure functor $f^* : \mathscr{M}_Q \longrightarrow \mathscr{M}_R$ specify a Quillen equivalence of model categories.*

Theorem 6. *If R is a product of commutative S-algebras R_i with projections $\varepsilon_i : R \longrightarrow R_i$, then the functor that sends an R-module M to the tuple $(\varepsilon_{i*}M)$ is the left adjoint of a Quillen equivalence from $\mathscr{M}_R$ to the product of the categories $\mathscr{M}_{R_i}$. The right adjoint sends (N_i) to the product of the R-modules $\varepsilon_i^* N_i$.*

Theorem 4 shows that the homotopy theory of R-modules entirely decomposes into the homotopy theories of the modules over the e_iR. The ring spectra that algebraic topologists usually work with have no non-trivial idempotents. However, interesting examples do arise naturally in algebraic K-theory, as observed in [13].

Remark 7. If R is connective, we have a map $R \longrightarrow HR_0$ that induces an isomorphism on π_0 [6, IV.3.1]. Here, if $X \subset R_0$ and we apply the functor $(-) \wedge_R HR_0$ to $\lambda : R \longrightarrow R[X^{-1}]$, we obtain a model for the localization

$$\lambda : HR_0 \cong R \wedge_R HR_0 \longrightarrow R[X^{-1}] \wedge_R HR_0 \cong (HR_0)[X^{-1}] \cong H(R_0[X^{-1}]).$$

In particular, for an idempotent $e \in R_0$, $eR \wedge_R HR_0$ is equivalent to $H(eR_0)$. This observation is the starting point of [13, 14].

Interesting examples also arise in equivariant algebraic topology. The results above generalize directly to the equivariant setting of commutative S_G-algebras and their modules [6, 9, 12], where G is a compact Lie group and S_G is the sphere G-spectrum. Here, for a commutative S_G-algebra R, we take $R_* = \pi_*(R^G)$. In particular, $(S_G)_*$ is the equivariant stable homotopy groups of spheres and $(S_G)_0$ is isomorphic to the Burnside ring $A(G)$. The ring $A(G)$, and more so its localizations at subrings of the rationals, usually does have non-trivial idempotents [5, 8].

The splittings of Theorem 4 give model theoretic refinements of splittings in equivariant stable homotopy theory that are discussed in [8, V] and [12, XVII§6]. Those sources describe splittings of homology and cohomology theories, and it is now apparent that these splittings arise from splittings of corresponding equivariant stable categories. The splittings involve change of group functors, and these are discussed model theoretically in the contexts both of S_G-modules and of orthogonal G-spectra in [9]. Briefly, by [9, VI.1.2], for an inclusion $\iota : H \subset G$, there is a Quillen adjoint pair $(G_+ \wedge_H (-), \iota^*)$ relating $H\mathscr{M}$ to $G\mathscr{M}$. Let $WH = NH/H$ and let $\varepsilon : NH \longrightarrow WH$ be the quotient homomorphism. By [9, 3.12], there is also a Quillen adjoint pair relating $NH\mathscr{M}$ to $WH\mathscr{M}$. This remains true after localization at a prime or rationalization. Thus we can split localized stable categories along idempotents and identify the pieces as equivalent to stable categories over subquotient groups.

We now turn to a completely different topic, but one that also arises naturally from consideration of spectra constructed from MU, namely the Landweber exact functor theorem. In fact, that result has the following more structured version in the category of MU-modules. We say that an MU_*-module M_* is *Landweber exact* if, for each prime p, the set $\{v_i | i \geq 0\}$ is a regular sequence for M_*. Here $v_0 = p$ and the v_i for $i > 0$ are indecomposable elements of degree $2p^i - 2$ with Chern numbers divisible by p.

Theorem 8. *If M_* is a Landweber exact MU_*-module, then there is an MU-module M such that $\pi_*(M) = M_*$ and, for any finite cell MU-module X,*

$$\pi_*(X) \otimes_{MU_*} M_* \cong \pi_*(X \wedge_{MU} M).$$

As a matter of algebra, Landweber [7, 2.6] proved the following result. Let $\mathscr{MU}$ denote the category of comodules over $MU_*(MU)$ that are finitely presented as MU-modules.

Theorem 9 (Landweber)**.** *The functor $(-) \otimes_{MU_*} M_*$ on the category $\mathscr{MU}$ is exact if and only if the MU_*-module M_* is Landweber exact.*

By the following two results, MU-modules naturally gives rise to objects of $\mathscr{MU}$.

Lemma 10. *If X is a finite cell MU-module, then $\pi_*(X)$ is a finitely presented MU_*-module.*

Proof. This is proven by exactly the same induction on the number of cells as in the classical special case $X = MU \wedge Y$, where Y is a finite CW spectrum. Of course, in that case $\pi_*(X) = MU_*(Y)$. For example, the proof is clear from the algebraic argument given by Adams [1, pp. 132–133]. $\square$

Lemma 11. *If X is an R-module, where R is a commutative S-algebra such that R_*R is R_*-flat, then the Hurewicz map gives X_* a structure of R_*R-comodule.*

Proof. This is proven by diagram chasing as in Adams [1]. It is the starting point of the development of an Adams spectral sequence in brave new algebra [3]. The main point is that

$$R_*R \otimes_{R_*} X_* \cong \pi_*((R \wedge R) \wedge_R X) \cong \pi_*(R \wedge_R X). \quad \square$$

Of course, this applies with $R = MU$. The previous three results imply the following conclusion.

Proposition 12. *Let M_* be a Landweber exact MU_*-module. Then the functor $\pi_*(X) \otimes_{MU_*} M_*$ specifies a homology theory on finite cell MU-modules X.*

Applying Adams' variant [2] of Brown's representability theorem, which applies since MU_* is countable [6, III.2.13], we obtain the MU-module M promised in Theorem 8. The construction of M is non-uniquely functorial: given a map $f_* : M_* \longrightarrow N_*$ of Landweber exact MU-modules, there is a map $f : M \longrightarrow N$ of MU-modules that realizes f_*, but f will not be unique unless the relevant $\lim^1$ groups vanish.

Example 13. Recall that $KU_* = \mathbb{Z}[u, u^{-1}]$, where $\deg(u) = 2$, and give it the MU_*-module structure specified by the ring homomorphism $MU_* \longrightarrow KU_*$ that sends $[M^{2n}]$ to $Td(M^{2n})u^n$. We know by the methods of [6, V§4] that KU is an MU-module and in fact an MU-ring spectrum. There results an isomorphism

$$\pi_*(X) \otimes_{MU_*} KU_* \longrightarrow KU_*(X)$$

for finite cell MU-modules X. Alternatively, granting that there is a unique ring spectrum KU with the cited homotopy groups, we can construct KU as an MU-module by Theorem 8 and then show that it is an MU-ring spectrum by the methods of [6, V§4]. The resulting map $Td : MU \longrightarrow KU$ is a map of MU-ring spectra. The calculation of Td_* in terms of the Todd genius is evident from the present approach, but is not clear from the approach of [6, V§4]. In any case, this gives a generalization to MU-modules of the Conner-Floyd theorem that MU-theory determines KU-theory.

Of course, if BP is a commutative S-algebra, then the Landweber exact functor theorem will admit a precisely analogous and more useful version for BP_*-modules.

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