CATEGORIES OF OPERATORS AND STRICTIFICATION

Abstract. Categories of operators and their algebras generalize operads and their algebras and mediate between operadic and Segal style infinite loop space machines. We explain here how 2-monadic strictification theory specializes to strictify algebras over categories of operators and how this relates to the specialization to algebras over operads. While this theory is of independent interest, it is essential to the development of equivariant and multiplicative infinite loop space theory starting from categorical input.

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Introduction and statements of results

In this brief sequel to [8], we generalize the categorical results there to algebras over categories of operators. This is geared towards multiplicative infinite loop space theory and will be exploited in [7, 6, 5], but it is not devoid of independent interest. On the space level, categories of operators, as first defined in [15], mediate between operadic [12] and Segal style [17] approaches to infinite loop space theory. This was generalized to G-spaces for finite groups G in [14].

To generalize input from spaces to categories, we must strictify the data that appear naturally in terms of symmetric monoidal categories to data specified in terms of permutative categories. For the equivariant theory, the starting point is the notion of a symmetric monoidal G-category as defined operadically and interpreted monadically in [8]. We here interpolate a reinterpretation in terms of categories of operators. While we shall say nothing about it in this note, the interpolation is central to generalizing the multiplicative version of equivariant infinite loop space

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theory from G-spaces, which are handled in [4], to G-categories, which will be handled in [7, 6, 5].

As in [8], equivariance plays no role at all beyond motivation in this paper, and we work in the cartesian monoidal 2-category $\mathbf{Cat}(\mathscr{V})$ of internal categories in any suitable cartesian closed category $\mathscr{V}$. We establish the general categorical context of $\mathscr{V}$-2-categories and $\mathscr{V}$-pseudofunctors in Section 1.1.

Let $\mathscr{F}$ be the category of finite based sets $\mathbf{n} = \{0, 1, \cdots, n\}$ and based functions, and let $\Pi \subset \mathscr{F}$ be the subcategory generated by injections, projections, and permutations. By Remark 2.2, we may regard Π and $\mathscr{F}$ as categories enriched in $\mathbf{Cat}(\mathscr{V})$. A category of operators $\mathscr{D}$ has objects $\mathbf{n}$ and comes with $\mathscr{V}$-2-functors $\iota\colon \Pi \longrightarrow \mathscr{D}$ and $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ such that $\xi \circ \iota$ is the inclusion $\Pi \longrightarrow \mathscr{F}$; see Definition 2.4. In §2.1 we define $\mathscr{D}$-pseudoalgebras to be pseudofunctors from $\mathscr{D}$ to $\mathbf{Cat}(\mathscr{V})$, but with restrictions (Assumption 2.7) analogous to those in our definition of $\mathscr{O}$-pseudoalgebras for operads $\mathscr{O}$ in [8, Section 1.2]. The $\mathscr{D}$-pseudoalgebras simultaneously generalize $\mathscr{F}$-algebras (also known as Γ-categories) and $\mathscr{O}$-pseudoalgebras.

We introduce names for three 2-categories of algebras over a category of operators $\mathscr{D}$ in $\mathbf{Cat}(\mathscr{V})$. These are analogous to the 2-categories of algebras over operads and monads defined in [8].

- $\mathscr{D}$-$\mathbf{PsAlg}$: $\mathscr{D}$-pseudoalgebras and $\mathscr{D}$-pseudomorphisms.
- $\mathscr{D}$-$\mathbf{AlgPs}$: $\mathscr{D}$-algebras and $\mathscr{D}$-pseudomorphisms.
- $\mathscr{D}$-$\mathbf{AlgSt}$: $\mathscr{D}$-algebras and (strict) $\mathscr{D}$-maps.

In all of them, the 2-cells are the algebra 2-cells.

As in [8], we view $\mathscr{D}$-$\mathbf{AlgPs}$ as a happy medium, but we shall make no use of it. While the 0-cells, 1-cells, and 2-cells of $\mathscr{O}$-$\mathbf{PsAlg}$ (introduced in [8]) are categories, functors and natural transformations with extra structure, the 0-cells, 1-cells, and 2-cells of $\mathscr{D}$-$\mathbf{PsAlg}$ are pseudofunctors, pseudonatural transformations, and modifications with extra structure. This requires us to be careful categorically, but the precise parallel helps us organize the categorical data.

We show in §2.3 that the same general theorem of Power and Lack that specializes to prove the strictification theorem [8, Theorem 0.5] also specializes to prove the following analogue.

Theorem 0.1. *Let $\mathscr{D}$ be a category of operators in $\mathbf{Cat}(\mathscr{V})$. Then the inclusion of 2-categories*

$$\mathbb{J}\colon \mathscr{D}\text{-}\mathbf{AlgSt} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}$$

has a left adjoint

$$\mathbb{S}t\colon \mathscr{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{AlgSt}.$$

For $\mathscr{X} \in \mathscr{D}$-$\mathbf{PsAlg}$, the unit $\mathscr{X} \longrightarrow \mathbb{J}\mathbb{S}t(\mathscr{X})$ of the adjunction is an (internal) equivalence in $\mathscr{D}$-$\mathbf{PsAlg}$.

This result is independent of operads and in principle more general, but our interest is in the categories of operators associated to operads. We show in §3.1 that there is a category of operators $\mathscr{D} = \mathscr{D}(\mathscr{O})$ associated to any operad $\mathscr{O}$ in $\mathscr{V}$. For example, if we take $\mathscr{V} = \mathbf{Set}$, then $\mathscr{F} = \mathscr{D}(\mathbf{Com})$, where $\mathbf{Com}$ is the operad that defines commutative monoids, so that $\mathbf{Com}(n) = *$ for $n \geq 0$. A $\mathscr{D}$-pseudoalgebra $\mathscr{X}$ gives a $\mathscr{V}$-category $\mathscr{X}(\mathbf{n})$ for each n, and one of our restrictions on $\mathscr{D}$-pseudoalgebras requires $\mathscr{X}$ to be reduced in the sense that $\mathscr{X}(\mathbf{0}) = *$.[1] A $\mathscr{V}$-category $\mathscr{A}$ is based if we are given a $\mathscr{V}$-functor $* \longrightarrow \mathscr{A}$, and we let $\mathbf{Cat}(\mathscr{V})_*$ denote the 2-category of based $\mathscr{V}$-categories. When $\mathscr{A}$ is based, we define $(\mathbb{R}\mathscr{A})(\mathbf{n}) = \mathscr{A}^n$. Then $\mathbb{R}\mathscr{A}$ determines a functor $\Pi \longrightarrow \mathbf{Cat}(\mathscr{V})$. For a Π-algebra $\mathscr{X}$, we define $\mathbb{L}\mathscr{X} = \mathscr{X}(\mathbf{1})$. It is based, with $* = \mathscr{X}(\mathbf{0}) \longrightarrow \mathscr{X}(\mathbf{1})$ induced by the unique map $\mathbf{0} \longrightarrow \mathbf{1}$ in Π. We prove the following analogue of [15, 4.2] in §3.2.

Theorem 0.2. *Let $\mathscr{O}$ be an operad in $\mathbf{Cat}(\mathscr{V})$ and let $\mathscr{D} = \mathscr{D}(\mathscr{O})$. Then*

$$\mathbb{R}\colon \mathbf{Cat}(\mathscr{V})_* \longrightarrow \Pi\text{-}\mathbf{AlgSt}$$

induces a 2-functor

$$\mathbb{R}\colon \mathscr{O}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}.$$

Its image is the full sub 2-category of those $\mathscr{D}$-pseudoalgebras $\mathscr{X}$ such that the unit $\mathscr{X} \longrightarrow \mathbb{R}\mathbb{L}\mathscr{X}$ of the adjunction is the identity. It restricts on $\mathscr{O}$-algebras to a 2-functor

$$\mathbb{R}\colon \mathscr{O}\text{-}\mathbf{AlgSt} \longrightarrow \mathscr{D}\text{-}\mathbf{AlgSt},$$

and the composites $\mathbb{S}\mathrm{t} \circ \mathbb{R}$ and $\mathbb{R} \circ \mathbb{S}\mathrm{t}$ are isomorphic.

When we take $\mathscr{V}$ to be G-spaces and apply the classifying space functor, we arrive at input data for the additive infinite loop space machine of [14]. In effect, the results here compatibly generalize the categorical input both from strict $\mathscr{D}$-algebras and $\mathscr{D}$-maps to $\mathscr{D}$-pseudoalgebras and $\mathscr{D}$-pseudomorphisms and from operadic input to the more general input allowed by categories of operators. These generalizations are essential for the multiplicative infinite loop space theory of the sequels [7, 6, 5].

One source of relevant generalized input comes from changes of categories of operators. Note that pullback along a map $\xi\colon \mathscr{D} \longrightarrow \mathscr{E}$ of categories of operators[2] gives a 2-functor

$$\xi^*\colon \mathscr{E}\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})] \longrightarrow \mathscr{D}\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})].$$

The case $\mathscr{E} = \mathscr{F}$ is especially relevant.

Theorem 0.3. *Let $\xi\colon \mathscr{D} \longrightarrow \mathscr{E}$ be a map of categories of operators in $\mathbf{Cat}(\mathscr{V})$. The 2-functor ξ^* has a left adjoint*

$$\xi_*\colon \mathscr{D}\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})] \longrightarrow \mathscr{E}\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})].$$

Note that Theorem 0.3 is weaker than Theorem 0.1 in that the analogue of the last statement of Theorem 0.1 is false. It is not true that the unit $\mathscr{X} \longrightarrow \xi^*\xi_*\mathscr{X}$ of the adjunction (ξ_*, ξ^*) is an equivalence for all $\mathscr{X} \in \mathscr{D}\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})]$. This suggests the interest of starting with $\mathscr{F}$-pseudoalgebras rather than $\mathscr{D}$-pseudoalgebras. Strictification as in Theorem 0.1 applies to both $\mathscr{F}$ and $\mathscr{D}$, and we have the following consistency statement relating the two.

Proposition 0.4. *Let $\mathscr{X}$ be an $\mathscr{F}$-pseudoalgebra in $\mathbf{Cat}(\mathscr{V})$. Then there is a natural equivalence*

$$\iota\colon \mathbb{S}\mathrm{t}\xi^*\mathscr{X} \longrightarrow \xi^*\mathbb{S}\mathrm{t}\mathscr{X}$$

of strict $\mathscr{D}$-algebras over the $\mathscr{D}$-pseudoalgebra $\xi^*\mathscr{X}$.

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[1] With based enrichment, this holds automatically as in [14, Lemma 1.13].

[2] A map is given by a $\mathscr{V}$-2-functor under Π and over $\mathscr{F}$.

Just as in [8], the results above do not mention 2-monads, but we prove them using the formal theory of monads developed in the categorical papers [1, 10, 16, 18]. We are interested in the 2-categories named above, but they can be described equivalently in terms of algebras and pseudoalgebras over 2-monads. In §2.2, we describe how to associate a 2-monad to a category of operators in such a way that the resulting 2-categories of pseudoalgebras are isomorphic. Notationally, we will write $\mathbb{D}$ for the 2-monad associated to a category of operators $\mathscr{D}$.

1. Categorical Preliminaries

1.1. **Preliminaries on $\mathscr{V}$-2-categories.** We introduce some categorical language needed for dealing with categories of operators. It parallels and motivates the language we used when defining the 2-category $\mathscr{O}$-**PsAlg** of $\mathscr{O}$-pseudoalgebras in [8, 1.2]. Let $\mathscr{V}$ be a cartesian closed category and let $\mathbf{Cat}(\mathscr{V})$ be the 2-category of categories internal to $\mathscr{V}$. When considering 2-categorical data in this section, we insist on a set (rather than an object of $\mathscr{V}$) of 0-cells. We can then enrich in **Cat** or in $\mathbf{Cat}(\mathscr{V})$, in which case we have an internal $\mathscr{V}$-category giving the 1-cells and 2-cells between each pair of objects.

Definition 1.1. A $\mathscr{V}$-2-category $\mathscr{D}$ is a category enriched in $\mathbf{Cat}(\mathscr{V})$. We have the concomitant notions of a $\mathscr{V}$-2-functor and a $\mathscr{V}$-2-transformation. By definition, these are $\mathbf{Cat}(\mathscr{V})$-enriched functors and natural transformations.

More explicitly, $\mathscr{D}$ has an object set of 0-cells and its 1-cells and 2-cells are given by the objects $\mathbf{Ob}\mathscr{D}(\mathbf{m},\mathbf{n})$ and $\mathbf{Mor}\mathscr{D}(\mathbf{m},\mathbf{n})$ in $\mathscr{V}$ of the $\mathscr{V}$-categories $\mathscr{D}(\mathbf{m},\mathbf{n})$. We have identity $\mathscr{V}$-functors $I\colon * \longrightarrow \mathscr{D}(\mathbf{n},\mathbf{n})$ for each $\mathbf{n}$ and composition $\mathscr{V}$-functors

$$C\colon \mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{m},\mathbf{p}).$$

We think of the underlying sets of $\mathbf{Ob}\mathscr{D}(\mathbf{m},\mathbf{n})$ and $\mathbf{Mor}\mathscr{D}(\mathbf{m},\mathbf{n})$ as specifying the underlying 2-category of $\mathscr{D}$. We need more general kinds of $\mathscr{V}$-functors and $\mathscr{V}$-transformations.

Definition 1.2. Let $\mathscr{D}$ and $\mathscr{E}$ be $\mathscr{V}$-2-categories. Then a $\mathscr{V}$-pseudofunctor $F\colon \mathscr{D} \longrightarrow \mathscr{E}$ consists of a function F on objects, $\mathscr{V}$-functors

$$F\colon \mathscr{D}(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{E}(F(\mathbf{m}),F(\mathbf{n})),$$

and invertible unit and composition $\mathscr{V}$-transformations

$$\iota\colon I \Longrightarrow F\circ I$$

and

$$\begin{tikzcd}
\mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \arrow[r, "F\times F"] \arrow[d, "C"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{E}(F(\mathbf{n}),F(\mathbf{p})) \times \mathscr{E}(F(\mathbf{m}),F(\mathbf{n})) \arrow[d, "C"] \\
\mathscr{D}(\mathbf{m},\mathbf{p}) \arrow[r, "F"'] & \mathscr{E}(F(\mathbf{m}),F(\mathbf{p}))
\end{tikzcd}$$

that are unital and associative in the sense that the relevant equalities of pasting diagrams hold (see for example [11, Section 1.1]). We say that a $\mathscr{V}$-pseudofunctor is normal if ι and the composites $\varphi\circ(I\times \mathrm{id})$ and $\varphi\circ(\mathrm{id}\times I)$ are the identity; that is, F strictly preserves identity functors. *Henceforward, we restrict attention to normal $\mathscr{V}$-pseudofunctors F.* We say that F is a $\mathscr{V}$-2-functor if the φ are also identity maps. It is a lax $\mathscr{V}$-2-functor if the ι and φ are not required to be invertible.

These will specialize to give analogues of our operadic $\mathscr{O}$-pseudoalgebras.

Definition 1.3. Let F and G be $\mathscr{V}$-pseudofunctors $\mathscr{D} \longrightarrow \mathscr{E}$, where $\mathscr{D}$ and $\mathscr{E}$ are $\mathscr{V}$-2-categories. Then a $\mathscr{V}$-pseudotransformation[3] $\zeta\colon F \Longrightarrow G$ consists of 1-cells $\zeta_n\colon F(\mathbf{n}) \longrightarrow G(\mathbf{n})$ and invertible $\mathscr{V}$-transformations

$$\begin{tikzcd}
\mathscr{D}(\mathbf{m},\mathbf{n}) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\not\Downarrow\zeta_{m,n}"] & \mathscr{E}(F(\mathbf{m}),F(\mathbf{n})) \arrow[d, "(\zeta_n)_*"] \\
\mathscr{E}(G(\mathbf{m}),G(\mathbf{n})) \arrow[r, "(\zeta_m)^*"'] & \mathscr{E}(F(\mathbf{m}),G(\mathbf{n}))
\end{tikzcd}$$

such that the evident coherence diagram expressing compatibility with composition commutes (see for example [11, Section 1.2]).

Just as we require $\mathscr{V}$-pseudofunctors to be normal, we require $\mathscr{V}$-pseudotransformations to be normal in the sense that $\zeta_{n,n} \circ I$ is the identity for all objects $\mathbf{n}$. We say that ζ is a $\mathscr{V}$-2-transformation if the $\zeta_{m,n}$ are all identity maps. It is a lax $\mathscr{V}$-2-transformation if the $\zeta_{m,n}$ are not required to be invertible.

Certain $\mathscr{V}$-pseudotransformations will give analogues of the $\mathscr{O}$-pseudomorphisms of [8], and it is for this comparison that we require the 2-cells $\zeta_{m,n}$ to be invertible.

Remark 1.4. Our language here is non-standard. We have chosen it to precisely parallel the more standard language of 2-functor, pseudofunctor, and lax 2-functor. The term pseudotransformation is meant to emphasize that the constraints $\zeta_{m,n}$ are invertible; the term strong $\mathscr{V}$-2-transformation would be a synonym.[4]

Definition 1.5. Define a *$\mathscr{V}$-modification* χ between $\mathscr{V}$-pseudotransformations ζ and ξ to be a collection of underlying 2-cells $\chi_n\colon \zeta_n \Rightarrow \xi_n$, not required to be invertible, satisfying a suitable compatibility condition with respect to the $\mathscr{V}$-transformations $\zeta_{m,n}$ and $\xi_{m,n}$ (see Remark 1.7 below or [3, Section 3.10]).

To make these notions more concrete, we express the $\mathscr{V}$-pseudotransformations as examples of $\mathscr{V}$-pseudofunctors and express the $\mathscr{V}$-modifications as examples of lax $\mathscr{V}$-2-functors.

Remark 1.6. As in [8, Section 1.1], let $\mathscr{I}$ be the category with two objects $[0]$ and $[1]$ and one non-identity morphism $I\colon [0] \longrightarrow [1]$, and continue to write $\mathscr{I}$ for the 2-category obtained by adjoining an identity 2-cell. Then a $\mathscr{V}$-pseudotransformation $\zeta\colon F \longrightarrow G$ is the same thing as a $\mathscr{V}$-pseudofunctor $\zeta\colon \mathscr{D} \times \mathscr{I} \longrightarrow \mathscr{E}$ that restricts to F on $\mathscr{D} \times [0]$ and to G on $\mathscr{D} \times [1]$.

Remark 1.7. Define $\mathscr{J}$ to be the 2-category with two 0-cells $[0]$ and $[1]$, their identity 1 and 2-cells, and a copy $\mathscr{I}([0],[1])$ of $\mathscr{I}$ as the category of 1-cells and 2-cells $[0] \to [1]$. Write I_0 and I_1 for the 1-cells of $\mathscr{J}([0],[1])$ and J for its 2-cell.

$$\begin{tikzcd}
{[0]} \arrow[r, bend left, "I_0"] \arrow[r, bend right, "I_1"'] \arrow[r, phantom, "\Downarrow J"] & {[1]},
\end{tikzcd}$$

[3]This is meant as an abbreviation of $\mathscr{V}$-pseudonatural transformation.
[4]That clashes with the term weak functor that is sometimes used for pseudofunctors.

Let $i_0 \colon \mathscr{I} \longrightarrow \mathscr{J}$ and $i_1 \colon \mathscr{I} \to \mathscr{J}$ be the identity on 0-cells and map $I \colon [0] \longrightarrow [1]$ to $[I_0]$ and $[I_1]$ respectively. Then a $\mathscr{V}$-modification between $\mathscr{V}$-pseudotransformations ζ and ξ is a lax $\mathscr{V}$-2-functor $\mathscr{D} \times \mathscr{J} \longrightarrow \mathscr{E}$ that restricts to ζ along i_0 and to ξ along i_1. This encodes the unspecified compatibility conditions in Definition 1.5.

Remark 1.8. In the examples of $\mathscr{V}$ that we are most interested in, $\mathscr{V}$-2-categories are just 2-categories with extra structure (topology and/or G-action) on the sets of 1-cells and 2-cells such that all of the source, target, identity and composition maps respect the given structure. In these examples, the definitions of $\mathscr{V}$-pseudofunctors, $\mathscr{V}$-pseudotransformations and $\mathscr{V}$-modifications reduce to standard 2-categorical notions, taking into account that all of the pieces must respect the extra structure.

1.2. **$\mathscr{V}$-pseudofunctors with target $\mathbf{Cat}(\mathscr{V})$.** We are interested primarily in the case when $\mathscr{E} = \mathbf{Cat}(\mathscr{V})$, which so far has not been given an enrichment, and we first explain how it can be viewed as a $\mathscr{V}$-2-category. It is a standard categorical fact that $\mathbf{Cat}(\mathscr{V})$ is a closed cartesian monoidal 2-category. That means that it has an internal hom functor given by internal $\mathscr{V}$-categories $\underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{X},\mathscr{Y})$ that give an adjunction

$$\underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{X}\times\mathscr{Y},\mathscr{Z}) \cong \underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{X},\underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{Y},\mathscr{Z})).$$

for each triple of $\mathscr{V}$-categories. Passing to hom sets, the adjunction gives an adjunction on the level of underlying categories,

$$\mathbf{Cat}(\mathscr{V})(\mathscr{X}\times\mathscr{Y},\mathscr{Z}) \cong \mathbf{Cat}(\mathscr{V})(\mathscr{X},\underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{Y},\mathscr{Z})).$$

With 0-cells the internal $\mathscr{V}$-categories, the $\underline{\mathbf{Cat}}(\mathscr{V})(\mathscr{X},\mathscr{Y})$ give the required enrichment making $\mathbf{Cat}(\mathscr{V})$ a $\mathscr{V}$-2-category. It would be digressive to describe the internal hom functor in detail, but we can use the adjunction to reexpress $\mathscr{V}$-functors solely in terms of their adjoint evaluation maps, which are $\mathscr{V}$-functors

$$\theta\colon \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n}).$$

This is very convenient for comparison with operadic structures. We spell out the previous definitions more explicitly in this adjoint form. Remember that we restrict to normal structures, so that we always have strict unit data.

Definition 1.9. A $\mathscr{V}$-pseudofunctor $\mathscr{X}: \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$ consists of a $\mathscr{V}$-category $\mathscr{X}(\mathbf{n})$ for each object $\mathbf{n}$ of $\mathscr{D}$, a $\mathscr{V}$-functor

$$\theta\colon \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n})$$

for each pair of objects, and invertible composition $\mathscr{V}$-transformations

$$\begin{tikzcd}
\mathscr{D}(\mathbf{n},\mathbf{p})\times\mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{X}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[d, "C\times\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{D}(\mathbf{n},\mathbf{p})\times\mathscr{X}(\mathbf{n}) \arrow[d, "\theta"] \\
\mathscr{D}(\mathbf{m},\mathbf{p})\times\mathscr{X}(\mathbf{m}) \arrow[r, "\theta"'] & \mathscr{X}(\mathbf{p}).
\end{tikzcd}$$

We require the normality unit conditions to hold and we require equalities of pasting diagrams precisely analogous to those displayed monadically in [16, (2.1)-(2.3)]. We say that $\mathscr{X}$ is a $\mathscr{V}$-2-functor if the φ are identity maps; it is a lax $\mathscr{V}$-2-functor if they are not required to be invertible.

Definition 1.10. Let $\mathscr{X}, \mathscr{Y} \colon \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$ be $\mathscr{V}$-pseudofunctors. A $\mathscr{V}$-pseudo-transformation $\zeta \colon \mathscr{X} \longrightarrow \mathscr{Y}$ consists of $\mathscr{V}$-functors $\zeta_n \colon \mathscr{X}(\mathbf{n}) \longrightarrow \mathscr{Y}(\mathbf{n})$ and invertible $\mathscr{V}$-transformations

$$\begin{tikzcd}
\mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\mathrm{id} \times \zeta_m"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow \zeta_{m,n}"] & \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\theta"] \\
\mathscr{X}(\mathbf{n}) \arrow[r, "\zeta_n"'] & \mathscr{Y}(\mathbf{n}).
\end{tikzcd}$$

We require the normality unit conditions to hold, and we require equalities of pasting diagrams precisely analogous to those displayed monadically in [16, (2.4)-(2.5)]. We say that ζ is a $\mathscr{V}$-2-transformation if all $\zeta_{m,n}$ are identity maps; it is lax if the $\zeta_{m,n}$ are not required to be invertible.

2. Categories of operators and their pseudoalgebras

2.1. **Categories of operators in $\mathbf{Cat}(\mathscr{V})$.** In the previous subsections, $\mathscr{D}$ was a general $\mathscr{V}$-2-category. However, we are only interested in the case when $\mathscr{D}$ is a $\mathbf{Cat}(\mathscr{V})$-category of operators. To abbreviate notation, we first define a $\mathscr{W}$-category of operators for a general $\mathscr{W}$ and then specialize to $\mathscr{W} = \mathbf{Cat}(\mathscr{V})$. We assume that $\mathscr{W}$ satisfies our original hypotheses on $\mathscr{V}$, as $\mathbf{Cat}(\mathscr{V})$ does.

Definition 2.1. Let $\mathscr{F}$ be the category of based sets $\mathbf{n} = \{0, 1, \ldots, n\}$ with basepoint 0 and let Π be its subcategory of morphisms $\phi \colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $|\phi^{-1}(j)| = 0$ or 1 for $1 \leq j \leq n$.

Remark 2.2. As observed in [8, Definition 1.5], the underlying set functor $\mathbb{U} \colon \mathscr{V} \longrightarrow \mathbf{Set}$ specified by $\mathbb{U}X = \mathscr{V}(*, X)$ has left adjoint $\mathbb{V} \colon \mathbf{Set} \longrightarrow \mathscr{V}$ specified by $\mathbb{V}S = \coprod_{s \in S} *$, the coproduct of copies of the terminal object $*$ indexed on the elements of the set S. Thus

$$\mathscr{V}(\mathbb{V}S, X) \cong \mathbf{Set}(S, \mathbb{U}X). \tag{2.3}$$

Application of $\mathbb{V}$ to hom sets allows us to change the enrichment of $\mathscr{F}$ and Π from $\mathbf{Set}$ to $\mathbf{Cat}(\mathscr{V})$. In the following definition, we assume that $\mathscr{F}$ and Π have been enriched to our ambient category $\mathscr{W}$.

Definition 2.4. A $\mathscr{W}$-category of operators $\mathscr{D}$ over $\mathscr{F}$, abbreviated $\mathscr{W}$-**CO** over $\mathscr{F}$, is a category enriched in $\mathscr{W}$ whose objects are the based sets $\mathbf{n}$ for $n \geq 0$ together with $\mathscr{W}$-functors

$$\Pi \xrightarrow{\iota} \mathscr{D} \xrightarrow{\xi} \mathscr{F}$$

such that ι and ξ are the identity on objects and $\xi \circ \iota$ is the inclusion. We restrict attention to reduced $\mathscr{W}$-**CO**s over $\mathscr{F}$, for which $\mathscr{D}(\mathbf{m}, \mathbf{n}) = *$ when either $m = 0$ or $n = 0$. A morphism $\nu \colon \mathscr{D} \longrightarrow \mathscr{E}$ of $\mathscr{W}$-**CO**s over $\mathscr{F}$ is a $\mathscr{W}$-functor over $\mathscr{F}$ and under Π.

Remark 2.5. As will be relevant in [6], note that while we do not have k-fold products of $\mathscr{W}$-**CO**s over $\mathscr{F}$, we do have k-fold products over $\mathscr{F}^k$ and under Π^k.

Remark 2.6. For equivariant applications, we also need the notion of a category of operators over $\mathscr{F}_G$, where we replace $\mathscr{F}$ and Π by the category $\mathscr{F}_G$ of finite G-sets and its subcategory Π_G specified in [14, Definition 2.24]. Their common sets of objects are the G-sets $\mathbf{n} = \{0, 1, \ldots, n\}$ with action specified by a homomorphism

$\alpha\colon G \longrightarrow \Sigma_n$. We will not consider categories of operators over $\mathscr{F}_G$ in this paper, but they will be relevant in the sequels, just as they were in [14], where we needed them to keep track of equivariant homotopy types. As we will expand on in [7, 6], the theory in this paper applies verbatim after making these replacements.

As said, we are mainly interested in the special case in which $\mathscr{W} = \mathbf{Cat}(\mathscr{V})$, and in this case we must of course think 2-categorically. Thus we regard Π and $\mathscr{F}$ as 2-categories with only identity 2-cells and we apply the functor $\mathbb{V}$ of Remark 2.2 to their hom objects so as to view them as $\mathscr{V}$-2-categories. Thus a $\mathbf{Cat}(\mathscr{V})$-$\mathbf{CO}$ $\mathscr{D}$ over $\mathscr{F}$ must be a $\mathscr{V}$-2-category and ι and ξ must be $\mathscr{V}$-2-functors (and not just $\mathscr{V}$-pseudofunctors).

Assumption 2.7. In line with our requirement that $\mathscr{D}$ be reduced, *we require all $\mathscr{V}$-2-functors and $\mathscr{V}$-pseudofunctors $\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$ to be reduced in the sense that $\mathscr{X}(\mathbf{0}) = *$.* The $\mathscr{X}(\mathbf{n})$ are then naturally based, and we write 0 for the resulting natural map $* \longrightarrow \mathscr{X}(\mathbf{n})$ in $\mathbf{Cat}(\mathscr{V})$.

We now define the categories of algebras and pseudoalgebras over a $\mathbf{Cat}(\mathscr{V})$-category of operators $\mathscr{D}$ over $\mathscr{F}$. Recall from Definition 1.2 that we require all $\mathscr{V}$-pseudofunctors $\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$ to be normal in the sense defined there.

Definition 2.8. Define a $\mathscr{D}$-algebra to be a (reduced) $\mathscr{V}$-2-functor $\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$. Define a $\mathscr{D}$-pseudoalgebra to be a (reduced and normal) $\mathscr{V}$-pseudofunctor $\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$ for which the following further assumptions hold.

(i) When $\mathscr{D} = \Pi$, we require $\mathscr{X}$ to be a $\mathscr{V}$-2-functor. Then $\mathscr{X}$ is just a $\mathscr{V}$-functor since its domain has only identity 2-cells. In particular, for a general $\mathscr{D}$, we require $\mathscr{X}$ to restrict to a $\mathscr{V}$-functor $\Pi \longrightarrow \mathbf{Cat}(\mathscr{V})$.

(ii) More generally, we require the $\mathscr{V}$-transformation

$$\begin{tikzcd}
\mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[d, "C\times\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{X}(\mathbf{n}) \arrow[d, "\theta"] \\
\mathscr{D}(\mathbf{m},\mathbf{p}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\theta"'] & \mathscr{X}(\mathbf{p})
\end{tikzcd}$$

to be the identity when $\mathscr{D}(\mathbf{n},\mathbf{p})$ is restricted to $\Pi(\mathbf{n},\mathbf{p})$ or when $\mathscr{D}(\mathbf{m},\mathbf{n})$ is restricted to $\Pi(\mathbf{m},\mathbf{n})$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{D}$-pseudoalgebras. A *$\mathscr{D}$-map* $\zeta\colon \mathscr{X} \longrightarrow \mathscr{Y}$ is a $\mathscr{V}$-2-transformation. A *$\mathscr{D}$-pseudomorphism* $\zeta\colon \mathscr{X} \longrightarrow \mathscr{Y}$ is a $\mathscr{V}$-pseudotransformation such that the $\mathscr{V}$-transformation

$$\begin{tikzcd}
\mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\zeta_m"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\Downarrow\zeta_{m,n}"] & \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\theta"] \\
\mathscr{X}(\mathbf{n}) \arrow[r, "\zeta_n"'] & \mathscr{Y}(\mathbf{n}).
\end{tikzcd}$$

is the identity when $\mathscr{D}(\mathbf{m},\mathbf{n})$ is restricted to $\Pi(\mathbf{m},\mathbf{n})$. An *algebra 2-cell or $\mathscr{D}$-modification* between $\mathscr{D}$-maps or between $\mathscr{D}$-pseudomorphisms is a $\mathscr{V}$-modification between them.

Remark 2.9. The assumptions in the definition may appear ad-hoc. They are imposed in order to ensure that the notions of $\mathscr{D}$-algebras and pseudoalgebras precisely correspond to the notions of algebras and pseudoalgebras over the 2-monad $\mathbb{D}$ constructed from $\mathscr{D}$ in §2.2.

With these definitions of $\mathscr{D}$-algebras, $\mathscr{D}$-pseudoalgebras, $\mathscr{D}$-maps, $\mathscr{D}$-pseudomorphisms and $\mathscr{D}$-modifications, we obtain the 2-categories

$$\mathscr{D}\text{-}\mathbf{AlgSt} \subseteq \mathscr{D}\text{-}\mathbf{AlgPs} \subseteq \mathscr{D}\text{-}\mathbf{PsAlg}$$

specified in the introduction. Ignoring the 2-cells, these categories are bicomplete, with limits and colimits constructed termwise. We make the standard caveat that colimits in $\mathbf{Cat}(\mathscr{V})$ are poorly behaved and generally to be avoided. In particular, the classifying space functor B does not commute with colimits when $\mathscr{V} = \mathscr{U}$.

2.2. **The 2-monads associated to categories of operators.** Recall the discussion of the 2-category $\mathbf{Cat}(\mathscr{V})$ and its enrichment to a $\mathscr{V}$-2-category from §1.1.

Definition 2.10. Let $\mathscr{D}$ be a (small) $\mathscr{V}$-2-category with object set Ω and a subcategory Ψ with the same objects. Write $\mathbf{Cat}(\mathscr{V})^{\Psi}$ for the 2-category of functors $\Psi \longrightarrow \mathbf{Cat}(\mathscr{V})$ and write $\mathbf{Cat}(\mathscr{V})^{\Omega}$ for the 2-category of functors $\Omega \longrightarrow \mathbf{Cat}(\mathscr{V})$, where Ω is regarded as a discrete category. We have an underlying objects 2-functor $\mathbf{Cat}(\mathscr{V})^{\Psi} \longrightarrow \mathbf{Cat}(\mathscr{V})^{\Omega}$.

We define a monad $\mathbb{D}^{\times}$ in $\mathbf{Cat}(\mathscr{V})^{\Omega}$ and a monad $\mathbb{D}^{\Psi}$ in $\mathbf{Cat}(\mathscr{V})^{\Psi}$. For $\mathscr{Y}$ in $\mathbf{Cat}(\mathscr{V})^{\Omega}$ and $n \in \Omega$, define

$$(\mathbb{D}^{\times}\mathscr{Y})(n) = \coprod_{m\in\Omega} \mathscr{D}(m,n) \times \mathscr{Y}(m). \tag{2.11}$$

The superscript $\times$ is a reminder that we use products only, with no identifications. For $\mathscr{Y}$ in $\mathbf{Cat}(\mathscr{V})^{\Psi}$ and $n \in \Omega$, define

$$(\mathbb{D}^{\Psi}\mathscr{Y})(n) = \mathscr{D}(-,n) \otimes_{\Psi} \mathscr{Y}. \tag{2.12}$$

Here we are using the usual coequalizer construction of the tensor product of a contravariant and a covariant functor over a category Ψ. Let $q\colon \mathbb{D}^{\times}\mathscr{Y} \longrightarrow \mathbb{D}^{\Psi}\mathscr{Y}$ be the quotient 2-functor. The unit $\eta\colon \mathscr{Y} \longrightarrow \mathbb{D}^{\times}\mathscr{Y}$ is given by the $\mathscr{V}$-functors

$$I \times \mathrm{id}\colon \mathscr{Y}(t) = * \times \mathscr{Y}(n) \longrightarrow \mathscr{D}(n,n) \times \mathscr{Y}(n).$$

The product $\mu\colon \mathbb{D}^{\times}\mathbb{D}^{\times}\mathscr{Y} \longrightarrow \mathbb{D}^{\times}\mathscr{Y}$ is given by the $\mathscr{V}$-functors

$$C \times \mathrm{id}\colon \mathscr{D}(n,p) \times \mathscr{D}(m,n) \times \mathscr{Y}(m) \longrightarrow \mathscr{D}(m,p) \times \mathscr{Y}(m).$$

The unit and associativity properties of $\mathbb{D}^{\times}$ are inherited from the unit and associativity properties of $\mathscr{D}$. The 2-category of $\mathbb{D}^{\times}$-pseudoalgebras has 0-cells, 1-cells, and 2-cells the $\mathscr{V}$-pseudofunctors $\mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$, the $\mathscr{V}$-pseudotransformations, and the $\mathscr{V}$-modifications. The unit η and product μ of $\mathbb{D}^{\Psi}$ are induced similarly from the identity and composition of $\mathscr{D}$. The $\mathbb{D}^{\Psi}$-pseudoalgebras are those $\mathbb{D}^{\times}$-pseudoalgebras $\mathscr{Y}$ such that $\mathscr{Y}$ is in $\mathscr{Y} \in \mathbf{Cat}(\mathscr{V})^{\Psi}$ and all pseudostructure is strict when restricted to Ψ.

These monads are motivated by the fact that a $\mathscr{D}$-pseudoalgebra $\mathscr{Y}$ can be described in terms of its evaluation $\mathscr{V}$-functors

$$\mathscr{D}(m,n) \times \mathscr{Y}(m) \longrightarrow \mathscr{Y}(n).$$

With the appropriate definition of a morphism of monads defined on related 2-categories, $q\colon \mathbb{D}^{\times} \longrightarrow \mathbb{D}^{\Psi}$ is a morphism of monads.

The most relevant examples of Ψ and Ω are Π with its object set N of natural numbers and Π_G with its object set N_G of finite G-sets. The relevant $\mathscr{D}$ are categories of operators over $\mathscr{F}$ and categories of operators over $\mathscr{F}_G$. We unpack the definition, focusing on the category of operators $\mathscr{D} = \mathscr{D}(\mathscr{O})$ over $\mathscr{F}$ associated to an operad $\mathscr{O}$ in $\mathbf{Cat}(\mathscr{V})$. A similar unpacking applies to a category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$ associated to a G-operad $\mathscr{O}$. We are modifying the construction of $\mathbb{D}^{\times}$ to fit our precise definition of a $\mathscr{D}$-pseudoalgebra, adding in morphisms of Π to strictify structure. We have two relevant subcategories of Π:

$$\Sigma \subset \Upsilon \subset \Pi.$$

Here Σ denotes the subcategory of permutations of natural numbers, which we focused on in [4], and Υ adds in the projections. Then Π is obtained from Υ by adding in the injections.

Example 2.13. To add in strict action by permutations, for $\mathscr{Y} \in \mathbf{Cat}^{\Sigma}$ define

$$(\mathbb{D}^{\Sigma}\mathscr{Y})(\mathbf{n}) = \coprod_{m} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times_{\Sigma_m} \mathscr{Y}(\mathbf{m}).$$

When we allow morphisms $\mathbf{m} \longrightarrow \mathbf{n}$ for $m \neq n$, we want a monad that has the effect of restricting attention to the effective morphisms $\mathscr{E} \subset \mathscr{F}$, namely those $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $\phi^{-1}(0) = 0$. This does not use "basepoint identifications" and gives the correct analogue of the monad $\mathbb{O}_+$ defined in [8, (2.4)].

Example 2.14. For $\mathscr{Y} \in \mathbf{Cat}^{\Upsilon}$, define

$$(\mathbb{D}^{\Upsilon}\mathscr{Y})(n) = \mathscr{D}(-, \mathbf{n}) \otimes_{\Upsilon} \mathscr{Y}.$$

Explicitly,

$$(\mathbb{D}^{\Upsilon}\mathscr{Y})(n) = \coprod_{\phi \in \mathscr{E}(\mathbf{m},\mathbf{n})} \big(\prod_{1 \le j \le n} \mathscr{O}(\phi_j) \big) \times_{\Sigma(\phi)} \mathscr{Y}_m,$$

where $\phi_j = |\phi^{-1}(j)|$ and $\Sigma(\phi) = \prod_j \Sigma_{\phi_j} \subset \Sigma_m$.

As said, $\mathbb{D}^{\Upsilon}$ is analogous to the operadic monad $\mathbb{O}_+$. Just as we added in the injections to the permutations to construct the category Λ used to define $\mathbb{O}$ from $\mathbb{O}_+$ in [8, Section 2.2], we now add in the injections to the subcategory Υ of Π to define $\mathbb{D}^{\Pi}$ from $\mathbb{D}^{\Upsilon}$. This idea when $\mathbf{Cat}(\mathscr{V})$ is replaced by $\mathscr{U}$ is discussed in detail in [13, Section 2]. In the categorical context here, to tailor the monad to fit with Assumption 2.7 in our definition of $\mathscr{D}$-pseudoalgebras, we must work with $\mathbb{D}^{\Pi}$ and not $\mathbb{D}^{\Upsilon}$.

Example 2.15. We construct $\mathbb{D}^{\Pi}$ from $\mathbb{D}^{\Upsilon}$ just as was done with $\mathbf{Cat}(\mathscr{V})$ replaced by spaces in [15, Section 5] and by G-spaces in [14, 5.1]. For a Π-category $\mathscr{Y}$,

$$(\mathbb{D}^{\Pi}\mathscr{Y})(\mathbf{n}) = \mathscr{D}(-, \mathbf{n}) \otimes_{\Pi} \mathscr{Y}.$$

Therefore $(\mathbb{D}^{\Pi}\mathscr{Y})_n$ is constructed from $(\mathbb{D}^{\Upsilon}\mathscr{Y})_n$ by successive pushouts in $\mathbf{Cat}(\mathscr{V})$ exactly as in [15, Lemma 5.5].

Notation 2.16. From now on, we abbreviate notation by writing $\mathbb{D}$ for the monad $\mathbb{D}^{\Pi}$ on $\mathbf{Cat}(\mathscr{V})^{\Pi}$.

Recall the notion of pseudoalgebra over a 2-monad from [8, Section 2.1]. We have the following monadic identifications of the 2-categories in **??**

Proposition 2.17. *With $\mathbb{D} = \mathbb{D}^{\Pi}$, there are canonical identifications of 2-categories*

- $\mathbb{D}$-$\mathbf{PsAlg} = \mathscr{D}$-$\mathbf{PsAlg}[\mathbf{Cat}(\mathscr{V})]$: *$\mathbb{D}$-pseudoalgebras and $\mathbb{D}$-pseudomorphisms.*
- $\mathbb{D}$-$\mathbf{AlgPs} = \mathscr{D}$-$\mathbf{AlgPs}[\mathbf{Cat}(\mathscr{V})]$: *$\mathbb{D}$-algebras and $\mathbb{D}$-pseudomorphisms.*
- $\mathbb{D}$-$\mathbf{AlgSt} = \mathscr{D}$-$\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})]$: *$\mathbb{D}$-algebras and (strict) maps.*

Proof. If X is a $\mathscr{D}$-pseudoalgebra, there results a transformation $\mathbb{D}X \longrightarrow X$ because the transformation φ of Definition 2.8 is the identity when $\mathscr{D}(\mathbf{m}, \mathbf{n})$ is restricted to Π. The required invertible 2-cell $\varphi : \theta \circ \mathbb{D}^{\Pi}\theta \cong \theta \circ \mu$ is induced from the φ of the $\mathscr{D}$-pseudoalgebra structure, using the equality of pasting diagrams in Definition 2.8. Adding in the $\mathbb{D}$-transformations completes the identifications of 2-categories. □

2.3. **The proof of the strictification theorem.** Power [16] and Lack [10] proved a strictification theorem for pseudoalgebras over 2-monads. This also appears as [8, Theorem 2.15]), where it is treated in more detail and is generalized from $\mathbf{Cat}$ to $\mathbf{Cat}(\mathscr{V})$. In this section, we show that the Power-Lack theorem specializes to give a proof of Theorem 0.1. In order to apply [8, Theorem 2.15], we must check that the monad $\mathbb{D}$ satisfies its hypotheses.

The notion of a rigid enhanced factorization system (EFS) on a 2-category $\mathscr{K}$ is defined in [8, Definiition 2.9]. It consists of a distinguished pair of classes of 1-cells $(\mathscr{E}, \mathscr{M})$ satisfying certain conditions. The notion of a product-preserving EFS is defined in [8, Definition 2.13]. A monad $\mathbb{D}$ on $\mathscr{K}$ is said to preserve $\mathscr{E}$ if $\mathbb{D}e$ is a 1-cell in $\mathscr{E}$ whenever e is a 1-cell in $\mathscr{E}$ ([8, Definition 2.14]). With this language, the hypotheses of [8, Theorem 2.15] are:

(i) The ground category $\mathbf{Cat}(\mathscr{V})^{\Pi}$ for the 2-monad $\mathbb{D}$ must have a rigid EFS.
(ii) The monad $\mathbb{D}$ must preserve $\mathscr{E}$.

The strictification 2-functor $\mathbb{S}t$ behaves well with respect to products if the EFS is also product-preserving [8, Corollary 2.17].

In [8, Section 2.4], we obtained a rigid EFS of "bijective on objects" and "fully faithfull" functors $(\mathscr{B}\mathscr{O}, \mathscr{F}\mathscr{F})$ on $\mathbf{Cat}(\mathscr{V})$; it is defined in [8, Definition 2.22] and proven to be a product-preserving rigid EFS in [8, Theorem 2.24]. As noted by Power [16], when applied levelwise, the conclusion of [8, Theorem 2.24] applies equally well to any category of the form $\mathbf{Cat}(\mathscr{V})^S$ for a set S. More generally, he wrote "It is straightforward to generalise this result to functor 2-categories $[\mathscr{C}, Cat]$ for small $\mathscr{C}$. However, it requires a succession of pasting diagrams, and it is not the case of primary interest; so I omit the proof." Here "this result" refers to his version of [8, Theorem 2.24].[5] The required painstaking generalization that allows one to replace S with any small 2-category Ψ (even allowing non-identity 2-cells), was carried out by Peter Haine [9]. In our context, the following specialization of his theorem gives that hypothesis (i) is satisfied.

Theorem 2.18. *([9, Theorem 2.9]) Let $\mathbf{Cat}(\mathscr{V})^{\Pi}$ be the 2-category of functors $\Pi \longrightarrow \mathbf{Cat}(\mathscr{V})$, natural transformations, and modifications. Applied levelwise on the set of objects of Π, the classes $(\mathscr{B}\mathscr{O}, \mathscr{F}\mathscr{F})$ specify a product-preserving rigid enhanced factorization system on $\mathbf{Cat}(\mathscr{V})^{\Pi}$.*

Finally, we check hypothesis (ii) and so complete the proof that [8, Theorem 2.15] implies Theorem 0.1. The following proof is a slight elaboration of the proof of [8, Proposition 2.28].

Proposition 2.19. *The monad $\mathbb{D}$ in $\mathbf{Cat}(\mathscr{V})^{\Pi}$ preserves $\mathscr{B}\mathscr{O}$.*

[5] In his version, $\mathscr{V}$ is the category of sets.

Proof. Recall that a 1-cell $\mathscr{X}$ in $\mathbf{Cat}(\mathscr{V})^{\Pi}$ is in $\mathscr{B}\mathscr{O}$ if the $\mathscr{V}$-map $\mathbf{Ob}\mathscr{X}(\mathbf{n}) \longrightarrow \mathbf{Ob}\mathscr{Y}(\mathbf{n})$ is an isomorphism for each object $\mathbf{n} \in \Pi$. Here $\mathbf{Ob}\colon \mathbf{Cat}(\mathscr{V}) \longrightarrow \mathscr{V}$ is the object functor. The functor $\mathbf{Ob}$ commutes with colimits since it is a left adjoint (with right adjoint the chaotic category functor), and it clearly commutes with products. By inspection, this implies that $\mathbf{Ob}\mathbb{D}$ is a monad in $\mathscr{V}$ that satisfies

$$\mathbf{Ob}(\mathbb{D}\mathscr{X}) \cong (\mathbf{Ob}\mathbb{D})(\mathbf{Ob}\mathscr{X}).$$

Since $\mathbf{Ob}\mathbb{D}$ preserves isomorphisms by functoriality, the conclusion follows. □

3. Operads and their associated categories of operators

3.1. **Operads and categories of operators.** In infinite loop space theory, we are only interested in those categories of operators that arise from operads.

Definition 3.1. Let $\mathscr{O}$ be a (reduced) operad in $\mathbf{Cat}(\mathscr{V})$. We construct a $\mathbf{Cat}(\mathscr{V})$-**CO** over $\mathscr{F}$, which we denote by $\mathscr{D}(\mathscr{O})$ and abbreviate to $\mathscr{D}$ when there is no risk of confusion. The morphism $\mathscr{V}$-categories of $\mathscr{D}$ are

$$\mathscr{D}(\mathbf{m},\mathbf{n}) = \coprod_{\phi\in\mathscr{F}(\mathbf{m},\mathbf{n})} \prod_{1\le j\le n} \mathscr{O}(|\phi^{-1}(j)|).$$

To specify composition

$$\mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \xrightarrow{\circ} \mathscr{D}(\mathbf{m},\mathbf{p}),$$

write $\phi_j = |\phi^{-1}(j)|$. It suffices to specify $\circ$ on coproduct summands. Fix morphisms $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $\psi\colon \mathbf{n} \longrightarrow \mathbf{p}$. Then $\circ$ sends the summand of (ψ,ϕ) in the domain to the summand of $\psi\circ\phi$ in the target. Observe that, for $1 \le k \le p$,

$$(\psi\circ\phi)_k = \sum_{\psi(j)=k} \phi_j.$$

Let $\rho(\psi,\phi)_k$ be that permutation of $(\psi\circ\phi)_k$-letters which converts the natural ordering of $(\psi\circ\phi)^{-1}(k)$ as a subset of $\{1,\cdots,m\}$ to its ordering obtained by regarding it as $\coprod_{\psi(j)=k}\phi^{-1}(j)$, so ordered that elements of $\phi^{-1}(j)$ precede elements of $\phi^{-1}(j')$ if $j<j'$ and each $\phi^{-1}(j)$ has its natural ordering as a subset of $\{1,\cdots,m\}$. Then the (ψ,ϕ)-summand of $\circ$ is the composite

$$\begin{tikzcd}
\prod_k \mathscr{O}(\psi_k) \times \prod_j \mathscr{O}(\phi_j) \arrow[d, "\cong"] \\
\prod_k \mathscr{O}(\psi_k) \times (\prod_{\psi(j)=k} \mathscr{O}(\phi_j)) \arrow[d, "\prod_k \gamma"] \\
\prod_k \mathscr{O}((\psi\circ\phi)_k) \arrow[d, "\prod_k \rho(\psi,\phi)_k"] \\
\prod_k \mathscr{O}((\psi\circ\phi)_k).
\end{tikzcd}$$

Here γ is the structural functor of the operad $\mathscr{O}$ and the bottom map $\rho(\psi,\phi)_k$ is the right action of $\rho(\psi,\phi)_k$. The unit object of $\mathscr{D}(\mathbf{n},\mathbf{n})$ is (id, id^n), where id on the right is the unit element in $\mathscr{O}(1)$. The map $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ sends the summand indexed on ϕ to ϕ. The inclusion $\iota\colon \Pi \longrightarrow \mathscr{D}$ sends $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ to (ϕ, c), where

$c_i = \mathrm{id} \in \mathscr{O}(1)$ if $\phi(i) = 1$ and $c_i = * \in \mathscr{O}(0)$ if $\phi(i) = 0$. Since $\mathscr{O}$ is reduced as a $\mathscr{V}$-operad, $\mathscr{D}$ is reduced as a $\mathbf{Cat}(\mathscr{V})$-$\mathbf{CO}$ over $\mathscr{F}$.

3.2. **From $\mathscr{O}$-pseudoalgebras to $\mathscr{D}$-pseudoalgebras.** Recall from the introduction that the 2-functor

$$\mathbb{R}\colon \mathbf{Cat}(\mathscr{V})_* \longrightarrow \Pi\text{-}\mathbf{AlgSt}[\mathbf{Cat}(\mathscr{V})]$$

is given by

$$(\mathbb{R}\mathscr{A})(\mathbf{n}) = \mathscr{A}^n$$

for a based $\mathscr{V}$-category $\mathscr{A}$. As explained in [13, Section 1], since we restrict to reduced Π-algebras, $\mathbb{R}$ is right adjoint to the 2-functor $\mathbb{L}$ defined by $\mathbb{L}\mathscr{X} = \mathscr{X}(\mathbf{1})$ for a Π-$\mathscr{V}$-category $\mathscr{X}$.

We here use $\mathbb{R}$ to compare $\mathscr{O}$-pseudoalgebras to $\mathscr{D}$-pseudoalgebras. By [8, Definition 1.23], an $\mathscr{O}$-pseudoalgebra $\mathscr{A}$ is a $\mathscr{V}$-category equipped with action functors

$$\theta = \theta(n) : \mathscr{O}(n) \times \mathscr{A}^n \longrightarrow \mathscr{A}$$

and invertible transformations

$$\begin{tikzcd}
\mathscr{O}(n) \times (\times_r \mathscr{O}(m_r) \times \mathscr{A}^{m_r}) \arrow[r, "\mathrm{id} \times (\times_r \theta(m_r))"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{O}(n) \times \mathscr{A}^n \arrow[dr, "\theta(n)"] & \\
\mathscr{O}(n) \times (\times_r \mathscr{O}(m_r)) \times \mathscr{A}^m \arrow[r, "\gamma \times \mathrm{id}"'] & \mathscr{O}(m) \times \mathscr{A}^m \arrow[r, "\theta(m)"'] & \mathscr{A}
\end{tikzcd}$$

satisfying assumptions precisely analogous to those of Definition 2.8.

Proof of Theorem 0.2. We must prove that, when restricted to $\mathscr{O}$-pseudoalgebras, $\mathbb{R}$ gives a 2-functor

$$\mathbb{R}\colon \mathscr{O}\text{-}\mathbf{PsAlg}[\mathbf{Cat}(\mathscr{V})] \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}[\mathbf{Cat}(\mathscr{V})].$$

Thus we must prove that if $\mathscr{A}$ is an $\mathscr{O}$-pseudoalgebra, then $\mathbb{R}\mathscr{A}$ is a $\mathscr{D}$-pseudoalgebra.

For strict algebras, there is a quick trick [15, 4.2]. An action of $\mathscr{O}$ on $\mathscr{A}$ is a map of operads $\lambda : \mathscr{O} \longrightarrow \mathbf{End}(\mathscr{A})$, where $\mathbf{End}(\mathscr{A})$ is the endomorphism operad of $\mathscr{A}$ with nth $\mathscr{V}$-category $\mathbf{Cat}(\mathscr{V})(\mathscr{A}^n, \mathscr{A})$. Applying $\mathscr{D}$ gives a 2-functor $\mathscr{D}(\lambda)\colon \mathscr{D}(\mathscr{O}) \longrightarrow \mathscr{D}(\mathbf{End}(\mathscr{A}))$. The composition

$$\mathscr{D}(\mathscr{O}) \xrightarrow{\mathscr{D}(\lambda)} \mathscr{D}(\mathbf{End}(\mathscr{A})) \xrightarrow{\mathbb{R}\mathscr{A}} \mathbf{Cat}(\mathscr{V})$$

exhibits $\mathscr{A}$ as a $\mathscr{D}(\mathscr{O})$-algebra.

Analogously, an $\mathscr{O}$-pseudoalgebra structure on $\mathscr{A}$ gives rise to a pseudofunctor $\mathscr{D}(\lambda)\colon \mathscr{D}(\mathscr{O}) \longrightarrow \mathscr{D}(\mathbf{End}(\mathscr{A}))$. The transformations φ of [8, Definition 1.23] give rise to the transformations φ of Definition 1.9. The operadic unit axiom ([8, Axiom 1.29]) ensures that this pseudofunctor is normal. Moreover, [8, Properties 1.30–1.33] ensure that the transformation φ is the identity in the cases specified in Definition 2.8(ii). To see this, consider a function $\rho \in \Pi(\mathbf{m}, \mathbf{n})$. Since $|\rho^{-1}(j)|$ is 0 or 1 for $1 \leq j \leq n$, the function ρ is sent to a tuple of operadic zero's (in $\mathscr{O}(0)$) and operadic identities (in $\mathscr{O}(1)$) in $\mathscr{D}(\mathbf{m}, \mathbf{n})$. Thus [8, Properties 1.31 and 1.32] ensure that φ is the identity.

If $f : \mathscr{A} \longrightarrow \mathscr{B}$ is an $\mathscr{O}$-pseudomorphism in the sense of [8, Definition 1.35], then the invertible transformations ∂_n providing the pseudomorphism structure give rise to a pseudotransformation. As ∂_n is required to be the identity at the operadic identity, this ensures that the pseudotransformation satisfies the normality condition. Finally, an $\mathscr{O}$-transformation in the sense of [8, Definition 1.37] defines a modification.

Conversely, let $\mathscr{A}$ be a based $\mathscr{V}$-2-category such that $\mathscr{X} = \mathbb{R}\mathscr{A}$ has a given structure of a $\mathscr{D}$-pseudoalgebra. Then restriction of the action maps

$$\theta\colon \mathscr{D}(\mathbf{n}, \mathbf{1}) \times \mathscr{A}^n \longrightarrow \mathscr{A}$$

to the component of the map $\phi\colon \mathbf{n} \longrightarrow \mathbf{1}$ such that $\phi(j) = 1$ for $1 \leq j \leq n$ gives action maps $\theta(n)\colon \mathscr{O}(n) \times \mathscr{A}^n \longrightarrow \mathscr{A}$. Together with a similar restriction of the $\mathscr{V}$-transformation ϕ of $\mathscr{X}$, these give $\mathscr{A}$ a structure of $\mathscr{O}$-pseudoalgebra such that $\mathbb{R}\mathscr{A} = \mathscr{X}$ is a $\mathscr{D}$-pseudoalgebra. Here the assumptions in Definition 2.8 show that [8, Properties 1.30–1.33] hold.

Finally, to prove that $\mathbb{R}\mathbb{S}t$ is isomorphic to $\mathbb{S}t\mathbb{R}$, recall from [8, Section 3.2] that, for an $\mathbb{O}$-pseudoalgebra $\mathscr{A}$, the strictification $\mathbb{S}t\mathscr{A}$ is obtained by factoring the structure map $\theta\colon \mathbb{O}\mathscr{A} \longrightarrow \mathscr{A}$ as the composite of a map $e\colon \mathbb{O}\mathscr{A} \longrightarrow \mathbb{S}t\mathscr{A}$ in $\mathscr{B}\mathscr{O}$ and a map $m\colon \mathbb{S}t\mathscr{A} \longrightarrow \mathscr{A}$ in $\mathscr{F}\mathscr{F}$, and analogously for $\mathbb{D}$-pseudoalgebras. Just as for space level categories of operators [15, Lemma 5.7], we see that $\mathbb{D}\mathbb{R}$ is naturally isomorphic to $\mathbb{R}\mathbb{O}$ and the action of $\mathbb{D}$ on $\mathbb{R}\mathscr{A}$ is obtained by applying $\mathbb{R}$ to the action θ, and similarly for the coherence isomorphism φ. Since the factorization for categories of operators is constructed levelwise on the objects $\mathbf{n} \in \Pi$, we see that application of $\mathbb{R}$ to the factorization of θ gives the factorization of the structure map of $\mathbb{R}\mathscr{A}$. $\square$

4. The construction of the 2-functor ξ_*

We treat the formal construction of ξ_* in considerable generality. We start with any map $\xi\colon \mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads in any 2-category $\mathscr{K}$ and we define ν to be the composite

$$\mathbb{E}\mathbb{D} \xrightarrow{\mathbb{E}\xi} \mathbb{E}\mathbb{E} \xrightarrow{\mu} \mathbb{E}.$$

We have an evident pull back of action forgetful 2-functor

$$\xi^*\colon \mathbb{E}\text{-}\mathbf{AlgSt} \longrightarrow \mathbb{D}\text{-}\mathbf{AlgSt}.$$

Under mild hypotheses on $\mathscr{K}$, explained in [1, Theorem 3.9] and satisfied in our examples, the 2-functor ξ^* has a left adjoint ξ_*. On a (strict) $\mathbb{D}$-algebra Y, it can be constructed naively as the coequalizer in $\mathscr{K}$ displayed in the diagram

$$\mathbb{E}\mathbb{D}Y \underset{\mathbb{E}\theta}{\overset{\nu}{\rightrightarrows}} \mathbb{E}Y \xrightarrow{\pi} \xi_* Y. \tag{4.1}$$

We use the alternative notation $\mathbb{E} \otimes_{\mathbb{D}} Y$ for this categorical tensor product. Note that we have the map $\mathbb{E}\eta\colon \mathbb{E}Y \longrightarrow \mathbb{E}\mathbb{D}Y$ such that

$$\mathbb{E}\theta \circ \mathbb{E}\eta = \mathrm{id} = \nu \circ \mathbb{E}\eta.$$

This says that we have a reflexive coequalizer. A similar 1-categorical situation is studied in [2, pp. 46-47], which readily adapts to our 2-categorical setting. It is easily checked using a comparison of coequalizer diagrams that

$$\mathbb{E}(\mathbb{E} \otimes_{\mathbb{D}} Y) \cong (\mathbb{E}\mathbb{E}) \otimes_{\mathbb{D}} Y,$$

which says that $\mathbb{E}$ preserves the relevant reflexive coequalizers. Granting that, [2, Lemmas 6.6 and 6.7] give that if Y is a $\mathbb{D}$-algebra, then $\mathbb{E} \otimes_{\mathbb{D}} Y$ is an $\mathbb{E}$-algebra such that $\pi\colon \mathbb{E}Y \longrightarrow \xi_* Y$ is a map of $\mathbb{E}$-algebras. Using the universal property of coequalizers, we see that ξ_* is left adjoint to ξ^*. Indeed, if Z is an $\mathbb{E}$-algebra and $\alpha\colon Y \longrightarrow \xi^* Z$ is a map of $\mathbb{D}$-algebras, then the diagram

(4.2)
$$\begin{tikzcd}
\mathbb{E}\mathbb{D}Y \arrow[rr, "\mathbb{E}\theta"] \arrow[dr, "\mathbb{E}\mathbb{D}\alpha"] \arrow[dd, "\mathbb{E}\xi"'] & & \mathbb{E}Y \arrow[dd, "\mathbb{E}\alpha"] \\
& \mathbb{E}\mathbb{D}Z \arrow[d, "\mathbb{E}\xi"] & \\
\mathbb{E}\mathbb{E}Y \arrow[r, "\mathbb{E}\mathbb{E}\alpha"] \arrow[d, "\mu"] & \mathbb{E}\mathbb{E}Z \arrow[r, "\mathbb{E}\theta"] \arrow[d, "\mu"] & \mathbb{E}Z \arrow[d, "\theta"] \\
\mathbb{E}Y \arrow[r, "\mathbb{E}\alpha"] & \mathbb{E}Z \arrow[r, "\theta"] & Z
\end{tikzcd}$$

shows that the composite of $\mathbb{E}\alpha$ and the action map of Z coequalizes $\mathbb{E}\theta$ and ν. The resulting map $\widetilde{\alpha}\colon \xi_* Y \longrightarrow Z$ is the required adjoint map of $\mathbb{E}$-algebras.

A formal argument using the diagram

$$\begin{tikzcd}
Y \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}"'] \arrow[d, "\mathbb{D}\eta\circ\eta"'] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "\eta"] & Y \arrow[d, equal] \\
\mathbb{D}\mathbb{D}Y \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathbb{D}\theta"'] & \mathbb{D}Y \arrow[r, "\theta"] & Y.
\end{tikzcd}$$

shows that $Y \cong \mathrm{id}_* Y = \mathbb{D} \otimes_{\mathbb{D}} Y$. The unit of the adjunction (ξ_*, ξ^*) is just

$$\xi \otimes_{\mathbb{D}} \mathrm{id}\colon Y \cong \mathbb{D} \otimes_{\mathbb{D}} Y \longrightarrow \mathbb{E} \otimes_{\mathbb{D}} Y = \xi^* \xi_* Y.$$

Proof of Proposition 0.4. Let $\mathscr{X}$ be an $\mathbb{E}$-pseudoalgebra. Again using the definition of $\mathbb{S}t\mathscr{X}$ from [8, Section 3.2], we apply [8, Lemma 2.11] to obtain ι making the following diagram commute.

$$\begin{tikzcd}
\mathbb{D}\xi^*\mathscr{X} \arrow[r, "\xi"] \arrow[d, "e"'] & \xi^*\mathbb{E}\mathscr{X} \arrow[r, "\xi^* e"] & \xi^*\mathbb{S}t\mathscr{X} \arrow[d, "\xi^* m"] \\
\mathbb{S}t\xi^*\mathscr{X} \arrow[urr, "\iota"] \arrow[rr, "m"'] & & \xi^*\mathscr{X}.
\end{tikzcd}$$

On the right, we are applying ξ^*, which does not change underlying maps in $\mathscr{K}$, to a factorization (e, m) of the action $\theta\colon \mathbb{F}\xi^*\mathscr{X} \longrightarrow \mathscr{X}$. On the left, (e, m) is a factorization of the induced action $\xi^*\theta \circ \xi\colon \mathbb{D}\xi^*\mathscr{X} \longrightarrow \xi^*\mathscr{X}$. The map ξ at the top left is $\xi\colon \mathbb{D}\mathscr{X} \longrightarrow \mathbb{E}\mathscr{X}$; insertion of the notation ξ^* just records how we are thinking about its domain and target. An argument as in the proof of [8, Theorem 2.15] shows that c is a map of $\mathbb{D}$-algebras, and it is an equivalence since m and $\xi^* m$ are equivalences by Theorem 0.1. $\square$

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