FROM STRUCTURED G-CATEGORIES TO STRUCTURED G-SPECTRA

Contents

1. Introduction and the statement of the main theorem

We develop aspects of multiplicative equivariant infinite loop space theory and illustrate their use by giving the central proofs needed to complete the work of [2], which describes genuine G-spectra as presheaves of nonequivariant spectra.

The Barratt-Priddy-Quillen (BPQ) theorem describes suspension spectra as outputs of infinite loop space machines. It comes in several variant forms, and a categorical equivariant version is central to [2]. The most geometrically explicit form is given in terms of operads, and several variants of the operadic version are worked out equivariantly, but only additively, in [3]. There is another version, due to Segal [16], one interpretation of which is explained equivariantly in [13]. It is quite elegant, but quite different from the operadic variant and less well suited to our purposes; it does not start from the free algebras over operads that are used in [3].

In [2], we need a form of the equivariant BPQ theorem that is well-behaved categorically, homotopically, and multiplicatively. It must accept free permutative G-categories as input, it must give suspension G-spectra up to homotopy as output, and it must be precisely compatible with multiplicative structure. We give such a form of the BPQ theorem here. We collate ingredients from the topological prequels [5, 13] and the categorical prequels [8, 4, 7, 6], laying out multiplicative equivariant infinite loop space theory as we go. We emphasize that what is needed to complete the proofs in [2] is just a fragment of the overall theory, which, for example, also gives one starting point for equivariant algebraic K-theory.

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We start out in the category $\mathrm{Cat}(G\mathscr{U})$ of topological G-categories, by which we understand categories internal to the cartesian monoidal category $G\mathscr{U}$ of unbased G-spaces.[1] This category is enriched in G-categories. For topological G-categories X and Y, the internal hom G-category $\mathrm{Cat}_G(X,Y)$ has objects and morphisms the continuous functors $X \longrightarrow Y$ and their natural transformations, with G acting by conjugation. To simplify care of basepoints, we understand a based G-category to be the disjoint union of the trivial G-category $*$ and an unbased G-category. For the BPQ theorem, we need only consider finite G-sets with disjoint basepoints, viewed as based G-categories that are discrete in both the categorical sense (identity morphisms only) and the topological sense.

Let $\mathscr{P}_G$ be the operad of G-categories whose strict algebras are the genuine permutative G-categories [3, Definitions 4.4 and 4.5]. We recall it briefly in Section 2.3. Its pseudoalgebras are the genuine symmetric monoidal G-categories [8, Definition 0.1 and Section 1.3]. These are not just the familiar nonequivariant structures with compatible group actions. Those are the naive variants of these notions. As we also recall in Section 2.3, for any finite group G every naive symmetric monoidal G-category has an associated genuine symmetric monoidal G-category and an associated genuine G-spectrum. Our concern here is with additional multiplicative structure on such categorical input.

For the application to the BPQ-theorem, we start with the functor

$$i\,\mathbb{P}_{G,+}\colon G\mathscr{U} \longrightarrow \mathscr{P}_G\text{-}\mathbf{AlgSt}.$$

The target is the category of permutative topological G-categories; the notation **AlgSt** indicates restriction to strict rather than pseudo $\mathscr{P}_G$ -algebras in $\mathbf{Cat}(G\mathscr{U})$. Given an unbased G-space X, we first adjoin a disjoint basepoint, next apply the free permutative G-category functor $\mathbb{P}_G$ on based G-spaces, as defined in [3, Section 5.1] (or [7, Section 7.1]), and finally apply the functor $i\colon G\mathscr{U} \longrightarrow \mathbf{Cat}(G\mathscr{U})$ of [8, Remark 1.4] that sends a G-space X to X regarded as a constant internal category in $G\mathscr{U}$ (object and morphism G-space both X). Since

$$(X \times Y)_+ \cong X_+ \wedge Y_+,$$

the functor $(-)_+$ is symmetric monoidal, converting cartesian products to smash products. The only smash products of categories that we shall deal with are those of this simple form, so the fact that the classifying space functor B does not commute with colimits, which are needed to define smash products in general, need not concern us here. As observed in [3, Section 5.1], $\mathbb{P}_G(\mathscr{A}_+)$ can be identified with $\mathbb{P}_{G,+}(\mathscr{A})$, where $\mathbb{P}_{G,+}$ is the free $\mathscr{P}_G$-algebra functor defined on $G\mathscr{U}$ rather than on $G\mathscr{T}$; disjoint basepoints are then built into the output of the functor. Explicitly,

$$\mathbb{P}_{G,+}\mathscr{A} = \coprod_{j\geq 0} \mathscr{P}_G(j) \times_{\Sigma_j} \mathscr{A}^j.$$

One goal of this paper is to analyze multiplicative properties of the passage from these free $\mathscr{P}_G$-algebras to suspension G-spectra. This analysis requires use of pseudofunctors between 2-categories and most of the work must be carried out categorically, before passage to topology. The relevant category theory is quite general, a priori having nothing to do with equivariance or even topology, and is

[1] As usual, spaces are understood to be compactly generated and weak Hausdorff. We write $\mathscr{U}_*$ and $\mathscr{T}$ for the categories of based and of nondegenerately based spaces, and similarly equivariantly.

worked out in [8, 4, 7, 6]. We compose $i\,\mathbb{P}_{G+}$ with either a lax monoidal composite algebraic K-theory functor

$$\mathbb{K}_G\colon \mathscr{P}_G\text{-}\mathbf{AlgSt}[\mathbf{Cat}(G\mathscr{U})] \longrightarrow G\mathscr{S} \tag{1.1}$$

or, for better control of homotopy type, a more general such composite

$$\mathbb{K}_G\colon \mathscr{P}_G\text{-}\mathbf{AlgSt}[\mathbf{Cat}^2(G\mathscr{U})] \longrightarrow G\mathscr{S}, \tag{1.2}$$

where $\mathbf{Cat}^2(G\mathscr{U})$ is the 2-category of double categories internal to $G\mathscr{U}$, namely internal categories in $\mathbf{Cat}(G\mathscr{U})$. Generalizing from strict $\mathscr{P}_G$-algebras to $\mathscr{P}_G$-pseudoalgebras, we shall outline the construction of the composite $\mathbb{K}_G$ in (1.1) in Section 2 and of the composite (1.2) in Section 3.

It is well-known that the suspension G-spectrum functor

$$\Sigma^\infty_G\colon G\mathscr{T} \longrightarrow G\mathscr{S}$$

is symmetric monoidal. A proof is given in [5, Lemma 6.1]. First adjoining disjoint basepoints and then applying Σ^∞_G gives a symmetric monoidal functor

$$\Sigma^\infty_{G,+}\colon G\mathscr{U} \longrightarrow G\mathscr{S}.$$

We shall prove the following theorem in Section 4. It is the heart of what is needed in [2].

Theorem 1.3. *There is a lax monoidal natural transformation*

$$\alpha : \Sigma^\infty_{G,+} \longrightarrow \mathbb{K}_G i\,\mathbb{P}_{G,+}$$

of functors $G\mathscr{U} \longrightarrow G\mathscr{S}$ such that α is a weak homotopy equivalence of G-spectra for all input G-spaces X.

2. The construction of $\mathbb{K}_G$

Before turning to the proof of Theorem 1.3, we describe the construction of $\mathbb{K}_G$. In general, to deal categorically with the preservation of multiplicative structure, the obvious starting point is lax symmetric monoidal functors between symmetric monoidal categories, and that is all that is really relevant in this paper. However, it is best to work more generally with multicategories, which can be viewed as many object (or colored) operads. Just as operads come with symmetric group actions, we understand multicategories to be symmetric multicategories.[2] Any symmetric monoidal category $\mathscr{C}$ has an associated multicategory $\mathbf{Mult}(\mathscr{C})$ whose k-morphisms $(a_1, \cdots, a_k) \longrightarrow b$ are just morphisms $a_1 \otimes \cdots \otimes a_k \longrightarrow b$ in $\mathscr{C}$. Any lax symmetric monoidal functor $\mathscr{C} \longrightarrow \mathscr{C}'$ induces a symmetric multifunctor $\mathbf{Mult}(\mathscr{C}) \longrightarrow \mathbf{Mult}(\mathscr{C}')$.

There are two main approaches to infinite loop space theory, one operadic and the other Segalic. Additively, they are redeveloped equivariantly and proven to be equivalent in [13]. Multiplicatively, there are again nonequivariant operadic and Segalic approaches [1, 10, 11], with different advantages and disadvantages [12], but they have not been proven to be equivalent. The only equivariant version of multiplicative infinite loop space currently available is given in [5], which gives a Segalic approach that is new even nonequivariantly. We summarize it first before turning to our categorical data.

In (1.1), $\mathbb{K}_G$ is a composite

[2] See [7, Section 1.1] for relevant preliminaries on multicategories.

$$\mathbb{K}_G = \mathbb{S}_G \circ B \circ \xi_{\#} \circ \mathbb{R}_G.$$

We describe these functors, out of order, in the following subsections.

2.1. $\mathbb{S}_G$**: From $\mathscr{F}_G$-G-spaces to G-spectra.** Let $\mathscr{F}$ be the category of finite based sets $\mathbf{n} = \{\mathbf{0}, \mathbf{1}, \cdots, \mathbf{n}\}$, with basepoint 0. Let $\mathscr{F}_G$ be the category of finite based G-sets $(\mathbf{n}, \alpha)$, where $\alpha\colon G \longrightarrow \Sigma_n$ is a homomorphism fixing an action of G on $\mathbf{n}$; maps are just maps $\mathbf{m} \longrightarrow \mathbf{n}$, and G acts by conjugation on hom sets. An $\mathscr{F}$-G-space is a $\mathscr{U}_*$-functor $\mathscr{F} \longrightarrow G\mathscr{T}$ and an $\mathscr{F}_G$-G-space is a $G\mathscr{U}_*$-functor $\mathscr{F}_G \longrightarrow \mathscr{T}_G$.[3] Let $\mathrm{Fun}(\mathscr{F}, G\mathscr{T})$ denote the category of $\mathscr{F}$-G-spaces and $\mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$ denote the category of $\mathscr{F}_G$-G-spaces, where the morphisms of the former are the $\mathscr{U}_*$-natural transformations and those of the latter are the $G\mathscr{U}_*$-natural transformations. Since $\mathscr{F}$ and $\mathscr{F}_G$ have zero objects, functors X defined on them are automatically based, and the based enrichment ensures that they are reduced, meaning that $X(\mathbf{0}) = *$ [13, Lemma 1.13]. Use of the notations $\mathscr{T}$ and $\mathscr{T}_G$ indicates that we require each $X(\mathbf{n})$ or $X(\mathbf{n}, \alpha)$ to be a nondegenerately based G-space.

See [13, Sections 2.1 and 2.4] or [5, Section 1] for details of these definitions and for the notions of (genuinely) special $\mathscr{F}$-G-spaces and $\mathscr{F}_G$-G-spaces. By [13, Theorem 2.30], the evident forgetful and prolongation functors $\mathbb{U}_{\mathscr{F}}^{\mathscr{F}_G}$ and $\mathbb{P}_{\mathscr{F}}^{\mathscr{F}_G}$ specify an adjoint equivalence between $\mathrm{Fun}(\mathscr{F}, G\mathscr{T})$ and $\mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$, and this equivalence preserves the relevant notions of special and the relevant notions of (level) equivalence. In the case of $\mathscr{F}_G$, these homotopical properties are defined only in terms of weak equivalence of G-spaces, whereas families of subgroups of products $G \times \Sigma_n$ are needed in the case of $\mathscr{F}$. For this and other reasons, we find it more convenient to work with $\mathscr{F}_G$ rather than $\mathscr{F}$ in this paper. However, [5] shows that we can use the two interchangeably: anything proven for $\mathscr{F}_G$-G-spaces applies equally well to $\mathscr{F}$-G-spaces, by transport along the adjoint equivalence $(\mathbb{P}_{\mathscr{F}}^{\mathscr{F}_G}, \mathbb{U}_{\mathscr{F}}^{\mathscr{F}_G})$.

Since $\mathscr{F}$ and $\mathscr{F}_G$ are symmetric monoidal under the smash product, Day convolution gives the categories $\mathrm{Fun}(\mathscr{F}, G\mathscr{T})$ and $\mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$ symmetric monoidal structures. The left adjoint $\mathbb{P}_{\mathscr{F}}^{\mathscr{F}_G}$ is strong symmetric monoidal, hence the right adjoint $\mathbb{U}_{\mathscr{F}}^{\mathscr{F}_G}$ is lax symmetric monoidal. The notion of a Segal infinite loop space machine $\mathbb{S}_G$ is defined in [5, Definition 1.4]. There are equivalent versions starting from $\mathscr{F}$-G-spaces or from $\mathscr{F}_G$-G-spaces. In the latter version, a Segal machine is a functor $\mathbb{S}_G\colon \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \longrightarrow \mathscr{S}_G$ with the appropriate group completion property on special $\mathscr{F}_G$-G-spaces. We have the following result [5, Theorem 5.5].

Theorem 2.1. *There is a lax symetric monoidal Segal machine*

$$\mathbb{S}_G\colon \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \longrightarrow \mathscr{S}_G.$$

2.2. B**: From $\mathscr{F}_G$-G-categories to $\mathscr{F}_G$-G-spaces.** The problem with Theorem 2.1 is that its space level input is inadequate to the applications, and we need more general categorical input. It is convenient to start with the evident categorical precursors of $\mathscr{F}_G$-G-spaces, namely $\mathscr{F}_G$-G-categories. These are GCat-functors $X\colon \mathscr{F}_G \longrightarrow \mathrm{Cat}_G(\mathscr{U})$. Thus X is given by G-categories $X(\mathbf{n}, \beta)$ and G-functors

$$X\colon \mathscr{F}_G((\mathbf{m}, \alpha), (\mathbf{n}, \beta)) \longrightarrow \mathrm{Cat}_{\mathbf{G}}(\mathbf{X}(\mathbf{m}, \alpha), \mathbf{X}(\mathbf{n}, \beta)).$$

[3] $\mathscr{V}$-functors must be $\mathscr{V}$-maps on object and morphism objects in $\mathscr{V}$.

The notation for the target indicates that are letting G-act by conjugation on the category $\mathrm{Fun}(X(\mathbf{m},\alpha),\mathbf{X}(\mathbf{n},\beta))$ of nonequivariant continuous functors and natural transformations.

We work with categories of unbased G-spaces, but all of our functors will be basepoint preserving. This ensures that our composite categorical functors starting with based categories land in based spaces. That allows us to ignore basepoints and work throughout with categories of unbased spaces with theiir cartesian products. We write $\mathrm{Fun}(\mathscr{F}_G,\mathrm{Cat}_G(\mathscr{U}))$ for the category of $\mathscr{F}_G$-G-categories; its morphisms are the $\mathrm{Cat}(G\mathscr{U})$-natural transformations. Again using Day convolution but now using products rather than smash products of (unbased) G-categories, we see that $\mathrm{Fun}(\mathscr{F}_G,\mathrm{Cat}_G(\mathscr{U}))$ is symmetric monoidal.

We write $B\colon \mathrm{Cat}(G\mathscr{U})\longrightarrow G\mathscr{U}$ for the standard classifying G-space functor, the composite of the nerve and geometric realization functors. Applying B levelwise to $\mathscr{F}_G$-G-categories and using that B is product-preserving, we obtain the following result.[4]

Theorem 2.2. *Applied levelwise, the functor B induces a symmetric monoidal functor*

$$B\colon \mathrm{Fun}(\mathscr{F}_G,\mathrm{Cat}_G(\mathscr{U}))\longrightarrow \mathrm{Fun}(\mathscr{F}_G,\mathscr{U}_G).$$

Remark 2.3. We are only interested in categorical input on which B takes values in $\mathrm{Fun}(\mathscr{F}_G,\mathscr{T}_G)$.

2.3. $\mathbb{R}_G$**: From $\mathscr{P}_G$-pseudoalgebras to $\mathscr{D}_G$-pseudoalgebras.** Our problem now is to start with 2-categorical operadic input, which multiplicatively must involve pseudofunctors and not just 2-functors, and transform it to suitably equivalent $\mathscr{F}_G$-G-category output to which the composite symmetric monoidal functor $\mathbb{S}_G\circ B$ or $\mathbb{S}_G\circ B^2$ can be applied. The input we are most interested in is symmetric monoidal G-categories. By definition [8, Definition 0.1], these are the $\mathscr{P}_G$-pseudoalgbras.

Briefly, with details in [8, Section 1.3], to define the permutativity operad $\mathscr{P}_G$, we start with the associativity operad **Assoc** of sets, which has kth set the symmetric group Σ_k. The kth category of the permutativity operad $\mathscr{P}$ is the chaotic (or indiscrete) category with object set Σ_k. For a G-category $\mathscr{A}$, possibly G-trivial, let $\mathrm{Cat}(\mathcal{E}G,\mathscr{A})$ denote the functor category $\mathrm{Fun}(\mathcal{E}G,\mathscr{A})$ with (left) G-action given by conjugation, where $\mathcal{E}G$ is the chaotic G-category with object set G. The kth G-category of $\mathscr{P}_G$ is $\mathrm{Cat}(\mathscr{E}G,\mathscr{P}(j))$. Pseudoalgebras over operads of G-categories are defined in [8, Definition 1.23]. For a naive permutative (resp. symmetric monoidal) G-category $\mathscr{A}$, $\mathbf{Cat}(\mathcal{E}G,\mathscr{A})$ is a genuine permutative (resp. symmetric monoidal) G-category, so examples abound.

For the purely additive theory, we can apply Power-Lack [9, 15, 8] (or Street [17]) strictification to pass directly from symmetric monoidal to permutative G-categories, and we can then apply the classifying space functor to pass from categorical data to topological input to a choice of several equivalent infinite loop G-space machines. See [8, 13] for details. However, for the multiplicative theory of interest here, we must interpolate pseudoalgebras over equivariant categories of operators between operadic data and $\mathscr{F}$-G-categories. We recall that, on the nonequivariant space level, categories of operators were introduced in [14] to give a common home for operadic and Segalic input and thus to allow comparisons.

[4]For the enrichment, we implicitly apply B to hom G-categories $\mathrm{Cat}_G(-.-)$.

On the space level, we defined G-categories of operators in [13, Section 4], following [14]. They are defined in the same way on the category level [4, Definition 1.11], just replacing $G\mathscr{U}$ by $\mathbf{Cat}(G\mathscr{U})$. As explained in full multiplicative detail in [7, Section 6], we have two variants. One starts from $\mathscr{F}$ and the other starts from $\mathscr{F}_G$. We let $\mathscr{D}$ be the G-category of operators over $\mathscr{F}$ associated to the operad $\mathscr{P}_G$, and we let $\mathscr{D}_G$ be the G-category of operators over $\mathscr{F}_G$ associated to $\mathscr{P}_G$. It is obtained as a prolongation of $\mathscr{D}$. Briefly, just to give the idea,

$$\mathscr{D}(\mathbf{m},\mathbf{n}) = \coprod_{\phi\in\mathscr{F}(\mathbf{m},\mathbf{n})}\prod_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}} \mathscr{P}_{\mathbf{G}}(|\phi^{-\mathbf{1}}(\mathbf{j})|);$$

$\mathscr{D}_G$ is defined similarly. Ignoring the action of G, $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) = \mathscr{D}(\mathbf{m},\mathbf{n})$, but with G acting by conjugation [7, Section 6.1]. Even if we start with strict $\mathscr{P}_G$-algebras, alias permutative G-categories, the multiplicative structure intrinsically involves pseudofunctors and not just strict 2-functors between them.

As described in detail in [8, 4], we have the 2-categories

$$\mathscr{P}_G\text{-}\mathbf{PsAlg},\quad \mathscr{D}\text{-}\mathbf{PsAlg},\quad \text{and}\quad \mathscr{D}_{\mathbf{G}}\text{-}\mathbf{PsAlg}$$

of pseudoalgebras and pseudomorphisms of $\mathscr{P}_G$-categories, $\mathscr{D}$-categories, and $\mathscr{D}_G$-categories in $\mathbf{Cat}(\mathbf{G}\mathscr{U})$. These have sub 2-categories

$$\mathscr{P}_G\text{-}\mathbf{AlgSt},\quad \mathscr{D}\text{-}\mathbf{AlgSt},\quad \text{and}\quad \mathscr{D}_{\mathbf{G}}\text{-}\mathbf{AlgSt}$$

of strict algebras and strict morphisms. As proven in [7, Theorems 0.2, 0.3, 0.5], we have multicategories

$$\mathbf{Mult}(\mathscr{P}_{\mathbf{G}}),\quad \mathbf{Mult}(\mathscr{D}),\quad \text{and}\quad \mathbf{Mult}(\mathscr{D}_{\mathbf{G}})$$

with underlying 2-categories $\mathscr{P}_G$-**PsAlg**, $\mathscr{D}$-**PsAlg**, and $\mathscr{D}_G$-**PsAlg** and these have submulticategories

$$\mathbf{Mult_{st}}(\mathscr{P}_{\mathbf{G}}),\quad \mathbf{Mult_{st}}(\mathscr{D}),\quad \text{and}\quad \mathbf{Mult_{st}}(\mathscr{D}_{\mathbf{G}})$$

with underlying 2-categories $\mathscr{P}_G$-**AlgSt**, $\mathscr{D}$-**AlgSt**, and $\mathscr{D}_G$-**AlgSt**. Unlike $\mathscr{F}$ and $\mathscr{F}_G$, the smash product on $\mathscr{D}$ and $\mathscr{D}_G$ is only a pseudofunctor, hence the multimorphisms $(\mathscr{A}_1,\cdots,\mathscr{A}_j)\longrightarrow\mathscr{B}$ for $j\ge 2$ require pseudofunctors in all six of these multicategories. the cited multicategories are given by specialization of [7, Theorems 0.2, 0.3, 0.5]. The constructions there work with arbitrary "pseudocommutative operads" $\mathbb{O}$ and "pseudocommutative categories of operators" $\mathbb{D}$ and $\mathbb{D}_G$ in general ground categories $G\mathscr{V}$ rather than just in $G\mathscr{U}$.

Just as on the space level in [13, Section 6], we have a commutative diagram of 2-functors

$$\begin{tikzcd} \mathscr{P}_G\text{-}\mathbf{PsAlg} \arrow[r, "\mathbb{R}"] \arrow[dr, "\mathbb{R}_G"'] & \mathscr{D}\text{-}\mathbf{PsAlg} \arrow[d, "\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}"] \\ & \mathscr{D}_G\text{-}\mathbf{PsAlg} \end{tikzcd} \tag{2.4}$$

that restricts to a commutative diagram

$$\begin{tikzcd} \mathscr{P}_G\text{-}\mathbf{AlgSt} \arrow[r, "\mathbb{R}"] \arrow[dr, "\mathbb{R}_G"'] & \mathscr{D}\text{-}\mathbf{AlgSt} \arrow[d, "\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}"] \\ & \mathscr{D}_G\text{-}\mathbf{AlgSt}. \end{tikzcd} \tag{2.5}$$

Moreover $\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}$ is an equivalence of 2-categories. Here we need not define $\mathbb{R}_G$ independently since we may as well just define it to be the composite.

On the space level, [13, Theorem 4.11] shows that the equivalence $\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}$ preserves the appropriate respective notions of specialness and equivalence. For $\mathscr{D}_G$-G-categories these notions are defined in terms of level G-equivalences, where a map f of $\mathscr{D}_G$-G-categories is a level G-equivalence if Bf is a level G-equivalence of G-spaces. Equivalences are a good deal easier to recognize using $\mathscr{D}_G$ than using $\mathscr{D}$, and we therefore work with $\mathscr{D}_G$ rather than $\mathscr{D}$ in what follows.

From here, [7, Theorems 0.4 and 0.6] give the following comparisons. They show how to go from $\mathscr{P}_G$-pseudoalgebras to $\mathscr{D}_G$-pseudoalgebras while retaining all multiplicative structure.

Theorem 2.6. *There is a commutative diagram of multifunctors*

$$\begin{tikzcd} \mathbf{Mult}(\mathscr{P}_{\mathbf{G}}) \arrow[r, "\mathbb{R}"] \arrow[dr, "\mathbb{R}_G"'] & \mathbf{Mult}(\mathscr{D}) \arrow[d, "\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}"] \\ & \mathbf{Mult}(\mathscr{D}_{\mathbf{G}}) \end{tikzcd} \tag{2.7}$$

that restricts to a commutative diagram

$$\begin{tikzcd} \mathbf{Mult_{st}}(\mathscr{P}_{\mathbf{G}}) \arrow[r, "\mathbb{R}"] \arrow[dr, "\mathbb{R}_G"'] & \mathbf{Mult_{st}}(\mathscr{D}) \arrow[d, "\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}"] \\ & \mathbf{Mult_{st}}(\mathscr{D}_{\mathbf{G}}) \end{tikzcd} \tag{2.8}$$

Moreover $\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}$ is an equivalence of multicategories in both diagrams.

Remember that we understand multicategories and multifunctors to be symmetric. Symmetry will be sacrificed in a later step, but that is irrelevant to Theorem 1.3.

2.4. **$\xi_\#$: From $\mathscr{D}_G$-pseudoalgebras to $\mathscr{F}_G$-G-categories.** A purely categorical route from $\mathscr{D}_G$-pseudoalgebras to $\mathscr{F}_G$-G-categories is developed in [6]. Briefly, one starts with the composite

$$\mathscr{D}_G\text{-}\mathbf{PsAlg} \xrightarrow{\mathbb{S}t} \mathscr{D}_G\text{-}\mathbf{AlgSt} \xrightarrow{\xi_*} \mathscr{F}_G\text{-}\mathbf{AlgSt} \tag{2.9}$$

of Power-Lack strictification $\mathbb{S}t$ and the left adjoint ξ_* to the pull back of action 2-functor $\xi^*\colon \mathscr{F}_G\text{-}\mathbf{AlgSt} \longrightarrow \mathscr{D}_{\mathbf{G}}\text{-}\mathbf{AlgSt}$ that is induced by the canonical map $\xi\colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$ of categories of operators. Define $\xi_\# = \xi_* \circ \mathbb{S}t$. A non-trivial categorical analysis [6] that starts from an alternative description of $\xi_\#$. A non-trivial categorical analysis [6] that starts from an alternative description of $\xi_\#$ in terms of the 2-categorical codescent objects defined by Lack [9] shows that the composite lifts to give a multifunctor

$$\mathbf{Mult}(\mathscr{D}_{\mathbf{G}}) \xrightarrow{\xi_\#} \mathbf{Mult}(\mathrm{Fun}(\mathscr{F}_{\mathbf{G}}, \mathrm{Cat}_{\mathbf{G}}(\mathscr{U})). \tag{2.10}$$

The key nontrivial point is that the codescent description allows the proof that the composite lifts to a construction that strictifies the intrinsically 2-categorical structure used to define the source multicategory, landing in the target multicategory associated to the symmetric monoidal category $\mathrm{Fun}(\mathscr{F}_G, \mathrm{Cat}_G)$. This means

that any multiplicative structure defined in terms of pseudofunctors in the source is transported to the corresponding structure defined in terms of strict 2-functors in the target. On a formal level, this route is ideal.

For $\mathscr{D}_G$-pseudoalgebras $\mathscr{X}$, the 2-functor $\mathbb{S}t$ comes with a natural level G-equivalence $m\colon \mathbb{S}t\mathscr{X} \longrightarrow \mathscr{X}$ of $\mathscr{D}_G$-pseudoalgebras. Ignoring structure, the classifying space functor sends this map to a level G-equivalence of underlying Π_G-G-spaces. Thus $\mathbb{S}t$ preserves homotopy type. For strict $\mathscr{D}_G$-algebras $\mathscr{X}$, the 2-functor ξ_* comes with a natural map $\varepsilon\colon \xi_*\xi^*\mathscr{X} \longrightarrow \mathscr{X}$ of strict $\mathscr{D}_G$-algebras. However, ε may or may not be a level G-equivalence, and we do not have a satisfactory criterion for describing those $\mathscr{X}$ for which it is so.

We next give a derived elaboration of this route which always gives the correct homotopy type, but at the price of losing symmetry.

3. The variant derived construction of $\mathbb{K}_G$

We obtain the $\mathbb{K}_G$ of (1.2) by interpolating a general categorical approximation functor $\mathbb{G}r$ which replaces $\mathscr{D}_G$-pseudofunctors $\mathscr{Y}$ by derived approximations $\mathbb{G}r\mathscr{Y}$ given by familiar double categories internal to $G\mathscr{U}$. Essentially, this route just replicates the topological generalized Segal machine of [13] on the categorical level. The $\mathbb{K}_G$ of (1.2) is the composite

$$\mathbb{K}_G = \mathbb{S}_G \circ B^2 \circ \xi_\# \circ \mathbb{G}r \circ \mathbb{R}_G.$$

The construction of $\mathbb{G}r$ has nothing to do with equivariance or topology and is carried out in full categorical generality in [6, Section 5]. It gives a categorical precursor of the two-sided bar construction that played a central role in the additive space level theory of [13, 5]. Its nerve is a simplicial internal category in $G\mathscr{U}$, and the geometric realization of that is a two-sided bar construction that is equivalent to a bar construction $B^\times(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y})$ exploited in [13]. The categorical theory in [6] works out the multiplicative behavior of $\mathbb{G}r$, and we show how to pass from there to topology in Section 3.2. We use this to prove Theorem 1.3 in Section 4.

3.1. Grothendieck categories.

Our starting point is the "Grothendieck category of elements", which we recall from [13, Section 3.1]. Let $\mathscr{W}$ be a bicomplete cartesian monoidal category, such as the category $G\mathscr{U}$ of G-spaces. In this section we are mainly thinking of $\mathscr{W} = \mathrm{Cat}(G\mathscr{U})$, the category of categories internal to $G\mathscr{U}$, meaning categories with G-spaces of objects and G-spaces of morphisms with source, target, identity and composition G-maps S, T, I and C. For a category $\mathscr{C}$ enriched in $\mathscr{W}$ and contravariant and covariant $\mathscr{W}$-functors $\mathscr{X}$ and $\mathscr{Y}$, we have a Grothendieck category internal to $\mathscr{W}$ with object and morphism objects

$$\coprod_n \mathscr{X}(n) \times \mathscr{Y}(n)$$

and

$$\coprod_{(m,n)} \mathscr{X}(n) \times \mathscr{C}(m,n) \times \mathscr{Y}(n)$$

in $\mathscr{W}$, where m and n range over the objects of $\mathscr{C}$. On components, the evaluation maps of $\mathscr{X}$ and $\mathscr{Y}$ give the source and target maps

$$S\colon \mathscr{X}(n) \times \mathscr{C}(m,n) \times \mathscr{Y}(m) \longrightarrow \mathscr{X}(m) \times \mathscr{Y}(m)$$

and

$$T\colon \mathscr{X}(n) \times \mathscr{C}(m,n) \times \mathscr{Y}(m) \longrightarrow \mathscr{X}(n) \times \mathscr{Y}(n)$$

in $\mathscr{W}$. On components. the identity maps $* \longrightarrow \mathscr{C}(n,n)$ give

$$I\colon \mathscr{X}(n)\times\mathscr{Y}(n)\longrightarrow \mathscr{X}(n)\times\mathscr{C}(n,n)\times\mathscr{Y}(n).$$

The evident pullbacks

$$\begin{tikzcd}
\mathscr{X}(n)\times\mathscr{C}(m,n)\times\mathscr{C}(\ell,m)\times\mathscr{Y}(\ell) \arrow[r, "ev_{\mathscr{X}}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times ev_{\mathscr{Y}}"'] & \mathscr{X}(m)\times\mathscr{C}(\ell,m)\times\mathscr{Y}(\ell) \arrow[d, "T"] \\
\mathscr{X}(n)\times\mathscr{C}(m,n)\times\mathscr{Y}(m) \arrow[r, "S"'] & \mathscr{X}(m)\times\mathscr{Y}(m)
\end{tikzcd}$$

define the components of the domain of composition, and composition in $\mathscr{C}$ induces

$$C\colon \mathscr{X}(n)\times\mathscr{C}(m,n)\times\mathscr{C}(\ell,m)\times\mathscr{Y}(\ell)\longrightarrow \mathscr{X}(n)\times\mathscr{C}(\ell,n)\times\mathscr{Y}(\ell).$$

When $\mathscr{W}=\mathrm{Cat}(G\mathscr{U})$, these Grothendieck categories live in $\mathrm{Cat}(\mathscr{W})=\mathrm{Cat}^2(G\mathscr{U})$, the 2-category of double categories internal to $G\mathscr{U}$.

Now take $\mathscr{C}=\mathscr{D}_G$. We take $\mathscr{Y}$ to be a $\mathscr{D}_G$-algebra for simplicity here, as we may by [4, theorem 0.1]. but the construction is generalized to allow $\mathscr{Y}$ to be a $\mathscr{D}_G$-pseudoalgebra in [6, Section 5]; see Remark 3.23 below. We take $\mathscr{X}$ to be either the represented contravariant functor $\mathscr{D}_G(-,(\mathbf{n},\alpha))$ or the contravariant functor on $\mathscr{D}_G$ obtained by composing the represented functor $\mathscr{F}_G(-,(\mathbf{n},\alpha))$ with $\xi\colon \mathscr{D}_G\longrightarrow\mathscr{F}_G$. Letting $(\mathbf{n},\alpha)$ vary, composition in $\mathscr{D}_G$ or in $\mathscr{F}_G$ gives functors

$$\mathbb{G}r(\mathscr{D}_G,\mathscr{D}_G,\mathscr{Y}) \quad\text{and}\quad \mathbb{G}r(\mathscr{F}_G,\mathscr{D}_G,\mathscr{Y}).$$

The first is a $\mathscr{D}_G$-algebra and the second is an $\mathscr{F}_G$-algebra, both in the 2-category $\mathrm{Cat}^2(G\mathscr{U})$. That is, for the first, we have object and morphism $\mathscr{D}_G$-algebras in $\mathrm{Cat}(G\mathscr{U})$ such that S, T, I, and C are maps of $\mathscr{D}_G$-algebras, and similarly for the second. We abbreviate notation by setting

$$\mathbb{G}r\mathscr{Y}=\mathbb{G}r(\mathscr{D}_G,\mathscr{D}_G,\mathscr{Y}). \tag{3.1}$$

Composition and the action of $\mathscr{D}_G$ give a map

$$\varepsilon\colon \mathbb{G}r\mathscr{Y}\longrightarrow\mathscr{Y} \tag{3.2}$$

of $\mathscr{D}_G$-algebras in $\mathrm{Cat}^2(G\mathscr{U})$, where $\mathscr{Y}$ is viewed as the constant double category internal to $G\mathscr{U}$; its object and morphism objects are $\mathscr{Y}$, with S, T, I, and C the identity. Using identity morphisms in $\mathscr{D}_G$ in an obvious way, we obtain a map

$$\eta\colon \mathscr{Y}\longrightarrow\mathbb{G}r\mathscr{Y} \tag{3.3}$$

of Π_G-algebras in $\mathbf{Cat^2}(\mathbf{G}\mathscr{U})$ such that $\varepsilon\circ\eta=\mathrm{id}$; η is not a map of $\mathscr{D}_G$-algebras.

By [6, Remark 5.18], we have a natural isomorphism of $\mathscr{F}_G$-algebras in $\mathrm{Cat}^2(G\mathscr{U})$

$$\xi_*\mathbb{G}r\mathscr{Y}\cong\mathbb{G}r(\mathscr{F}_G,\mathscr{D}_G,\mathscr{Y}), \tag{3.4}$$

where, as in Section 2.4, ξ_{G*} is left adjoint to the pull back of action functor ξ^* from $\mathscr{D}_G$-algebras to $\mathscr{F}_G$-algebras. Then the map of $\mathscr{D}_G$-algebras

$$\mathbb{G}r(\xi_G,\mathrm{id},\mathrm{id})\colon \mathbb{G}r(\mathscr{D}_G,\mathscr{D}_G,\mathscr{Y})\longrightarrow\xi^*\mathbb{G}r(\mathscr{F}_G,\mathscr{D}_G,\mathscr{Y}) \tag{3.5}$$

can be identified with the unit

$$\nu\colon \mathbb{G}r\mathscr{Y}\longrightarrow\xi^*\xi_{G*}\mathbb{G}r\mathscr{Y} \tag{3.6}$$

of the adjunction.

Now return to Section 2.4. There (2.9) and (2.10) implicitly work in the ground 2-category $\mathrm{Cat}(G\mathscr{U})$. However, the general theory of [7, 6] works equally well if

we replace $\mathrm{Cat}(G\mathscr{U})$ by $\mathrm{Cat}^2(G\mathscr{U})$. Let $\mathrm{Cat}^2_G(\mathscr{U})$ denoting the equivariant avatar in which G acts by conjugation on the underlying double categories internal to $\mathscr{U}$ of given double categories internal to $G\mathscr{U}$. Then another non-trivial categorical analysis, which is an elaboration of that constructing the multifunctor (2.10) and is given in [6, section 5.2], proves that the composite $\xi_{G\#}\circ\mathbb{G}r$ extends to a multifunctor

$$\mathbf{Mult}(\mathscr{D}_{\mathbf{G}})[\mathrm{Cat}(\mathbf{G}\mathscr{U})] \longrightarrow \mathbf{Mult}\big(\mathrm{Fun}(\mathscr{F}_{\mathbf{G}}, \mathrm{Cat}^{\mathbf{2}}_{\mathbf{G}}(\mathscr{U}))\big).$$

Warning 3.7. While this composite is a multifunctor, it is not a *symmetric* multifunctor. The reason is explained in [6, Warning 5.13].

3.2. **The double bar construction.** We have a generalization of the bar construction in which we replace categories internal to $G\mathscr{U}$ by double categories internal to $G\mathscr{U}$. It gives a double classifying G-space functor $B^2\colon \mathrm{Cat}^2(G\mathscr{U}) \longrightarrow G\mathscr{U}$. We define it here and use it to prove that the maps ε and ν of (3.2) and (3.6) induce levelwise G-homotopy equivalences on passage to topology. It can be described in several equivalent ways. We choose to view it as the composite

$$\mathrm{Cat}^2(G\mathscr{U}) \xrightarrow{B} \mathrm{Cat}(G\mathscr{U}) \xrightarrow{B} G\mathscr{U}. \tag{3.8}$$

Here the first B is obtained by applying the second B, namely the standard classifying G-space functor B, to the object and morphism $G\mathscr{U}$-categories and to the structure maps S, T, I, and C of a double $G\mathscr{U}$-category. Since B^2 is defined in terms of B, Theorem 2.2 implies the following analog.

Theorem 3.9. *Applied levelwise, the functor B^2 induces a symmetric monoidal functor*

$$B^2\colon \mathrm{Fun}(\mathscr{F}_G, \mathrm{Cat}^2_G(\mathscr{U}_G)) \longrightarrow \mathrm{Fun}(\mathscr{F}_G, \mathscr{U}_G).$$

The following remark is trivial but will play an essential role.

Remark 3.10. If $\mathscr{Y}$ is a category internal to $G\mathscr{U}$ viewed as a constant double category internal to $G\mathscr{U}$, then the first functor B in (3.8) is the identity and we can identify B^2 with the second functor B.

Write $\mathscr{D}^{top}_G$ for the category of operators $B\mathscr{D}_G$ in $G\mathscr{U}$. Applying the Grothendieck category construction with $\mathscr{W} = G\mathscr{U}$ rather than $\mathscr{W} = \mathrm{Cat}(G\mathscr{U})$, we again abbreviate notation by writing

$$\mathbb{G}r\mathscr{X} = \mathbb{G}r(\mathscr{D}^{top}_G, \mathscr{D}^{top}_G, \mathscr{X}) \tag{3.11}$$

for a $\mathscr{D}^{top}_G$-algebra $\mathscr{X}$ in $G\mathscr{U}$. The nerve of $\mathbb{G}r\mathscr{X}$ is the simplicial two-sided bar construction whose geometric realization is the topological two-sided bar construction $B^{\times}(\mathscr{D}^{top}_G, \mathscr{D}^{top}_G, \mathscr{Z})$. The superscript $\times$ registers that we are using cartesian products rather than smash products in the construction; there are quite a few relevant variants, as is discussed in [4, Section 3].

Since the classifying G-space construction commutes with finite limits, we have an identification of Grothendieck categories

$$B\mathbb{G}r\mathscr{Y} \cong \mathbb{G}rB\mathscr{Y}$$

for a $\mathscr{D}_G$-algebra $\mathscr{Y}$ in $\mathrm{Cat}(G\mathscr{U})$. Thus we have natural isomorphisms

$$B^2\mathbb{G}r\mathscr{Y} \cong B\mathbb{G}rB\mathscr{Y} \cong B^{\times}(\mathscr{D}^{top}_G, \mathscr{D}^{top}_G, B\mathscr{Y}). \tag{3.12}$$

By a standard extra degeneracy argument, the map $B^2\varepsilon$ of $\mathscr{D}^{top}$-algebras induces a levelwise G-homotopy equivalence

$$B^\times(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) \longrightarrow B\mathscr{Y} \tag{3.13}$$

with homotopy inverse induced by $B^2\eta$.

Notice that $\mathscr{F}_G = \mathscr{F}_G^{top}$ since $\mathscr{F}_G$ is categorically discrete. Arguing as for (3.12) and using (3.4), we obtain natural isomorphisms

$$B^2\xi_{G*}\mathbb{G}r\mathscr{Y} \cong B\mathbb{G}r(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}) \cong B^\times(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}). \tag{3.14}$$

By [13, Theorem 4.25], $\xi^{top}\colon \mathscr{D}_G^{top} \longrightarrow \mathscr{F}_G$ induces a levelwise G-homotopy equivalence of $\mathscr{D}_G$-algebras

$$B^\times(\xi^{top}, \mathrm{id}, \mathrm{id})\colon B^\times(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) \longrightarrow \xi^{top*}B^\times(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}), \tag{3.15}$$

where ξ^{top*} is the pullback of action functor from $\mathscr{F}_G$-algebras in $G\mathscr{U}$ to $\mathscr{D}_G^{top}$-algebras in $G\mathscr{U}$.

Moreover, as shown in [13, Section 4.5],[5] we have a natural isomorphism

$$B^\times(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}) \cong \xi_{G*}^{top}B^\times(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) \tag{3.16}$$

where ξ^{top} is left adjoint to ξ^{top*}. In view of(3.4), (3.5), and (3.6), the identifications (3.12), (3.14), and (3.16) allow us to identify the levelwise G-homotopy equivalence of (3.15) with

$$B^2\nu\colon B^2\mathbb{G}r\mathscr{Y} \longrightarrow B^2\xi^*\xi_{G*}\mathbb{G}r\mathscr{Y}. \tag{3.17}$$

The bar constructions $B^\times$ are not reduced but, as in [13, Section 3.1],[6] they map via compatible G-homotopy equivalences to reduced quotients

$$\tilde{B}(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) = B^\times(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y})/B^\times(*, \mathscr{D}_G^{top}, B\mathscr{Y}) \tag{3.18}$$

and

$$\tilde{B}(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}) = B^\times(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y})/B^\times(*, \mathscr{D}_G^{top}, B\mathscr{Y}). \tag{3.19}$$

Here (3.16) induces an isomorphism

$$\tilde{B}(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}) \cong \xi^{top}\tilde{B}(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) \tag{3.20}$$

of $\mathscr{F}_G$-algebras in $G\mathscr{U}$.

In sum, after passage to quotients as in (3.18) and (3.19), applying B^2 to the diagram

$$\mathscr{Y} \xleftarrow{\varepsilon} \mathbb{G}r(\mathscr{D}_G, \mathscr{D}_G, \mathscr{Y}) \xrightarrow{\mathbb{G}r(\xi,\mathrm{id},\mathrm{id})} \xi^*\mathbb{G}r(\mathscr{F}_G, \mathscr{D}_G, \mathscr{Y}). \tag{3.21}$$

of double categories internal to $G\mathscr{U}$ gives the diagram of levelwise G-homotopy equivalences

$$B\mathscr{Y} \xleftarrow{\varepsilon} B(\mathscr{D}_G^{top}, \mathscr{D}_G^{top}, B\mathscr{Y}) \xrightarrow{B(\xi,\mathrm{id},\mathrm{id})} \xi^{top*}B(\mathscr{F}_G, \mathscr{D}_G^{top}, B\mathscr{Y}). \tag{3.22}$$

obtained by combining (3.13) and (3.15).

[5] In [13] we used the notation ξ_* for the derived functor $\tilde{B}(\mathscr{F}_G, \mathscr{D}_G^{top}, -)$.

[6] We used the unadorned notation B in [13].

Remark 3.23. We can start more generally with a $\mathscr{D}_G$-pseudoalgebra $\mathscr{Y}$. It has a strictification $\mathbb{S}t\mathscr{Y}$, and it follows from [6, Remarks 0.19, 5.17, and 5.18] that everything we have said so far in this section carries over equivalently and without change with $\mathscr{Y}$ replaced by $\mathbb{S}t\mathscr{Y}$. Applying $\mathbb{S}t$ to the objects, morphisms, and structural maps of $\mathbb{G}r\mathscr{Y}$ to obtain $\mathbb{S}t\mathbb{G}r\mathscr{Y}$, the cited remarks give that the natural map $\mathbb{S}t\mathbb{G}r\mathscr{Y} \longrightarrow \mathbb{G}r\mathscr{Y}$ is an equivalence of $\mathscr{D}_G$-algebras. We denote its inverse by μ. The cited remarks also give a map $\xi_{G*}\mathbb{G}r\mathscr{Y} \simeq \mathbb{G}r(\mathscr{F}_G, \mathscr{D}_G, \mathscr{Y})$ of $\mathscr{F}_G$-algebras. Both of these induce levelwise G-homotopy equivalences on application of B^2. Thus we have homotopical control even if we start with $\mathscr{D}_G$-pseudoalgebras.

Remark 3.24. Since B^2 is defined in terms of B, it is clear that Theorem 3.9 is a direct consequence of Theorem 2.2. Therefore, in the multiplicative theory that is our focus, we can just as well take the categorical target to be the multicategory associated to the symmetric monoidal 2-category $\mathrm{Fun}(\mathscr{F}_G, \mathrm{Cat}^2(G\mathscr{U}))$, applying the symmetric monoidal functor $\mathbf{S_G} \circ B^2$ to go from there to the multicategory associated to the symmetric monoidal category of orthogonal G-spectra.

3.3. **From $\mathscr{P}_G$-transformations to homotopies of maps of G-spectra.** A bit digressively, we here prove the following addendum that shows how homotopies behave. As we shall see, $\mathscr{P}_G$-transformations should be regarded as homotopies between pseudomaps of symmetric monoidal G-categories.

Theorem 3.25. *The infinite loop space machines $\mathbb{K}_G$ from symmetric monoidal G-categories to orthogonal G-spectra take $\mathscr{P}_G$-transformations to homotopies.*

It is classical that the classifying space functor takes G-categories, G-functors and G-natural transformations to G-spaces, G-maps, and G-homotopies. For the last, G-natural transformations are functors $\mathscr{X} \times \mathscr{I} \longrightarrow \mathscr{Y}$, where $\mathscr{I}$ is the category with two objects $[0]$ and $[1]$ and one non-identity morphism $[0] \longrightarrow [1]$. The functor B commutes with products, and it satisfies $B\mathscr{I} = I$.

Theorem 3.25 shows how this theory adapts to our infinite loop space machine. We showed in [5, Proposition 6.16] that the topological Segal machine $\mathbb{S}_G$ preserves homotopies. If we start with $\mathscr{F}_G$-G-categories, which of course are themselves G-functors, then maps between them are G-natural transformations and maps between those are G-modifications. However, levelwise we are seeing G-categories, G-functors, and G-natural transformations. Since B commutes with products, it converts G-modifications to level homotopies between maps of $\mathscr{F}_G$-spaces. The same argument works for B^2.

Now consider the passage from $\mathscr{P}_G$-pseudoalgebras to $\mathscr{F}_G$-G-categories. The 2-morphisms of the 2-category $\mathscr{P}_G$-**PsAlg** are the $\mathscr{P}_G$-transformations as defined in [8, Definition 1.37]. The 2-morphisms of the 2-categories $\mathscr{D}$-**PsAlg** and $\mathscr{D}_G$-**PsAlg** are the $\mathscr{D}$ and $\mathscr{D}_G$-modifications as defined in [4, Definition 2.8]. As explained in [8, 4], these can be viewed as categorical homotopies. By [4, Theorem 0.3], $\mathbb{R}$ is a 2-functor and the prolongation $\mathbb{P}^{\mathscr{D}_G}_{\mathscr{D}}$ is an equivalence of 2-categories. Thus we may view $\mathbb{R}_G$ as homotopy preserving. General categorical observations of the same nature given in [6, Section 5.3] show that, for a $\mathscr{D}_G$-pseudoalgebra $\mathscr{Y}$ and a small category $\mathscr{C}$,

$$\mathbb{G}r(\mathscr{Y} \times \mathscr{C}) \cong \mathbb{G}r(\mathscr{Y}) \times \mathscr{C}$$

and, working monadically, there is a natural map

$$(\mathbb{S}t\mathscr{Y}) \times \mathscr{C} \longrightarrow \mathbb{S}t(\mathscr{Y} \times \mathscr{C})$$

of $\mathscr{D}_G$-pseudoalgebras and a natural map

$$(\xi_{l*}(\mathscr{Y})) \times \mathscr{C} \longrightarrow \xi_*(\mathscr{Y} \times \mathscr{C})$$

of $\mathscr{F}_G$-algebras. Again taking $\mathscr{C} = \mathscr{I}$, we may also view $\mathbb{G}r$, $\mathbb{S}t$, and ξ_* as homotopy preserving.

4. The proof of the multiplicative BPQ theorem

We shall exploit the fact that $\Sigma^\infty_G \colon G\mathscr{T} \longrightarrow G\mathscr{S}$ is left adjoint to the zeroth G-space functor $(-)_0$. In the notation of [5, Section 4.1], for an $\mathscr{F}_G$-G-space $\mathscr{Z}$ we have

$$(\mathbb{S}_G\mathcal{Z})_0 = B^{\Sigma_G}(S^0, \mathbb{F}_G, \mathcal{Z}),$$

which is the geometric realization of a simplicial G-space. There is a natural map

$$\zeta \colon \mathcal{Z}(\mathbf{1}) \cong S^0 \wedge \mathcal{Z}(\mathbf{1}) \longrightarrow (\mathbb{S}_G\mathcal{Z})_0$$

given by inclusion into the G-space of 0-simplices.

Since $\mathbb{K}_G$ is the composite $B\xi_{G\#}\mathbb{G}r\mathbb{R}_G$, to define $\alpha : \Sigma^\infty_{G,+} \longrightarrow \mathbb{K}_G i\,\mathbb{P}_{G,+}$ in Theorem 1.3, it suffices by adjunction to define a map of based G-spaces

$$\alpha_X : X_+ \longrightarrow \mathbb{S}_G\big(B\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G,+}X\big)_0$$

for each unbased G-space X. We define α_X to be the composite displayed in the diagram

$$\begin{tikzcd}
X_+ \arrow[rrr, "\alpha_X"] \arrow[d, "\cong"'] & & & \mathbb{S}_G\big(B^2\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G,+}X\big)_0 \\
Bi\,X_+ \arrow[d, "Bi\,\eta"'] & & & \\
Bi\,\mathbb{P}_{G,+}X \arrow[rr, equal] & & \big(B^2\mathbb{R}_G i\,\mathbb{P}_{G,+}X\big)(\mathbf{1} \arrow[r, "(B^2\iota)(\mathbf{1})"'] & \big(B^2\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G,+}X\big)(\mathbf{1}). \arrow[uu, "\zeta"']
\end{tikzcd}$$

The top left isomorphism is an immediate inspection, and η on the left is the unit of the monad $\mathbb{P}_{G,+}$. For the bottom left equality, it is true by definition that $\mathscr{X} = (\mathbb{R}_G\mathscr{X})(\mathbf{1})$ for any $G\mathscr{U}$-category $\mathscr{X}$, such as $\mathscr{X} = i\,\mathbb{P}_{G+}X$, and that $B(\mathscr{Z}(\mathbf{1})) = (\mathbf{B}\mathscr{Z})(\mathbf{1})$ for any Π_G-G-category $\mathscr{Z}$, such as $\mathscr{Z} = \mathbb{R}_G i\,\mathbb{P}_{G+}X$. We also use Remark 3.10 to replace B by B^2. We must explain the map

$$\iota \colon \mathbb{R}_G i\,\mathbb{P}_{G,+}X \longrightarrow \xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G,+}X \tag{4.1}$$

to which $B^2(-)(\mathbf{1})$ is applied at the bottom right. For any $\mathscr{D}_G$-G-category $\mathscr{Y}$, we define ι to be the composite map of Π_G-algebras in $\mathrm{Cat}^2(G\mathscr{U})$

$$\mathscr{Y} \xrightarrow{\eta} \mathbb{G}r\mathscr{Y} \xrightarrow{\nu} \xi_*\mathbb{G}r\mathscr{Y} \xrightarrow{\xi_{G*}\zeta} \xi_*\mathbb{S}t\mathbb{G}r\mathscr{Y} \tag{4.2}$$

given by (3.3)), (3.6)), and Remark 3.23. Taking $\mathscr{Y} = \mathbb{R}_G i\,\mathbb{P}_{G,+}X$, this gives the map ι of (4.1)).

The adjoints of the α_X define the natural transformation α, and we must verify the formal property that α is monoidal and the homotopical property that α is a level G-equivalence.

We start with the first. Since the adjoint of α_{S^0} is easily seen to be the unit map of the monoidal functor $\mathbb{K}_G i\,\mathbb{P}_{G+}$, it suffices to verify that the following diagram commutes for G-spaces X and Y. Recall again that $(X\times Y)_+\cong X_+\wedge Y_+$.

$$\begin{tikzcd}
\Sigma^\infty_G X_+\wedge\Sigma^\infty_G Y_+ \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\cong"'] & \mathbb{K}_G i\,\mathbb{P}_{G,+}X\wedge\mathbb{K}_G i\,\mathbb{P}_{G+}Y \arrow[d] \\
\Sigma^\infty_G(X\times Y)_+ \arrow[r, "\alpha"'] & \mathbb{K}_G i\mathbb{P}_{G,+}(X\times Y)
\end{tikzcd}$$

The right map is the pairing constructed by use of the monoidal composite implied by our morphisms of multicategories. The left map is the isomorphism induced by the isomorphism of functors $\mathscr{I}\times\mathscr{I}\longrightarrow G\mathscr{T}$ given by the evident G-homeomorphisms

$$\Sigma^V X_+\wedge\Sigma^W Y_+\cong\Sigma^{V\oplus W}(X_+\wedge Y_+)$$

for inner product G-spaces V and W. We implicitly consider its inverse, so that we are now comparing two maps out of $\Sigma^\infty_G(X\times Y)_+$. By adjointness, we have reduced to showing that the following diagram of maps of G-spaces commutes.

$$\begin{tikzcd}
X_+\wedge Y_+ \arrow[r, "\alpha_X\wedge\alpha_Y"] \arrow[d, "\cong"'] & (\mathbb{K}_G i\,\mathbb{P}_{G,+}X)_0\wedge(\mathbb{K}_G i\,\mathbb{P}_{G+})_0 \arrow[d] \\
(X\times Y)_+ \arrow[r, "\alpha_{X\times Y}"'] & (\mathbb{K}_G i\,\mathbb{P}_{G+}(X\times Y))_0.
\end{tikzcd}$$

The diagram factors into two diagrams which commute by direct comparisons of definitions. We observe that while B need not commute with smash products in general, its commutation with products and a little diagram chase show that it is lax monoidal, so that there is a natural map

$$\pi\colon B\mathscr{X}\wedge B\mathscr{Y}\longrightarrow B(\mathscr{X}\wedge\mathscr{Y})$$

for G-categories $\mathscr{X}$ and $\mathscr{Y}$. Similarly, since the composite $\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G+}$ is monoidal, by our morphisms of multicategories, a little diagram chase shows that we have a natural map

$$\psi\colon\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G+}X\wedge\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G+}Y\longrightarrow\xi_{G\#}\mathbb{G}r\mathbb{R}_G i\,\mathbb{P}_{G,+}(X\times Y)$$

for G-spaces X and Y. Since $\mathbb{S}_G$ is monoidal, we also have a natural map

$$\varphi\colon\mathbb{S}_G Z\wedge\mathbb{S}_G W\longrightarrow\mathbb{S}_G(Z\wedge W)$$

for $\mathscr{F}_G$-G-spaces Z and W. Abbreviating notation by setting $\mathbb{T}=\xi_{G\#}\mathbb{G}r\mathbb{R}_G$, the factored diagram takes the form

$$\begin{tikzcd}
X_+\wedge Y_+ \arrow[r, "B\eta\wedge B\eta"] \arrow[d, "B\eta"'] & Bi\,\mathbb{P}_{G+}X\wedge Bi\,\mathbb{P}_{G+}Y \arrow[r, "B\iota\wedge B\iota"] & B\mathbb{T}i\,\mathbb{P}_{G,+}X(\mathbf{1}\wedge B\mathbb{T}i\,\mathbb{P}_{G+}Y(\mathbf{1} \arrow[dl, "\pi"'] \arrow[d, "\zeta\wedge\zeta"] \\
Bi\,\mathbb{P}_{G+}(X\times Y) \arrow[d, "B\iota"'] & B\big(\mathbb{T}i\,\mathbb{P}_{G+}X\wedge\mathbb{T}i\,\mathbb{P}_{G+}Y\big)(\mathbf{1} \arrow[dl, "B\psi(\mathbf{1})"'] & \big(\mathbb{S}_G B\mathbb{T}i\,\mathbb{P}_{G,+}X\big)_0\wedge\big(\mathbb{S}_G B\mathbb{T}i\,\mathbb{P}_{G+}Y\big)_0 \arrow[d, "\mathbb{S}_G\pi\circ\phi"] \\
B\mathbb{T}i\,\mathbb{P}_{G+}(X\times Y) \arrow[r, "\zeta"'] & \big(\mathbb{S}_G B\mathbb{T}i\,\mathbb{P}_G(X\times Y)_+\big)_0 & \mathbb{S}_G B\big(\mathbb{T}i\,\mathbb{P}_G X\wedge\mathbb{T}i\,\mathbb{P}_G Y\big)_0 \arrow[l, "\mathbb{S}_G B\psi"]
\end{tikzcd}$$

The verification of commutativity is straightforward, impeded only by the notation.

To see that α is an equivalence, we exploit the version of the BPQ theorem proven in [3, Theorem 6.2] using the operadic infinite loop space machine $\mathbb{E}_G$ together with the equivalence of the operadic and Segalic infinite loop space machines proven in [13, Theorem VI.1.1]. Here we interpret these results in terms of categories of operators over $\mathscr{F}_G$ rather than over $\mathscr{F}$, as we may. We exhibit α as a composite zigzag of equivalences displayed schematically in the diagram

$$\begin{tikzcd}
\Sigma^\infty_G X_+ \arrow[r, "\alpha"] \arrow[d, "\boxed{1}"'] & \mathbb{S}_G B^2 \xi_\# \mathbb{G}r \mathbb{R}_G i\, \mathbb{P}_{G+} X \\
\mathbb{E}_G \mathbb{P}^{top}_G X_+ \arrow[d, "\cong"'] & \mathbb{S}_G B^2 \mathbb{R}_G i \mathbb{P}_{G+} X \arrow[u, dashed, "\boxed{3}"'] \\
\mathbb{E}_G \mathbb{R}_G \mathbb{P}^{top}_G X_+ \arrow[r, dashed, "\boxed{2}"'] & \mathbb{S}_G \mathbb{R}_G \mathbb{P}^{top}_G X_+ \arrow[u, "\cong"']
\end{tikzcd} \tag{4.3}$$

We have written $\mathbb{P}^{top}_G$ for the monad on based G-spaces obtained from the operad $\mathscr{P}^{top}_G$. For an unbased G-space X, we then have an identification

$$\mathbb{P}^{top}_G X_+ \cong Bi\, \mathbb{P}_{G,+} X.$$

We also have $\mathbb{R}_G B \cong B\mathbb{R}_G$, and the composite of these isomorphisms, together with Remark 3.10, gives the right hand isomorphism in (4.3)).

The BPQ theorem [3, Theorem 6.2] gives the equivalence labelled $\boxed{1}$ in (4.3)). The machine $\mathbb{E}_G$ is defined compatibly on $\mathscr{P}^{top}_G$-G-spaces and on $\mathscr{D}^{top}_G$-G-spaces, where $\mathscr{D}^{top}_G$ is the topological category of operators over $\mathscr{F}_G$ associated to $\mathscr{P}^{top}_G$. As explained in [13, Section V.4], for $\mathscr{P}^{top}_G$-spaces A, we have a natural isomorphism between $\mathbb{E}_G A$ defined operadically and $\mathbb{E}_G \mathbb{R}_G A$ defined using $\mathscr{D}^{top}_G$. Taking $A = \mathbb{P}^{top}_G X_+$, this gives the left-hand isomorphism in the diagram (4.3)).

The machine $\mathbb{E}_G$ and the Segal machine $\mathbb{S}_G$ are both defined on $\mathscr{D}^{top}$-G-spaces $\mathcal{Y}$, but they are defined very differently. Homotopies intrinsic to the Steiner operads were shown in [13, Theorem VI.1.1] to mediate between these machines, giving a natural zigzag of equivalences between $\mathbb{E}_G \mathcal{Y}$ and $\mathbb{S}_G \mathcal{Y}$. Taking $\mathcal{Y} = \mathbb{R}_G \mathbb{P}^{top}_G X_+$, that zigzag gives the dotted arrow $\boxed{2}$ in (4.3)).

Finally, by [2, Theorem 4.32(ii)] and the equivalence of the machine $\mathbb{S}_G$ used there with the new variant defined in [5], the G-spectra $\mathbb{S}_G Y$ and $\mathbb{S}_G \xi^{top}_* Y$ are equivalent for any $\mathscr{D}_G$-G-space Y. Taking $Y = B\mathscr{Y}$ for a $\mathscr{D}_G$-G-category $\mathscr{Y}$, we see by (3.14) and (3.16)) that the dotted arrow equivalence $\boxed{3}$ in (4.3) results by taking $\mathscr{Y} = \mathbb{R}_G i\, \mathbb{P}_{G,+} X$, taking the dotted arrow to be $B^2\iota$, and using the cited comparisons of categorical and topological maps. Said another way, replacing η in (4.2)), which is not a map of $\mathscr{D}_G$-algebras, with $\varepsilon\colon \mathbb{G}r\mathscr{Y} \longrightarrow \mathscr{Y}$, which is, we see that (4.1) itself gives a zigzag on the categorical level that gives the equivalence $\boxed{3}$ on application of $\mathbb{S}_G B^2$.

Of course, given the zigzags, it is not meaningful to ask that the diagram commute. However, since everything is completely explicit on the point set level and the identification of the 0th spaces of the machine built entries is straightforward, it is not hard to check by inspection of definitions that we obtain adjoint commutative diagrams mapping X to the zeroth spaces of our machine built constructions

corresponding to all of the zigzag maps. Therefore the diagram commutes up to zigzags and α is an equivalence.

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