THE HARE AND THE TORTOISE

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It is a pleasure to be here to help celebrate Mike Boardman's 60th birthday.[1]

I have just finished writing a history of stable algebraic topology from the end of World War II through 1966 [18]. The starting point was natural enough. The paper of Eilenberg and Mac Lane [6] that introduced the categorical language we now all speak appeared in 1945, and so did the paper of Eilenberg and Steenrod [7] that announced the axiomatic treatment of homology and cohomology.

The ending point was more artificial, at first dictated by constraints of time and energy and the fact that Steenrod's compendium of Mathematical Reviews in topology contained all reviews published through 1967 and thus all papers published through 1966. It also made it easy for me to be modest and impersonal. Although I got my PhD in 1964, I only plugged into the circuit and began to know what was going on when I arrived at Chicago, at the end of 1966.

Mike also got his PhD in 1964. Since he is two years older than I am, I guess he was a little slow. But then, his thesis was a lot more important than mine was, although people at the time didn't seem to understand that. Its results became available in a mimeographed summary in 1966. So maybe 1966 wasn't such a bad stopping point mathematically. It is amazing how much we didn't know then, how many familiar names had not yet made their mark.

In fact, a complete list of the people who made sustained and important contributions to the development of stable algebraic topology in the years 1945 through 1966 would have no more than around 40 names on it. On the other hand, the caliber of the people working in the field was extraordinary.

The towering figures were Adams, Atiyah, Borel, Bott, Cartan, Eilenberg, Hirzebruch, Mac Lane, Milnor, Serre, Steenrod, Thom, and J.H.C. Whitehead. Others who made major contributions were Adem, Araki, Barratt, Brown, Conner, Dold, Dyer, Eckmann, Floyd, Heller, Hilton, James, Kan, Kudo, Lashof, Liulevicius, Moore, Peterson, Pontryagin, Postnikov, Puppe, Spanier, Stong, Swan, Toda, Wall, G.W. Whitehead, and Wu.

Several people attempted to build good stable homotopy categories. In 1953, Spanier and J.H.C. Whitehead [22] gave a start by introducing the "S-category", which we now understand to be the full subcategory of finite CW complexes in the stable homotopy category. In 1962, G.W. Whitehead [24] gave the definitive category of Ω-spectra equivalent to the category of cohomology theories on spaces. It is definitely not triangulated, however, and therefore cannot be the stable homotopy category as we know it today.

Adams [1] and Puppe [19, 20] gave constructions with roughly the same starting point in the early 1960's. They consider CW prespectra, which are sequences of CW complexes T_n and inclusions of subcomplexes $\Sigma T_n \to T_{n+1}$, subject to certain connectivity restrictions. The term "prespectrum" rather than "spectrum" agrees

[1] This is a revised and shortened version of my talk at Baltimore

with Mike's terminology; this distinction was first made by Dan Kan [11]. There is an analogy with presheaves that was made precise later.

Adams [1] described his point of view in an amusing analogy, well worth recalling.

"The hare is an idealist: his preferred position is one of elegant and all embracing generality. He wants to build a new heaven and a new earth and no half-measures. The tortoise, on the other hand, takes a much more restrictive view. He says that his modest aim is to make a cleaner statement of known theorems, and he'd like to put a lot of restrictions on his stable objects so as to be sure that his category has all the good properties he may need. Of course, the tortoise tends to put on more restrictions than are necessary, but the truth is that the restrictions give him confidence.

You can decide which side you're on by contemplating the Spanier-Whitehead dual of an Eilenberg-Mac Lane object. This is a "complex" with "cells" in all stable dimensions from $-\infty$ to $-n$. According to the hare, Eilenberg-Mac Lane objects are good, Spanier-Whitehead duality is good, therefore this is a good object: And if the negative dimensions worry you, he leaves you to decide whether you are a tortoise or a chicken. According to the tortoise, on the other hand, the first theorem in stable homotopy theory is the Hurewicz isomorphism theorem, and this object has no dimension at all where that theorem is applicable, and he doesn't mind the hare introducing this object as long as he is allowed to exclude it. Take the nasty thing away!"

Frank was a great mathematician and my closest friend, but to my mind this is not the best attitude to take towards matters of foundations: that was not his strongest suit. His category had only connective spectra in it, so excluded even the periodic Bott spectra, and his spectra therefore could not be desuspended. Puppe's spectra were bounded below, still excluding the Bott spectra, but at least they could be desuspended.

Mike's introduction [4] gave persuasive propaganda for his version of the stable homotopy category.

"This introduction attempts to give some criteria for a stable category. It is addressed without compromise to the experts. The novice has the advantage of not having been misled by previous theories. The place of our theory on Adams' tortoise-hare scale is obvious."

Mike was the prototypical "hare". He was operating at a level of categorical sophistication that, while commonplace and even fashionable now, was well ahead of its time. Unfortunately for the subject, this prototypical hare was and is a tortoise when it comes to publication. His treatment of the stable category has never been published. Mike, let me again ask you, as I think I have done every year for the last decade: please publish your paper on conditionally convergent spectral sequences.[2] For those not in the know, the cited paper gives the definitive treatment of convergence of spectral sequences.

The introduction goes on: "In this advertisement we compare our category $\mathcal{S}$ of CW spectra, or rather its homotopy category $\mathcal{S}_h$, with competing products. We find the comparison quite conclusive, because the more good properties the competitors have, the closer they are to $\mathcal{S}_h$. Findings subject to verification by an independent consumer agency."

[2] It's in this volume — thanks, Mike.

All consumers are now in agreement: Mike's stable homotopy category is definitively the right one, up to equivalence. However, the really fanatical hare demands a good category even before passage to homotopy, with all of the modern bells and whistles. The ideal category of spectra should be a complete and cocomplete Quillen model category, tensored and cotensored over the category of based spaces (or simplicial sets), and closed symmetric monoidal under the smash product. Its homotopy category (obtained by inverting the weak equivalences) should be equivalent to Mike's original stable homotopy category.

We now have such a category, namely the category of S-modules of [9]; an introductory account is given in [8]. Indeed, we have several such categories and know how to compare them [10, 14, 17, 13, 21].

Mike was brought up in the English tradition of J.H.C. Whitehead, whose students were reared on beer and CW complexes. I have no objection to beer, but I early came to the conclusion that to restrict to CW complexes, as Adams, Puppe, and Boardman all did, is to err on the side of the tortoise. At a conference in 1968 [15], I advertised the idea that spectra really should be defined as sequences of based spaces E_n and *homeomorphisms* $E_n \cong \Omega E_{n+1}$. Many constructions, such as products, limits, and function spectra, that are impossible with any kind of CW spectra become easy with spectra of that sort.

The S-modules of [9] are spectra with additional structure. It took a number of developments over nearly thirty years to arrive at the category of S-modules. One key idea was to introduce coordinate-free spectra [16], indexing them on finite dimensional inner product spaces rather than on the non-negative integers: those just give the canonical inner product spaces $\mathbb{R}^n$. This led to a formal development of the theory of smash products of coordinate-free spectra that is closely analogous to Mike's original treatment of smash products of CW spectra, despite a great difference between the definitions of the two kinds of spectra.

A brief summary of Mike's theory may be of interest, since younger algebraic topologists are unlikely to have seen it. Details are in Mike's preprints [3, 4] and in an exposition by his student Rainer Vogt [23]. For each countably infinite dimensional real inner product U, Mike constructs a category $\mathcal{S}(U)$ of CW spectra. He starts with a copy $\mathcal{F}_A$ of the category of finite CW complexes for each finite dimensional subspace A of U. He first constructs a category $\mathcal{F}(U)$ from the $\mathcal{F}_A$ by a certain colimit of categories construction. Intuitively, $\mathcal{F}(U)$ is a copy of the category of finite CW spectra, constructed by stabilization from the category of finite CW complexes. Then $\mathcal{S}(U)$ is constructed from $\mathcal{F}(U)$ by another categorical colimit construction. Intuitively, the objects of $\mathcal{S}(U)$ are all of the colimits of diagrams of inclusions between objects of $\mathcal{F}(U)$. The stable homotopy category $\mathcal{S}_h(U)$ is obtained from $\mathcal{S}(U)$ by passage to homotopy classes of maps. The canonical stable homotopy category is $\mathcal{S}_h = \mathcal{S}_h(\mathbb{R}^\infty)$.

There is an obvious external smash product $\overline{\wedge} : \mathcal{S}(U) \times \mathcal{S}(U') \to \mathcal{S}(U \oplus U')$. For a linear isometry $f : U \to U'$, there is a functor $f_* : \mathcal{S}(U) \to \mathcal{S}(U')$. In particular, each linear isometry $f : \mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$ can be composed with the external smash product $\overline{\wedge}$ to give an internal smash product $\mathcal{S}(\mathbb{R}^\infty) \times \mathcal{S}(\mathbb{R}^\infty) \to \mathcal{S}(\mathbb{R}^\infty)$. More generally, for any finite CW complex K and any map $k : K \to \mathcal{I}(U, U')$, where $\mathcal{I}(U, U')$ is the space of linear isometries $U \to U'$, there is a functor $k_* : \mathcal{S}(U) \to \mathcal{S}(U')$. If L is a subcomplex and deformation retract of K and ℓ is the restriction of k to L, there is a natural equivalence $\ell_* \to k_*$. Using this and the contractibility of the spaces $\mathcal{I}(U, U')$, it follows formally that the internal smash products all become canonically equivalent on passage to the stable homotopy category $\mathcal{S}_h$ and $\mathcal{S}_h$ is a symmetric monoidal category under the internal smash product.

The theory that Gaunce Lewis and I developed later [12] looks formally the same. We also have categories $\mathcal{S}(U)$, external smash products $\bar{\wedge}$, functors f_* and k_*, and so on; we call the functors k_* twisted half-smash products. However, although the formal structure of our theory closely follows Mike's blueprint, the realization of the formal structure is entirely different: whereas Mike's categories $\mathcal{S}(U)$ are constructed categorically out of finite CW complexes, the objects of our categories $\mathcal{S}(U)$ are the coordinate-free spectra indexed on the finite dimensional subspaces of U, and CW complexes play no role in the basic definitions. The generality allows us to define twisted half-smash products k_* for maps $k : K \to \mathcal{I}(U, U')$ defined on arbitrary spaces K. In particular, we can take k to be the identity map of $\mathcal{I}(U, U')$, in which case we write $k_*E = \mathcal{I}(U, U') \ltimes E$.

Let $\mathcal{L}(j) = \mathcal{I}((\mathbb{R}^\infty)^j, \mathbb{R}^\infty)$. As Boardman and Vogt observed and exploited in [5], the spaces $\mathcal{L}(j)$ give an E_∞ operad. More precisely, since I had not yet introduced operads, they observed that the $\mathcal{L}(j)$ are some of the morphism spaces of a PROP (product and permutation category). In [16], the linear isometries operad $\mathcal{L}$ was used to define E_∞-ring spaces and E_∞-ring spectra. We did not then have the general twisted half-smash products. However, once we had them, we could form the extended powers $\mathcal{L}(j) \ltimes E^j$ of a spectrum E, where E^j is the j-fold external smash power of E. An E_∞-ring spectrum is just a spectrum E together with an action of $\mathcal{L}$ given by maps $\mathcal{L}(j) \ltimes E^j \to E$ such that the appropriate unit, asssociativity, and equivariance diagrams commute.

The twisted half-smash product later became the starting point of the theory of S-modules. We have the canonical j-fold internal smash product $\mathcal{L}(j) \ltimes E_1 \bar{\wedge} \cdots \bar{\wedge} E_j$. The j-fold internal smash products determined by the linear isometries $f \in \mathcal{L}(j)$ all map into this canonical j-fold smash power. Imposing extra structure on spectra, defined in terms of maps $\mathcal{L}(1) \ltimes E \to E$, we can construct a quotient $E \wedge_{\mathcal{L}} E'$ of $\mathcal{L}(2) \ltimes E \wedge E'$. The smash product $\wedge_{\mathcal{L}}$ is commutative and associative. A slight variant gives the notion of an S-module E and a commutative, associative, and unital smash product $E \wedge_S E'$ of S-modules. Up to canonical weak equivalence, E_∞ ring spectra are exactly the commutative monoids in the symmetric monoidal category of S-modules. This, roughly, is the starting point of the theory of Elmendorf, Kriz, Mandell, and myself [8, 9].

As this brief sketch indicates, the theory of S-modules owes a great deal to ideas and insights due to Boardman. It is typical of Mike's generosity to the ideas of others that he began his review of [8] with the sentence "This is the paper that promises to revolutionize stable homotopy theory" and did not mention the influence or relevance of his original approach to the stable homotopy category. I am happy to have had this occasion to rectify the omission.

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