STABLE ALGEBRAIC TOPOLOGY
AND
STABLE TOPOLOGICAL ALGEBRA

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Algebraic topology is a young subject, and its foundations are not yet firmly in place. I shall give some history, examples, and modern developments in that part of the subject called stable algebraic topology, or stable homotopy theory. This is by far the most calculationally accessible part of algebraic topology, although it is also the least intuitively grounded in visualizable geometric objects. It has a great many applications to such other subjects as algebraic geometry and geometric topology. Time will not allow me to say as much as I would like about that. Rather I will emphasize some foundational issues that have been central to this part of algebraic topology since the early 1960's, but that have only been satisfactorily resolved in the last few years.

It was only in 1952, with Eilenberg and Steenrod's book "Foundations of algebraic topology" [9], that the nature of ordinary homology and cohomology theories was reasonably well understood. Even then, the modern way of thinking about cohomology as represented by Eilenberg-Mac Lane spaces was nowhere mentioned. It may have been known by then, but it certainly was not known to be important. The subject changed drastically with a series of extraordinary advances in the 1950's and early 1960's. By around 1960, it had become apparent that algebraic topology divides naturally into two rather different major branches: unstable homotopy theory and stable homotopy theory. The former concerns space level invariants, such as the fundamental group, that are more or less invisible to homology and cohomology theories. The latter concerns invariants that are in a sense independent of dimension. More precisely, it concerns invariants that are stable under suspension, such as homology and cohomology groups. It had also became apparent that many interesting phenomena that a priori seemed to depend on a dimension could be translated into questions in stable algebraic topology. Three fundamental examples have set the tone for a great deal of modern algebraic topology. They occurred nearly simultaneously in the late 1950's and early 1960's. The order I will give is not chronological.

First, Adams [1] proved that the only possible dimensions of a normed linear algebra over $\mathbb{R}$ are 1, 2, 4, or 8 by translating the problem into one in stable homotopy theory. More precisely, the problem translated into a problem in ordinary mod 2 cohomology theory that involved only the Steenrod cohomology operations

$$Sq^n : H^q(X;\mathbb{Z}_2) \longrightarrow H^{n+q}(X;\mathbb{Z}_2)$$

and not the cup product. The Steenrod operations are stable, in the sense that

$$\Sigma Sq^n = Sq^n \Sigma$$

This is essentially the text of the 1997 Hardy Lecture delivered to the Society on June 20, 1997.

where

$$\Sigma : \tilde{H}^q(X;\mathbb{Z}_2) \longrightarrow \tilde{H}^{q+1}(\Sigma X;\mathbb{Z}_2)$$

is the suspension isomorphism. Cup products, in contrast are unstable: the cup product of two cohomology classes in $\tilde{H}^*(\Sigma X)$ is zero. We explain the reason. For based spaces X, the analog of cartesian product is the smash product

$$X \wedge Y = X \times Y/\{*\} \times Y \cup X \times \{*\}.$$

The suspension ΣX is $X \wedge S^1$. The cup products on the reduced cohomology of ΣX are induced by the reduced diagonal map $\Sigma X \longrightarrow \Sigma X \wedge \Sigma X$, and this map is null homotopic because the diagonal map $S^1 \longrightarrow S^1 \wedge S^1 \cong S^2$ is null homotopic. Nevertheless, since it is visible to cohomology, cup products should also be visible to stable homotopy theory. This dichotomy, the semi-stable nature of products, is the starting point for the theme of this talk.

Second, Thom [24] succeeded in classifying smooth compact n-manifolds up to cobordism for every n by translating the problem into one in stable homotopy theory. Here two such n-manifolds M and M' are cobordant if their disjoint union is the boundary of an $(n+1)$-manifold W. The set $\mathscr{N}_n$ of cobordism classes is isomorphic to $\pi_{n+q}(TO(q))$ for any sufficiently large q, where $TO(q)$ is the "Thom space" associated to the universal q-plane bundle $\xi_q : EO(q) \longrightarrow BO(q)$. $TO(q)$ is obtained by forming the fiberwise one-point compactification of $EO(q)$ and then identifying all of the points at infinity. We won't need to know the precise definition of $BO(q)$. Universality means that any q-plane bundle over any space X is equivalent to pullback of the universal bundle along some map $f : X \longrightarrow BO(q)$; homotopic maps pull back to equivalent bundles. That is, the set of equivalence classes of q-plane bundles over X is in bijective correspondence with the set of homotopy classes of maps $X \longrightarrow BO(q)$. Whitney sum, that is, fiberwise direct sum, of vector bundles corresponds on the classifying space level to a system of maps $BO(q) \times BO(r) \longrightarrow BO(q+r)$. On passage to Thom complexes, there results a system of maps $TO(q) \wedge TO(r) \longrightarrow TO(q+r)$. These induce pairings of homotopy groups

$$\pi_{m+q}(TO(q)) \otimes \pi_{n+r}(TO(r)) \longrightarrow \pi_{m+n+q+r}(TO(q+r)).$$

Under the Thom isomorphism, this corresponds to Cartesian product of manifolds.

Third, Atiyah and Hirzebruch [3] followed up Bott's proof [7] of his periodicity theorem by inventing topological K-theory as the prime example of a generalized cohomology theory. Bott's theorem states that $\pi_n(BO) \cong \pi_{n+8}(BO)$ for $n > 0$, where BO is the union of the classifying spaces $BO(q)$ under the sequence of maps $i_q : BO(q) \longrightarrow BO(q+1)$ that correspond to adding a 1-dimensional trivial bundle to a q-plane bundle. For a compact space X, the set of bundles over X of all dimensions is a semi-group under Whitney sum, and $KO(X)$ is the Grothendieck group obtained by formally adjoining inverses. There is a reduced version of K-theory on based spaces X, and $\widetilde{KO}(X)$ is isomorphic to the set $[X, BO \times \mathbb{Z}]$ of homotopy classes of based maps from X to $BO \times \mathbb{Z}$. Crossing with $\mathbb{Z}$ has the effect of making periodicity hold for $n \geq 0$, and Bott periodicity implies that $KO(X)$ is the degree zero part of a cohomology theory that satisfies

$$\widetilde{KO}^n(X) \cong \widetilde{KO}^{n+8}(X).$$

Here the tensor product of bundles gives rise to a multiplication in KO-theory analogous to the cup products in cohomology. It arises from a system of pairings of classifying spaces $BO(q) \times BO(r) \longrightarrow BO(qr)$.

It quickly became apparent that these three examples have a great deal in common. Long before the work just described, Eilenberg and MacLane had constructed the Eilenberg-MacLane spaces $K(\pi, n)$ for abelian groups π. They are characterized by

$$\pi_q(K(\pi, n)) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{if } q \neq n. \end{cases}$$

For based spaces X and Y, we let $[X, Y]$ denote the set of homotopy classes of based maps $X \to Y$. Then, for an abelian group n and based space X,

$$\tilde{H}^n(X; \pi) \cong [X, K(\pi, n)].$$

The suspension isomorphism corresponds to the fact that

$$K(\pi, n) \simeq \Omega K(\pi, n+1),$$

where ΩY is the space of loops at the basepoint in Y, the implication coming from the adjunction

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Replacing π by a commutative ring R, the cup product on $\tilde{H}^*(X; R)$ is induced from the diagonal map $X \to X \wedge X$ and a system of pairings

$$K(R, m) \wedge K(R, n) \longrightarrow K(R, m+n).$$

For cobordism, the map of bundles over $i_q : BO(q) \longrightarrow BO(q+1)$ induces a map $\Sigma TO(q) \longrightarrow TO(q+1)$. Taking adjoints and iterating, we obtain inclusions

$$TO(n) \longrightarrow \Omega TO(n+1) \longrightarrow \Omega^2 TO(n+2) \longrightarrow \cdots .$$

The union of these spaces is called $MO(n)$, and we have homeomorphisms

$$MO(n) \cong \Omega MO(n+1).$$

We define a cohomology theory on based spaces by

$$\widetilde{MO}^n(X) = \begin{cases} [X, MO(n)] & \text{if } n \geq 0 \\ [X, \Omega^{-n} MO(0)] & \text{if } n < 0. \end{cases}$$

We didn't need to define ordinary cohomology in negative degrees this way, since, if we did, we would obtain zero groups because $K(\pi, 0)$ is discrete. However, that is not true in cobordism, where

$$MO^{-n}(pt) \equiv \widetilde{MO}^{-n}(S^0) = \mathscr{N}_n.$$

The Whitney sum pairing of Thom spaces induces pairings

$$MO(m) \wedge MO(n) \longrightarrow MO(m+n)$$

and these give rise to a product in the cobordism theory $\widetilde{MO}^*(X)$.

For K-theory, we define

$$KO(8j - i) = \Omega^i(BO \times \mathbb{Z})$$

for $j \geq 1$ (or $j \in \mathbb{Z}$) and $1 \leq i \leq 8$. Bott periodicity gives homotopy equivalences

$$KO(n) \simeq \Omega KO(n+1)$$

and reduced real K-theory is given by

$$\widetilde{KO}^n(X) = [X, KO(n)],\ n \in \mathbb{Z}.$$

The tensor product pairing gives rise to maps

$$KO(m) \wedge KO(n) \longrightarrow KO(m+n),$$

that represent the products in $\widetilde{KO}^*(X)$.

In all three cases, we obtain unreduced theories by adjoining disjoint basepoints to unbased spaces; for example,

$$H^*(X; R) = \tilde{H}^*(X_+; R).$$

Then all three of our cohomology theories take values in $\mathbb{Z}$-graded associative, commutative, and unital rings.

With these examples in mind, Brown in 1962 [8] proved a representability theorem to the effect that every cohomology theory on spaces arises in the fashion illustrated by our examples. By the early 1960's, algebraic topologists realized that serious calculational analysis required a good "stable homotopy category" whose objects should be some sort of stabilized analog of spaces, each of which should represent a cohomology theory. The essential need was to be able to work effectively with multiplicative structures given by rings and modules in the stable homotopy category. I will give a capsule introduction to one version of what the stable category is, but you shouldn't take this part too seriously: I am just trying to give a vague impression of what the objects we study really are. This category was first constructed by Boardman in his 1964 Warwick thesis [5]. An exposition was later given by Adams [2]. In historical perspective, I regard their construction as a stopgap. It had very serious intrinsic limitations that I will describe shortly. The construction I will sketch leads to an equivalent category.

We define a prespectrum to be a sequence of spaces T_n and maps $\sigma : \Sigma T_n \longrightarrow T_{n+1}$. The Thom prespectrum $TO = \{TO(n)\}$ is an example. The sequence of suspensions $\{\Sigma^n X\}$ of a based space X is another. A map $f : T \longrightarrow T'$ of prespectra is a sequence of maps $f_n : T_n \longrightarrow T'_n$ that are strictly compatible with the structure maps, in the sense that the following diagrams commute:

$$\begin{tikzcd} \Sigma T_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma T'_n \arrow[d, "\sigma'_n"] \\ T_{n+1} \arrow[r, "f_{n+1}"'] & T'_{n+1}. \end{tikzcd}$$

A spectrum E is a prespectrum whose adjoint structure maps $\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1}$ are homeomorphisms. A map $f : E \longrightarrow E'$ of spectra is just a map of underlying prespectra. The forgetful functor $\ell : \mathscr{S} \longrightarrow \mathscr{P}$ from the category of spectra to the category of prespectra has a left adjoint $L : \mathscr{P} \longrightarrow \mathscr{S}$. For example, $MO = \{MO(n)\}$ is LTO. Similarly, for a based space X, define $QX = \cup \Omega^q \Sigma^q X$, where the inclusions are given by suspension of maps. Then the suspension spectrum of X is

$$\Sigma^\infty X = L\{\Sigma^n X\} = \{Q\Sigma^n X\}.$$

We define $S = \Sigma^\infty S^0$ to be the sphere, or zero sphere, spectrum.

The zeroth space E_0 of a spectrum is denoted $\Omega^\infty E$; such spaces are called infinite loop spaces. These are not the kind of spaces one encounters in nature.

They are monstrously large. Even loop spaces are quite large: there are lots of maps from a circle into a given space. This is one reason that this definition of a spectrum, which I first gave in 1968 [19], was slow to catch on. Most experts now accept that this is really the right definition. For one thing, there is a very simple relationship between maps of spaces and maps of spectra. The functors Σ^∞ and Ω^∞ are left and right adjoint. More generally, there is a shift desuspension functor Σ^∞_n that is left adjoint to the nth space functor; that is,

$$\mathscr{S}(\Sigma^\infty_n X, E) \cong \mathscr{T}(X, E_n),$$

where $\mathscr{T}$ is the category of based spaces. We define sphere spectra for integers n by

$$S^n = \Sigma^\infty S^n \text{ if } n \geq 0 \quad \text{ and } \quad S^{-n} = \Sigma^\infty_n S^0 \text{ if } n > 0.$$

We define the smash product of a prespectrum T and a based space X by

$$(T \wedge X)_n = T_n \wedge X,$$

with the obvious structure maps. We then define $E \wedge X = L(\ell E \wedge X)$. Taking $X = I_+$, this gives us a notion of homotopy between maps of spectra. We define homotopy groups of spectra by

$$\pi_n(E) = [S^n, E],\ n \in \mathbb{Z}.$$

By our adjunctions, they are computable in terms of homotopy groups of spaces. We say that a map of spectra is a weak equivalence if it induces an isomorphism of homotopy groups. We have a homotopy category $h\mathscr{S}$, in which homotopic maps are identified. The desired stable homotopy category $\bar{h}\mathscr{S}$ is obtained from $h\mathscr{S}$ by adjoining formal inverses to the weak equivalences. Every spectrum is weakly equivalent to a CW spectrum, and $\bar{h}\mathscr{S}$ is equivalent to the homotopy category of CW spectra. This category is stable in the sense that the suspension functor $\Sigma : \bar{h}\mathscr{S} \longrightarrow \bar{h}\mathscr{S}$ is an equivalence of categories.

The essential point is to define a smash product $E \wedge E'$ of spectra. Looking on the prespectrum level, there is no obvious choice for $(T \wedge T')_n$, and naive attempts to go from pairings $T_m \wedge T_n \longrightarrow T_{m+n}$ of the sort we had in our examples to some sort of honest map of prespectra run into problems of permutations of suspension coordinates. Up to homotopy, Boardman solved this in his 1964 thesis, and Adams gave an equivalent solution. Starting with different definitions than I have given, they obtained a smash product that is homotopy associative, commutative, and unital with unit S. That is, one has these relations in $\bar{h}\mathscr{S}$, although one does not have them on the point-set level. This allows the definition of a ring spectrum E and E-module M in terms of unit and product maps $S \longrightarrow E$, $E \wedge E \longrightarrow E$, and $E \wedge M \longrightarrow M$. The homotopy groups of such a ring spectrum E are graded rings.

Before going further, let us record the homotopy groups of some of the spectra we have on hand. For a based space X,

$$\pi_n(\Sigma^\infty X) = \operatorname{colim} \pi_{n+q}(\Sigma^q X)$$

is the nth stable homotopy group of X. For an Abelian group π we can choose our Eilenberg-MacLane spaces so that $\{K(\pi, n)\}$ is a spectrum, denoted $H\pi$, and then $\pi_0(H\pi) = \pi$ and $\pi_i(H\pi) = 0$ for $i \neq 0$. For a ring R, HR is a ring spectrum and $\pi_0(HR) = R$ as rings. Rather than record the homotopy groups of MO and KO, let us consider their complex analogs MU and KU. They are constructed like MO and KO, but using complex vector bundles instead of real ones. Manifold theoretically, $\pi_*(MU)$ is isomorphic to the cobordism groups of weakly almost complex manifolds, namely those with complex structures on their stable normal bundles. These are ring spectra, and we have

$$\pi_*(MU) = \mathbb{Z}[x_i \,|\, i \geq 1],$$

where $deg\, x_i = 2i$, and

$$\pi_*(KU) = \mathbb{Z}[u, u^{-1}],$$

where $deg\, u = 2$.

For each prime p and each $i \geq 1$, there is a ring spectrum $K(i)$ such that

$$\pi_*(K(i)) = \mathbb{F}_p[v_i, v_i^{-1}],$$

where $deg\, v_i = 2p^i - 2$. These are the Morava K-theory spectra. Together with the HF for fields F, they are the only ring spectra whose homotopy rings are graded fields. They have played a vital role in stable homotopy theory since the 1970's. It has been an embarrassment to the subject that there has been no good construction of them. Originally, they were obtained by the Baas-Sullivan theory of manifolds with singularities, which relied on difficult geometric topology to kill unwanted generators of $\pi_*(MU)$ by making them cobordant to zero in a weaker sense than the usual one. Moreover, this procedure makes it very hard to show that the $K(i)$ are actually ring spectra. It has long been understood that if one could obtain a sufficiently precise theory of MU-module spectra, then one could mimic commutative algebra to construct the $K(i)$ as if one were doing pure algebra, forming $K(i)$ as the MU-ring spectrum, or MU-algebra up to homotopy, specified by

$$(MU/I)[x_{p^i-1}^{-1}]$$

where $I \subset \pi_*(MU)$ is the ideal generated by p and the x_j for $j \neq p^i - 1$. One problem is that, with the usual homotopical notion of ring and module spectra, the cofiber $N \cup_f CM$ of a map $f : M \longrightarrow N$ of MU-modules need not be an MU-module.

This is one of very many motivations for trying to obtain a good point-set topological notion of a commutative ring spectrum E and its modules M. One wants E to be associative, commutative, and unital on the point-set level. However, that is impossible even to formulate if the underlying smash product is itself only associative, commutative, and unital up to homotopy. Until quite recently, I would have said that there could be no category of spectra with an associative, commutative, and unital point-set level smash product from which a category equivalent to the stable homotopy category could be constructed. I am happy to say that I was wrong.

The construction is due to Elmendorf, Kriz, Mandell and myself [10, 11] and I find it quite beautiful. If I were addressing an audience of algebraic topologists, this would be the starting point of my talk, and I would give some details. Instead, I will just state that there is a notion of an S-module, which is a spectrum with additional structure. The category of S-modules is symmetric monoidal under a smash product $\wedge_S$ with unit S, and there is even a function S-module functor F_S such that

$$\mathscr{M}_S(M \wedge_S N, P) \cong \mathscr{M}_S(M, F_S(N, P)),$$

where $\mathscr{M}_S$ is the category of S-modules. Thus the properties of $\mathscr{M}_S$ are exactly like those of the category of modules over a commutative ring k, with its tensor product and Hom functors. We define the derived category $\mathscr{D}_S$ by adjoining formal inverses to the weak equivalences, and we prove that $\mathscr{D}_S$ is equivalent to the stable homotopy category $\bar{h}\mathscr{S}$ and that the equivalence preserves smash products and function spectra.

Given the category of S-modules, we define an S-algebra R by requiring a unit $S \longrightarrow R$ and product $R \wedge_S R \longrightarrow R$ such that the evident associativity and unit diagrams commute. We say that R is a commutative S-algebra if the evident commutativity diagram also commutes. We define a right R-module similarly, by requiring a map $R \wedge_S N \longrightarrow N$ such that the evident associativity diagram commutes. For a right R-module M and left R-module N, we define $M \wedge_R N$ by a coequalizer diagram

$$M \wedge_S R \wedge_S N \underset{\mathrm{Id} \wedge_S \nu}{\overset{\mu \wedge_S \mathrm{Id}}{\rightrightarrows}} M \wedge_S N \longrightarrow M \wedge_R N,$$

where μ and ν are the given actions of R on M and N. If R is commutative, then the smash product of R-modules is an R-module, the category $\mathscr{M}_R$ of R-modules is symmetric monoidal with unit R, and there is a function R-module functor $F_R(M, N)$ with the usual adjunction. We can define R-algebras exactly as we defined S-algebras, via unit and product maps $R \longrightarrow A$ and $A \wedge_R A \longrightarrow A$. This is beginning to look like genuine algebra, isn't it? Lots of formal properties of modules, rings, and algebras go over directly to the new subject of stable topological algebra. For example, the smash product $A \wedge_R A'$ of commutative R-algebras A and A' is their coproduct in the category of commutative R-algebras.

Thinking homotopically, $\mathscr{M}_R$ has a derived category $\mathscr{D}_R$ that is obtained by inverting the maps of R-modules that are weak equivalences of underlying spectra. For an S-algebra R, the category $\mathscr{D}_R$ is not just a tool for the the study of classical algebraic topology, but an interesting new subject of study in its own right.

What about examples? There are notions of A_∞ and E_∞ ring spectra that Frank Quinn, Nigel Ray, and I defined in 1972 [20]. These earlier definitions turn out to be essentially equivalent to the new definitions of S-algebras and commutative S-algebras. Therefore earlier work gives a host of examples. In particular, multiplicative infinite loop space theory provides a tool for constructing A_∞ and E_∞ ring spectra, and therefore S-algebras, from space level data.

This theory constructs HR as an S-algebra for any discrete ring R, and HR is commutative if R is. With their usual constructions, the Thom spectra MO and MU are E_∞ ring spectra and thus commutative S-algebras by direct inspection of definitions. The new theory has led to a beautiful conceptual proof that KO and KU are commutative S-algebras. The question of proving that they are E_∞ ring spectra had been an unsolved open problem for the last twenty years. Infinite loop space theory showed much earlier that the connective versions kO and kU are E_∞ ring spectra. Here

$$\pi_*(ku) = \mathbb{Z}[u].$$

Similarly, the algebraic K-theory spectrum KR of a discrete ring R is an E_∞ ring spectrum. By inspection, the suspension spectrum $\Sigma^\infty G_+$ of a topological monoid G is an S-algebra.

We have used the word "derived" in analogy with algebra. For a discrete ring R, the category $\mathscr{M}_R$ of chain complexes over R has an associated derived category $\mathscr{D}_R$ that is obtained from the homotopy category $h\mathscr{M}_R$ by adjoining formal inverses to the quasi-isomorphisms (= homology isomorphisms), which are analogous to weak equivalences in topology. This process is made rigorous by replacing chain complexes by appropriate projective resolutions. I have suppressed the fact that, in topology, the construction of our derived categories is made rigorous by replacing R-modules by weakly equivalent cell R-modules. In fact, one can carry out the algebraic construction most efficiently by mimicking the topological theory of cell R-modules. In any case, working in the derived category of HR-modules for a discrete ring R, we find that

$$\mathrm{Tor}_*^R(M,N) \cong \pi_*(HM \wedge_{HR} HN)$$

and

$$\mathrm{Ext}_R^*(M,N) \cong \pi_{-*}(F_{HR}(HM,HN)).$$

Moreover,

$$\mathscr{D}_R \text{ is equivalent to } \mathscr{D}_{HR}.$$

Thus our new topological derived categories subsume a lot of classical algebra.

As the first topologically interesting example, the algebraic structure of the categories $\mathscr{D}_{kU}$, $\mathscr{D}_{KU}$, $\mathscr{D}_{kO}$, and $\mathscr{D}_{KO}$ has been analyzed by Jerome Wolbert [25, 26].

For another topological analog of a classical algebraic construction, consider a commutative S-algebra R and an R-algebra A. We can define $A^e = A \wedge_R A^{op}$. For an (A,A)-bimodule M, we can then define topological Hochschild homology by

$$THH_*^R(A;M) = \pi_*(M \wedge_{A^e} A).$$

Comparing with algebra, if R is a discrete commutative ring, A is an R-algebra such that A is a flat R-module, and M is an (A,A)-bimodule, then the (relative) algebraic Hochschild homology is realized topologically as

$$HH_*^R(A;M) \cong THH_*^{HR}(HA;HM).$$

However, the topology offers something new: we can take the ground ring to be S and define the topological Hochschild homology groups of R, $THH_*(R)$, by

$$THH_*(R) = THH_*^S(HR).$$

These algebraic invariants of discrete rings, due originally to Bökstedt [6], are the starting point for recent applications of stable algebraic topology to extensive computations of the algebraic K-groups of number rings. See Madsen [16, 17] for surveys.

Thinking of $\mathscr{D}_{MU}$ as a category in which to do homotopy theory, we easily construct $K(i)$ as an MU-ring spectrum in the homotopical sense, as I indicated earlier, and similarly for many other interesting spectra associated to MU. Moreover, the new constructions come with more structure and their possible ring structures are far more easily studied than was possible with earlier constructions. Here the work of [11] has recently been carried significantly further by Neil Strickland [23].

There are also applications in algebraic K-theory and algebraic geometry. Categories of modules in algebra have associated algebraic K-theories, and so do our new categories of modules over S-algebras. For a discrete ring R, the new algebraic K-theory of HR agrees with the old algebraic K-theory of R constructed by Quillen. Similarly, for a based space X, $\Sigma^\infty(\Omega X)_+$ is an S-algebra, and the algebraic K-theory associated to its modules is Waldhausen's algebraic K-theory of X. This is work of Mike Mandell.

The whole theory started from the development of a parallel algebraic theory that studies differential graded algebras up to homotopy, that is, k-chain complexes with actions by operads of chain complexes for some commutative ground ring k. These are called A_∞ and E_∞ k-algebras. When $k = \mathbb{Q}$, A_∞ and E_∞ algebras are equivalent to DGA's and commutative DGA's, but this is false for $\mathbb{Z}$ or for the field $\mathbb{F}_p$. Igor Kriz and I [15] developed the algebraic theory in order to carry out a program suggested by Deligne for the construction and study of integral mixed Tate motives in algebraic geometry, but it has other applications.

Recently Mike Mandell [18] has proven that the categories of commutative Hk-algebras and E_∞ k-algebras are equivalent. While intuitively plausible, this is quite difficult to prove. This allows a really remarkable application of stable topological algebra: it gives rise to an algebraization of unstable p-adic homotopy theory analogous to the Quillen-Sullivan algebraization of rational homotopy theory. There is a functor that carries a space X to the commutative Hk-algebra $F(X_+, Hk)$, Here

$$H^*(X;k) \cong \pi_{-*}(F(X_+, Hk)).$$

An unpublished theorem of Dwyer and Hopkins, for which Mandell has given a proof, states that, when k is the algebraic closure $\bar{\mathbb{F}}_p$ of the field $\mathbb{F}_p$, this functor gives an equivalence from the homotopy category of p-adic spaces to a full subcategory of the homotopy category of commutative $H\bar{\mathbb{F}}_p$-algebras. Mandell identifies the composite of these two functors with the singular cochain functor $C^*(X;\bar{\mathbb{F}}_p)$ and thus shows that this functor algebraicizes p-adic homotopy theory.

Another application of the new theory involves the generalization of the entire theory to stable equivariant topological algebra. In fact, we can develop the entire theory with a compact Lie group G acting on everything in sight. The starting point is to index G-spectra not as sequences of G-spaces E_n but as systems of G-spaces E_V indexed on representations V of G. We require E_V to be G-homeomorphic to $\Omega^{W-V}E_W$ when $V \subset W$. This builds representation theory into the definition of G-spectra. Stability allows for spheres S^V associated to representations, and the associated cohomology theories are graded, not on $\mathbb{Z}$, but on the real representation ring $RO(G)$.

A theorem of Atiyah and Segal [4] asserts that, for a compact G-space X,

$$KU_G(X)\hat{_I} \cong KU_G(EG \times X),$$

where I is the augmentation ideal of the complex representation ring $R(G)$. Here

$$KU_G(EG \times X) \cong KU(EG \times_G X),$$

In particular, this calculates the K-theory of the classifying space BG:

$$R(G)\hat{_I} \cong K(BG).$$

When G is finite, John Greenlees and I observed that there is a spectrum level generalization of the Atiyah-Segal theorem that is given in terms of completions of R_G-modules at ideals of R_G^*, where R_G is a commutative S_G-algebra [12, 13]. In symbols,

$$(M_G)\hat{_I} \simeq F(EG_+, M_G)$$

for any KU_G-module M_G. We then proved the same theorem for equivariant cobordism [14], replacing KU_G by MU_G and I by the augmentation ideal of MU_G^*. For example, our new construction of $K(i)$ works just as well to give an equivariant Morava K-theory G-spectrum $K_G(i)$. More elegantly there is a construction

$MU_G \wedge_{MU} M$ of MU_G-modules from MU-modules [22]. As with K-theory, we have the relation

$$(F(EG_+, M_G))^G \simeq F(BG_+, M).$$

Thus our completion theorem shows that the equivariant homotopy groups of $(M_G)\hat{_I}$ compute the ordinary nonequivariant cohomology groups $M^*(BG)$.

Each of the last five applications could be the starting point of another talk, but I've already said more than enough in this one.

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