CATERADS AND ALGEBRAS OVER CATERADS

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Abstract. I define "symmetric monoids", and "caterads" in a closed symmetric monoidal category, and I define what it means for a symmetric monoid to be an algebra over a caterad. These notions codify formal structure that appears in motivic cohomology and should be of more general interest.

Contents

Introduction

The purpose of this note is to advertise some elementary categorical definitions that describe frequently encountered types of mathematical structures. In particular, I define categorical analogues of operads that I call "caterads" and others call "props" or "PROP's", together with algebras over them. In fact, caterads specialize to both the PROP's and the PACT's that were defined originally by Adams and Mac Lane [12]. However, even for these specializations, our algebras over caterads are more general structures than they had in mind and are quite different philosophically. It is these generalized algebras that I want to advertise here.

In the work of Adams and Mac Lane, and in later work, what we call caterads are thought of as encoding operations on an object, say a space or a chain complex, that have many inputs and many outputs. The applications that we envisage dictate a different perspective and a reanalysis of the structure that is encoded by caterads. The new philosophy arose in my efforts [20] to understand the formal structure that is enjoyed by the motivic cochain complexes that define motivic cohomology in the work of Voevodsky, Suslin, and Friedlander [22, 23]. The disparity between the appropriate general context and the specifics relevant to that theory led me to separate out the categorical framework in this short, content free, note.

We define symmetric sequences and symmetric monoids in §1, caterads in §2, and algebras over caterads in §3. We compare caterads to operads in §4, where we

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explain some of the relevant structure that is implied by an action of a caterad. We describe the hybrid notion of a catoperad in §5. It is relevant to hypercohomology.

We observe in §6 that symmetric monoids are equivalent to commutative monoids in the category of symmetric sequences, endowed with a suitable symmetric monoidal structure. However, the word "commutative" must be taken with a grain of salt. For example, as we shall see, tensor algebras give rise to commutative monoids in the category of symmetric sequences of vector spaces. Nevertheless, this idea of commutativity is important in recent work in stable homotopy theory [9, 15].

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1. Symmetric sequences and symmetric monoids

Recall that a permutative category $\mathscr{P}$ is a category with a strictly associative and unital product and a natural commutativity isomorphism τ satisfying the usual coherence axioms [13, XI§1]. In other words, $\mathscr{P}$ is a strictly associative and unital symmetric monoidal category. The standard skeleton of the category of finite sets and isomorphisms is a permutative category Σ.

Definition 1.1. Let Σ be the category whose objects are the sets $\mathbf{q} = \{1, \dots, q\}$ and whose morphisms are the symmetric groups. Thus there are no morphisms $\mathbf{q} \longrightarrow \mathbf{r}$ unless $q = r$, and $\Sigma(\mathbf{q}, \mathbf{q})$ is the symmetric group Σ_q (or $\mathrm{id}_{\mathbf{0}}$ if $q = 0$). This category is permutative under concatenation of sets, $(\mathbf{q}, \mathbf{r}) \mapsto \mathbf{q} + \mathbf{r}$, and block sum of permutations $\Sigma_q \times \Sigma_r \longrightarrow \Sigma_{q+r}$, with $\mathbf{0}$ as unit and with commutativity isomorphism τ given by the block transpositions $\tau_{q,r} \in \Sigma_{q+r}$.

Definition 1.2. A *(left) symmetric sequence* in a category $\mathscr{C}$ is a covariant functor $F \colon \Sigma \longrightarrow \mathscr{C}$, and a map of symmetric sequences is a natural transformation. We write $F(q)$ for the value of F on $\mathbf{q}$ and $F(\sigma)$ for the value of F on a permutation $\sigma \in \Sigma_q$. Thus F is just a sequence of objects $F(q)$ with left actions by Σ_q, the covariance dictating left rather than right actions. A *right symmetric sequence* in $\mathscr{C}$ is a contravariant functor $\Sigma \longrightarrow \mathscr{C}$.

Such objects appear throughout mathematics. They underly operads, in which context they are often called "collections", and they play a serious role in stable homotopy theory, where symmetric sequences of simplicial sets (or spaces) are the starting point for Jeff Smith's notion of a symmetric spectrum [9]; see also [15].

Now assume that the category $\mathscr{C}$ is symmetric monoidal with product $\otimes$, unit object κ, and commutativity isomorphism τ. As usual, we think of the unit and associativity isomorphisms of $\mathscr{C}$ as less essential, treating them by abuse as if they were identity maps. When $\mathscr{C}$ is permutative, they are identity maps.

Definition 1.3. A *symmetric monoid* in $\mathscr{C}$ is a symmetric sequence F that is a lax symmetric monoidal functor. This means that there is a unit map $\lambda \colon \kappa \longrightarrow F(0)$ and $(\Sigma_q \times \Sigma_r)$-equivariant product maps

$$\phi \colon F(q) \otimes F(r) \longrightarrow F(q + r)$$

such that the evident unit, associativity, and commutativity diagrams commute (strictly, not just up to isomorphism). We display the last of these.

$$\begin{array}{ccc} F(q)\otimes F(r) & \xrightarrow{\phi} & F(q+r) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle F(\tau_{q,r})} \\ F(r)\otimes F(q) & \xrightarrow[\phi]{} & F(q+r) \end{array}$$

We say that F is *reduced* if $F(0)=\kappa$ and λ is the identity map. A map $\alpha\colon F\longrightarrow G$ of symmetric monoids is a symmetric monoidal natural transformation; that is, α must commute with the unit and product maps for F and G, in the strict sense that $\alpha\circ\lambda_F=\lambda_G$ and $\alpha\circ\phi_F=\phi_G\circ(\alpha\otimes\alpha)$.

Tensor algebras provide perhaps the most familiar examples.

Example 1.4. Let $\mathscr{C}$ be the category of vector spaces over a field k. Regard the tensor algebra $T(X)$ on a vector space X as graded, letting $T(X)(q)$ be the q-fold tensor power X^q, with Σ_q acting by permutations of factors. The unit of $T(X)$ is given by $k=T(X)(0)$. The product $T(X)(q)\otimes T(X)(r)\longrightarrow T(X)(q+r)$ gives $T(X)$ a structure of reduced symmetric monoid in $\mathscr{C}$.

This example generalizes to construct free symmetric monoids.

Definition 1.5. For an object X in the symmetric monoidal category $\mathscr{C}$, define the *free symmetric monoid* $T(X)$ by letting $T(X)(q)$ be the q-fold $\otimes$-power X^q, with Σ_q acting by permutations. By convention, the 0th power of X is κ and $T(X)$ is reduced. The product $T(X)(q)\otimes T(X)(r)\longrightarrow T(X)(q+r)$ is the evident juxtaposition isomorphism.

Lemma 1.6. *For a symmetric monoid F and a map $f\colon X\longrightarrow F(1)$ in $\mathscr{C}$, there is a unique map $\tilde{f}\colon T(X)\longrightarrow F$ of symmetric monoids such that $\tilde{f}(1)=f$.*

2. Caterads

For purposes of motivation, assume that our given symmetric monoidal category $\mathscr{C}$ is closed, with internal hom functor Hom. We then have the adjunction

$$\mathrm{Hom}(X\otimes Y,Z)\cong\mathrm{Hom}(X,\mathrm{Hom}(Y,Z)) \tag{2.1}$$

with evaluation map

$$\varepsilon\colon\ \mathrm{Hom}(X,Y)\otimes X\longrightarrow Y. \tag{2.2}$$

We also have the composition pairing

$$\mu\colon\ \mathrm{Hom}(Y,Z)\otimes\mathrm{Hom}(X,Y)\longrightarrow\mathrm{Hom}(X,Z) \tag{2.3}$$

and the $\otimes$-product pairing

$$\phi\colon\ \mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(X',Y')\longrightarrow\mathrm{Hom}(X\otimes X',Y\otimes Y'). \tag{2.4}$$

All of this structure is given by maps in $\mathscr{C}$. Now consider the situation in which all objects that appear in any of the last three displays are $\otimes$-powers of a given object X of $\mathscr{C}$. The structure on the set of objects $\mathrm{Hom}(X^q,X^r)$ of $\mathscr{C}$ that is given by permutations and by (2.3) and (2.4) provides the model for the definition of a caterad. Then $T(X)$ and the action maps of (2.2) provide the model for the definition of an algebra over a caterad. In these examples, all of the structure comes from the symmetric monoidal structure and the adjunction (2.1). However, similar structure often arises in situations where the caterad and the symmetric monoid on which it acts are not so intimately intertwined. The definitions in this section and the next codify the structure seen in such situations.

The definitions are most clearly and succinctly expressed in the language of enriched category theory. However, I will unpack exactly what they mean for the reader who may not be familiar with that language, providing something of an introduction to it. Standard references are [4, 7, 10].

Definition 2.5. A *caterad in* $\mathscr{C}$ is an enriched permutative category $\mathscr{A}$ over $\mathscr{C}$ together with a permutative functor ι from Σ to the underlying category $\mathscr{A}_0$ that is a bijection on objects. A map $\psi\colon \mathscr{A} \longrightarrow \mathscr{B}$ of caterads is an enriched permutative functor such that $\psi \circ \iota_{\mathscr{A}} = \iota_{\mathscr{B}}$.

We proceed to unpack this definition. Since ι is a bijection on objects, $\mathscr{A}$ has object set $\{\mathbf{q} \,|\, q \geq 0\}$. To say that $\mathscr{A}$ is a *category enriched over* $\mathscr{C}$ is to say that there are morphism objects $\mathscr{A}(q,r)$ in $\mathscr{C}$ and identity and composition morphisms

$$\mathrm{id}_q\colon \kappa \longrightarrow \mathscr{A}(q,q)$$

$$\mu\colon \mathscr{A}(q,r) \otimes \mathscr{A}(p,q) \longrightarrow \mathscr{A}(p,r)$$

in $\mathscr{C}$ that satisfy the evident unit and associativity laws. The hom sets of the *underlying category* $\mathscr{A}_0$ are specified by

$$\mathscr{A}_0(q,r) = \mathscr{C}(\kappa, \mathscr{A}(q,r)).$$

The elements of these sets can be manipulated just like morphisms in an ordinary category. For example, for maps $f\colon \kappa \longrightarrow \mathscr{A}(p,q)$ and $g\colon \kappa \longrightarrow \mathscr{A}(r,s)$ in $\mathscr{C}$, the induced map

$$\mathscr{A}(f,g)\colon \mathscr{A}(q,r) \longrightarrow \mathscr{A}(p,s)$$

in $\mathscr{C}$ is defined as the composite displayed in the commutative diagram

$$\begin{array}{ccc} \mathscr{A}(q,r) & \overset{(unit\ iso)}{=\!=\!=\!=} & \kappa \otimes \mathscr{A}(q,r) \otimes \kappa \\ {\scriptstyle \mathscr{A}(f,g)}\downarrow & & \downarrow{\scriptstyle g \otimes \mathrm{id} \otimes f} \\ \mathscr{A}(p,s) & \overset{\mu}{\longleftarrow} & \mathscr{A}(r,s) \otimes \mathscr{A}(q,r) \otimes \mathscr{A}(p,q). \end{array}$$

For each q, we have a homomorphism ι mapping the group Σ_q into the monoid $\mathscr{A}_0(q,q)$, and $\iota(1) = \mathrm{id}_q$. We often omit ι from the notation, regarding permutations as elements of hom sets. For $\tau \in \Sigma_r$ and $\sigma \in \Sigma_q$, we have the induced map

$$\mathscr{A}(\tau,\sigma)\colon \mathscr{A}(q,r) \longrightarrow \mathscr{A}(q,r).$$

Since the functor ι is permutative, the product on the enriched permutative category $\mathscr{A}$ is given on objects by $(\mathbf{q},\mathbf{r}) \mapsto \mathbf{q}+\mathbf{r}$. The identity object is $\mathbf{0}$ and the commutativity isomorphism $\tau_{q,r}\colon \kappa \longrightarrow \mathscr{A}(q+r,q+r)$ is the map $\iota(\tau_{q,r})$ in $\mathscr{C}$. The product is given on morphisms by a strictly associative and unital system of maps

$$\phi\colon \mathscr{A}(q,r) \otimes \mathscr{A}(q',r') \longrightarrow \mathscr{A}(q+q',r+r')$$

in $\mathscr{C}$. When $q = r$ and $q' = r'$, ϕ restricts along ι to block sum on permutations. These maps specify an enriched bifunctor ϕ, which means that the following diagrams commute in $\mathscr{C}$.

$$\begin{array}{ccc}
\mathscr{A}(q,r)\otimes\mathscr{A}(p,q)\otimes\mathscr{A}(q',r')\otimes\mathscr{A}(p',q') & \xrightarrow{\mu\otimes\mu} & \mathscr{A}(p,r)\otimes\mathscr{A}(p',r') \\
{\scriptstyle (\phi\otimes\phi)(\mathrm{id}\otimes\tau\otimes\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathscr{A}(q+q',r+r')\otimes\mathscr{A}(p+p',q+q') & \xrightarrow{\mu} & \mathscr{A}(p+p',r+r')
\end{array}$$

The enriched naturality of the commutativity isomorphism means that the following diagrams commute in $\mathscr{C}$.

$$\begin{array}{ccc}
\mathscr{A}(q,r)\otimes\mathscr{A}(q',r') & \xrightarrow{\phi} & \mathscr{A}(q+q',r+r') \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \mathscr{A}(\tau_{q',q},\tau_{r,r'})} \\
\mathscr{A}(q',r')\otimes\mathscr{A}(q,r) & \xrightarrow{\phi} & \mathscr{A}(q'+q,r'+r)
\end{array}$$

A map $\psi\colon \mathscr{A}\longrightarrow\mathscr{B}$ consists of maps $\psi\colon \mathscr{A}(r,q)\longrightarrow\mathscr{B}(r,q)$ in $\mathscr{C}$ that commute with all structure in sight. This completes our unpacking of Definition 2.5.

The portmanteau[1] word "caterad" is meant to bring to mind both category and operad, just as the portmanteau word "operad" was meant to bring to mind both operation and monad. See Lewis Carroll [6, Ch.VI] for a philosophical discussion. However, the reader may prefer one of the existing acronyms.

Remark 2.6. Adams and Mac Lane defined PROP's (PROduct and Permutation categories) and PACT's (which have Permutations, Addition, Composition, and Tensor product) well before enriched category had been developed [12]. Caterads in the category of sets are almost exactly PROP's, and caterads in the category of chain complexes are almost exactly PACT's. Adams and Mac Lane insisted that ι be an inclusion and they omitted the unit condition for $\otimes$ on morphisms, but otherwise their definitions agree with these cases of ours. Similarly, caterads of spaces are essentially the topological PROP's used by Boardman and Vogt [1, 2].

The following promised example is paradigmatic.

Definition 2.7. For an object X of $\mathscr{C}$, define the *endomorphism caterad* $\mathscr{E}(X)$ in $\mathscr{C}$ by letting $\mathscr{E}(X)(q,r)=\mathrm{Hom}(X^q,X^r)$, where X^q denotes the q-fold $\otimes$-power, and letting $\iota(\sigma)$ be the permutation σ of X^q obtained from the commutativity isomorphism in $\mathscr{C}$. The required composition μ and pairing ϕ are obtained by specialization of (2.3) and (2.4).

Warning 2.8. Clearly, we can rewrite

$$\mathscr{E}(X)(q,r)=\mathrm{Hom}(T(X)(q),T(X)(r)).$$

It is tempting to replace the symmetric monoid $T(X)$ by any symmetric monoid F and to try to define a similar endomorphism caterad $\mathscr{E}(F)$. This fails due to the contravariance of Hom in the first variable. That is, we cannot compose the ϕ of (2.4) with application of Hom to the ϕ that define the product on F, since the latter only makes sense when the maps $\phi\colon F(q)\otimes F(r)\longrightarrow F(q+r)$ are isomorphisms.

Because we have not insisted that $\iota\colon \Sigma\longrightarrow\mathscr{A}$ be an inclusion in our definition of a caterad, we have the following elementary example of a caterad in any $\mathscr{C}$.

[1] "You see it's like a portmanteau – there are two meanings packed up into one word" [6].

Definition 2.9. The commutativity caterad in $\mathscr{C}$, denoted either $\mathscr{N}$ or $\mathscr{C}om$, has $\mathscr{C}om(p,q) = \kappa$, with all structure given by identity maps.

We also have endomorphism caterads of functors, which are defined analogously to the endomorphism operads of functors specified in [19, 2.3].

Construction 2.10. Let $\mathscr{D}$ be a small category and $\Lambda\colon \mathscr{D} \longrightarrow \mathscr{C}$ be a covariant functor. Define the *endomorphism caterad* $\mathrm{End}(\Lambda)$ in $\mathscr{C}$ by setting

$$\mathrm{End}(\Lambda)(q,r) = \mathrm{Hom}_{\mathscr{D}}(\Lambda^r, \Lambda^q).$$

The permutations $\sigma\colon \kappa \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\Lambda^q, \Lambda^q)$ are adjoint to the permutations of the power functors Λ^q, the composition μ is obtained from (2.3), after transposition, and the product ϕ is obtained from the product (2.4).

As in [19, 3.1], we have the following example. It is a modern version of a PACT that Adams and Mac Lane had in mind in their unpublished work.

Definition 2.11. The *Eilenberg–Zilber caterad* $\mathscr{Z}$ in the category $\mathrm{Ch}\mathscr{M}_R$ of chain complexes over a commutative ring R is the endomorphism caterad of the normalized chain complex functor $\Lambda = C_*(\Delta^\bullet, R)$ on Δ.

As observed after [19, 3.1] in the context of operads, we have an augmentation $\varepsilon\colon \mathscr{Z} \longrightarrow \mathscr{C}om$. The theory of acyclic models [8] implies the analogue of [19, 3.2].

Proposition 2.12. *Each* $\varepsilon\colon \mathscr{Z}(p,q) \longrightarrow \mathscr{C}om(p,q)$ *is a quasi-isomorphism.*

3. Algebras over caterads

Definition 3.1. Let $\mathscr{A}$ be a caterad in $\mathscr{C}$. An *algebra* F over $\mathscr{A}$ is a symmetric monoid F in $\mathscr{C}$ and an enriched product-preserving functor $F\colon \mathscr{A} \longrightarrow \mathscr{C}$ that restricts along $\iota\colon \Sigma \longrightarrow \mathscr{A}_0$ to the symmetric sequence $F\colon \Sigma \longrightarrow \mathscr{C}$.

We need to unpack this definition too. The enriched functor F is given by evaluation maps

$$\varepsilon\colon \mathscr{A}(q,r) \otimes F(q) \longrightarrow F(r)$$

in $\mathscr{C}$ such that the following diagrams commute in $\mathscr{C}$.

$$\begin{array}{ccc} \kappa \otimes F(q) & & \\ {\scriptstyle \mathrm{id}_q \otimes \mathrm{id}}\downarrow & \searrow\!\!= & \\ \mathscr{A}(q,q) \otimes F(q) & \xrightarrow[\varepsilon]{} & F(q) \end{array} \qquad \begin{CD} \mathscr{A}(q,r) \otimes \mathscr{A}(p,q) \otimes F(p) @>{\mu\otimes\mathrm{id}}>> \mathscr{A}(p,r) \otimes F(p) \\ @V{\mathrm{id}\otimes\varepsilon}VV @VV{\varepsilon}V \\ \mathscr{A}(q,r) \otimes F(q) @>>{\varepsilon}> F(r) \end{CD}$$

To say that F is product-preserving is to say that the following diagrams commute.

$$\begin{CD} \mathscr{A}(q,r) \otimes \mathscr{A}(q',r') \otimes F(q) \otimes F(q') @>{\phi\otimes\phi}>> \mathscr{A}(q+q', r+r') \otimes F(q+q') \\ @V{(\varepsilon\otimes\varepsilon)(\mathrm{id}\otimes\tau\otimes\mathrm{id})}VV @VV{\varepsilon}V \\ F(r) \otimes F(r') @>>{\phi}> F(r+r') \end{CD}$$

To say that F restricts along ι to the given symmetric sequence F is to say that the following diagram commutes for $\sigma \in \Sigma_q$; it is the unit diagram when $\sigma = \mathrm{id}_q$.

$$\begin{array}{ccc} \kappa \otimes F(q) & = & F(q) \\ {\scriptstyle \iota(\sigma)\otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle F(\sigma)} \\ \mathscr{A}(q,q) \otimes F(q) & \xrightarrow[\varepsilon]{} & F(q) \end{array}$$

As is easily checked, this implies the following general equivariance lemma.

Lemma 3.2. *The following diagram commutes, where $\sigma \in \Sigma_q$ and $\tau \in \Sigma_r$.*

$$\begin{array}{ccc} \mathscr{A}(q,r) \otimes F(q) & \xrightarrow{\mathscr{A}(\sigma,\tau)\otimes F(\sigma^{-1})} & \mathscr{A}(q,r) \otimes F(q) \\ {\scriptstyle \varepsilon} \downarrow & & \downarrow {\scriptstyle \varepsilon} \\ F(r) & \xrightarrow[F(\tau)]{} & F(r) \end{array}$$

Remark 3.3. The $\mathscr{A}(0,r)$ form a symmetric monoid with unit id_0 and product ϕ, and the composites

$$\varepsilon \circ (\mathrm{id} \otimes \lambda)\colon \mathscr{A}(0,r) \cong \mathscr{A}(0,r) \otimes \kappa \longrightarrow \mathscr{A}(0,r) \otimes F(0) \longrightarrow F(r)$$

specify a map of symmetric monoids.

We have the promised paradigmatic example and a related observation.

Lemma 3.4. *The free symmetric monoid $T(X)$ is an algebra over $\mathscr{E}(X)$.*

Proof. The identity map of the caterad $\mathscr{E}(X)$ transforms under the adjunction (2.1) to the required action. □

Lemma 3.5. *In adjoint form, an action of a caterad $\mathscr{A}$ on $T(X)$ is a morphism of caterads $\mathscr{A} \longrightarrow \mathscr{E}(X)$. We then say that $\mathscr{A}$ acts on X, rather than on $T(X)$.*

We can now explain the philosophy behind our definitions. In most, if not all, existing work that employs operads, PROP's, or PACT's, the purpose has been to define and study algebraic structure on an object X of $\mathscr{C}$, and the only underlying symmetric monoids considered were of the form $T(X)$. The idea has been to think of $\mathscr{A}(q,r)$ as codifying operations on X with q inputs and r outputs. That is not our point of view here or in the sequel [20].

Rather, we are interested in symmetric monoids F, regarded as graded (or weighted) objects $\{F(q)\}$, that are *not* of the restricted form $T(X)$. We regard the $\mathscr{A}(q,r)$ as specifying internal structure that maps the part of F in grading q to the part of F in grading r. Note that $\mathscr{A}(q,r)$ should be assigned grading $r-q$, so that the action ε is homogeneous. This point of view feels very different from the operational one. Warning 2.8 highlights a reason that the greater generality of our definitions is genuinely different mathematically.

Just as for operads and their algebras, a major reason for wanting a definition of caterads and their algebras that applies in any symmetric monoidal category $\mathscr{C}$ is to have functoriality in $\mathscr{C}$. As in the theory of operads, that allows one to transfer structure from one context to another, typically from topology or geometry to homological algebra.

Proposition 3.6. *Let $\Phi\colon \mathscr{C} \longrightarrow \mathscr{C}'$ be a lax symmetric monoidal functor. If $\mathscr{A}$ is a caterad in $\mathscr{C}$, then $\Phi\mathscr{A}$ is a caterad in $\mathscr{C}'$. If F is an $\mathscr{A}$-algebra, then ΦF is a $\Phi\mathscr{A}$-algebra.*

Proof. Since Φ is lax symmetric monoidal, we are given a map $\kappa_{\mathscr{C}'} \longrightarrow \Phi\kappa_{\mathscr{C}}$ in $\mathscr{C}'$ and a natural map $\Phi X \otimes_{\mathscr{C}'} \Phi Y \longrightarrow \Phi(X \wedge_{\mathscr{C}} Y)$ in $\mathscr{C}'$. These need not be isomorphisms, but they commute strictly with the unit, associativity, and commutativity isomorphisms of $\mathscr{C}$ and $\mathscr{C}'$. Since the definitions of caterads and their algebras are given solely in terms of the symmetric monoidal structure on $\mathscr{C}$, it is clear how to define induced structures after application of Φ. □

Remark 3.7. Although the examples we have given rely on Hom in $\mathscr{C}$, the definitions of caterads and their algebras depend only on the symmetric monoidal structure and do not require an internal hom functor.

4. Some comparisons between operads and caterads

The definition of an operad deliberately sacrifices generality by focusing on those algebraic structures that can be defined in terms of operations with only one output (see [18] for more discussion of this choice). The sacrifice in generality results in a gain of simplicity that has proven its value in a variety of contexts. A complete analysis of the relationship between operads and caterads is complicated by the wealth of internal structure that is present in caterads.

The gain in simplicity is especially apparent when one tries to construct algebras over caterads. This is generally a quite difficult undertaking. For example, as we saw in [19], it is very easy to construct actions of the Eilenberg-Zilber *operad*. However, it is very hard to construct actions of the Eilenberg-Zilber *caterad*. In fact, historically, as both Adams and Mac Lane told me, the main reason that their extensive collaboration on PACT's did not result in published work was their failure to construct the algebras over PACT's that they had hoped for.

For a caterad $\mathscr{A}$ we have the following composite structural maps γ, where $p = p_1 + \cdots + p_k$ and $q = q_1 + \cdots + q_k$.

(4.1)

$$\gamma\colon \mathscr{A}(q,r) \otimes \mathscr{A}(p_1,q_1) \otimes \cdots \otimes \mathscr{A}(p_k,q_k) \xrightarrow{\mathrm{id}\otimes\phi} \mathscr{A}(q,r) \otimes \mathscr{A}(p,q) \xrightarrow{\mu} \mathscr{A}(p,r)$$

When $k = 1$, these maps reduce to the composition μ, and when $q = r$ and we restrict along id_q, these maps with $k = 2$ reduce to ϕ. Therefore caterads could be defined in terms of these maps together with commutative equivariance, unit, and associativity diagrams relating them.

Similarly, if $\mathscr{A}$ acts on F, we have the following composite action maps θ, where $q = q_1 + \cdots + q_k$.

$$\theta\colon \mathscr{A}(q,r) \otimes F(q_1) \otimes \cdots \otimes F(q_k) \xrightarrow{\mathrm{id}\wedge\phi} \mathscr{A}(q,r) \otimes F(q) \xrightarrow{\varepsilon} F(r) \tag{4.2}$$

When $k = 1$, these maps reduce to the evaluation ε, and when $q = r$ and we restrict along id_q, these maps with $k = 2$ reduce to ϕ. Therefore actions can be defined in terms of these maps together with suitable commutative diagrams relating them.

This looks just like the definitions of an operad and of an action by an operad [16, 17]. When $q_1 = \cdots = q_k = 1 = r$, the maps γ of (4.1) are the structure maps of an operad $\mathscr{A}_1 = \{\mathscr{A}(k,1)\}$ and the maps θ of (4.2) specify an action of $\mathscr{A}_1$ on $F(1)$. In the case of PROP's, it is known, although not conveniently documented, that all operads arise from caterads in this way; see [2, 2.43] or [21, 4.1]. The proof adapts to prove that this holds for operads in $\mathscr{C}$ when the ambient category $\mathscr{C}$ is cocomplete. However, quite different caterads can give rise to the same operad $\mathscr{A}_1$.

Parenthetically, we note that the study of caterads simplifies considerably when the ambient symmetric monoidal category $\mathscr{C}$ is *cartesian* monoidal, as in the case of (topological) PROP's. In the literature, PROP's are usually studied in relationship with Lawvere's theories [11], and the latter only make sense in cartesian monoidal categories. Projections on coordinates play a major role in the study of PROP's.

There are many further operads and operad like structures hiding in a caterad. For example, taking $q_1 = \cdots = q_k = r$ and renaming it q, the maps γ of (4.1) are the structural maps of an operad $\mathscr{A}_q = \{\mathscr{A}(qk, q)\}$, where we use block permutations to give the required action of Σ_k on $\mathscr{A}(qk, q)$. If F is an $\mathscr{A}$-algebra, then the operad $\mathscr{A}_q$ acts on $F(q)$ via the maps $\theta\colon \mathscr{A}(qk, q) \otimes F(q)^k \longrightarrow F(q)$ of (4.2). When we take the point of view that $\mathscr{A}(q, r)$ parametrizes operations with q inputs and r outputs on an $\mathscr{A}$-algebra $T(X)$, these operad actions for $q > 1$ are not terribly interesting. However, when we take the point of view that $\mathscr{A}(q, r)$ parametrizes relations between grading q and grading r on a general $\mathscr{A}$-algebra F, these actions may be of real interest. They describe internal structure in grading q.

The maps $\varepsilon\colon \mathscr{A}(q, q) \otimes F(q) \longrightarrow F(q)$ give the "degree zero" part of an action by a caterad. They fit together to give a different operad like situation defined in terms of the structural maps

$$\gamma\colon \mathscr{A}(qj, qj) \otimes \mathscr{A}(qj_1, qj_1) \otimes \cdots \otimes \mathscr{A}(qj_k, qj_k) \longrightarrow \mathscr{A}(qj, qj) \tag{4.3}$$

and the action maps

$$\theta\colon \mathscr{A}(qj, qj) \otimes F(qj_1) \otimes \cdots \otimes F(qj_k) \longrightarrow F(qj), \tag{4.4}$$

where $j = j_1 + \cdots j_k$.

The following consequence of Lemma 3.2 is inserted for quotation in [20].

Lemma 4.5. *Let F be an $\mathscr{A}$-algebra. Let $\sigma \in \Sigma_\ell$ act on the object $\mathbf{q}\ell$ of Σ by permutation of blocks. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{A}(q\ell, r) \otimes F(q)^\ell & \xrightarrow{\mathscr{A}(\sigma,\mathrm{id})\otimes\sigma^{-1}} & \mathscr{A}(q\ell, r) \otimes F(q)^\ell \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & F(r) & \end{array}$$

That is, θ is Σ_ℓ-equivariant, where Σ_ℓ acts trivially on $F(r)$.

When ℓ is a prime, $\mathscr{C}$ is the category of mod ℓ chain complexes, and $\mathscr{A}(q\ell, r)$ is acyclic, this allows use of these action maps to define Steenrod type operations.

5. Catoperads and hypercohomology

Riddle: What do you get when you cross a caterad with an operad?

Answer: A "catoperad".

Alas, this feeble joke is meant quite literally, as the following example makes clear.

Example 5.1. Let $\mathscr{A}$ be a caterad in $\mathscr{C}$ and let $\mathscr{Z}$ be an operad in $\mathscr{C}$. Define the product "catoperad" $\mathscr{B} = \mathscr{A} \otimes \mathscr{Z}$ by letting

$$\mathscr{B}(q, r; k) = \mathscr{A}(q, r) \otimes \mathscr{Z}(k).$$

If $q = q_1 + \cdots + q_j$, $p = p_1 + \cdots + p_j$ and $j = j_1 + \cdots + j_k$, the structure maps

$$\gamma\colon \mathscr{B}(q, r; k) \otimes \mathscr{B}(p_1, q_1; j_1) \otimes \cdots \otimes \mathscr{B}(p_k, q_k; j_k) \longrightarrow \mathscr{B}(p, r; j)$$

are obtained by first shuffling factors so that the factors from $\mathscr{A}$ are at the left and the factors from $\mathscr{Z}$ are at the right and then applying $\gamma \otimes \gamma$.

The relevance of this example is that, after making the definitions precise, we expect to have naturally occurring examples of algebras over catoperads in the context of the hypercohomology groups that define motivic cohomology [20]. Fortunately, at least under resolution of singularities and restriction to quasi-projective schemes, the passage to hypercohomology and thus the need to take our riddle seriously disappear, so we shall be brief.

We return to the context of [19, 5.8], but using the generalization of the Čech construction of [19, 4.1–4.3] to covers $\mathscr{U}$ of an object X in a site $\mathscr{S}$, as in [19, 5.10]. In [19, 5.8], we started with a presheaf of algebras over an operad. However, in [20], we will start with a sheaf of algebras over a caterad $\mathscr{A}$. Let $\mathscr{Z}$ be the Eilenberg-Zilber operad [19, 3.1]. After completing the definition of a catoperad and defining algebras over them, we will have the following analogue of [19, 5.8]. Recall the composite structure maps θ of (4.2).

Theorem 5.2. *Let $\mathscr{A}$ be a caterad and let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-algebras on a site $\mathscr{S}$. Then $\check{C}^*(\mathscr{U}, \mathscr{F})$ and $\check{C}^*(X, \mathscr{F})$ are $\mathscr{A} \otimes \mathscr{Z}$-algebras. The action maps*

$$\theta\colon \mathscr{A}(q, r) \otimes \mathscr{Z}(k) \otimes \check{C}^*(\mathscr{U}, \mathscr{F}(q_1)) \otimes \cdots \otimes \check{C}^*(\mathscr{U}, \mathscr{F}(q_k)) \longrightarrow \check{C}^*(\mathscr{U}, \mathscr{F}(r)),$$

where $q = q_1 + \cdots + q_k$, are defined in terms of the action maps θ of $\mathscr{F}$ as the composites

$$\begin{array}{c}
\mathscr{A}(q, r) \otimes \mathrm{Hom}_\Delta(\Lambda, \Lambda^k) \otimes \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}(q_1)^\bullet_\mathscr{U}) \otimes \cdots \otimes \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}(q_k)^\bullet_\mathscr{U}) \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \xi} \\
\mathscr{A}(q, r) \otimes \mathrm{Hom}_\Delta(\Lambda, \mathscr{F}(q_1)^\bullet_\mathscr{U} \otimes \cdots \otimes \mathscr{F}(q_k)^\bullet_\mathscr{U}) \\
\Big\downarrow {\scriptstyle \zeta} \\
\mathrm{Hom}_\Delta(\Lambda, \mathscr{A}(q, r) \otimes \mathscr{F}(q_1)^\bullet_\mathscr{U} \otimes \cdots \otimes \mathscr{F}(q_k)^\bullet_\mathscr{U}) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_\Delta(\mathrm{id}, \theta)} \\
\mathrm{Hom}_\Delta(\Lambda, \mathscr{F}(r)^\bullet_\mathscr{U}).
\end{array}$$

Here ξ is an Eilenberg-Zilber map provided by [19, 2.6] *and ζ is the evident map moving the tensor factor inside* Hom, *as in* [19, 5.7].

We shall not carry out the implicit development of definitions, thus leaving an honest answer to the riddle to the reader. The point is that the equivariance, unit, and associativity diagrams that enter into the definitions of operads, caterads, and their algebras imply corresponding diagrams relating the structure maps displayed in Example 5.1 and the action maps displayed in Theorem 5.2. The missing definitions are given by abstracting the diagrams. Since we have no further examples in mind, we desist. The essential point is that algebraic structure implied by the operad and caterad actions will carry over appropriately to hypercohomology.

6. Some structure on diagram categories $\mathscr{D}\mathscr{C}$

We return to the context of §§1–2. We again let $\mathscr{C}$ be a closed symmetric monoidal category, but we now assume further that $\mathscr{C}$ is complete and cocomplete. For example, $\mathscr{C}$ might be the category of presheaves on a site $\mathscr{S}$. We let $\mathscr{D}$ be any small category. For present purposes, we have in mind $\mathscr{D} = \Sigma$. Let $\mathscr{D}\mathscr{C}$ denote the category of covariant functors $\mathscr{D} \longrightarrow \mathscr{C}$ and natural transformations. We are thinking of $\mathscr{D}\mathscr{C}$ as the category of $\mathscr{D}$-shaped diagrams, abbreviated $\mathscr{D}$-diagrams, in $\mathscr{C}$. We can also think of it as the category of presheaves on $\mathscr{D}^{\mathrm{op}}$ with values in $\mathscr{C}$. There is a great deal of structure on this category, some of which seems distracting when trying to understand motivic cochains. However, the structure is there, and it is useful for other purposes and in other contexts. Of course, since $\mathscr{C}$ is complete, $\mathscr{D}\mathscr{C}$ is cartesian monoidal [3, 2.15.2]. At least when $\mathscr{C}$ is the category of sets, $\mathscr{D}\mathscr{C}$ also has accompanying hom's that make it closed cartesian monoidal, as in [4, 2.3.4]. However, that is not the product structure that we have in mind.

For $\mathscr{D}$-diagrams F and G, we have an external product

$$\boxtimes\colon \mathscr{D}\mathscr{C} \times \mathscr{D}\mathscr{C} \longrightarrow (\mathscr{D} \times \mathscr{D})\mathscr{C}.$$

It is defined by

$$(F \boxtimes G)(d,e) = F(d) \otimes G(e).$$

Since $\mathscr{C}$ is complete, we have the internal hom $\mathrm{Hom}_{\mathscr{D}}(F,G)$ in $\mathscr{C}$, which is defined in [19, 2.1] in terms of equalizers between products of internal hom's in $\mathscr{C}$. We use this internal hom to obtain an external hom functor

$$\mathrm{Hom}_{\boxtimes}\colon (\mathscr{D}\mathscr{C})^{\mathrm{op}} \times (\mathscr{D} \times \mathscr{D})\mathscr{C} \longrightarrow \mathscr{D}\mathscr{C}$$

to accompany the external product $\boxtimes$. It is defined by

$$\mathrm{Hom}_{\boxtimes}(G,K)(d) = \mathrm{Hom}_{\mathscr{D}}(G, K\langle d\rangle)$$

for G in $\mathscr{D}\mathscr{C}$ and K in $(\mathscr{D}\times\mathscr{D})\mathscr{C}$, where $K\langle d\rangle(e) = K(d,e)$. We have the adjunction

$$(\mathscr{D} \times \mathscr{D})\mathscr{C}(F \boxtimes G, K) \cong \mathscr{D}\mathscr{C}(F, \mathrm{Hom}_{\boxtimes}(G,K)). \tag{6.1}$$

Now assume that $\mathscr{D}$ is symmetric monoidal, with product $\oplus$ and unit object $\mathbf{0}$. Then the diagram category $\mathscr{D}\mathscr{C}$ has a closed symmetric monoidal structure, with product and internal hom that we shall write $\odot$ and $\mathrm{Hom}_{\odot}$. The unit object, which we call κ_0, sends $\mathbf{0}$ to κ and sends all other d to the zero object in $\mathscr{C}$ (the coproduct of the empty set of objects). Since $\mathscr{C}$ has colimits, there is a left Kan extension functor that assigns a product $\mathscr{D}$-diagram $F \odot G$ to $\mathscr{D}$-diagrams F and G. It is characterized by the universal property that, for a $\mathscr{D}$-diagram H,

$$\mathscr{D}\mathscr{C}(F \odot G, H) \cong (\mathscr{D} \times \mathscr{D})\mathscr{C}(F \boxtimes G, H \circ \oplus); \tag{6.2}$$

see [5] or, in variant versions, [15, 21.4] or [14, I.2.11] for details of the general definition. In the case $\mathscr{D} = \Sigma$, we shall shortly display the product $\odot$ explicitly.

The internal hom $\mathrm{Hom}_{\odot}$ is defined by

$$\mathrm{Hom}_{\odot}(G,H) = \mathrm{Hom}_{\boxtimes}(G, H \circ \oplus). \tag{6.3}$$

The adjunctions (6.1) and (6.2) directly imply the adjunction

$$\mathscr{D}\mathscr{C}(F \odot G, H) \cong \mathscr{D}\mathscr{C}(F, \mathrm{Hom}_{\odot}(G,H)). \tag{6.4}$$

In turn, by a standard Yoneda lemma argument, this implies the adjunction

$$\mathrm{Hom}_{\odot}(F \odot G, H) \cong \mathrm{Hom}_{\odot}(F, \mathrm{Hom}_{\odot}(G,H)), \tag{6.5}$$

which has an evaluation map

(6.6) $$\varepsilon\colon \operatorname{Hom}_{\odot}(X,Y)\odot X \longrightarrow Y.$$

With the usual categorical specification of their adjoints, the adjunction (6.5) gives rise to a composition pairing

(6.7) $$\mu\colon \operatorname{Hom}_{\odot}(G,H)\odot \operatorname{Hom}_{\odot}(F,G) \longrightarrow \operatorname{Hom}_{\odot}(F,H)$$

and a $\odot$-product pairing

(6.8) $$\phi\colon \operatorname{Hom}_{\odot}(X,Y)\odot \operatorname{Hom}_{\odot}(X',Y') \longrightarrow \operatorname{Hom}_{\odot}(X\odot X',Y\odot Y').$$

The formal structure given by (6.5)–(6.8) is an example of the general formal structure recalled in (2.1)–(2.4). Using this structure, we have endomorphism caterads just as in §2. In particular, we have an endomorphism caterad $\operatorname{End}(\Lambda)$ in $\mathscr{D}\mathscr{C}$ for any covariant functor $\Lambda\colon \mathscr{D} \longrightarrow \mathscr{C}$, as in Construction 2.10. It is unclear to me whether or not such caterads (or operads) or their variants available when $\mathscr{C}$ is a presheaf category have a useful role to play in the context of motivic cohomology.

Specializing to the case $\mathscr{D}=\Sigma$, we display the product $\odot$. Let $\kappa[\Sigma_q]$ denote the "$\mathscr{C}$-group ring" of Σ_q; it is the coproduct of copies of κ indexed by permutations in Σ_q. Using coequalizers to define orbits, $F\odot G$ is specified explicitly by

(6.9) $$(F\odot G)(s) = \coprod_{q+r=s} \kappa[\Sigma_s]\otimes_{\Sigma_q\times\Sigma_r} (F(q)\otimes G(r)).$$

The commutativity isomorphism in this case is induced from the maps

$$\tau_{q,r}\otimes\tau\colon \kappa[\Sigma_{q+r}]\otimes F(q)\otimes G(r) \longrightarrow \kappa[\Sigma_{q+r}]\otimes G(r)\otimes F(q).$$

Proposition 6.10. *The categories of symmetric monoids in $\mathscr{C}$ and of commutative monoids in $\Sigma\mathscr{C}$ are isomorphic.*

Proof. Let F be a symmetric sequence. If F is a symmetric monoid, its product maps $\phi\colon F(q)\otimes F(r) \longrightarrow F(q+r)$ specify an external product $\phi\colon F\boxtimes F \longrightarrow F\circ\oplus$, which is a map of $(\Sigma\times\Sigma)$-diagrams. Under the isomorphism

$$\Sigma\mathscr{C}(F\odot F,F) \cong (\Sigma\times\Sigma)\mathscr{C}(F\boxtimes F, F\circ\oplus),$$

this product internalizes to a product $F\odot F \longrightarrow F$ that gives F a structure of commutative monoid in $\Sigma\mathscr{C}$. Conversely, the product $F\odot F \longrightarrow F$ of a commutative monoid in $\Sigma\mathscr{C}$ externalizes to a product $F\boxtimes F \longrightarrow F\circ\oplus$ that gives F a structure of symmetric monoid. □

Applied to symmetric sequences of simplicial sets, these definitions are the starting point for Smith's definitions [9] of the smash product of symmetric spectra and of symmetric ring spectra. It is curious that, after passing from symmetric sequences to symmetric spectra, the "commutativity" of commutative monoids with respect to the product $\odot$ can be taken seriously in that situation, whereas the opposite would seem to be true intuitively. Indeed, as promised, we return to Example 1.4 and explain in what sense tensor algebras are commutative monoids.

Example 6.11. The product (6.9) on symmetric sequences of vector spaces over a field k is given by inducting up representations of $\Sigma_q\times\Sigma_r$ to representations of Σ_{q+r}. Here the commutative monoid in $\Sigma\mathscr{C}$ associated to the tensor algebra symmetric monoid $T(X)$ is defined by the evident maps

$$k[\Sigma_{q+r}]\otimes_{\Sigma_q\times\Sigma_r} X^q\otimes X^r \longrightarrow X^{q+r}$$

from the induced representations of tensor products $X^q \otimes X^r$ of power representations X^q and X^r of the vector space X to the power representation X^{q+r}.

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