MINIMAL ATOMIC COMPLEXES

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WITH AN APPENDIX BY R. PEREIRA

Abstract. We define *minimal atomic complexes* and *irreducible complexes*, and we prove that they are the same. The irreducible complexes admit homological characterizations that make them easy to recognize. These concepts apply both to spaces and to spectra. On the spectrum level, our characterizations allow us to show that such familiar spectra as ko, eo_2, and BoP at the prime 2, all $BP\langle n\rangle$ at any prime p, and the indecomposable wedge summands of $\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$ and $\Sigma^\infty\mathbb{H}\mathrm{P}^\infty$ at any prime p are irreducible and therefore minimal atomic. Up to equivalence, the minimal atomic complexes admit descriptions as CW complexes with restricted attaching maps, called *nuclear complexes*, and this description can be refined further to *nuclear minimal complexes*, which are nuclear and have zero differential on their mod p chains. As an illustrative example, we construct BoP as a nuclear complex.

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Introduction

Atomic spaces and spectra have long been studied. They are so tightly bound together that a self-map which induces a isomorphism on homotopy in the Hurewicz dimension must be an equivalence. Atomic spaces and spectra can often be shrunk to ones with smaller homotopy groups. Minimal ones can be shrunk no further. Clearly, these are very natural objects of study. They seem to have been first introduced in [10]. Spheres, 2-cell complexes that are not wedges, and $K(\pi, n)$'s for cyclic groups π are obviously minimal atomic, but there are many much more interesting examples. It is not at all clear to us how important this notion will turn

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out to be, but it is certainly intriguing. Although the illustrative examples that we consider in this paper are spectra, it appears to us that minimal atomic spaces are likely to be even more interesting, since atomic spaces have generally played a far more important role in algebraic topology than atomic spectra.

We think of atomic complexes (spaces or spectra) as analogues of "atomic modules", namely modules for which a non-trivial self-map is an isomorphism. We think of minimal atomic complexes as analogues of irreducible modules. We give a definition of an irreducible complex (different from that in [10] and elsewhere) that makes this analogy more transparent, and we prove that the irreducible complexes are precisely the minimal atomic complexes. In one direction, the implication is a homotopical analogue of Schur's lemma. Just as in algebra, we suggest that the irreducible, or minimal atomic, complexes are more basic mathematical objects than the atomic complexes. Carrying the analogy further, we see that the spectra we consider admit (dual) composition series of infinite length, constructed in terms of irreducible complexes.

In Section 1, even before tying in atomic complexes, we use one proof supplied by the referee and another supplied by Rochelle Pereira, who gives details in Appendix A, to characterize irreducible complexes in homological terms. They are the Hurewicz complexes with no homotopy detectable by mod p homology, and there is an equivalent condition expressed in terms of the k-invariants of Postnikov towers. This gives a powerful criterion for showing that complexes are irreducible.

To tie in atomic complexes, we need a general method for constructing them. This is where nuclear complexes enter. These are atomic complexes that are built up in an especially economical, but noninvariant, way. The definition was implicit in Priddy's paper [20], in which he gave an elegant homotopy theoretic construction of the Brown-Peterson spectrum at a prime p. It was made explicit by the second author, who showed how to construct a "core" of a complex Y, namely a "monomorphism" from a nuclear complex X into Y. He hoped the construction was unique, but Hu and Kriz showed that it is not. Details are in [10], which gives the starting point of our work.

With our present understanding of nuclear complexes as equivalents of irreducible complexes, it is clear by analogy with algebra that it is unreasonable to expect uniqueness. We prove in Section 2 that nuclear complexes are minimal atomic, as was conjectured in [10], and that every minimal atomic complex is equivalent to a nuclear complex. The invariant notion of a minimal atomic complex is perhaps the more fundamental, but the combinatorial notion of a nuclear seems essential to proving that enough minimal atomic complexes exist to give an interesting theory.

There is a different and much more elementary notion of minimality, implicitly due to Cooke [5], such that *any* complex is equivalent to a minimal complex. This notion is also combinatorial and noninvariant. In Section 3, we prove that a Hurewicz complex that is minimal in this sense is nuclear if and only if it has no homotopy that is detected by mod p homology. This is an invariant characterization of a combinatorial structure, and it implies that any minimal atomic complex is equivalent to a complex that is both minimal in Cooke's sense and nuclear. Such restricted CW complexes might seem to be quite rare, were it not that our theory says that they are ubiquitous.

We show how minimal atomic complexes behave under several familiar constructions in Section 4. Restricting to spectra, we turn to examples in Section 5. We show that ko and eo_2 at the prime 2, $BP\langle n\rangle$ at any prime p, and the indecomposable wedge summands of $\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$ and $\Sigma^\infty\mathbb{H}\mathrm{P}^\infty$ at any prime p are minimal atomic. We give a few other examples and remarks, but we regard this section as just a beginning. Our results imply that minimal atomic complexes exist in abundance, and something closer to a classification of them would be desirable.

In Section 6, we describe Pengelley's 2-local spectrum BoP as a nuclear complex and thereby give it a new construction that is independent of [17]. This is in the same spirit as Priddy's construction of BP [20], which is recalled in Section 5. The key step in the proof that our construction does give BoP is deferred to Section 7.

The brief Appendix B corrects minor errors in one of the proofs in [10].

We are very grateful to the referee and to Rochelle Pereira, who unwittingly collaborated to give the characterization of irreducible complexes in Theorem 1.3. We are also grateful to the referee for Example 5.6.

1. Definitions and invariant characterization theorems

Here we give the definitions needed to make sense of the introduction and give characterizations of irreducible and minimal atomic complexes that are invariant under equivalence. In order to write things so that the stable reader can view our results as statements about spectra and the unstable reader can view them as statements about (based) spaces, we adopt the following conventions throughout. They allow us to treat spaces and spectra uniformly and to avoid repeated mention of the fact that we are working p-locally under connectivity and finite type hypotheses.

We agree once and for all that all spaces and spectra X are to be localized at a fixed prime p. Thus S^n, for example, means a p-local sphere. We also agree that all spaces and spectra are to be p-local CW spaces or spectra, so that the domains of their attaching maps are p-local spheres. Spaces are to be simply connected, and their attaching maps are to be based. Spectra are to be bounded below. In either case, we say that X *has Hurewicz dimension* n_0 if X is (n_0-1)-connected, but not n_0-connected. Thus $n_0 \geqslant 2$ in the case of spaces, and there is no real loss of generality if we take $n_0 = 0$ in the case of spectra. We say that X is a *Hurewicz complex* if $\pi_{n_0}(X)$ is a cyclic module over $\mathbb{Z}_{(p)}$.

We may assume without loss of generality that X has no cells (except the base vertex) of dimension less than n_0. If X is a Hurewicz complex, we may assume that it has a single cell in dimension n_0. We assume further that X has only finitely many cells in each dimension. We agree to use the ambiguous term "complex" to mean such a p-local CW space or spectrum. We write X_n for the n-skeleton of X. We take $X_{n_0-1} = *$ and, if X is a Hurewicz complex, $X_{n_0} = S^{n_0}$. For $n \geq n_0$, X_{n+1} is the cofiber of a map $j_n : J_n \longrightarrow X_n$, where J_n is a finite wedge of (p-local) n-spheres S^n. We use these notations generically.

By $H_*(X)$, we always understand (reduced) homology with p-local coefficients. Any (n_0-1)-connected space or spectrum such that each $H_n(X)$ is a finitely generated $\mathbb{Z}_{(p)}$-module is weakly equivalent to a complex in the sense that we have just specified. If, further, $H_{n_0}(X;\mathbb{F}_p) = \mathbb{F}_p$ or, equivalently, $\pi_{n_0}(X)$ is a cyclic $\mathbb{Z}_{(p)}$-module, then X is weakly equivalent to a Hurewicz complex.

We begin with definitions of concepts that are invariant under equivalence.

Definition 1.1. Consider complexes X and Y of Hurewicz dimension n_0. Think of Y as fixed but X as variable.

(i) A map $f : X \longrightarrow Y$ is a *monomorphism* if $f_*\colon \pi_{n_0}(X)\otimes\mathbb{F}_p \longrightarrow \pi_{n_0}(Y)\otimes\mathbb{F}_p$ and all $f_*\colon \pi_n(X) \longrightarrow \pi_n(Y)$ are monomorphisms
(ii) Y is *irreducible* if any monomorphism $f : X \longrightarrow Y$ is an equivalence.
(iii) Y is *atomic* if it is a Hurewicz complex and a self-map $f : Y \longrightarrow Y$ that induces an isomorphism on π_{n_0} is an equivalence.
(iv) Y is *minimal atomic* if it is atomic and any monomorphism $f : X \longrightarrow Y$ from an atomic complex X to Y is an equivalence.
(v) *Y has no homotopy detected by mod p homology* if Y is a Hurewicz complex and the mod p Hurewicz homomorphism $h : \pi_n(Y) \longrightarrow H_n(Y;\mathbb{F}_p)$ is zero for all $n > n_0$.
(vi) *The k-invariants of Y detect its homotopy* if Y is a Hurewica complex and each k-invariant $k^{n+2}\colon Y[n] \longrightarrow K(\pi_{n+1}(Y), n+2)$, $n \geq n_0$, of a Postnikov tower $\{Y[n]\}$ induces an epimorphism

$$H_{n+2}(Y[n];\mathbb{F}_p) \longrightarrow H_{n+2}(K(\pi_{n+1}(Y), n+2);\mathbb{F}_p) \cong \pi_{n+1}(Y)\otimes\mathbb{F}_p.$$

(vii) Y is *H^*-monogenic* if $H^*(Y;\mathbb{F}_p)$ is a cyclic algebra (in the case of spaces) or module (in the case of spectra) over the mod p Steenrod algebra A.

Remarks 1.2. We offer several comments on these notions.

(i) The structure theory for finitely generated modules over a PID implies that if $f\colon X \longrightarrow Y$ is a monomorphism, then $f_*(\pi_{n_0}(X))$ is a direct summand of $\pi_{n_0}(Y)$. If X is a Hurewicz complex, this summand is cyclic. If X and Y are both Hurewicz complexes, then f induces an isomorphism on π_{n_0}.
(ii) In [10, 1.1], following [25] and other early sources, Y was defined to be irreducible if it has no non-trivial retracts. On the space level, that concept has its uses, but we think that "irreducible" is the wrong name for it. We suggest "irretractible". On the spectrum level, irretractibility is equivalent to wedge indecomposability. However, just as in algebra, irreducibility should be stronger rather than weaker than atomic. That is, there should be implications irreducible $\Longrightarrow$ atomic $\Longrightarrow$ indecomposable. One could avoid the conflict with the earlier literature by using the word "simple" instead of "irreducible", the two being synonymous in algebra, but that risks confusion with the standard use of the term "simple" in topology.
(iii) A complex that does not have cyclic homotopy in its Hurewicz dimension can still have the property that a self-map that induces an isomorphism on π_{n_0} is an equivalence. A particularly interesting example is given in [2, §4]. It might be sensible to delete the requirement that Y be a Hurewicz complex from the definition of atomic. By Theorem 2.4 below, the notion of minimal atomic would not change.
(iv) It turns out that the phrase "is atomic and" in the definition of "minimal atomic" is redundant, but the definition is best understood as it stands.
(v) Since our methods are cellular, we definitely mean to consider p-local rather than p-complete spaces and spectra. However, Definition 1.1 makes just as much sense in the p-complete case as the p-local case, and it is well worth studying there. Since a finite type p-complete space or spectrum is the p-completion of a finite type p-local space or spectrum, one can easily deduce conclusions in the p-complete case from the results here. We leave the details to the interested reader.

(vi) A Hurewicz complex Y has no homotopy detected by mod p homology if and only if there are no permanent cycles in dimension greater than n_0 on the zeroth row of the classical (unstable or stable) mod p Adams spectral sequence for Y. It is a much more computable condition than the others.

We begin work on the relationships among these concepts with two results that, a priori, have nothing to do with atomic complexes. The first characterizes irreducible complexes. Here we are indebted to the referee for the proof that (i) is equivalent to (ii), which led us to reorganize our original arguments. The result that (ii) is equivalent to (iii) is due to Rochelle Pereira, and her proof of that is given in Appendix A below.

Theorem 1.3. *The following conditions on a Hurewicz complex Y are equivalent.*

(i) *Y is irreducible.*
(ii) *Y has no homotopy detected by mod p homology.*
(iii) *The k-invariants of Y detect its homotopy groups.*

Proof. To see that (i) implies (ii), assume (i) and assume for a contradiction that $h\colon \pi_n(Y) \longrightarrow H_n(Y;\mathbb{F}_p)$ is non-zero, where $n > n_0$. Then there is a map $S^n \longrightarrow Y$ that is non-zero on mod p cohomology, hence there is a map $g\colon Y \longrightarrow K(\mathbb{Z}/p\mathbb{Z}, n)$ that is non-zero on homotopy. Let $f\colon X \longrightarrow Y$ be the fiber of g. Then f is a monomorphism that is not an equivalence, which contradicts (i). To see that (ii) implies (i), assume (ii) and let $f\colon X \longrightarrow Y$ be a monomorphism. We must show that f is an equivalence. Let $g\colon Y \longrightarrow Z$ be the cofiber of f. In both the space and spectrum contexts, our standing assumptions ensure that f is an equivalence if and only if $\pi_*(Z) = 0$. Suppose that $\pi_*(Z) \neq 0$ and let n be minimal such that $\pi_n(Z) \neq 0$. Then $h\colon \pi_n(Z) \longrightarrow H_n(Z;\mathbb{F}_p)$ is non-zero. Since Y is a Hurewicz complex and f is a monomorphism, $n > n_0$ and g induces an epimorphism on homotopy groups. The commutative diagram

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{g_*} & \pi_n(Z) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_n(Y;\mathbb{F}_p) & \xrightarrow[g_*]{} & H_n(Z;\mathbb{F}_p) \end{array}$$

shows that the left arrow h is non-zero, which contradicts (ii). □

Remark 1.4. Theorem 1.8 below shows that irreducible complexes are Hurewicz complexes, but that is not apparent at this stage.

Theorem 1.5. *If Y is H^*-monogenic, then Y has no homotopy detected by mod p homology.*

Proof. Certainly Y must be a Hurewicz complex. For spectra, the long exact sequence on Ext arising from the epimorphism $\Sigma^{n_0} A \longrightarrow H^*(Y;\mathbb{F}_p)$ implies that the zeroth row $\mathrm{Hom}_A^{0,*}(H^*(Y;\mathbb{F}_p), \mathbb{F}_p)$ of the Adams spectral sequence is $\mathbb{F}_p$ concentrated in degree n_0, and similarly for spaces □

Remark 1.6. The converse of Theorem 1.5 fails. For $q \geqslant 2$, Moore spaces and spectra $M(\mathbb{Z}/p^q, n)$ and Eilenberg-Mac Lane spaces and spectra $K(\mathbb{Z}/p^q, n)$ give elementary counterexamples.

To see the logic of our definitions, let us accept the following result for a moment.

Theorem 1.7. *For any complex Y, there is a monomorphism $f\colon X \longrightarrow Y$ such that X is atomic.*

Then we have the following characterization theorem.

Theorem 1.8. *A complex Y is irreducible if and only if it is minimal atomic.*

Proof. If Y is irreducible, then it is atomic and therefore minimal atomic by Theorem 1.7 and the definitions. If Y is minimal atomic and $f\colon X \longrightarrow Y$ is a monomorphism, let $g\colon W \longrightarrow X$ be a monomorphism such that W is atomic. Then the composite $f \circ g$ is an equivalence, by the definition of minimal atomic, and thus f is an epimorphism on homotopy groups and therefore an equivalence. □

The fact that irreducible complexes are atomic should be viewed as a homotopical analogue of Schur's lemma since its intuitive content is that a non-trivial self-map of an irreducible complex must be an equivalence. Of course, it is consistent with the analogy that not all atomic complexes are irreducible. When [10] was written, examples of minimal atomic spectra seemed hard to come by. Theorems 1.3 and 1.5 combine with Theorem 1.8 to make it easy to find examples.

2. Nuclear complexes and minimal atomic complexes

We are perhaps more interested in describing minimal atomic complexes in terms of special kinds of CW complexes than in Theorem 1.8 itself, since these descriptions seem to shed new light on homotopy types. In any case, we need them to explain Theorem 1.7. We recall the notions of nuclear complexes and cores from [10].

Definition 2.1. A *nuclear complex* is a Hurewicz complex X such that

$$\mathrm{Ker}(j_{n*} : \pi_n(J_n) \longrightarrow \pi_n(X_n)) \subset p \cdot \pi_n(J_n) \tag{2.2}$$

for each n. Observe that X is nuclear if and only if each X_n for $n \geqslant n_0$ is nuclear. A *core* of a complex Y is a nuclear complex X together with a monomorphism $f : X \longrightarrow Y$.

This notion of a core is more general than that of [10, 1.7], where it was assumed that $\pi_{n_0}(Y)$ is cyclic; that is, the definition there was restricted to Hurewicz complexes Y. Since cores are not unique even when Y is a Hurewicz complex, the more general notion seems preferable. With the present language, the following results are proven in [10, 1.5, 1.6]. Together, they immediately imply Theorem 1.7.

Proposition 2.3. *A nuclear complex is atomic.*

Theorem 2.4. *If Y has Hurewicz dimension n_0 and C is a cyclic direct summand of $\pi_{n_0}(Y)$, then there is a core $f : X \longrightarrow Y$ such that $f_*(\pi_{n_0}(X)) = C$.*

The idea is to start with a map $f_{n_0}\colon S^{n_0} \longrightarrow Y$ that realizes C and then proceed inductively. Given $f_n\colon X_n \longrightarrow Y$, we construct the $(n+1)$-skeleton X_{n+1} by killing the kernel of $f_{n*} : \pi_n(X_n) \longrightarrow \pi_n(Y)$ in a minimal way. We use null homotopies to extend f_n to $f_{n+1}\colon X_{n+1} \longrightarrow Y$, and we obtain $f\colon X \longrightarrow Y$ by passage to colimits.

This allows us to construct homotopical analogues of composition series. That is, for spectra, we can shrink homotopy groups inductively by successively taking cofibers of cores, as discussed in [10, 1.8]. This works less well for spaces, where we would have to take fibers and so gradually decrease the Hurewicz dimension. To make the analogy with algebra precise, recall that a (countably infinite) composition series of a module Y is a sequence of monomorphisms

$$Y = Y_0 \xleftarrow{i_0} Y_1 \longleftarrow \cdots \longleftarrow Y_n \xleftarrow{i_n} Y_{n+1} \longleftarrow \cdots$$

such that the cokernels of the i_n are irreducible and $\lim Y_n = 0$. A *dual composition series* of Y is a sequence of epimorphisms

$$Y = Y_0 \xrightarrow{p_0} Y_1 \longrightarrow \cdots \longrightarrow Y_n \xrightarrow{p_n} Y_{n+1} \longrightarrow \cdots$$

such that the kernels of the p_n are irreducible and $\operatorname{colim} Y_n = 0$. For a spectrum Y, we construct an analogous sequence by letting $p_n : Y_n \longrightarrow Y_{n+1}$ be the cofiber of a core $f_n : X_n \longrightarrow Y_n$. Each p_n induces an epimorphism on all homotopy groups, we kill $\pi_{n_0}(Y)$ in finitely many steps, then kill $\pi_{n_0+1}(Y)$ in finitely many steps, and so on. If Y has non-zero homotopy groups in only finitely many dimensions, then this sequence has only finitely many terms.

The proof of Proposition 2.3 in [10, 1.5] starts with a cellular self-map f of X that induces an isomorphism on π_{n_0} and shows inductively that the self-maps f_n of the skeleta X_n are equivalences for all n. We show below that the proof adapts to give the following analogue.

Proposition 2.5. *Let X and Y be nuclear complexes of Hurewicz dimension n_0 and let $f\colon X \longrightarrow Y$ be a core of Y. Then f is an equivalence.*

This implies the following strengthening of Proposition 2.3, which was conjectured in [10, 1.12].

Theorem 2.6. *A nuclear complex is a minimal atomic complex.*

Proof. Let Y be a nuclear complex and let $f\colon X \longrightarrow Y$ be a monomorphism, where X is atomic. By Proposition 2.5, the composite of f and a core $g\colon W \longrightarrow X$ is an equivalence, hence so is f. □

In turn, this implies the following description of minimal atomic complexes.

Theorem 2.7. *The following conditions on a complex Y are equivalent.*

(i) *Y is minimal atomic.*
(ii) *Any core of Y is an equivalence.*
(iii) *Y is equivalent to a nuclear complex.*

Proof. Theorem 2.6 gives that (iii) implies (i), and it is trivial that (i) implies (ii) and (ii) implies (iii). □

Remark 2.8. With these implications in place, it is perhaps better to redefine the notion of core invariantly, taking X to be minimal atomic but not necessarily nuclear. There is no substantive difference.

Proof of Proposition 2.5. Take $f\colon X \longrightarrow Y$ to be cellular. Since f is a monomorphism between Hurewicz complexes, $f : X_{n_0} \longrightarrow Y_{n_0}$ is an equivalence. Assume that $f : X_n \longrightarrow Y_n$ is an equivalence. We must show that $f : X_{n+1} \longrightarrow Y_{n+1}$ is an equivalence. The attaching maps of X and Y give rise to the following map of cofiber sequences.

$$\begin{array}{ccccc} J_n & \xrightarrow{j_n} & X_n & \longrightarrow & X_{n+1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ K_n & \xrightarrow[k_n]{} & Y_n & \longrightarrow & Y_{n+1} \end{array}$$

Passing to homology, this gives rise to a commutative diagram with exact rows.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_{n+1}(X_{n+1}) & \longrightarrow & H_n(J_n) & \xrightarrow{j_{n*}} & H_n(X_n) & \longrightarrow & H_n(X_{n+1}) & \longrightarrow & 0 \\ & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} & & \cong\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ 0 & \longrightarrow & H_{n+1}(Y_{n+1}) & \longrightarrow & H_n(K_n) & \xrightarrow[k_{n*}]{} & H_n(Y_n) & \longrightarrow & H_n(Y_{n+1}) & \longrightarrow & 0 \end{array}$$

It suffices to prove that the left and right vertical arrows are isomorphisms. By the five lemma and the Hurewicz theorem, this holds if $f_*\colon \pi_n(J_n) \longrightarrow \pi_n(K_n)$ is an isomorphism. To see that this is so, consider the following diagram.

$$\begin{array}{ccccccc} \pi_n(J_n) & \xrightarrow{j_{n*}} & \pi_n(X_n) & \longrightarrow & \pi_n(X_{n+1}) & \longrightarrow & 0 \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \pi_n(K_n) & \xrightarrow[k_{n*}]{} & \pi_n(Y_n) & \longrightarrow & \pi_n(Y_{n+1}) & \longrightarrow & 0 \end{array}$$

The rows are exact, and a chase of the diagram shows that the right arrow f_* is an epimorphism. Now consider the following diagram.

$$\begin{array}{ccc} \pi_n(X_{n+1}) & \xrightarrow{\cong} & \pi_n(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_n(Y_{n+1}) & \xrightarrow[\cong]{} & \pi_n(Y) \end{array}$$

Since its right arrow f_* is a monomorphism, its left arrow f_* is a monomorphism and therefore an isomorphism. This implies that the right vertical arrow is an isomorphism in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\ j_{n*} & \xrightarrow{i} & \pi_n(J_n) & \xrightarrow{\cong} & \mathrm{Im}\, j_{n*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathrm{Ker}\ k_{n*} & \xrightarrow[i]{} & \pi_n(K_n) & \xrightarrow[\cong]{} & \mathrm{Im}\, k_{n*} & \longrightarrow & 0 \end{array}$$

In view of (2.2), both maps i become 0 after tensoring with $\mathbb{F}_p$. This implies that $f_* \otimes \mathbb{F}_p$ is an isomorphism, and therefore so is f_*. □

3. Minimal complexes and nuclear complexes

We have another description of minimal atomic complexes that involves a quite different notion of minimality of a complex X. Of course, our complexes have p-local chain complexes specified by $C_n(X) = H_n(X_n/X_{n-1})$.

Definition 3.1. A complex X is *minimal* if the differential on its mod p chain complex $C_*(X;\mathbb{F}_p)$ is zero. It is *minimal Hurewicz* if it is minimal and Hurewicz. Observe that X is minimal if and only if each X_n is minimal.

A simple inductive argument gives a homological reformulation of this notion.

Lemma 3.2. *A complex X is minimal if and only if the inclusion $X_n \longrightarrow X_{n+1}$ of skeleta induces an isomorphism*

$$H_n(X_n;\mathbb{F}_p) \longrightarrow H_n(X_{n+1};\mathbb{F}_p) = H_n(X;\mathbb{F}_p)$$

for each n.

The following result is implicit in Cooke's paper [5, Theorem A], which gives an integral space level version. Cooke described the result as "a well-known, basic fact". For a recent reappearance, see [8, 4.C.1]. The proof is easy, but we shall run through it below since the result is not as well known as it should be.

Theorem 3.3. *For any complex Y, there is a minimal complex X and an equivalence $f\colon X \longrightarrow Y$.*

We shall shortly prove the following result, which explains the relevance of minimal complexes to the present theory.

Theorem 3.4. *A minimal complex X is nuclear if and only if it has no homotopy detected by mod p homology.*

The point is that the invariant condition about the mod p Hurewicz homomorphism gives noninvariant information about the cellular structure of X. This implies the following description of minimal atomic complexes.

Theorem 3.5. *The following conditions on a complex Y are equivalent.*

(i) *Y is minimal atomic.*
(ii) *Any equivalence $f\colon X \longrightarrow Y$ from a minimal complex X to Y is a core of Y.*
(iii) *A minimal complex equivalent to Y is nuclear.*

Proof. Clearly (ii) and (iii) are equivalent, and they imply (i) by Theorem 2.6. Theorems 1.8 and 3.4 show that (i) implies (ii). □

We need the following lemma, which is based on an observation of Priddy [20], to prove Theorem 3.4. It gives a recasting of the definition of a nuclear complex in terms of the Hurewicz homomorphisms of its skeleta.

Lemma 3.6. *A Hurewicz complex of dimension n_0 is nuclear if and only if the mod p Hurewicz homomorphism $h : \pi_n(X_n) \longrightarrow H_n(X_n;\mathbb{F}_p)$ is zero for $n > n_0$.*

Proof. Recall the defining property (2.2) of a nuclear complex. In the case of spaces, our assumption that X is simply connected allows us to quote the relative and absolute Hurewicz theorem to deduce that

$$\pi_{n+1}(X_{n+1}, X_n) \cong \pi_{n+1}(\Sigma J_n) \cong \pi_n(J_n)$$

from the trivial analogue in p-local homology. In either the space or the spectrum context, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\pi_{n+1}(X_n) & \longrightarrow & \pi_{n+1}(X_{n+1}) & \longrightarrow & \pi_n(J_n) & \xrightarrow{j_*} & \pi_n(X_n) \\
 & & \downarrow h & & \downarrow h & & \downarrow h \\
0 & \longrightarrow & H_{n+1}(X_{n+1};\mathbb{F}_p) & \longrightarrow & H_n(J_n;\mathbb{F}_p) & \xrightarrow[j_*]{} & H_n(X_n;\mathbb{F}_p)
\end{array}$$

An easy diagram chase gives that (2.2) holds for n if and only if the left arrow h is zero. The conclusion follows. □

Proof of Theorem 3.4. The following naturality diagram relates the Hurewicz homomorphisms of X_n and X, where $n > n_0$.

$$\begin{array}{ccc}
\pi_n(X_n) & \longrightarrow & \pi_n(X) \\
h\downarrow & & \downarrow h \\
H_n(X_n;\mathbb{F}_p) & \longrightarrow & H_n(X;\mathbb{F}_p)
\end{array}$$

Since X is minimal, the bottom arrow is an isomorphism by Lemma 3.2, and the top arrow is an epimorphism. Therefore the left arrow h is zero if and only if the right arrow h is zero. By Lemma 3.6, this gives the conclusion. □

Proof of Theorem 3.3. We are given a complex Y. Recall that our complexes are simply connected in the case of spaces and bounded below in the case of spectra and that everything is p-local. We have assumed that $H_*(Y)$ is of finite type, so that each $H_n(Y)$ is a direct sum of finitely many cyclic $\mathbb{Z}_{(p)}$-modules $A_{n,i}$. We must construct a minimal complex X and an equivalence $f : X \longrightarrow Y$, and it suffices for the latter to ensure that f induces an isomorphism on H_*. The complex X will have an n-cell $j_{n,i}$ for each free cyclic summand $A_{n,i}$ and an n-cell $j_{n,i}$ and an $(n+1)$-cell $k_{n,i}$ with differential $q_i j_{n,i}$ for each summand $A_{n,i}$ of order q_i. Since each q_i must be a power of p, it will be immediate that the differential on $C_*(X;\mathbb{F}_p)$ is zero. The cells $j_{n,i}$ will map to cycles that represent the generators of the $A_{n,i}$, and the cells $k_{n,i}$ will map to chains with boundary $q_i f_*(j_{n,i})$.

Assume inductively that we have constructed the n-skeleton X_n together with a (based) map $f_n : X_n \longrightarrow Y$ that induces an isomorphism on homology in dimensions less than n and an epimorphism on H_n. More precisely, assume that $H_n(X_n)$ is $\mathbb{Z}_{(p)}$-free on basis elements given by cells $j_{n,i}$ that map to chosen generators of the $A_{n,i}$. Let Cf_n be the cofiber of f_n. Then $H_m(Cf_n) = 0$ for $m \leqslant n$. The kernel of $f_* : H_n(X_n) \longrightarrow H_n(Y)$ is free on the basis $q_i j_{n,i}$ for those i such that $A_{n,i}$ has finite order. These elements are the images of elements $k''_{n,i}$ in $H_{n+1}(Cf_n)$, and $k''_{n,i} = h(k'_{n,i})$ for unique elements $k'_{n,i}$ in $\pi_{n+1}(Cf_n)$. Similarly, the chosen generators of the $A_{n+1,i} \subset H_{n+1}(Y)$ map to elements $j''_{n+1,i}$ in $H_{n+1}(Cf_n)$ with $j''_{n+1,i} = h(j'_{n+1,i})$. For spectra, we have the connecting homomorphism $\pi_{n+1}(Cf_n) \longrightarrow \pi_n(X_n)$. For spaces, the relative Hurewicz theorem gives $\pi_{n+1}(Mf_n, X_n) \cong \pi_{n+1}(Cf_n)$, and we have the connecting homomorphism $\pi_{n+1}(Mf_n, X_n) \longrightarrow \pi_n(X_n)$. Thus in either case the elements $k'_{n,i}$ and $j'_{n+1,i}$ determine elements of $\pi_n(X_n)$. Choose maps $S^n \longrightarrow X_n$ that represent these elements and use them as attaching maps for the construction of X_{n+1} from X_n by attaching cells $k_{n,i}$ and $j_{n+1,i}$.

Since the sequence $\pi_{n+1}(Cf_n) \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(Y)$ is exact, these attaching maps become null homotopic in Y, and there is an extension $f_{n+1} : X_{n+1} \longrightarrow Y$ of f_n. Thus we can construct the following map of cofiber sequences.

$$\begin{array}{ccccccc}
X_n & \longrightarrow & X_{n+1} & \longrightarrow & X_{n+1}/X_n & \longrightarrow & \Sigma X_n \\
\| & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow & & \| \\
X_n & \xrightarrow[f_n]{} & Y & \longrightarrow & Cf_n & \longrightarrow & \Sigma X_n
\end{array}$$

This gives rise to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_{n+1}(X_{n+1}) & \longrightarrow & H_{n+1}(X_{n+1}/X_n) & \longrightarrow & H_n(X_n) & \longrightarrow & H_n(X_{n+1}) & \longrightarrow & 0 \\
 & & {\scriptstyle (f_{n+1})_*}\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle (f_{n+1})_*} & & \\
0 & \longrightarrow & H_{n+1}(Y) & \longrightarrow & H_{n+1}(Cf_n) & \longrightarrow & H_n(X_n) & \longrightarrow & H_n(Y) & \longrightarrow & 0
\end{array}$$

Of course, the differential on $C_{n+1}(X_{n+1})$ is the composite

$$H_{n+1}(X_{n+1}/X_n) \longrightarrow H_n(X_n) \longrightarrow H_n(X_n/X_{n-1}),$$

where the second arrow is a monomorphism. By construction, the first arrow sends the basis elements $k_{n,i}$ to $q_i j_{n,i}$ and the basis elements $j_{n+1,i}$ to zero, so that $H_{n+1}(X_{n+1})$ is $\mathbb{Z}_{(p)}$-free on the basis elements $j_{n+1,i}$. By construction and a chase of the diagram, the map f_{n+1} induces an isomorphism on H_n and sends the basis elements $j_{n+1,i}$ to generators of the groups $A_{n+1,i}$. This completes the inductive step in the construction of $f : X \longrightarrow Y$. □

4. Constructions on minimal atomic complexes

We indicate briefly how the collection of minimal atomic complexes behaves with respect to some basic topological constructions. The proofs are direct consequences of the "no homotopy detected by mod p homology" characterization, so the results are really about irreducible complexes. The following triviality may help the reader see the various implications.

Lemma 4.1. *Consider a commutative diagram*

$$\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\
B & \xrightarrow{g} & B'
\end{array}$$

of Abelian groups. If f is an epimorphism and $h = 0$, then $h' = 0$. If g is a monomorphism and $h' = 0$, then $h = 0$.

We begin by recording an immediate consequence of Theorems 3.4 and 2.7.

Proposition 4.2. *If Y is minimal atomic, then Y is equivalent to a complex X whose skeleta X_n for $n \geqslant n_0$ are minimal atomic.*

There is no reason to believe that the skeleta of Y itself are minimal atomic. We have a more invariant analogue for Postnikov sections, which we denote by $Y[n]$.

Proposition 4.3. *A complex Y is minimal atomic if and only if $Y[n]$ is minimal atomic for each $n \geqslant n_0$.*

Proof. We have the following commutative diagram.

$$\begin{array}{ccc} \pi_q(Y) & \longrightarrow & \pi_q(Y[n]) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_q(Y;\mathbb{F}_p) & \longrightarrow & H_q(Y[n];\mathbb{F}_p) \end{array}$$

Since $\pi_q(Y[n])$ is zero for $q > n$ and the horizontal arrows are isomorphisms for $q \leqslant n$, the conclusion is immediate from Lemma 4.1. □

In the following result, which is only of interest for spaces, we consider the loop and suspension functors.

Proposition 4.4. *If either ΩY or ΣY is minimal atomic, then so is Y.*

Proof. This is immediate from the following commutative diagrams.

$$\begin{array}{ccc} \pi_q(\Omega Y) & \xrightarrow{\cong} & \pi_{q+1}(Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_q(\Omega Y;\mathbb{F}_p) & \xrightarrow[\sigma]{} & H_{q+1}(Y;\mathbb{F}_p) \end{array} \qquad \begin{array}{ccc} \pi_q(Y) & \xrightarrow{\Sigma} & \pi_{q+1}(\Sigma Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_q(Y;\mathbb{F}_p) & \xrightarrow[\cong]{} & H_{q+1}(\Sigma Y;\mathbb{F}_p) \end{array}$$

□

This result has an analogue that relates minimal atomicity for spaces and spectra. Here, exceptionally, we must distinguish the two contexts notationally.

Proposition 4.5. *If E is a spectrum of Hurewicz dimension $n_0 \geqslant 2$ whose 0th space $\Omega^\infty E$ is minimal atomic, then E is minimal atomic. If Y is a simply connected space whose suspension spectrum $\Sigma^\infty Y$ is minimal atomic, then Y is minimal atomic.*

Proof. This is immediate from the following commutative diagrams.

$$\begin{array}{ccc} \pi_q(\Omega^\infty E) & \xrightarrow{\cong} & \pi_q(E) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_q(\Omega^\infty E;\mathbb{F}_p) & \xrightarrow[\sigma]{} & H_q(E;\mathbb{F}_p) \end{array} \qquad \begin{array}{ccc} \pi_q(Y) & \longrightarrow & \pi_q(\Sigma^\infty Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_q(Y;\mathbb{F}_p) & \xrightarrow[\cong]{} & H_q(\Sigma^\infty Y;\mathbb{F}_p) \end{array}$$

□

5. Spectrum level examples

We recall from [10, 1.12] that if Y has homotopy groups and cohomology groups concentrated in even degrees, then the core of Y is unique. We begin with examples of this sort. We first revisit Priddy's construction [20] of a nuclear spectrum equivalent to BP. Although it was the motivating example for [10], it was not explicitly discussed there. We work with p-local spectra in this section. Unless otherwise stated, p is unrestricted.

Example 5.1. BP is a minimal atomic spectrum, hence the canonical monomorphism $BP \longrightarrow MU$ is the core of MU.

Proof. $H^*(BP;\mathbb{F}_p)$ is a cyclic A-module, hence Theorem 1.5 applies. □

Proposition 5.2. *Let X be the nuclear complex of [20] defined by starting with S^0 and inductively killing the homotopy groups in odd degrees. Then there is an equivalence $X \longrightarrow BP$.*

Proof. A minimal complex equivalent to BP has cells only in even degrees and is nuclear. By construction, X also has cells only in even degrees and is nuclear, and its non-zero homotopy groups only occur in even degrees. Obstruction theory gives maps $f\colon X \longrightarrow BP$ and $g\colon BP \longrightarrow X$ that extend the identity on the bottom cell. The composites $g \circ f\colon X \longrightarrow X$ and $f \circ g\colon BP \longrightarrow BP$ are equivalences since X and BP are atomic. □

Recall that, for an odd prime p, there is a splitting of ku with $BP\langle 1\rangle$ as a wedge summand. In [10, 1.18] it is conjectured that the core of ku is $BP\langle 1\rangle$. Here $ku = BP\langle 1\rangle$ if $p = 2$. Since $H^*(BP\langle 1\rangle; \mathbb{F}_p)$ is a cyclic A-module, this is now immediate.

Example 5.3. The spectrum $BP\langle 1\rangle$ is minimal atomic, hence the canonical monomorphism $BP\langle 1\rangle \longrightarrow ku$ is a core.

More generally, $H^*(BP\langle n\rangle)$ is a cyclic A-module for all $n \geqslant -1$, the extreme cases being $BP\langle -1\rangle = H\mathbb{F}_p$ and $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$.

Example 5.4. For $n \geqslant -1$, $BP\langle n\rangle$ is a minimal atomic spectrum.

The following example of the non-uniqueness of cores generalizes [10, 1.17].

Example 5.5. For $n \geqslant 0$, the canonical maps

$$BP \longrightarrow BP \wedge BP\langle n\rangle \longleftarrow BP\langle n\rangle$$

induced by the units of BP and $BP\langle n\rangle$ are both cores of $BP \wedge BP\langle n\rangle$.

Proof. The left map is a monomorphism since it factors the (p-local) Hurewicz homomorphism of BP. The right map is a monomorphism since it is split by the BP-action $BP \wedge BP\langle n\rangle \longrightarrow BP\langle n\rangle$. □

The referee supplied the following more sophisticated example of a Hurewicz complex with infinitely many cores.

Example 5.6. Define a Hurewicz complex Y by

$$Y = SP^\infty(S) \vee \bigvee_{k \geq 1} SP^{p^k}(S)/SP^{p^{k-1}}(S).$$

Let $X_k = SP^{p^k}(S)$ and define $f\colon X_k \longrightarrow Y$ to be the sum of the inclusion of X_k in $SP^\infty(S)$ and the quotient map from X_k to $SP^{p^k}(S)/SP^{p^{k-1}}(S)$. Since $H^*(X_k; \mathbb{F}_p)$ is a cyclic A-module, X_k is minimal atomic, and f_k is a monomorphism by the Whitehead conjecture [12, 13]. Therefore f_k is a core of Y for each $k \geq 1$.

Since $H^*(ko; \mathbb{F}_2) = A//A(1)$ and $H^*(eo_2; \mathbb{F}_2) = A//A(2)$ are cyclic A-modules, we have the following complement to Proposition 5.3.

Proposition 5.7. *At $p = 2$, ko and eo_2 are minimal atomic spectra.*

Some well-known Thom complexes give further examples.

Proposition 5.8. *Let X be $\mathbb{R}\mathrm{P}^\infty_{-1}$, $\mathbb{C}\mathrm{P}^\infty_{-1}$, or $\mathbb{H}\mathrm{P}^\infty_{-1}$, that is, the Thom spectrum of the negative of the canonical real, complex, or quaternionic line bundle. At $p = 2$, X is minimal atomic.*

Proof. Let $d = 1$, 2, and 4 and $P = \mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^\infty$, and $\mathbb{H}\mathrm{P}^\infty$ in the respective cases. Then $H^*(P;\mathbb{F}_2) = \mathbb{F}_2[x]$, where $x \in H^d(P;\mathbb{F}_2)$ is the dth Stiefel-Whitney class of the canonical line bundle. Since X is a Thom spectrum, $H^*(X;\mathbb{F}_2)$ is the free $H^*(P;\mathbb{F}_2)$-module generated by the Thom class μ in degree $-d$. A standard calculation shows that $\mathrm{Sq}^{nd}\mu = x^n\mu$ for $n \geqslant 1$, so $H^*(X;\mathbb{F}_2)$ is cyclic over A. □

To give examples where we must check the "no homotopy detected by mod p homology" condition directly, we consider a few suspension spectra and another Thom spectrum. Let $\xi_3 \longrightarrow \mathbb{H}\mathrm{P}^\infty$ be the bundle associated to the adjoint representation of S^3. Its Thom complex $M\xi_3$ is known as a "quaternionic quasi-projective space". It has one cell in each positive dimension congruent to 3 (mod 4).

By [9, 6, 3], for each odd prime p, there is a splitting of p-local spaces

$$\Sigma\mathbb{C}\mathrm{P}^\infty \simeq W_1 \vee W_2 \vee \cdots \vee W_{p-1},$$

where W_r has cells in all dimensions of the form $2(p-1)k + 2r + 1$ with $k \geqslant 0$.

Proposition 5.9. *At the prime* 2*,* $\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$*,* $\Sigma^\infty\mathbb{H}\mathrm{P}^\infty$ *and* $\Sigma^\infty M\xi_3$ *are minimal atomic spectra. At an odd prime* p*, each* $\Sigma^\infty W_r$ *is minimal atomic.*

Proof. Let $a(n) = 1$ if n is even and $a(n) = 2$ if n is odd. By [21], the Hurewicz homomorphisms

$$h\colon \pi_{2n}(\Sigma^\infty\mathbb{C}\mathrm{P}^\infty) \longrightarrow H_{2n}(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z} \quad \text{and} \quad h\colon \pi_{4n}(\Sigma^\infty\mathbb{H}\mathrm{P}^\infty) \longrightarrow H_{4n}(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$$

have images of index $n!$ and $(2n)!/a(n)$, respectively. Thus, for $n > 1$, the corresponding mod 2 Hurewicz homomorphisms are trivial. By [23], the Hurewicz homomorphism

$$h\colon \pi_{4n+3}(\Sigma^\infty M\xi_3) \longrightarrow H_{4n+3}(M\xi_3) \cong \mathbb{Z}$$

has image of index $a(n)(2n-1)!$, so for each $n \geqslant 1$ the associated mod 2 Hurewicz homomorphism is also trivial. The odd primary results follow similarly from the calculation of h for $\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$. □

Remark 5.10. We raise a few questions here.

(i) There are many basic results in the literature in which interesting spaces are split p-locally into products of indecomposable factors and interesting spectra are split p-locally into wedges of indecomposable summands. (The notion of wedge indecomposability is less interesting in the case of spaces). It is a very interesting set of problems to revisit these splittings and determine which of the summands are atomic rather than just indecomposable, and which are minimal atomic rather than just atomic. The results above just give particularly elementary examples.

(ii) The suspension spectrum of $\mathbb{R}\mathrm{P}^\infty$ presents an interesting challenge. It is a standard observation that $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$ is an atomic, but not cyclic, A-module, in the sense that any A-endomorphism which is the identity on $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$ is an isomorphism. This implies that $\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$ is atomic. However, since the top cell of $\mathbb{R}\mathrm{P}^3$ splits off stably, the stable Hurewicz homomorphism $\pi_3(\Sigma^\infty\mathbb{R}\mathrm{P}^\infty) \longrightarrow H_3(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$ is non-trivial, hence $\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$ cannot be minimal atomic. It would be interesting to identify a core of $\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$. For an odd prime p, similar remarks apply to the $(p-1)$ wedge summands of $\Sigma^\infty B\mathbb{Z}/p$, one of which is $\Sigma^\infty B\Sigma_p$.

(iii) A related question, posed by Priddy and Fred Cohen, is whether $K(\mathbb{Z}/2, n)$ is stably atomic for $n \geqslant 2$. This was just recently answered in the affirmative by Powell [18]. Presumably the $p-1$ summands of $\Sigma K(\mathbb{Z}/p, n)$, $p > 2$, are also stably atomic for $n \geqslant 2$.

(iv) Example 5.6 is somewhat artificial in that Y is obviously not atomic. The referee asks whether or not an atomic complex must have only finitely many cores. We guess not.

6. A construction of the spectrum BoP

In this section, all spectra are understood to be localized at 2, and $S = S^0$. Recall the spectrum BoP of Pengelley [17]. It has no homotopy detected by mod 2 homology [17, 5.5] and it is a retract of $M\mathrm{SU}$, so we have a monomorphism $j : BoP \longrightarrow M\mathrm{SU}$.

Example 6.1. The monomorphism $j\colon BoP \longrightarrow M\mathrm{SU}$ is a core of MSU.

We recall a further property of BoP, proven in Pengelley [17, 6.15, 6.16].

Proposition 6.2. *There is a map $p : BoP \longrightarrow ko$ that induces an epimorphism on homotopy groups in all degrees and an isomorphism in odd degrees.*

Corollary 6.3. *The odd degree homotopy groups of the fiber Fp are zero.*

We now give a description of BoP as a nuclear spectrum, thus providing a simple construction of it that is independent of [17]. Guided by Proposition 6.2, we construct a nuclear spectrum X and a map $q : X \longrightarrow ko$ that induces a monomorphism on homotopy groups in odd degrees, and we prove that it induces an epimorphism on homotopy groups. That turns out to imply that X is equivalent to BoP.

We begin with $X_0 = S$, and we inductively attach even dimensional cells, letting $X_{2n} = X_{2n+1}$ for all $n \geqslant 0$. Suppose that we have factored the unit $\iota : S \longrightarrow ko$ through a map $q_n\colon X_{2n-1} \longrightarrow ko$. We enlarge X_{2n-1} to X_{2n} by attaching $2n$-cells minimally, so that (2.2) is satisfied. We do this so as to kill the kernel of

$$q_{n*} : \pi_{2n-1}(X_{2n-1}) \longrightarrow \pi_{2n-1}(ko).$$

Thus, in the resulting cofiber sequence

$$J_{2n-1} \longrightarrow X_{2n-1} \longrightarrow X_{2n},$$

$$\mathrm{Im}(\pi_{2n-1}(J_{2n-1}) \longrightarrow \pi_{2n-1}(X_{2n-1})) = \mathrm{Ker}(\pi_{2n-1}(X_{2n-1}) \longrightarrow \pi_{2n-1}(ko)).$$

Clearly q_n extends to a map

$$q_{n+1}\colon X_{2n} = X_{2n+1} \longrightarrow ko.$$

In the limit we obtain a nuclear complex X and a map $q\colon X \longrightarrow ko$ that induces an isomorphism on π_0 and a monomorphism on π_* in odd degrees.

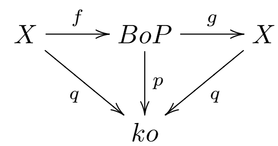
Proposition 6.4. *$q\colon X \longrightarrow ko$ induces an epimorphism on homotopy groups.*

Corollary 6.5. *The odd degree homotopy groups of the fiber Fq are zero.*

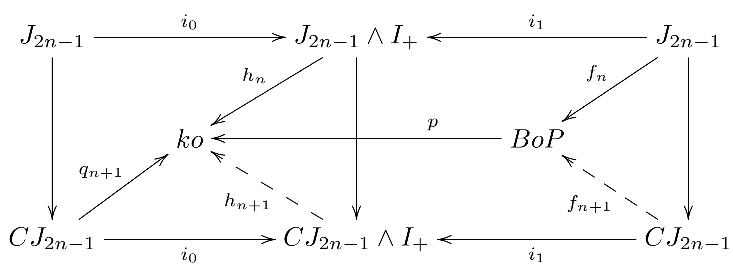
Let $\nu \in \pi_3(S)$ and $\sigma \in \pi_7(S)$ be the Hopf maps. If $x \in \pi_*(X)$ has even degree, then νx and σx are odd degree elements of the kernel of q_*, hence they are zero. The proposition is therefore a direct consequence of the following result, which is presumably known. Since we do not know of a reference for it, we will give a proof in the next section.

Proposition 6.6. *Let X be a Hurewicz complex of dimension 0 with inclusion of the bottom cell $i\colon S \longrightarrow X$ and let $q\colon X \longrightarrow ko$ be a map such that the composite $S \xrightarrow{i} X \xrightarrow{q} ko$ is the unit $\iota : S \longrightarrow ko$. If $\nu x = 0$ and $\sigma x = 0$ in $\pi_*(X)$ for every even degree element $x \in \pi_*(X)$, then $q_*\colon \pi_*(X) \longrightarrow \pi_*(ko)$ is an epimorphism.*

Theorem 6.7. *There are equivalences $f\colon X \longrightarrow BoP$ and $g\colon BoP \longrightarrow X$ such that the following diagram is homotopy commutative.*



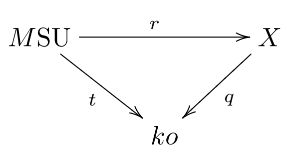
Proof. We construct maps f and g such that the diagram is homotopy commutative. The maps f and g, hence also the composites $g \circ f$ and $f \circ g$, then induce isomorphisms on π_0. Since X and BoP are atomic, these composites are equivalences and therefore f and g are equivalences. We may take BoP and ko to be Hurewicz complexes and take p to be the identity map on the bottom cell. Taking $f_0\colon X_0 = X_1 = S \longrightarrow BoP$ to be the identity map on the bottom cell and h_0 to be the constant homotopy at the identity map, we assume inductively that we have a map $f_n\colon X_{2n-1} \longrightarrow BoP$ and a homotopy $h_n\colon q_n \simeq p \circ f_n$. Consider the following diagram, where we implicitly precompose maps already specified with the map of cells $CJ_{2n-1} \longrightarrow X_{2n+1}$ that constructs X_{2n+1} from X_{2n-1}.



Since J_{2n-1} is a wedge of $(2n-1)$-spheres and $\pi_{2n-1}(Fp) = 0$, $[J_{2n-1}, Fp] = 0$. A standard result, given in just this form in [16, Lemma 1], shows that there are maps f_{n+1} and h_{n+1} that make the diagram commute. Passing to colimits, we obtain f and a homotopy $h\colon q \simeq p \circ f_n$. Since the homology groups of BoP are concentrated in even degrees [17], we can replace it by a minimal complex, with cells only in even degrees. This allows us to reverse the roles of X and BoP to construct g. □

A similar argument proves the following result.

Proposition 6.8. *There is a map $r\colon M\mathrm{SU} \longrightarrow X$ such that the following diagram is homotopy commutative.*



It is not clear that BoP is the only core of $M\mathrm{SU}$ up to equivalence, but we conjecture that it is. The following consequence of Lemma 6.6 may shed some light on this question.

Proposition 6.9. *If $Y \longrightarrow M\mathrm{SU}$ is a core, the composite $Y \longrightarrow M\mathrm{SU} \longrightarrow ko$ induces an epimorphism on homotopy groups.*

Remark 6.10. It might be of interest to revisit the results of [11, 17] from our present perspective. However, it is not clear how to construct a map $X \longrightarrow M\mathrm{SU}$ that induces the identity on π_0 and how the distinguished map of [11] fits in. It might be of more interest to revisit the results of [11, 17] from the perspective of S-modules [7]. Pengelley constructs BoP by first constructing another spectrum, which he denotes by X, and then taking a fiber to kill BP summands in it. His X is obtained from $M\mathrm{SU}$ by using the Baas-Sullivan theory of manifolds with singularities to kill a regular sequence of elements in $\pi_*(M\mathrm{SU})$. We can instead use the results of [7, Ch. V] to construct X as an $M\mathrm{SU}$-module together with a map of $M\mathrm{SU}$-modules $M\mathrm{SU} \longrightarrow X$. It seems plausible that the methods of [7, 22] can be used to construct BoP as a commutative $M\mathrm{SU}$-ring spectrum.

7. The proof of Proposition 6.6

We continue to work with spectra localized at 2. Recall that

$$\pi_*(ko) = \mathbb{Z}_{(2)}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta), \tag{7.1}$$

where $\deg \eta = 1$, $\deg \alpha = 4$, and $\deg \beta = 8$. We will describe elements of $\pi_*(X)$ that map to each of the additive generators of $\pi_*(ko)$. Note that, since we do not know that X is a ring spectrum, we cannot exploit the algebra structure of $\pi_*(ko)$. The essential point is to describe additive generators in terms of Toda brackets in $\pi_*(ko)$ that admit analogues in $\pi_*(X)$.

We are interested in Toda brackets of the form $\langle a, b, c\rangle$, where a and b are elements of $\pi_*(S)$ and c is an element of $\pi_*(Y)$ for a spectrum Y. We require $ab = 0$ and $bc = 0$, and then $\langle a, b, c\rangle$ is a coset of elements in $\pi_{|a|+|b|+|c|+1}(Y)$ with respect to the indeterminacy subgroup

$$\text{indeter}\,\langle a, b, c\rangle = a\pi_{|b|+|c|+1}(Y) + (\pi_{|a|+|b|+1}(S))c.$$

Such Toda brackets are natural with respect to maps $Y \longrightarrow Z$.

Remark 7.2. We remark parenthetically that the theory of Toda brackets simplifies greatly if one defines them in terms of the associative smash product in one of the modern categories of spectra, such as the category of S-modules of [7]. A systematic exposition would be of value. In brief, the conclusion must be that all of the results that are catalogued in [15] for matric Massey products in the homology of DGA's carry over verbatim to S-modules.

Now take X as in Proposition 6.6. Recall that $8\nu = 0$ and $16\sigma = 0$ in $\pi_*(S)$ and that, by hypothesis, ν and σ annihilate all even degree elements of $\pi_*(X)$. Let b_0 denote $i : S \longrightarrow X$ regarded as an element of $\pi_0(X)$ and choose coset representatives in iterated Toda products as follows:

$$a_1 \in \langle 8, \nu, b_0\rangle\,, \quad b_k \in \langle 16, \sigma, b_{k-1}\rangle\,, \quad \text{and} \quad a_{k+1} \in \langle 16, \sigma, a_k\rangle\,,$$

where $k \geqslant 1$. The indeterminacies are benign for our purposes since they are

$$\begin{aligned}
\text{indeter}\, a_1 &= (\pi_4(S))b_0 + 8\pi_4 X = 8\pi_4(X),\\
\text{indeter}\, b_k &= (\pi_8(S))b_{k-1} + 16\pi_{8k}(X) \equiv 16\pi_{8k}(X) \bmod \mathrm{Ker}(q_*)\\
\text{indeter}\, a_k &= (\pi_8(S))a_{k-1} + 16\pi_{8k-4}(X) \equiv 16\pi_{8k-4}(X) \bmod \mathrm{Ker}(q_*).
\end{aligned}$$

Here the congruences hold since $\pi_8(S)$ is 2-torsion and there are no torsion elements in the relevant degrees of $\pi_*(ko)$. For $k \geqslant 0$, we also have the elements

$$\mu_{8k+1}b_0 \in \pi_{8k+1}(X) \quad \text{and} \quad \mu_{8k+2}b_0 \in \pi_{8k+2}(X),$$

where μ_{8k+1} and μ_{8k+2} are the usual elements in $\pi_*(S)$. Now $q_* \colon \pi_*(X) \longrightarrow \pi_*(ko)$ maps these elements to elements of the same form in $\pi_*(ko)$, where $b_0 \in \pi_0(ko)$ is the unit of ko. In the familiar periodic pattern $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, the additive positive degree generators of $\pi_*(ko)$ are

$$\eta\beta^k = \mu_{8k+1}b_0, \quad \eta^2\beta^k = \mu_{8k+2}b_0, \quad \alpha\beta^k, \quad \text{and} \quad \beta^{k+1},$$

where $k \geqslant 0$. The following known result gives that $\alpha\beta^k = a_{k+1}$ and $\beta^{k+1} = b_{k+1}$ in $\pi_*(ko)$, and this completes the proof that q_* is an epimorphism.

Lemma 7.3. *In $\pi_*(ko)$,*

$$\alpha \in \langle 8, \nu, b_0 \rangle, \quad \beta^k \in \left\langle 16, \sigma, \beta^{k-1} \right\rangle, \quad \textit{and} \quad \alpha\beta^k \in \left\langle 16, \sigma, \alpha\beta^{k-1} \right\rangle$$

for $k \geqslant 1$, where the indeterminancy is $0 \bmod 2$ in each case.

An unstable version of the lemma is stated without proof in [24, p.64], where it is attributed to Barratt. One quick way to see the result is to use the convergence of Massey products to Massey products in the May spectral sequence and of Massey products to Toda brackets in the Adams spectral sequence, but the details would take us too far afield.

Appendix A. Irreducibility and k-invariants, by R. Pereira

We must prove that Y is irreducible if and only if

$$k_*^{n+2} \colon H_{n+2}(Y[n]; \mathbb{F}_p) \longrightarrow H_{n+2}(K(\pi_{n+1}(Y), n+2); \mathbb{F}_p)$$

is an epimorphism for each $n \geq n_0$. Write $K = K(\pi_{n+1}(Y), n+2)$ for brevity and observe that we have the following fibration sequence.

$$\Omega K \xrightarrow{\iota} Y[n+1] \longrightarrow Y[n] \xrightarrow{k^{n+2}} K$$

By the naturality of the Hurewicz homomorphism, the map ι gives rise to the following commutative diagram.

$$\begin{array}{ccc} \pi_{n+1}(\Omega K) & \xrightarrow{\cong} & \pi_{n+1}(Y[n+1]) \\ \downarrow & & \downarrow h \\ H_{n+1}(\Omega K; \mathbb{F}_p) & \xrightarrow[\iota_*]{} & H_{n+1}(Y[n+1]; \mathbb{F}_p) \end{array}$$

Here $H_{n+1}(\Omega K, \mathbb{F}_p) \cong \pi_{n+1}(Y) \otimes \mathbb{F}_p$ and the left arrow is just reduction mod p. Clearly h is zero if and only if ι_* is zero. By Proposition 4.3 and the equivalence of (i) and (ii) in Theorem 1.3, it suffices to show that ι_* is zero if and only if k_*^{n+2} is an epimorphism. To accomplish this, we look at the edge homomorphism of the Serre spectral sequence

$$H_*(Y[n]; H_*(\Omega K; \mathbb{F}_p)) \Longrightarrow H_*(Y[n+1]; \mathbb{F}_p).$$

Our map ι_* is the map $E^2_{0,n+1} \longrightarrow E^\infty_{0,n+1}$ given by taking successive quotients of $E^2_{0,n+1}$ by images of differentials. However, the only non-zero differential with image in $E^2_{0,n+1}$ is the transgression

$$d^{n+2}\colon H_{n+2}(Y[n];\mathbb{F}_p) \longrightarrow H_{n+1}(\Omega K;\mathbb{F}_p).$$

Thus ι_* is zero if and only if d^{n+2} is surjective. Essentially, this differential is k^{n+2}_* since the map of fibrations

$$\begin{array}{ccccc} \Omega K & \longrightarrow & Y[n+1] & \longrightarrow & Y[n] \\ \| & & \downarrow & & \downarrow{\scriptstyle k^{n+2}} \\ \Omega K & \longrightarrow & PK & \longrightarrow & K \end{array}$$

gives a commutative diagram

$$\begin{array}{ccc} H_{n+2}(Y[n];\mathbb{F}_p) & \xrightarrow{d^{n+2}} & H_{n+1}(\Omega K;\mathbb{F}_p) \\ {\scriptstyle k^{n+2}_*}\downarrow & & \| \\ H_{n+2}(K;\mathbb{F}_p) & \xrightarrow[d^{n+2}]{} & H_{n+1}(\Omega K;\mathbb{F}_p) \end{array}$$

in which the bottom arrow d^{n+2} is an isomorphism. Thus the top arrow d^{n+2} is surjective if and only if k^{n+2}_* is surjective and we have proven the result.

Appendix B. Errata to [10]

We take this opportunity to correct some minor errors in the proof of [10, 2.11]. In brief, the last two sentences of the cited proof should be replaced with the following two sentences. "If $p = 2$, then $Q^8(a_1) \equiv a_5$ mod decomposables, and, if $p > 2$, then $Q^{2p}(a_{p-1}) \equiv a_{(2p+1)(p-1)}$ mod decomposables, by [19] or [4, II.8.1]. Here a_{p-1} is in the image of $H_*(BP)$, but $H_*(BP)$ has no indecomposable elements in degree 10 if $p = 2$ or in degree $2(2p + 1)(p - 1)$ if $p > 2$."

References

[1] J. F. Adams. Stable Homotopy and Generalised Homology. University of Chicago Press (1974).

[2] J.F. Adams and N.J. Kuhn. Atomic spaces and spectra. Proc. Edinburgh Math. Society **32** (1989), 473-481.

[3] D. Carlisle, P. Eccles, S. Hilditch, N. Ray, L. Schwartz, G. Walker, and R. Wood. Modular representations of $GL(n,p)$, splitting $\Sigma(CP^\infty \times \cdots \times CP^\infty)$, and the β-family as framed hypersurfaces. Math. Z. **189** (1985), 239–261.

[4] F. R. Cohen, T. J. Lada, and J.P. May. The homology of iterated loop spaces. Lecture Notes in Mathematics **533**. Springer-Verlag. 1976.

[5] G. E. Cooke. Embedding certain complexes up to homotopy type in euclidean space. Ann. of Math. **90** (1969), 144–156.

[6] G. E. Cooke & L. Smith. Mod p decompositions of co H-spaces and applications. Math. Z. **157** (1977), 155–177.

[7] A. Elmendorf, I. Kriz, M. Mandell and J. P. May. Rings, modules, and algebras in stable homotopy theory. Mathematical Surveys and Monographs **47** (1997).

[8] A. Hatcher. Algebraic topology. Cambridge University Press (2002).

[9] R. Holzsager. Stable splitting of $K(G, 1)$. Proc. Amer. Math. Soc. **31** (1972), 305–306.

[10] P. Hu, I. Kriz, and J. P. May. Cores of spaces, spectra and E_∞ ring spectra. Homology, Homotopy and Applications **3** (2001), 341–54.

[11] S. O. Kochman. The ring structure of BoP_*. Contemp. Math. **146** (1993), 171–197.

[12] N.J. Kuhn. A Kahn-Priddy sequence and a conjecture of G. W. Whitehead. Math. Proc. Cambridge Philos. Soc. 92 (1982), 467–483. (Corrigenda. Math. Proc. Cambridge Philos. Soc. 95 (1984), 189–190.)

[13] N.J. Kuhn and S.B. Priddy. The transfer and Whitehead's conjecture. Math. Proc. Cambridge Philos. Soc. 98 (1985), 459–480.

[14] M. Mahowald and S. Priddy, editors. Homotopy theory via algebraic geometry and group representations. Contemporary Math. Vol. 220. Amer. Math. Soc. 1998.

[15] J.P. May. Matric Massey products. J. Algebra **12** (1969), 533–568.

[16] J.P. May. The dual Whitehead theorems. In Topological topics, edited by I.M. James. London Math. Society Lecture Note Series 86. 1983.

[17] D. J. Pengelley. The homotopy type of MSU. Amer. J. Math. **104** (1982), 1101–1123.

[18] G. Powell. On the rigidity of $H^*(K(F_2, n); F_2)$ in the category of unstable modules. Preprint (May 2003). Available from http://zeus.math.univ-paris13.fr/∼powell/home/preprints.html.

[19] S. Priddy. Dyer-Lashof operations for the classifying spaces of certain matrix groups. Quarterly J. Math. **26** (1975), 179–193.

[20] S. Priddy. A cellular construction of BP and other irreducible spectra. Math. Z. **173** (1980), 29–34.

[21] D.M. Segal. On the stable homotopy of quaternionic and complex projective spaces. Proc. Amer. Math. Soc. **25** (1970), 838–841.

[22] N.P. Strickland. Products on MU-modules. Trans. Amer. Math. Soc. **351** (1999), 2569–2606.

[23] G. Walker, Estimates for the complex and quaternionic James numbers, Quart. J. Math. **32** (1981), 467–489.

[24] G. W. Whitehead. Recent advances in homotopy theory. Amer. Math. Soc. Conf. Board of the Mathematical Sciences Regional Conference Series in Mathematics **5** (1970).

[25] C. Wilkerson. Genus and cancellation. Topology **14** (1975), 29–36.

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