CATERADS AND MOTIVIC COHOMOLOGY

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Abstract. We define "weighted multiplicative presheaves" and observe that there are several weighted multiplicative presheaves that give rise to motivic cohomology. By neglect of structure, weighted multiplicative presheaves give symmetric monoids of presheaves. We conjecture that a suitable stabilization of one of the symmetric monoids of motivic cochain presheaves has an action of a caterad of presheaves of acyclic cochain complexes, and we give some fragmentary evidence. This is a snapshot of work in progress.

Contents

Introduction

The first four sections set the stage for discussion of a conjecture about the multiplicative structure of motivic cochain complexes. In §1, we define the notion of a "weighted multiplicative presheaf", abbreviated WMP. This notion codifies formal structures that appear naturally in motivic cohomology and presumably elsewhere in algebraic geometry. We observe that WMP's determine symmetric monoids of presheaves, as defined in [11, 1.3], by neglect of structure.

In §§2–4, we give an outline summary of some of the constructions of motivic cohomology developed in [13, 17] and state the basic comparison theorems that relate them to each other and to Bloch's higher Chow complexes. There is a labyrinth of definitions and comparisons in this area, and we shall just give a brief summary overview with emphasis on product structures. The now standard construction is given in §2, a variant form of this construction is given in §3, and another construction is given in §4. There result three symmetric monoids of presheaves that give rise to motivic cohomology after cohomological reindexing and shifts of grading. All are sheaves in the Zariski topology, and they are quasi-isomorphic. For

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one of these motivic cochain complexes, there is a Mayer-Vietoris or cohomological descent theorem that says that hypercohomology can generally be computed as the naive cohomology of the cochains on the global sections. For the others, hypercohomology is essential.

Motivic cohomology is a bigraded algebra, as is the homology of any symmetric monoid of presheaves (see Corollary 1.7). Moreover, as Voevodsky proved [13, 15.9], it is commutative in the appropriate graded sense. This sets the stage. We want to understand the commutativity as a consequence of some conceptual internal structure on the motivic cochains. The paradigm is that the commutativity of the cohomology of spaces is just a first algebraic implication of the action of an acyclic operad on singular cochains. That action gives rise to Steenrod operations, among other implications. Similarly, the commutativity of motivic cohomology should be a first algebraic implication of analogous structure on motivic cochains that leads to Steenrod operations.

Originally, it was expected that the appropriate conceptual framework would involve an action of an acyclic operad on motivic cochains, which would imply that the motivic cochain complex is an E_∞ algebra. This seems especially plausible since Kriz and May proved some years ago [7, II§6] that Bloch's higher Chow cochain complexes [2, 3], which have partially defined commutative products, are quasi-isomorphic to E_∞ algebras. However, analysis of the definitions and comparison with Voevodsky's Steenrod operations leads us to believe that the caterad algebras of [11] are likely to provide a more appropriate structural framework, and it is the purpose of this note to explain that intuition. If the intuition is correct, it will result in much more algebraic structure than the expected operad action could provide. In §5 and §6, we give some idea of what should be involved in proving our conjecture that there is an acyclic caterad that acts on a suitably stabilized version of one of the motivic cochain complexes.

In §7, we speculate on Steenrod operations. We do not understand the situation. It seems that if our conjecture is right, it would give different families of operations than those developed by Voevodsky and used in his proof of the Milnor conjecture [14, 15]. In algebraic topology, one can obtain Steenrod operations either from an action of an acyclic operad on singular cochains or by inspection of the effect of the map

$$B\Sigma_p \times X = E\Sigma_p \times_{\Sigma_p} X \longrightarrow E\Sigma_p \times_{\Sigma_p} X^p$$

on mod p cohomology, where Σ_p is the symmetric group and the map is induced by the diagonal $X \longrightarrow X^p$. The operations agree up to sign by a direct comparison of definitions (e.g. [8, p. 206]). Voevodsky's definition of Steenrod operations proceeds by analogy with the second definition. Our definitions pursue the first definition, and it appears that it may give different operations.

May thanks Spencer Bloch, Mike Mandell, Fabien Morel, Madhav Nori, Vladimir Voevodsky, and Chuck Weibel for conversations and e-mails that helped him sort out what little he knows about this area of mathematics. His attempts to understand Joshua's unsuccessful attempt [6] at a motivic operad action led him to the original formulation of the conjecture here. He posted a first version of this note on his web page December 25, 2003. Guillou found a mistake in a parenthetical, but motivating, example (now deleted) and collaborated on working out the more precise and plausible version of the original conjecture described in §5 and §6.

1. Weighted multiplicative presheaves

Let $\mathscr{S}$ denote the category Sm/k of smooth separated schemes of finite type over a field k. Let $\mathbb{Z}$ denote the constant Abelian presheaf at $\mathbb{Z}$. Recall from [11, 1.2] that a symmetric sequence F in any category $\mathscr{C}$ is just a sequence of objects $F(q)$ with (left) actions by the symmetric group Σ_q. We write $F(\sigma)$ for the action of $\sigma \in \Sigma_q$. When $\mathscr{C}$ is symmetric monoidal with unit object κ, we have the notion of a symmetric monoid F in $\mathscr{C}$ [11, 1.3]. It is a symmetric sequence F together with a unit $\lambda\colon \kappa \longrightarrow F(0)$ and pairings $\phi = \phi_{q,r}\colon F(q) \otimes F(r) \longrightarrow F(q+r)$ such that the evident unit, associativity, and commutativity diagrams commute. In presheaf contexts, these arise naturally from the following structure.

Definition 1.1. A *weighted multiplicative presheaf* $\mathscr{F}$ on $\mathscr{S}$ (or any other site) is a symmetric sequence of Abelian presheaves on $\mathscr{S}$ together with a unit map $\lambda\colon \mathbb{Z} \longrightarrow \mathscr{F}(0)$, where $\mathbb{Z}$ is the constant presheaf at $\mathbb{Z}$, and "external" pairings

$$\phi = \phi_{q,r} : \mathscr{F}(q)(U) \otimes \mathscr{F}(r)(V) \longrightarrow \mathscr{F}(q+r)(U \times V)$$

of Abelian groups for schemes U and V in $\mathscr{S}$ which satisfy the following properties.

(i) The map $\phi_{q,r}$ is $\Sigma_q \times \Sigma_r$-equivariant.

(ii) The following unit diagrams commute.

$$\begin{array}{ccc} \mathscr{F}(q)(U) \otimes \mathbb{Z} & = & \mathscr{F}(q)(U) \\ {\scriptstyle \mathrm{id} \otimes \lambda}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(\pi_U)} \\ \mathscr{F}(q)(U) \otimes \mathscr{F}(0)(V) & \xrightarrow[\phi]{} & \mathscr{F}(q)(U \times V) \end{array} \qquad \begin{array}{ccc} \mathbb{Z} \otimes \mathscr{F}(q)(V) & = & \mathscr{F}(q)(V) \\ {\scriptstyle \lambda \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(\pi_V)} \\ \mathscr{F}(0)(U) \otimes \mathscr{F}(q)(V) & \xrightarrow[\phi]{} & \mathscr{F}(q)(U \times V) \end{array}$$

(iii) The following associativity diagrams commute.

$$\begin{array}{ccc} \mathscr{F}(p)(U) \otimes \mathscr{F}(q)(V) \otimes \mathscr{F}(r)(W) & \xrightarrow{\phi \otimes \mathrm{id}} & \mathscr{F}(p+q)(U \times V) \otimes \mathscr{F}(r)(W) \\ {\scriptstyle \mathrm{id} \otimes \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathscr{F}(p)(U) \otimes \mathscr{F}(q+r)(V \times W) & \xrightarrow[\phi]{} & \mathscr{F}(p+q+r)(U \times V \times W) \end{array}$$

(iv) The following commutativity diagrams commute.

$$\begin{array}{ccc} \mathscr{F}(q)(U) \otimes \mathscr{F}(r)(V) & \xrightarrow{\phi} & \mathscr{F}(q+r)(U \times V) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(\tau_{q,r})\mathscr{F}(q+r)(t)} \\ \mathscr{F}(r)(V) \otimes \mathscr{F}(q)(U) & \xrightarrow[\phi]{} & \mathscr{F}(q+r)(V \times U). \end{array}$$

On the left, $\tau(x \otimes y) = y \otimes x$. On the right, $t\colon U \times V \longrightarrow V \times U$ is the transposition. Since $\mathscr{F}(\tau_{q,r})$ is natural,

$$\mathscr{F}(\tau_{q,r})\mathscr{F}(q+r)(t) = \mathscr{F}(q+r)(t)\mathscr{F}(\tau_{q,r}).$$

We generally abbreviate weighted multiplicative presheaf to "WMP". The "weight" q will keep track of bigrading when we pass to homology.

The category $\mathrm{AbPre}(\mathscr{S})$ of Abelian presheaves is symmetric monoidal with respect to the diagonal or "internal" tensor product specified by

$$(\mathscr{F} \otimes \mathscr{G})(U) = \mathscr{F}(U) \otimes \mathscr{G}(U).$$

We also have the "external" tensor product $\mathscr{F} \boxtimes \mathscr{G}$ specified by

$$(\mathscr{F} \boxtimes \mathscr{G})(U \times V) = \mathscr{F}(U) \otimes \mathscr{G}(V).$$

It is a presheaf defined on $\mathscr{S} \times \mathscr{S}$. Similarly, for a presheaf $\mathscr{F}$, the $\mathscr{F}(U \times V)$ specify a presheaf, denoted $\mathscr{F} \circ \times$, on $\mathscr{S} \times \mathscr{S}$. The pairings ϕ of a WMP are maps $\mathscr{F}(q) \boxtimes \mathscr{F}(r) \longrightarrow \mathscr{F}(q+r) \circ \times$ of presheaves on $\mathscr{S} \times \mathscr{S}$.

Write $\triangle$ for the diagonal maps of objects of $\mathscr{S}$. The "external pairings" ϕ of a WMP $\mathscr{F}$ pull back along maps $\mathscr{F}(\triangle)$ to give "internal pairings", which we also denote by ϕ:

$$\phi : \mathscr{F}(q)(U) \otimes \mathscr{F}(r)(U) \longrightarrow \mathscr{F}(q+r)(U). \tag{1.2}$$

These give a map $\mathscr{F}(q) \otimes \mathscr{F}(r) \longrightarrow \mathscr{F}(q+r)$ of presheaves on $\mathscr{S}$. Comparing the definition of a WMP with the definition of a symmetric monoid in AbPre($\mathscr{S}$), as given in [11, 1.3], we have the following observation.

Proposition 1.3. *By pullback along diagonal maps, a WMP internalizes to a symmetric monoid in AbPre($\mathscr{S}$).*

Remark 1.4. Explicitly, after setting the variable schemes equal and using the maps $\mathscr{F}(\triangle)$, the unit, associativity, and commutativity diagrams in the definition of a WMP transform to the following commutative diagrams.

$$\begin{array}{ccc} \mathscr{F}(q)(U) \otimes \mathbb{Z} & & \\ \downarrow{\scriptstyle \mathrm{id} \otimes \lambda} & \searrow{=} & \\ \mathscr{F}(q)(U) \otimes \mathscr{F}(0)(U) & \xrightarrow[\phi]{} & \mathscr{F}(q)(U) \end{array} \qquad \begin{array}{ccc} \mathbb{Z} \otimes \mathscr{F}(q)(U) & & \\ \downarrow{\scriptstyle \lambda \otimes \mathrm{id}} & \searrow{=} & \\ \mathscr{F}(0)(U) \otimes \mathscr{F}(q)(U) & \xrightarrow[\phi]{} & \mathscr{F}(q)(U) \end{array}$$

$$\begin{array}{ccc} \mathscr{F}(p)(U) \otimes \mathscr{F}(q)(U) \otimes \mathscr{F}(r)(U) & \xrightarrow{\phi \otimes \mathrm{id}} & \mathscr{F}(p+q)(U) \otimes \mathscr{F}(r)(U) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \phi} & & \downarrow{\scriptstyle \phi} \\ \mathscr{F}(p)(U) \otimes \mathscr{F}(q+r)(U) & \xrightarrow[\phi]{} & \mathscr{F}(p+q+r)(U) \end{array}$$

$$\begin{array}{ccc} \mathscr{F}(q)(U) \otimes \mathscr{F}(r)(U) & \xrightarrow{\phi} & \mathscr{F}(q+r)(U) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathscr{F}(\tau_{q,r})} \\ \mathscr{F}(r)(U) \otimes \mathscr{F}(q)(U) & \xrightarrow[\phi]{} & \mathscr{F}(q+r)(U) \end{array}$$

The transposition (of U and V) has disappeared on the right, since $t\triangle = \triangle$, but $\mathscr{F}(\tau_{q,r})$ remains. This is the source of noncommutativity in this situation.

Our immediate goal is to explain the relevance of WMP's to motivic cohomology. Recall from [10, §6] that we have the composite "singular" chain complex functor $C_* = K \circ (-)_\bullet$ from presheaves of Abelian groups to presheaves of chain complexes. We shall also write C_* for its composite with the free Abelian group functor $\mathbb{Z}$ from presheaves of sets to presheaves of Abelian groups. As explained in [10, 1.4, 6.2], the functors $(-)_\bullet$, K, $\mathbb{Z}$, and therefore C_* (in both senses) are lax symmetric monoidal. For a smooth scheme X and an Abelian group A, we write

$$C_*(\mathscr{F}(X), A) = C_*(\mathscr{F})(X) \otimes A.$$

Definition 1.5. For a smooth scheme X, an Abelian presheaf $\mathscr{F}$, and an Abelian group A, let $h_p(X, \mathscr{F} \otimes A)$ denote the homology group in degree p of the chain complex $C_*(\mathscr{F}(X), A)$. Delete A from the notation when $A = \mathbb{Z}$.

The use of lower case h rather than upper case H is suggested by [17, IV§8]. It is meant to emphasize that we are just looking naively at the homology of the global sections functor and not taking any topology on $\mathscr{S}$ into account.

Since C_* is a lax symmetric monoidal functor, it preserves symmetric monoids.

Proposition 1.6. *A symmetric monoid $\mathscr{F}$ of Abelian presheaves determines a symmetric monoid $\mathscr{F}_\bullet$ of presheaves of simplicial Abelian groups and a symmetric monoid $C_*(\mathscr{F})$ of presheaves of chain complexes.*

Corollary 1.7. *Let $\mathscr{F}$ be a symmetric monoid of Abelian presheaves and R be a commutative ring. Then the groups $h_*(X, \mathscr{F} \otimes R)$ assemble into a bigraded ring with products*

$$h_p(X, \mathscr{F}(q) \otimes R) \otimes h_{p'}(X, \mathscr{F}(r) \otimes R) \longrightarrow h_{p+p'}(X, \mathscr{F}(q+r) \otimes R).$$

Remark 1.8. The product is made explicit in [10, 6.5]. Note that, because of $\mathscr{F}(\tau_{q,r})$ in the commutativity diagram, there is nothing in the definitions to ensure that this ring is (graded) commutative; if it were, the expected sign would be $(-1)^{pp'}$, independent of the weight grading. To see how real the non-commutativity is in general, take any Abelian presheaf $\mathscr{F}$ and consider the free "tensor algebra" symmetric monoid $T(\mathscr{F})$ in $\mathrm{AbPre}(\mathscr{S})$ given by [11, 1.5].

2. The standard motivic cochain complex

In this and the following two sections, we briefly review the definitions of cochain presheaves that define motivic cohomology. In fact, we define three symmetric monoids of Abelian presheaves that give rise to motivic cohomology. We follow [13, 17], where details may be found, and we largely adhere to the notations of those sources.

We first review the category SmCor/k of *smooth correspondences* over k, which is also denoted Cor_k. We abbreviate notation by writing $\mathscr{S}\mathscr{C}$ for this category. It is an additive symmetric monoidal category with the same objects as $\mathscr{S}$.

For smooth schemes U and V, $\mathscr{S}\mathscr{C}(U, V)$ is the free Abelian group generated by the closed integral subschemes $Z \subset U \times V$ that are finite and surjective over a component of U; via the projections, we think of Z as a diagram $U \longleftarrow Z \longrightarrow V$. The elements of $\mathscr{S}\mathscr{C}(U, V)$ are called *finite cycles* or *finite correspondences*, and the Z are called *elementary cycles* or *elementary correspondences*. As explained in [13, 1.4–1.7], pushforward along $p\colon U \times V \times W \longrightarrow U \times W$ of the intersection $Z \times W \cap U \times Z'$ gives a well-defined finite cycle $Z' \circ Z$ associated to elementary cycles Z' and Z, and this specifies the composition pairing

$$\mathscr{S}\mathscr{C}(V, W) \otimes \mathscr{S}\mathscr{C}(U, V) \longrightarrow \mathscr{S}\mathscr{C}(U, W).$$

We have a functor $\mathrm{Gr}\colon \mathscr{S} \longrightarrow \mathscr{S}\mathscr{C}$ that is the identity on objects and sends a map of schemes to its graph. Since we are ignoring that the morphism sets of $\mathscr{S}\mathscr{C}$ are Abelian groups, we are implicitly applying the forgetful functor from Abelian groups to sets on the right, and we often prefer to regard Gr equivalently as an additive functor $\mathbb{Z}\mathscr{S} \longrightarrow \mathscr{S}\mathscr{C}$ between additive categories, where $\mathbb{Z}\mathscr{S}$ is obtained by applying the free Abelian group functor to the morphism sets of $\mathscr{S}$.

The product for the symmetric monoidal structure on $\mathscr{SC}$ is given on objects by the cartesian product of schemes and has unit $* = \mathrm{Spec}(k)$. On morphisms, it is given by the pairing

$$\phi\colon \mathscr{SC}(U,V)\otimes\mathscr{SC}(U',V') \longrightarrow \mathscr{SC}(U\times U', V\times V') \tag{2.1}$$

specified on elementary cycles by $\phi(Z\otimes Z') = Z\times Z'$. The functor $\mathrm{Gr}\colon \mathbb{Z}\mathscr{S}\longrightarrow\mathscr{SC}$ is strong symmetric monoidal, where the product

$$\phi\colon \mathbb{Z}\mathscr{S}(U,V)\otimes\mathbb{Z}\mathscr{S}(U',V')\cong\mathbb{Z}(\mathscr{S}(U,V)\times\mathscr{S}(U',V'))\longrightarrow\mathbb{Z}\mathscr{S}(U\times U',V\times V')$$

is induced by the cartesian product pairing on $\mathscr{S}$. The naturality condition $\mathrm{Gr}\circ\phi = \phi\circ(\mathrm{Gr}\otimes\mathrm{Gr})$ on morphisms is easily checked.

Additive Abelian presheaves defined on the category $\mathscr{SC}$ are called *presheaves with transfers*, and we let $\mathscr{PST}$ denote the category of presheaves with transfers. It is an Abelian category with enough projectives and injectives. Presheaves with transfers restrict to Abelian presheaves on $\mathscr{S}$. Each smooth scheme X represents a projective presheaf with transfers, which is denoted $\mathbb{Z}_{\mathrm{tr}}(X)$. Thus

$$\mathbb{Z}_{\mathrm{tr}}(X)(U) = \mathscr{SC}(U,X). \tag{2.2}$$

The pairing (2.1) can be rewritten as a pairing

$$\phi\colon \mathbb{Z}_{\mathrm{tr}}(X)(U)\otimes\mathbb{Z}_{\mathrm{tr}}(Y)(V)\longrightarrow\mathbb{Z}_{\mathrm{tr}}(X\times Y)(U\times V). \tag{2.3}$$

An essential point is that $\mathbb{Z}_{\mathrm{tr}}(X)$ is a sheaf in the Zariski topology for all X. In particular, when $X = *$, this presheaf with transfers is the Zariski sheaf $\mathbb{Z}_{\mathrm{Zar}}$. Sheafification $\lambda\colon\mathbb{Z}\longrightarrow\mathbb{Z}_{\mathrm{Zar}}$ will later provide the unit maps required for our WMP's.

Following [13, 2.10], for based schemes X and Y with basepoints x and y, we mimic topology by defining $\mathbb{Z}_{\mathrm{tr}}(X\wedge Y)$ to be the presheaf with transfers

$$\mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}X\oplus\mathbb{Z}_{\mathrm{tr}}Y\longrightarrow\mathbb{Z}_{\mathrm{tr}}(X\times Y)),$$

where the maps are induced by $y\longrightarrow Y$ and $x\longrightarrow X$. This operation is associative and can be iterated. While $\mathbb{Z}_{\mathrm{tr}}(X\wedge Y)$ is not a represented presheaf, since $X\wedge Y$ is not a scheme, it is still a projective object of $\mathscr{PST}$. The pairing (2.3) induces a pairing

$$\phi\colon \mathbb{Z}_{\mathrm{tr}}(X)(U)\otimes\mathbb{Z}_{\mathrm{tr}}(Y)(V)\longrightarrow\mathbb{Z}_{\mathrm{tr}}(X\wedge Y)(U\times V). \tag{2.4}$$

For later use, we indicate a slightly different way of thinking about $\mathbb{Z}_{\mathrm{tr}}$. We can view $\mathbb{Z}_{\mathrm{tr}}$ as an additive functor $\mathscr{SC}\longrightarrow\mathscr{PST}$ or, by restriction along Gr, as an additive functor $\mathbb{Z}\mathscr{S}\longrightarrow\mathscr{PST}$. Any Abelian presheaf X is a "weighted" colimit of representables, in the sense that evaluation maps induce isomorphisms $\int^{\mathscr{S}}\mathbb{Z}\mathscr{S}(V,U)\otimes X(U)\cong X(V)$ for $V\in\mathscr{S}$. By commutation with weighted colimits (that is, with tensors by Abelian groups and ordinary colimits), of which the displayed coend is an example, we can extend this restriction to an additive functor

$$\mathbb{Z}_{\mathrm{tr}}\colon \mathrm{AbPre}(\mathscr{S})\longrightarrow\mathscr{PST}. \tag{2.5}$$

This functor is left adjoint to the forgetful functor

$$\mathbb{U}\colon \mathscr{PST}\longrightarrow\mathrm{AbPre}(\mathscr{S}) \tag{2.6}$$

that sends presheaves with transfers to Abelian presheaves by restriction along $\mathrm{Gr}\colon\mathscr{S}\longrightarrow\mathscr{SC}$. In particular, $\mathbb{U}$ forgets that presheaves with transfers are additive functors on an additive category.

The multiplicative group scheme $\mathbb{G}_m = \mathbb{A} - \{0\}$ has the basepoint 1, and we have presheaves with transfers $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})$; by convention, $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge 0}) = \mathbb{Z}_{\mathrm{Zar}}$. The pairings (2.4) induce pairings

$$\phi\colon \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})(U) \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge r})(V) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q+r})(U \times V). \tag{2.7}$$

Proposition 2.8. *Define $\mathscr{F}(q) = \mathbb{U}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$. Then the $\mathscr{F}(q)$ and the pairings ϕ specify a weighted multiplicative presheaf $\mathscr{F}$.*

The essential point is that we must take the permutations on the smash powers $\mathbb{G}_m^{\wedge q}$ into account to correctly formulate the commutativity diagram. We take the following fundamental definition from [13, 3.1].

Definition 2.9. For $q \geq 0$, define the standard *motivic complex* $\mathbb{Z}(q)$ to be the complex of presheaves with transfers

$$\mathbb{Z}(q) = C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q}))[-q].$$

Regrading cohomologically, this gives a bounded above cochain complex with

$$\mathbb{Z}(q)^p = C_{q-p}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})). \tag{2.10}$$

Pedantically, the chain functor C_* specifies a chain functor from $\mathscr{P}\mathscr{S}\mathscr{T}$ to the category of chain complexes of presheaves with transfer, and we have the relation $\mathbb{U}C_* = C_*\mathbb{U}$, so that $\mathbb{U}C_*(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})) = C_*(\mathscr{F}(q))$. Each $\mathbb{Z}(q)$ is a complex of Zariski sheaves of free Abelian groups, hence, for any Abelian group A, $A(q) = \mathbb{Z}(q) \otimes A$ is also a complex of Zariski sheaves. The same holds after restriction to the small Zariski site of any smooth scheme X.

Definition 2.11. The *motivic cohomology groups* $H^{p,q}(X, A)$ of smooth schemes X with coefficients in Abelian groups A are the hypercohomology groups of the motivic complexes $A(q)$ with respective to the Zariski topology:

$$H^{p,q}(X, A) = \mathbb{H}^p_{\mathrm{Zar}}(X, A(q)).$$

We can also define the naive homology of $C_*(\mathscr{F}(q))$, as in the previous section, and regrade it appropriately. We thus obtain groups

$$h^{p,q}(X, A) = h_{q-p}(C_*(\mathbb{U}\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})(X), A)) = h_{q-p}(C_*(\mathscr{F}(q)(X), A)). \tag{2.12}$$

If R is a commutative ring, $h^{*,*}(X, R)$ is a bigraded ring. We have a natural ring map from this naive motivic cohomology ring to the genuine motivic cohomology ring $H^{*,*}(X, R)$, but there is no reason to expect it to be an isomorphism.

Remark 2.13. The naive product structure of Corollary 1.7 agrees with the product specified in [13, 3.10]. It follows that the induced products in hypercohomology agree.

3. A variant of the standard motivic complex

Voevodsky's work gives an implicit variant of the the standard motivic complex that fits well into our point of view. We describe it here. In [13, §8], Voevodsky defines a tensor product $\otimes^{tr}$ on the category $\mathscr{P}\mathscr{S}\mathscr{T}$ such that

$$\mathbb{Z}_{tr}\colon \mathrm{AbPre}(\mathscr{S}) \longrightarrow \mathscr{P}\mathscr{S}\mathscr{T}$$

is symmetric monoidal. The definition of $\otimes^{tr}$ starts with

$$\mathbb{Z}_{tr}X \otimes^{tr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \times Y)$$

for $X, Y \in \mathscr{S}$ and extends from there to general presheaves with transfer by passage to colimits. The forgetful functor $\mathbb{U}\colon \mathscr{PST} \longrightarrow \mathrm{AbPre}(\mathscr{S})$ is lax symmetric monoidal [13, 8.9]. Indeed, taking $U = V$ in (2.3) and pulling back along the diagonal of U, we obtain a natural map

$$\phi\colon \mathbb{Z}_{\mathrm{tr}}(X)(U) \otimes \mathbb{Z}_{\mathrm{tr}}(Y)(U) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(X \times Y)(U)$$

that specifies

$$\phi\colon \mathbb{U}\mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathbb{U}\mathbb{Z}_{\mathrm{tr}}(Y) \longrightarrow \mathbb{U}(\mathbb{Z}_{tr}X \otimes^{tr} \mathbb{Z}_{tr}(Y)).$$

Similar definitions and observations apply starting with based schemes (X, x). Here we define $\mathbb{Z}_{\mathrm{tr}}(X, x) = \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(x)$ and find that

$$\mathbb{Z}_{tr}(X, x) \otimes^{tr} \mathbb{Z}_{tr}(Y, y)) \cong \mathbb{Z}_{tr}((X, x) \wedge (Y, y)).$$

Formally, the category $\mathrm{AbPre}\mathscr{S}_*$ of based presheaves is symmetric monoidal under the smash product, and $\mathbb{Z}_{tr}$ is strong symmetric monoidal.

Definition 3.1. Define a symmetric monoid F^V in $Pre\mathscr{S}_*$ by letting $F^V(q)$ be $\mathbb{A}^q/\mathbb{A}^q - 0$ with its evident Σ_q-action and letting the product $F^V(q) \wedge F^V(r) \longrightarrow F^V(q + r)$ be induced from the identification $\mathbb{A}^q \times \mathbb{A}^r = \mathbb{A}^{q+r}$. Applying $\mathbb{Z}_{tr}$ to F^V, we obtain a symmetric monoid in $\mathscr{PST}$; applying $\mathbb{U}$ to that, we obtain a symmetric monoid $\mathscr{F}^V$ in $\mathrm{AbPre}\mathscr{S}$.

Definition 3.2. For $q \geq 0$, define the variant *motivic complex* $\mathbb{Z}^V(q)$ to be the complex of presheaves with transfers

$$\mathbb{Z}^V(q) = C_*(\mathbb{Z}_{\mathrm{tr}}(F(q)))[-2q].$$

Regrading cohomologically, this gives a bounded above cochain complex with

$$\mathbb{Z}^V(q)^p = C_{2q-p}(\mathbb{Z}_{\mathrm{tr}}(F(q))). \tag{3.3}$$

We have the following basic comparison theorem of [13, 15.2].

Theorem 3.4. *If k is perfect, the motivic complexes $\mathbb{Z}(q)$ and $\mathbb{Z}^V(q)$ are quasi-isomorphic.*

Remark 3.5. A diagram chase shows that the products on these motivic complexes give rise to the same products on motivic cohomology. In [13, 15.9], the product $H^{p,q} \otimes H^{p',r} \longrightarrow H^{p+p',q+r}$ is proven to be commutative, with the usual sign $(-1)^{pp'}$. The proof goes by showing that, for $\sigma \in \Sigma_q$, the automorphism $\mathscr{F}^V(\sigma)$ of $\mathscr{F}^V(q)$ induces the identity map on homology. This implies that the homology ring of Corollary 1.7 is commutative, and a check of the signs introduced by the grading shifts then shows that the cohomology ring is also commutative.

Remark 3.6. After sheafification in the Nisnevich (or Zariski) topology, $F^V(q)$ is equivalent to $T^{\wedge q}$, where $T = \mathbb{A}^1/\mathbb{A}^1 - 0$.

4. The Suslin-Friedlander motivic cochain complex

We next recall the *Suslin-Friedlander motivic cochain complex*, denoted $\mathbb{Z}^{SF}(q)$, from [13, §16]. For a scheme X of finite type over k, we define an Abelian presheaf $\mathbb{Z}_{\mathrm{eq}}(X)$ on $\mathscr{S}$, denoted $z_{\mathrm{equi}}(X, 0)$ in [13, 17], by letting $\mathbb{Z}_{\mathrm{eq}}(X)(U)$ be the free Abelian group generated by the closed irreducible subvarieties Z of $U \times X$ which are quasi-finite and dominant over a component of U; see [13, 16.1]. It contains $\mathbb{Z}_{\mathrm{tr}}(X)$ as a subsheaf, and the two are equal if X is projective. Again, $\mathbb{Z}_{\mathrm{eq}}(X)$ is a sheaf in the Zariski topology, and it admits transfers that make it a presheaf with transfers.

Definition 4.1. Define $\mathbb{Z}^{SF}(q)$ by letting

$$\mathbb{Z}^{SF}(q) = C_*(\mathbb{Z}_{\rm eq}\mathbb{A}^q)[-2q].$$

Regrading cohomologically, this gives a bounded above cochain complex with

$$\mathbb{Z}^{SF}(q)(X)^p = C_{2q-p}(\mathbb{Z}_{\rm eq}(\mathbb{A}^q)(X)). \tag{4.2}$$

For an Abelian group A, let $A^{SF} = \mathbb{Z}^{SF} \otimes A$.

A major theorem of Voevodsky [16] (or [13, 16.7]) reads as follows.

Theorem 4.3. *There is a zigzag of quasi-isomorphisms in the Zariski topology*

$$\mathbb{Z}(q) \simeq \mathbb{Z}^{SF}(q).$$

Therefore $H^{p,q}(X,A) = \mathbb{H}^p_{Zar}(X, A(q)) \cong \mathbb{H}^p_{Zar}(X, A^{SF}(q))$.

We can again define the concomitant naive motivic cohomology groups

$$h^{p,q}_{SF}(X,A) = h_{2q-p}(C_*(\mathbb{Z}_{eq}(\mathbb{A}^q)(X), A)) = h_{2q-p}(C_*(\mathscr{F}^{SF}(q)(X), A)). \tag{4.4}$$

In contrast to the situation in the previous sections, another major theorem, due to Friedlander and Voevodsky [17, 4.8.1], gives the following result.

Theorem 4.5. *If k satisfies resolution of singularities and X is quasi-projective, then*

$$h^{p,q}_{SF}(X,A) \cong \mathbb{H}^p_{Zar}(X, A^{SF}(q)).$$

Remark 4.6. We do not know whether or not the more recent techniques of [13] suffice to prove this result without using resolution of singularities. There is an analogous cohomological descent result there, namely [13, 19.12], which does not rely on resolution of singularities, but it applies to Bloch's higher Chow complexes rather than to the motivic cochain complexes $\mathbb{Z}^{SF}(q)$.

Just as for $\mathbb{Z}_{\rm tr}$, taking cartesian products of sub-varieties gives a pairing

$$\phi\colon \mathbb{Z}_{\rm eq}(X)(U) \otimes \mathbb{Z}_{\rm eq}(Y)(V) \longrightarrow \mathbb{Z}_{\rm eq}(X\times Y)(U \times V). \tag{4.7}$$

Since $\mathbb{A}^q \times \mathbb{A}^r = \mathbb{A}^{q+r}$, the pairing (4.7) specializes to a pairing

$$\phi\colon \mathbb{Z}_{\rm eq}(\mathbb{A}^q)(U) \otimes \mathbb{Z}_{\rm eq}(\mathbb{A}^r)(V) \longrightarrow \mathbb{Z}_{\rm eq}(\mathbb{A}^{q+r})(U \times V). \tag{4.8}$$

This leads to the Suslin-Friedlander analogue of Proposition 2.8.

Proposition 4.9. *Define $\mathscr{F}^{SF}(q) = \mathbb{U}\mathbb{Z}_{eq}(\mathbb{A}^q)$. Then the $\mathscr{F}^{SF}(q)$ and the pairings ϕ specify a weighted multiplicative presheaf $\mathscr{F}^{SF}$.*

Theorems of Voevodsky, Friedlander, and Suslin ([17], [13, 19.1], [5]) give the following comparison between motivic cohomology and Bloch's higher Chow groups $CH^q(X, 2q-p)$ [2, 3].

Theorem 4.10. *For any smooth scheme X,*

$$\mathbb{H}^p_{Zar}(X, \mathbb{Z}^{SF}(q)) \cong CH^q(X, 2q-p).$$

Remark 4.11. It is natural to ask which of these isomorphisms preserve products. Weibel [18] has shown that the isomorphism of Theorem 4.10 preserves products, and he has compared these products with several other relevant products. Although this does not seem to be made explicit in the literature, it is implicit in the proof of Theorem 4.3 that the isomorphism there also preserves products.

5. Towards caterad actions

This sets the context. We defined caterads and algebras over caterads in [11], and our conjecture is that there is an acyclic caterad $\mathscr{A}$ in AbPre($\mathscr{S}$) that acts on a kind of stabilization of the symmetric monoid $\mathscr{F}^V$. One could ask instead for an analogous action starting from the symmetric monoid obtained from the WMP $\mathscr{F}$ or the WMP $\mathscr{F}^{SF}$.

Example 5.1. Since $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q}) \cong \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m)^{\otimes^{\mathrm{tr}} q}$, the endomorphism caterad $\mathrm{End}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m))$ acts on the tensor algebra symmetric monoid $T\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m)$ in $\mathscr{P}\mathscr{S}\mathscr{T}$ (see [11, 1.5]). Thus, since $\mathbb{U}$ is lax symmetric monoidal, the caterad $\mathbb{U}\mathrm{End}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m))$ in AbPre($\mathscr{S}$) acts on the symmetric monoid associated to the WMP $\mathscr{F}$.

However, we do not see how to go from this example to an action of an acyclic caterad, and observations on grading in §7 seem to indicate that $\mathscr{F}^V$ is more appropriate than $\mathscr{F}$ for this purpose. Note that $\mathbb{Z}_{\mathrm{tr}}(X)$ is covariantly functorial on X, whereas $\mathbb{Z}_{eq}(X)$ is only covariantly functorial on proper morphisms, such as closed immersions [17, 3.6.2 in Ch.2]. This suggests that $\mathscr{F}^V$ might be simpler to work with than $\mathscr{F}^{SF}$, but we will find it convenient to think about closed immersions in any case. There is another reason that $\mathscr{F}^V$ might be more convenient, namely the fact that it arises by application of the functor $\mathbb{Z}_{tr}(X)$ to the symmetric monoid F^V in $\mathrm{Pre}\mathscr{S}_*$, as explained in Definition 3.1.

If we can obtain an action of a caterad $\mathscr{A}$ on (a stabilization of) $\mathscr{F}^V$, then, since the chain functor C_* is lax symmetric monoidal, the caterad $C_*(\mathscr{A})$ in the category ChPre($\mathscr{S}$) of presheaves of chain complexes will act on $C_*(\mathscr{F}^V)$. If $\mathscr{A}(q,q)$ is homologically connected, in the sense that any two maps $\mathbb{Z} \longrightarrow \mathscr{A}(q,q)$ induce the same map on homology, it will follow that $\mathscr{F}^V(\sigma)$ induces the identity map on homology, where $\sigma = \tau_{1,1}$ is the transpostion, and this will give a cateradic proof that motivic cohomology is commutative. Higher homological connectivity of the $\mathscr{A}(q,r)$ will induce further structure on motivic cohomology.

We describe a sensible starting point by giving an action of a caterad, the *unstable linear inclusions caterad* I in $\mathscr{S}$, on the symmetric monoid F^V in $\mathrm{Pre}\mathscr{S}_*$. The action is given by maps of presheaves

$$I(p,q)_+ \wedge F^V(p) \longrightarrow F^V(q).$$

In fact, we take $I(p,q)$ to be the classical Stiefel variety $V_p(q)$. Its k-points are p-tuples of linearly independent vectors in $\mathbb{A}^q$, and we have a principal GL_p-bundle $V_p(\mathbb{A}^q) \longrightarrow Gr_p(\mathbb{A}^q)$ for each p and q. For example, $I(p,p) = GL_p$ and $I(1,q) = \mathbb{A}^q - 0$. Adjoining disjoint basepoints, we obtain a caterad in $\mathrm{Pre}\mathscr{S}_*$.

Although these constructions are familiar, we give some details to make clear how they fit into our cateradic context. We start with an explicit affine cover for $V_p(\mathbb{A}^q)$. For each subset $I \subseteq \{1, 2, \ldots, q\}$ of cardinality p, let U_I be the open subscheme

$$U_I = \mathrm{Spec}\ k\left[x_{11}, x_{12}, \ldots, x_{qp}, \frac{1}{\det_I}\right]$$

of $\mathbb{A}^{qp}$, where $\det_I$ is the Ith $p \times p$ minor of the $q \times p$ matrix (x_{ij}). We see that each U_I is isomorphic to $Gl_p \times \mathbb{A}^{p(q-p)}$. The U_I's glue together in the obvious way to give the scheme $V_p(\mathbb{A}^q)$. Note that $V_p(\mathbb{A}^q) = \emptyset$ for $q < p$, $V_p(\mathbb{A}^p) = GL_p$, and $V_1(\mathbb{A}^q) = \mathbb{A}^q - 0$.

This explicit affine cover allows us to write down explicit maps

$$\varphi : V_p(\mathbb{A}^q) \times \mathbb{A}^p \longrightarrow \mathbb{A}^q.$$

We do this for each $I \subseteq \{1, 2, \ldots, q\}$ of cardinality p. We define

$$\varphi_I : \text{Spec } k\left[x_{11}, x_{12}, \ldots, x_{qp}, \frac{1}{\det_I}\right] \times \text{Spec } k[y_1, \ldots, y_p] \longrightarrow \text{Spec } k[z_1, \ldots, z_q]$$

by

$$z_i \mapsto \sum_j x_{ij} \otimes y_j.$$

The φ_I's patch together to define $\varphi : V_p(\mathbb{A}^q) \times \mathbb{A}^p \longrightarrow \mathbb{A}^q$. The restriction of φ to $V_p(\mathbb{A}^q) \times \mathbb{A}^p - 0$ factors through $\mathbb{A}^q - 0$, and induces the desired map

$$V_p(\mathbb{A}^q)_+ \wedge \mathbb{A}^p/(\mathbb{A}^p - 0) \longrightarrow \mathbb{A}^q/(\mathbb{A}^q - 0).$$

We must check that the $I(p, q) = V_p(\mathbb{A}^q)$ determine a caterad. The unit map $\eta\colon * = \operatorname{Spec} k \longrightarrow GL_p$ is given by the identity matrix. The composition morphisms

$$\mu\colon V_q(\mathbb{A}^r) \times V_p(\mathbb{A}^q) \longrightarrow V_p(\mathbb{A}^r)$$

are given by matrix multiplication. The composition is then unital and associative. The underlying hom sets are given by the k-points, that is the classical Stiefel varieties over k. The homomorphism $\Sigma_q \longrightarrow GL_q$ is given by the usual embedding as the Weyl subgroup. Finally, the product on morphisms

$$\phi\colon V_q(\mathbb{A}^r) \times V_{q'}(\mathbb{A}^{r'}) \longrightarrow V_{q+q'}(\mathbb{A}^{r+r'})$$

is given by block sum of matrices. It is easily checked that these maps satisfy the compatibility conditions required of a caterad and that the maps φ define an action of I on F^V.

6. The stable linear inclusions caterad

Fix p, q, m, and n. We have a map $(\mathbb{A}^m)^q \longrightarrow (\mathbb{A}^{m+1})^q$ given by including via the first m coordinates on each factor. This induces a map $V_{pn}(\mathbb{A}^{qm}) \longrightarrow V_{pn}(\mathbb{A}^{q(m+1)})$, and so we can form

$$\operatorname{colim}_m V_{pn}(\mathbb{A}^{qm}).$$

Also, precomposing with analogous maps $\mathbb{A}^{pn} \longrightarrow \mathbb{A}^{p(n+1)}$ gives maps

$$\operatorname{colim}_m V_{p(n+1)}(\mathbb{A}^{qm}) \longrightarrow \operatorname{colim}_m V_{pn}(\mathbb{A}^{qm}).$$

We set

$$L(p, q) = \lim_n \operatorname{colim}_m V_{pn}(\mathbb{A}^{qm}).$$

This is meant to be reminiscent of the linear isometries caterad in topology, whose (p, q)th space $\mathscr{L}(p, q)$ is the space of linear isometries $(\mathbb{R}^\infty)^p \longrightarrow (\mathbb{R}^\infty)^q$.

As in the topological situation, the $L(p, q)$'s fit together to form a caterad in $\text{Pre}\mathscr{S}$. The unit map $\eta\colon * \longrightarrow L(p, p)$ is obtained by passage to colimits and limits from the maps $* \longrightarrow V_{pn}(\mathbb{A}^{pn})$ given by the identity matrix in $V_{pn}(\mathbb{A}^{pn}) = GL_{pn}$ The composition maps

$$\mu\colon L(q, r) \times L(p, q) \longrightarrow L(p, r),$$

that is,

$$(\lim_n \operatorname{colim}_m V_{qn}(\mathbb{A}^{rm})) \times (\lim_{n'} \operatorname{colim}_{m'} V_{pn'}(\mathbb{A}^{qm'})) \longrightarrow \lim_i \operatorname{colim}_j V_{pi}(\mathbb{A}^{rj}),$$

are obtained from the composition maps

$$V_{qm'}(\mathbb{A}^{rm}) \times V_{pi}(\mathbb{A}^{qm'}) \longrightarrow V_{pi}(\mathbb{A}^{rm})$$

by passage to colimits and limits. The verification that this works is routine, using that finite limits commute with filtered colimits. As in the unstable case, the homomorphism $\Sigma_p \hookrightarrow L(p,p))$ is obtained from the embedding $\Sigma_p \hookrightarrow GL_p$, and the product

$$\phi : L(q,r) \times L(q',r') \longrightarrow L(q+q',r+r')$$

is induced from block sum of matrices.

Moreover, we have the following standard lemma.

Lemma 6.1. *Each $L(p,q)$ with $q > 0$ is $\mathbb{A}^1$-contractible.*

Proof. We need some notation. Fix some $\gamma \in L(p,q)(k)$. Let $s\colon * \longrightarrow L(q,q)$ be the map that picks out the odd-numbered elements of the basis. More precisely, for fixed n we let

$$s_n\colon * \longrightarrow V_{qn}(\mathbb{A}^{q2n}) \longrightarrow \operatorname{colim}_m V_{qn}(\mathbb{A}^{q2m})$$

be the map that picks out the n odd coordinates in each factor of $\mathbb{A}^{2n}$, and these determine a map s as desired. Similarly, let $r\colon * \longrightarrow L(q,q)$ pick out the even coordinates.

We define a path $h_1 : \mathbb{A}^1 \longrightarrow L(q,q)$ by $h_1(t) = ts + (1-t)\mathrm{id}$. Let $i_1, i_2\colon * \longrightarrow L(p,2p)$ be the maps that pick out the first and last p copies of $\mathbb{A}^n$ in $\mathbb{A}^{2pn}$ and let $\beta\colon * \longrightarrow L(2q,q)$ be the map which applies r to the first summand and s to the second. We define a path $h_2 : \mathbb{A}^1 \longrightarrow L(p,2p)$ by $h_2(t) = ti_1 + (1-t)i_2$.

Now we define $H_1 : \mathbb{A}^1 \times L(p,q) \longrightarrow L(p,q)$ to be the composite $\mu \circ (h_1, \mathrm{id})$. Finally, we define $H_2 : L(p,q) \times \mathbb{A}^1 \longrightarrow L(p,q)$ to be the composite

$$\begin{aligned} L(p,q) \times \mathbb{A}^1 &\xrightarrow{\mathrm{id}\times h_2} L(p,q) \times L(p,2p) \cong * \times L(p,q) \times L(p,2p) \\ &\xrightarrow{\gamma\times\mathrm{id}\times\mathrm{id}} L(p,q) \times L(p,q) \times L(p,2p) \xrightarrow{\phi\times\mathrm{id}} L(2p,2q) \times L(p,2p) \\ &\xrightarrow{\mu} L(p,2q) \xrightarrow{\beta_*} L(p,q) \end{aligned}$$

H_1 gives a homotopy $\mathrm{id} \simeq s_*$ and H_2 gives a homotopy from s_* to the constant function at $r \circ \gamma$. □

This acyclic caterad is not directly useful to us since $L(p,q)_+$ does not act on our symmetric monoid F. It does act on a naive stabilization of F, namely the symmetric monoid given by

$$stF(p) = \operatorname{colim}_n \mathbb{A}^{pn}/(\mathbb{A}^{pn} - 0),$$

but again this does not seem to be of much use. Rather, we conjecture that, at least after passage to motivic cochain complexes, there is a stabilization that is related to the cancellation theorem and that gives motivic cochains with an action of the chain caterad $\mathscr{L} = C_*\mathbb{U}\mathrm{Gr}L$.

7. Steenrod operations?

We start with an observation about the construction of Steenrod like operations. Let $\mathscr{A}$ be any caterad in AbPre($\mathscr{S}$) and let $\mathscr{F}$ be any algebra over $\mathscr{A}$. By [11, 4.5], for any q and r we have Σ_ℓ-equivariant structure maps

$$C_*(\mathscr{A}(q\ell,r))\otimes C_*(\mathscr{F}(q))^\ell \longrightarrow C_*(\mathscr{F}(r)). \tag{7.1}$$

Let ℓ be a prime and reduce everything mod ℓ. If $C_*(\mathscr{A}(q\ell,r))$ is acyclic, so that its homology (on each smooth scheme X) is $\mathbb{F}_\ell$ concentrated in degree zero, we can crudely apply elementary homological algebra to map any $\mathbb{F}_\ell[\Sigma_\ell]$-free resolution W of $\mathbb{F}_\ell$ into $C_*(\mathscr{A}(q\ell,r))$, one X at a time. The result will be natural up to chain homotopy. We obtain chain maps

$$W\otimes_{\mathbb{F}_\ell[\Sigma_\ell]} C_*(\mathscr{F}(q))^\ell \longrightarrow C_*(\mathscr{A}(q\ell,r))\otimes_{\mathbb{F}_\ell[\Sigma_\ell]} C_*(\mathscr{F}(q))^\ell \longrightarrow C_*(\mathscr{F}(r))$$

which induce maps of homology presheaves. Such chain maps induce Steenrod like operations, as is explained in detail in [8], which gives methods for studying their properties. As explained in [8, 2.2] (where the operations are denoted P_s), there result operations

$$Q^s\colon h_n(X,\mathscr{F}(q)\otimes\mathbb{F}_\ell) \longrightarrow h_{n+2s(\ell-1)}(X,\mathscr{F}(r)\otimes\mathbb{F}_\ell)$$

when $\ell>2$ and similar operations Q^s of degree s when $\ell=2$. These operations are defined for all integers s, but they are zero if $\ell>2$ and $s<2n$ or if $\ell=2$ and $s<n$. When regrading cohomologically, one usually writes $P^s=Q^{-s}$, but to avoid confusion we shall instead write $R^s=Q^{-s}$.

Recall that Voevodsky [14, 15] constructed Steenrod operations

$$P^i\colon H^{p,q}(X;\mathbb{F}_\ell) \longrightarrow H^{p+2i(\ell-1),q+i(\ell-1)},$$

denoted Sq^{2i} if $\ell=2$.

Remark 7.2. Suppose for speculative purposes that we have an acyclic caterad action on either $\mathscr{F}$ or $\mathscr{F}^V$ (or $\mathscr{F}^{SF}$ which has the same grading conventions). Then passage to hypercohomology and inspection of the cohomological regrading would give operations as follows. For $\mathscr{F}$, we would have operations

$$R^s\colon H^{p,q}(X;\mathbb{F}_\ell) \longrightarrow H^{p+r-q+2s(\ell-1),r}(X;\mathbb{F}_\ell).$$

Setting $r=q+i(\ell-1)$, these would be operations

$$R^s\colon H^{p,q}(X;\mathbb{F}_\ell) \longrightarrow H^{p+(i+2s)(\ell-1),q+i(\ell-1)}(X;\mathbb{F}_\ell).$$

To retrieve Voevodsky's operations, we would have to have $2s=i$, which of course is impossible if i is odd. For $\mathscr{F}^V$, we would have operations

$$R^s\colon H^{p,q}(X;\mathbb{F}_\ell) \longrightarrow H^{p+2r-2q+2s(\ell-1),r}(X;\mathbb{F}_\ell).$$

Setting $r=q+i(\ell-1)$, these would be operations

$$R^s\colon H^{p,q}(X;\mathbb{F}_\ell) \longrightarrow H^{p+(2i+2s)(\ell-1),q+i(\ell-1)}(X;\mathbb{F}_\ell).$$

The only dimensional candidates for Voevodsky's operations occur when $s=0$, but then the Cartan formula for the R^s looks quite different from the Cartan formula for Voevodsky's operations.

For comparison, the originally expected E_∞ operad action, of the sort obtained for Bloch's higher Chow complexes in [7], would give us Σ_ℓ-equivariant maps

$$\mathscr{O}(\ell) \otimes C_*(\mathscr{F}^{SF}(q))^\ell \longrightarrow C_*(\mathscr{F}^{SF}(q\ell)), \tag{7.3}$$

where $\mathscr{O}(\ell)$ is Σ_ℓ-free and acyclic. Such maps would give Steenrod operations

$$P^i_B\colon H^{p,q}(X,\mathbb{F}_\ell) \longrightarrow H^{p+2i(\ell-1),q\ell}(X,\mathbb{F}_\ell)$$

if $\ell > 2$ and Sq^i_B of bidegree (i,q) if $\ell = 2$ that satisfy all of the usual properties. Using Theorem 4.10, the work of [7] already gives us such operations; see [7, I.7.2].[1] Operations of this sort would arise from the part $r = q\ell$ of an action by an acyclic caterad. Observe that these operations and Voevodsky's operations only overlap in the case $i = q$, and even there no comparison has been proven.

References

[1] J.M. Boardman and R.M. Vogt. Homotopy invariant structures in algebraic topology. Lecture Notes in Mathematics Vol. 347. Springer. 1973.

[2] S. Bloch. Algebraic cycles and higher K-theory. Advances in Math. 61(1986), 267–304.

[3] S. Bloch. The moving lemma for higher Chow groups. J. Algebraic Geometry 3(1994), 537–568.

[4] P. Brosnan. Steenrod operations in Chow theory. Trans. Amer. Math. Soc. 355(2003), 1869–1903.

[5] E. Friedlander and A. Suslin. The spectral sequence relating algebraic K-theory to motivic cohomology. Ann. Sci. cole Norm. Sup. (4)35(2002) 773–875.

[6] R. Joshua. The motivic DGA:I. Preprint, 2001. Available at http://front.math.ucdavis.edu/math.AG/0112275.

[7] I. Kriz and J.P. May. Operads, algebras, modules and motives. Astérisque No. 233 (1995).

[8] J.P. May. A general algebraic approach to Steenrod operations. Lecture Notes in Mathematics Vol. 168. Springer, 1970, 153–231.

[9] J.P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Lecture Notes in Mathematics Vol. 577. Springer. 1977.

[10] J.P. May. Operads and sheaf cohomology. Preprint, December 2003.

[11] J.P. May. Caterads and their algebras. Preprint, December 2003.

[12] G.M.L. Powell. Steenrod operations in motivic cohomology. Luminy lecture notes. 2002.

[13] C. Mazza, V. Voevodsky, and C. Weibel. Notes on motivic cohomology. Lecture notes, 2002. Available at http://math.rutgers.edu/ weibel/motiviclectures.html.

[14] V. Voevodsky. Reduced power operations in motivic cohomology Available at http://front.math.ucdavis.edu/math.AG/0107109.

[15] V. Voevodsky. On 2-torsion in motivic cohomology. Available at http://front.math.ucdavis.edu/math.AG/0107110.

[16] V. Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. Int. Math. Res. Not. 2002, no. 7, 351–355.

[17] V. Voevodsky, A. Suslin, and E.M. Friedlander. Cycles, transfers, and motivic homology theories. Annals of Mathematics Studies, 143. Princeton University Press. 2000.

[18] C. Weibel. Products in higher Chow groups and motivic cohomology. Algebraic K-theory (Seattle, WA, 1997), 305–315, Proc. Sympos. Pure Math. Vol. 67. Amer. Math. Soc., Providence, RI, 1999.

[1] In the cited result, $s \geq 0$ should read $s \in \mathbb{Z}$ and the first Adem relation should read

$$Q^tQ^s = \sum_i (-1)^{t+i}(pi, -t, t-(p-1)s-i-1)Q^{s+t-i}Q^i.$$