ISOMORPHISMS BETWEEN LEFT AND RIGHT ADJOINTS

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Abstract. There are many contexts in algebraic geometry, algebraic topology, and homological algebra where one encounters a functor that has both a left and right adjoint, with the right adjoint being isomorphic to a shift of the left adjoint specified by an appropriate "dualizing object". Typically the left adjoint is well understood while the right adjoint is more mysterious, and the result identifies the right adjoint in familiar terms. We give a categorical discussion of such results. One essential point is to differentiate between the classical framework that arises in algebraic geometry and a deceptively similar, but genuinely different, framework that arises in algebraic topology. Another is to make clear which parts of the proofs of such results are formal. The analysis significantly simplifies the proofs of particular cases, as we illustrate in a sequel discussing applications to equivariant stable homotopy theory.

Contents

1. Introduction

We shall give a categorical discussion of Verdier and Grothendieck isomorphisms on the one hand and formally analogous results whose proofs involve different issues on the other. Our point is to explain and compare the two contexts and to differentiate the formal issues from the substantive issues in each. The philosophy goes back to Grothendieck's "six operations" formalism. We give background in §2. We fix our categorical framework, explain what the naive versions of our theorems say, and describe which parts of their proofs are formal in §§3–6. This discussion does not require triangulated categories. Its hypotheses and conclusions make sense in general closed symmetric monoidal categories, whether or not triangulated. In

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practice, that means that the arguments apply equally well before or after passage to derived categories.

After giving some preliminary results about triangulated categories in §7, we explain the formal theorems comparing left and right adjoints in §8. Our "formal Grothendieck isomorphism theorem" is an abstraction of results of Amnon Neeman, and our "formal Wirthmüller isomorphism theorem" borrows from his ideas. His paper [25] has been influential, and he must be thanked for catching a mistake in a preliminary version by the third author. We thank Gaunce Lewis for discussions of the topological context, and we thank Sasha Beilinson and Madhav Nori for making clear that, contrary to our original expectations, the context encountered in algebraic topology is not part of the classical context familiar to algebraic geometers. We also thank Johann Sigurdsson for corrections and emendations.

2. Background and contexts

We start with an overview of the contexts that we have in mind. Throughout, we shall fix an adjoint pair of functors (f^*, f_*) relating closed symmetric monoidal categories. We shall always assume that the left adjoint f^* is strong symmetric monoidal, so that it preserves tensor products up to natural isomorphism.

The notation (f^*, f_*) meshes with standard notation in algebraic topology and algebraic geometry, where one starts with a map $f\colon A \longrightarrow B$ of suitably restricted spaces or schemes. Here f^* and f_* are pullback (or inverse image) and pushforward (or direct image) functors that relate the categories of sheaves on A and on B, or the categories of $\mathcal{O}_A$-modules and $\mathcal{O}_B$-modules, or that relate the respective derived categories. In such contexts, there is often another pair of adjoint functors $(f_!, f^!)$, where $f_!$ is direct image with compact or proper supports. Unless f is proper, the functor f_* is not well-behaved, not even preserving sums, and the functor $f_!$ remedies that defect. Here the construction of the right adjoint $f^!$ to $f_!$ is not obvious. Grothendieck's six functor formalism refers to the six functors $(\otimes, \mathrm{Hom}, f_*, f^*, f_!, f^!)$ and especially to base change as f varies. Even today, this intrinsically 2-categorical context cries out for a much more complete categorical study than exists in the literature.

However, we shall only be concerned with a fixed f. As noted by Lipman [17, p. 119], even here there is already a non-trivial "coherence problem", namely the problem of determining which compatibility diagrams relating the given data necessarily commute. An early reference for coherence in closed symmetric monoidal categories is [8], and the volume [14] contains several papers on the subject and many references. In particular, G. Lewis [15] gives a partial coherence theorem for closed monoidal functors. The categorical theory of coherence is relevant to the study of "compatibilities" that focuses on base change maps and plays an important role in the literature in algebraic geometry [1, 6, 7, 11] and especially Conrad [4]. A study of that is beyond the scope of this note. A full categorical coherence theorem is not known and would be highly desirable. A start on this has been made by Voevodsky [7]. Since his discussion focuses on base change relating quadruples $(f^*, f_*, f_!, f^!)$, ignoring $\otimes$ and Hom entirely, it is essentially disjoint from our discussion. Lipman [17, 18] takes a categorical point of view similar to ours, but he also leaves a full treatment of coherence as a problem for future work.

When f is proper, $f_* = f_!$ and thus $f^!$ is right adjoint to the right adjoint f_* of the strong symmetric monoidal functor f^*. Under favorable circumstances, the usually unfamiliar functor $f^!$ can be computed in terms of the usually familiar functor f^*. This gives one starting point for the vast literature of Grothendieck duality, initiated in Grothendieck's paper [10]; see also [1, 4, 6, 5, 11, 17, 18, 25, 29]. The theory extends to the general context with $f_* \neq f_!$. In the context of derived categories of sheaves over spaces, this development begins with Verdier's paper [28]; see also [2, 13, 27].

We shall introduce a categorical "Grothendieck context" that is modelled on the case $f_* = f_!$ and a categorical "Verdier-Grothendieck context" that is modelled on the general case of these sheaf theoretical contexts. We shall say a bit more about the sheaf theoretical specializations later. For now, we merely contrast them with another context, which feels both similar (or dual) and different. We shall be more precise about the similarities and differences later. Categorically, starting with two pairs of adjoint functors (f^*, f_*) and $(f_!, f^!)$ relating the same categories, with f^* strong symmetric monoidal, we will arrive at the Grothendieck context by assuming, in part, that $f_* = f_!$, so that f^* is a left adjoint of a left adjoint. We might instead assume that $f^* = f^!$, in which case the strong symmetric monoidal functor f^* has both a right adjoint f_* and a left adjoint $f_!$. Here we can seek to compute the right adjoint f_* in terms of the left adjoint $f_!$. We illustrate the idea with two examples.

Example 2.1. Consider a homomorphism $f : A \longrightarrow B$ of commutative rings, say a monomorphism. We have a pullback functor f^* from the category of B-modules to the category of A-modules, and similarly on passage to derived categories. Extension of scalars, $B \otimes_A X$, gives the left adjoint $f_!$ of f^*. Coextension of scalars, $\mathrm{Hom}_A(B, X)$, gives the right adjoint f_* of f^*. Here f^* is not strong symmetric monoidal since $f^*(Y \otimes_B Z)$ is not isomorphic to $f^*Y \otimes_A f^*Z$. It is instructive to compare this with the sheaf theoretical context that starts from the map $\mathrm{Spec}(f)$.

There is an interesting variant of this example for which f^* is strong symmetric monoidal. Indeed, assume that A and B are cocommutative Hopf algebras (not necessarily commutative) over a field k and let $f : A \longrightarrow B$ be a map of Hopf algebras. Then the category of A-modules is closed symmetric monoidal with unit object k under the tensor product and internal hom functor that send A-modules X and W to $X \otimes_k W$ and to $\mathrm{Hom}_k(X, W)$, with A-actions induced from the coproduct of A. The cocommutativity of the coproduct implies the symmetry of $\otimes_k$. The same holds for B. The pullback functor f^* is strong symmetric monoidal with respect to these tensor products, and it still has the left adjoint $f_!$ and right adjoint f_*. Observe that $f_!$ is not strong symmetric monoidal since $B \otimes_A (X \otimes_k W)$ is not isomorphic to $(B \otimes_A X) \otimes_k (B \otimes_A W)$. If B is finitely generated and free as an A-module, then $f_*X = \mathrm{Hom}_A(B, X)$ is naturally isomorphic to $f_!X = B \otimes_A X$.

Example 2.2. Consider a homomorphism of groups $f : H \longrightarrow G$, say a monomorphism. Let $\mathscr{G}$ be any complete and cocomplete closed symmetric monoidal category. We have a pullback functor f^* from the category of G-objects in $\mathscr{G}$ to the category of H-objects in $\mathscr{G}$, and f^* has both a left adjoint extension of group action functor $f_!$ and a right adjoint coextension of group action functor f_*. For example, if $\mathscr{G}$ is cartesian monoidal, then $f_!X = G \times_H X$ and $f_*X = \mathrm{Hom}_H(G, X)$. If $\mathscr{G}$ is additive and $f(H)$ has finite index in G, then f_*X is naturally isomorphic to $f_!X$.

When H and G are compact Lie groups, there are analogous functors f^*, $f_!$, and f_* relating the stable homotopy categories of G-spectra and of H-spectra, and there is an analogous description of f_* in terms of $f_!$. The first version of such a result was due to Wirthmüller [30]. In this paper, we will introduce a categorical "Wirthmüller context" that is modelled on this example, and we shall discuss its specialization to equivariant stable homotopy theory in the sequel [22].

In the general contexts that we shall introduce, there need be no underlying map "f" in sight. We give some simple illustrative examples in §5.

3. The starting point: the adjoint pair (f^*, f_*)

We fix closed symmetric monoidal categories $\mathscr{C}$ and $\mathscr{D}$ with unit objects $I_{\mathscr{C}}$ and $I_{\mathscr{D}}$. We write $\otimes$ and Hom for the tensor product and internal hom functor in either category, and we write X (or X', X'', or W) and Y (or Y' or Z) generically for objects of $\mathscr{C}$ and objects of $\mathscr{D}$, respectively. We write $\mathscr{C}(X, W)$ and $\mathscr{D}(Y, Z)$ for the categorical hom sets. We let $DX = \mathrm{Hom}(X, I_{\mathscr{C}})$ denote the dual of X, and similarly in $\mathscr{D}$. We let $\mathrm{ev}\colon \mathrm{Hom}(X, W) \otimes X \longrightarrow W$ denote the evaluation map, that is, the counit of the $(\otimes, \mathrm{Hom})$ adjunction

$$\mathscr{C}(X \otimes X', W) \cong \mathscr{C}(X, \mathrm{Hom}(X', W)).$$

We also fix a strong symmetric monoidal functor $f^*\colon \mathscr{D} \longrightarrow \mathscr{C}$. This means that we are given isomorphisms

$$f^* I_{\mathscr{D}} \cong I_{\mathscr{C}} \quad \text{and} \quad f^*(Y \otimes Z) \cong f^*Y \otimes f^*Z, \tag{3.1}$$

the second natural, that commute with the associativity, symmetry, and unit isomorphisms for $\otimes$ in $\mathscr{C}$ and $\mathscr{D}$. We assume throughout that f^* has a right adjoint f_*, and we write

$$\varepsilon\colon f^*f_*X \longrightarrow X \quad \text{and} \quad \eta\colon Y \longrightarrow f_*f^*Y$$

for the counit and unit of the adjunction. This general context is fixed throughout.

The assumption that f^* is strong symmetric monoidal has several basic, and well-known, implications. The adjuncts of the isomorphism $f^*I_{\mathscr{D}} \cong I_{\mathscr{C}}$ and the map

$$f^*(f_*W \otimes f_*X) \cong f^*f_*W \otimes f^*f_*X \xrightarrow{\varepsilon \otimes \varepsilon} W \otimes X$$

are maps

$$I_{\mathscr{D}} \longrightarrow f_* I_{\mathscr{C}} \quad \text{and} \quad f_*W \otimes f_*X \longrightarrow f_*(W \otimes X), \tag{3.2}$$

the second natural. These are not usually isomorphisms. This means that f_* is *lax* symmetric monoidal.

The adjunct of the map

$$f^*\,\mathrm{Hom}(Y, Z) \otimes f^*Y \cong f^*(\mathrm{Hom}(Y, Z) \otimes Y) \xrightarrow{f^*(\mathrm{ev})} f^*Z$$

is a natural map

$$\alpha\colon f^*\,\mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(f^*Y, f^*Z). \tag{3.3}$$

It may or may not be an isomorphism in general, and we say that f^* is *closed symmetric monoidal* if it is. However, the adjunct of the composite map

$$f^*\,\mathrm{Hom}(Y, f_*X) \xrightarrow{\alpha} \mathrm{Hom}(f^*Y, f^*f_*X) \xrightarrow{\mathrm{Hom}(\mathrm{id}, \varepsilon)} \mathrm{Hom}(f^*Y, X)$$

is a natural isomorphism

$$\mathrm{Hom}(Y, f_*X) \cong f_*\,\mathrm{Hom}(f^*Y, X). \tag{3.4}$$

In particular, $\mathrm{Hom}(Y, f_*I_{\mathscr{C}}) \cong f_*Df^*Y$. Indeed, we have the following two chains of isomorphisms of functors.

$$\mathscr{D}(Z, \mathrm{Hom}(Y, f_*X)) \cong \mathscr{D}(Z \otimes Y, f_*X) \cong \mathscr{C}(f^*(Z \otimes Y), X)$$

$$\mathscr{D}(Z, f_* \mathrm{Hom}(f^*Y, X)) \cong \mathscr{C}(f^*Z, \mathrm{Hom}(f^*Y, X)) \cong \mathscr{C}(f^*Z \otimes f^*Y, X)$$

By the Yoneda lemma and a check of maps, these show immediately that the assumed isomorphism of functors in (3.1) is *equivalent* to the claimed isomorphism of functors (3.4). That is, the isomorphism of left adjoints in (3.1) is adjunct, or "conjugate" to the isomorphism of right adjoints in (3.4). Systematic recognition of such conjugate pairs of isomorphisms can substitute for quite a bit of excess verbiage in the older literature. We call this a "comparison of adjoints" and henceforward leave the details of such arguments to the reader.

Using the isomorphism (3.4), we obtain the following natural map β, which is analogous to both α and the map of (3.2). Like the latter, it is not usually an isomorphism.

$$(3.5) \qquad \beta\colon f_* \mathrm{Hom}(X, W) \xrightarrow{f_* \mathrm{Hom}(\varepsilon, \mathrm{id})} f_* \mathrm{Hom}(f^*f_*X, W) \xrightarrow{\cong} \mathrm{Hom}(f_*X, f_*W).$$

Using (3.2), we also obtain a natural composite

$$(3.6) \qquad \pi\colon Y \otimes f_*X \xrightarrow{\eta \otimes \mathrm{id}} f_*f^*Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X).$$

Like α, it may or may not be an isomorphism in general. When it is, we say that the *projection formula* holds.

We illustrate the need for a systematic treatment of coherence by recording a particular diagram that commutes in the known examples and whose commutativity should be incorporated in such a treatment, namely

$$(3.7) \qquad \begin{array}{ccccc} f^*DY \otimes f^*Y & \xrightarrow{\cong} & f^*(DY \otimes Y) & \xrightarrow{f^*(ev)} & f^*I_{\mathscr{D}} \\ \downarrow{\scriptstyle \alpha \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \cong} \\ Df^*Y \otimes f^*Y & & \xrightarrow[ev]{\hspace{6em}} & & I_{\mathscr{C}}. \end{array}$$

We shall need a consequence of this diagram. There is a natural map

$$\nu\colon DX \otimes W \longrightarrow \mathrm{Hom}(X, W),$$

namely the adjunct of

$$DX \otimes W \otimes X \cong DX \otimes X \otimes W \xrightarrow{ev \otimes \mathrm{id}} I_{\mathscr{C}} \otimes W \cong W.$$

The commutativity of the diagram (3.7) implies the commutativity of the diagram

$$(3.8) \qquad \begin{array}{ccccc} f^*DY \otimes f^*Z & \xrightarrow{\cong} & f^*(DY \otimes Z) & \xrightarrow{f^*\nu} & f^* \mathrm{Hom}(Y, Z) \\ \downarrow{\scriptstyle \alpha \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \alpha} \\ Df^*Y \otimes f^*Z & & \xrightarrow[\nu]{\hspace{6em}} & & \mathrm{Hom}(f^*Y, f^*Z). \end{array}$$

We assume familiarity with the theory of "dualizable" (alias "strongly dualizable" or "finite") objects; see [20] for a recent exposition. The defining property is that X is dualizable if $\nu\colon DX \otimes X \longrightarrow \mathrm{Hom}(X, X)$ is an isomorphism. It follows that $\nu\colon DX \otimes W \longrightarrow \mathrm{Hom}(X, W)$ is an isomorphism if either X or W is dualizable.

It also follows that the natural map $\rho\colon X \longrightarrow DDX$ is an isomorphism, but the converse fails in general. When X' is dualizable, we have the duality adjunction

$$\mathscr{C}(X \otimes X', X'') \cong \mathscr{C}(X, DX' \otimes X''). \tag{3.9}$$

As observed in [16, III.1.9], (3.1) and the definitions imply the following result.

Proposition 3.10. *If $Y \in \mathscr{D}$ is dualizable, then DY, f^*Y, and Df^*Y are dualizable and, with $Z = I_{\mathscr{D}}$, the map α of (3.3) restricts to an isomorphism*

$$f^*DY \cong Df^*Y. \tag{3.11}$$

This implies that α and π are often isomorphisms for formal reasons.

Proposition 3.12. *If $Y \in \mathscr{D}$ is dualizable, then*

$$\alpha\colon f^*\operatorname{Hom}(Y,Z) \longrightarrow \operatorname{Hom}(f^*Y, f^*Z) \quad \text{and} \quad \pi\colon Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X)$$

are isomorphisms for all objects $X \in \mathscr{C}$ and $Z \in \mathscr{D}$. Thus, if all objects of $\mathscr{D}$ are dualizable, then f^* is closed symmetric monoidal and the projection formula holds.

Proof. For the first statement, α coincides with the composite

$$f^*\operatorname{Hom}(Y,Z) \cong f^*(DY \otimes Z) \cong f^*DY \otimes f^*Z \cong Df^*Y \otimes f^*Z \cong \operatorname{Hom}(f^*Y, f^*Z).$$

For the second statement, π induces the isomorphism of represented functors

$$\begin{aligned}\mathscr{D}(Z, Y \otimes f_*X) &\cong \mathscr{D}(Z \otimes DY, f_*X) \cong \mathscr{C}(f^*(Z \otimes DY), X) \cong \mathscr{C}(f^*Z \otimes f^*DY, X)\\ &\cong \mathscr{C}(f^*Z \otimes Df^*Y, X) \cong \mathscr{C}(f^*Z, f^*Y \otimes X) \cong \mathscr{D}(Z, f_*(f^*Y \otimes X)). \quad \square\end{aligned}$$

4. The general context: adjoint pairs (f^*, f_*) and $(f_!, f^!)$

In addition to the adjoint pair (f^*, f_*) of the previous section, we now assume given a second adjoint pair $(f_!, f^!)$ relating $\mathscr{C}$ and $\mathscr{D}$, with $f_!\colon \mathscr{C} \longrightarrow \mathscr{D}$ being the left adjoint. We write

$$\sigma\colon f_!f^!Y \longrightarrow Y \quad \text{and} \quad \zeta\colon X \longrightarrow f^!f_!X$$

for the counit and unit of the second adjunction.

The adjunction $\mathscr{D}(Y, f_*X) \cong \mathscr{C}(f^*Y, X)$ can be recovered from the more general "internal Hom adjunction" $\operatorname{Hom}(Y, f_*X) \cong f_*\operatorname{Hom}(f^*Y, X)$ of (3.4) by applying the functor $\mathscr{D}(I_{\mathscr{D}}, -)$ and using the assumption that $f^*I_{\mathscr{D}} \cong I_{\mathscr{C}}$. It seems reasonable to hope that the adjunction $\mathscr{D}(f_!X, Y) \cong \mathscr{C}(X, f^!Y)$ can be recovered by applying the functor $\mathscr{D}(I_{\mathscr{D}}, -)$ to an analogous internal Hom adjunction

$$\operatorname{Hom}(f_!X, Y) \cong f_*\operatorname{Hom}(X, f^!Y).$$

However, unlike (3.4), such an adjunction does not follow formally from our hypotheses. Motivated by different specializations of the general context, we consider two triads of basic natural maps that we might ask for relating our four functors. For the first triad, we might ask for either of the following two duality maps, the first of which is a comparison map for the desired internal Hom adjunction.

$$\gamma\colon f_*\operatorname{Hom}(X, f^!Y) \longrightarrow \operatorname{Hom}(f_!X, Y). \tag{4.1}$$

$$\delta\colon \operatorname{Hom}(f^*Y, f^!Z) \longrightarrow f^!\operatorname{Hom}(Y, Z). \tag{4.2}$$

We might also ask for a *projection formula map*

$$\hat{\pi}\colon Y \otimes f_!X \longrightarrow f_!(f^*Y \otimes X), \tag{4.3}$$

which should be thought of as a generalized analogue of the map π of (3.6). These three maps are not formal consequences of the given adjunctions, but rather must be constructed by hand. However, it suffices to construct any one of them.

Proposition 4.4. *Suppose given any one of the natural maps γ, δ, and $\hat{\pi}$. Then it determines the other two by conjugation. The map δ is an isomorphism for all dualizable Y if and only if its conjugate $\hat{\pi}$ is an isomorphism for all dualizable Y. If any one of the three conjugately related maps is a natural isomorphism, then so are the other two.*

Remark 4.5. In the contexts encountered in algebraic geometry, there is a natural map $\iota\colon f_!X \longrightarrow f_*X$ and $\hat{\pi}$ is a restriction of π along ι, in the sense that the following diagram commutes.

$$\begin{tikzcd}
Y \otimes f_!X \arrow[r, "\mathrm{id} \otimes \iota"] \arrow[d, "\hat{\pi}"'] & Y \otimes f_*X \arrow[d, "\pi"] \\
f_!(f^*Y \otimes X) \arrow[r, "\iota"'] & f_*(f^*Y \otimes X)
\end{tikzcd}$$

It is natural to restrict our general context by requiring such a map ι and requiring $\hat{\pi}$ to be such a restriction of π. This gives the context most relevant to algebraic geometry, and it is a sufficiently rigid context to give a sensible starting point for a categorical study of coherence that is applicable to base change functors.

The second triad results from the first simply by changing the direction of the arrows. That is, we can ask for natural maps in the following directions.

$$\bar{\gamma}\colon \operatorname{Hom}(f_!X, Y) \longrightarrow f_* \operatorname{Hom}(X, f^!Y). \tag{4.6}$$

$$\bar{\delta}\colon f^! \operatorname{Hom}(Y, Z) \longrightarrow \operatorname{Hom}(f^*Y, f^!Z). \tag{4.7}$$

$$\bar{\pi}\colon f_!(f^*Y \otimes X) \longrightarrow Y \otimes f_!X. \tag{4.8}$$

Here $\bar{\delta}$ is to be viewed as a generalized analogue of the map α of (3.3).

Proposition 4.9. *Suppose given any one of the natural maps $\bar{\gamma}$, $\bar{\delta}$, and $\bar{\pi}$. Then it determines the other two by conjugation. The map $\bar{\delta}$ is an isomorphism for all dualizable Y if and only if its conjugate $\bar{\pi}$ is an isomorphism for all dualizable Y. If any one of the three conjugately related maps is a natural isomorphism, then so are the other two.*

Of course, when the three maps are isomorphisms, the two triads of maps are inverse to each other and there is no real difference. However, we are interested in two quite different specializations: we might have $f_! = f_*$, or we might have $f^! = f^*$. Here these formulas should be interpreted formally, ignoring preassigned notational associations from particular contexts. The first means that the right adjoint of f^* is itself a left adjoint. The second means that f^* is both a left and right adjoint. It is entirely possible that both of these statements hold, but we shall not consider that situation. The first specialization occurs frequently in algebraic geometry, and is familiar. The second occurs in algebraic topology and elsewhere, but seems less familiar. However, it does also appear in algebraic geometry, in those base change situations where f^* has a left adjoint $f_!$; the latter is sometimes denoted $f_\#$, as in [23, 3.1.23], to avoid possible confusion. With the first specialization, the first triad of maps arises formally, taking $\hat{\pi}$ to be the map π of (3.6). With the second specialization, the second triad arises formally, taking $\bar{\delta}$ to be the map α of (3.3). Recall the isomorphism (3.4), the map β of (3.5), and Proposition 3.12.

Proposition 4.10. *Suppose $f_! = f_*$. Taking $\hat{\pi}$ to be the projection map π of (3.6), the conjugate map γ is the composite*

$$f_* \operatorname{Hom}(X, f^! Y) \xrightarrow{\beta} \operatorname{Hom}(f_* X, f_* f^! Y) \xrightarrow{\operatorname{Hom}(\mathrm{id},\sigma)} \operatorname{Hom}(f_* X, Y)$$

and the conjugate map δ is the adjunct of the map

$$f_* \operatorname{Hom}(f^* Y, f^! Z) \cong \operatorname{Hom}(Y, f_* f^! Z) \xrightarrow{\operatorname{Hom}(\mathrm{id},\sigma)} \operatorname{Hom}(Y, Z).$$

Moreover, π and δ are isomorphisms if Y is dualizable.

When $f^! = f^*$, passage to adjuncts from $I_{\mathscr{C}} \cong f^* I_{\mathscr{D}}$ and the natural map

$$W \otimes X \xrightarrow{\zeta\otimes\zeta} f^* f_! W \otimes f^* f_! X \cong f^*(f_! W \otimes f_! X)$$

gives maps, not usually isomorphisms,

$$(4.11) \qquad f_! I_{\mathscr{C}} \longrightarrow I_{\mathscr{D}} \quad \text{and} \quad f_!(W \otimes X) \longrightarrow f_! W \otimes f_! X,$$

the second natural. This means that $f_!$ is an *op-lax* symmetric monoidal functor.

Proposition 4.12. *Suppose $f^! = f^*$. Taking $\bar{\delta}$ to be the map α of (3.3), the conjugate map $\bar{\pi}$ is the composite*

$$f_!(f^* Y \otimes X) \longrightarrow f_! f^* Y \otimes f_! X \xrightarrow{\sigma\otimes\mathrm{id}} Y \otimes f_! X$$

and the conjugate map $\bar{\gamma}$ is the adjunct of the map

$$f^* \operatorname{Hom}(f_! X, Y) \xrightarrow{\alpha} \operatorname{Hom}(f^* f_! X, f^* Y) \xrightarrow{\operatorname{Hom}(\zeta,\mathrm{id})} \operatorname{Hom}(X, f^* Y).$$

Moreover α and $\bar{\pi}$ are isomorphisms if Y is dualizable.

Definition 4.13. We introduce names for the different contexts in sight. In all three, we start with an adjoint pair (f^*, f_*), with f^* strong symmetric monoidal.
(i) *The Verdier-Grothendieck context:* There is a second adjoint pair $(f_!, f^!)$ and a natural isomorphism $\hat{\pi}$ as in (4.3) (projection formula); there are then conjugately determined natural isomorphisms γ as in (4.1) and δ as in (4.2).
(ii) *The Grothendieck context:* The functor f_* has a right adjoint $f^!$ and the projection formula holds. That is, the map π of (3.6) is an isomorphism, hence so are the conjugate maps γ and δ specified in Proposition 4.10.
(iii) *The Wirthmüller context:* f^* has a left adjoint $f_!$ and is closed symmetric monoidal. That is, the map α of (3.3) is an isomorphism, hence so are the conjugate maps $\bar{\pi}$ and $\bar{\gamma}$ specified in Proposition 4.12.

We emphasize that these are abstract categorical concepts whose notations are dictated by consistency with our conceptual framework. Therefore, they cannot be expected to agree with standard notations in all contexts to which they apply. We remark that the coherence problem alluded to in §2 and Remark 4.5 should simplify considerably in either the Grothendieck or the Wirthmüller context, due to the canonicity of the maps in Propositions 4.10 and 4.12.

We repeat that our categorical results deduce formal conclusions from formal hypotheses and therefore work equally well before or after passage to derived categories. Much of the work in passing from categories of sheaves to derived categories can be viewed as the verification that formal properties in the category of sheaves carry over to the same formal properties in derived categories, although other properties only hold after passage to derived categories.

While the proofs of Propositions 4.4 and 4.10 are formal, in the applications to algebraic geometry they require use of *unbounded* derived categories, since otherwise we would not have closed symmetric monoidal categories to begin with. These were not available until Spaltenstein's paper [27], and he gave one of our formal implications, namely that an isomorphism (4.3) implies an isomorphism (4.2) [27, 6.19]. Unfortunately, as he makes clear, in the classical sheaf context his methods fail to give the $(f_!, f^!)$ adjunction for all maps f between locally compact spaces. It seems possible that a model theoretic approach to unbounded derived categories would allow one to resolve this problem. In any case, a complete reworking of the theory in model theoretical terms would be of considerable value.

While Wirthmüller contexts do sometimes arise in algebraic geometry when a base change functor f^* has a left adjoint $f_!$, we do not know of situations where there is a non-trivial question of proving that $f_!$ is isomorphic to a shift of f_*. In the examples of the Wirthmüller context that we have in mind, where there is such a question, we think of f^* as a forgetful functor that does not alter underlying structure, $f_!$ as a kind of extension of scalars functor, and f_* as a kind of coextension of scalars functor.

5. Isomorphisms in the Verdier–Grothendieck context

We place ourselves in the Verdier–Grothendieck context in this section.

Definition 5.1. For an object $W \in \mathscr{C}$, define $D_W X = \mathrm{Hom}(X, W)$, the *$W$-twisted dual of X*. Of course, if X or W is dualizable, then $D_W X \cong DX \otimes W$. Let $\rho_W \colon X \longrightarrow D_W D_W X$ be the adjunct of the evaluation map $D_W X \otimes X \longrightarrow W$. We say that X is *W-reflexive* if $\rho_W \colon X \longrightarrow D_W D_W X$ is an isomorphism.

Replacing Y by Z in (4.1) and letting $W = f^! Z$, the isomorphisms γ and δ take the following form:

$$f_* D_W X \cong D_Z f_! X \quad \text{and} \quad D_W f^* Y \cong f^! D_Z Y. \tag{5.2}$$

This change of notation and comparison with the classical context of algebraic geometry explains why we think of γ and δ as duality maps. If $f_! X$ is Z-reflexive, the first isomorphism implies that

$$f_! X \cong D_Z f_* D_W X. \tag{5.3}$$

If Y is isomorphic to $D_Z Y'$ for some Z-reflexive object Y', the second isomorphism implies that

$$f^! Y \cong D_W f^* D_Z Y. \tag{5.4}$$

These observations and the classical context suggest the following definition.

Definition 5.5. A *dualizing object* for a full subcategory $\mathscr{C}_0$ of $\mathscr{C}$ is an object W of $\mathscr{C}$ such that if $X \in \mathscr{C}_0$, then $D_W X$ is in $\mathscr{C}_0$ and X is W-reflexive. Thus D_W specifies an auto–duality of the category $\mathscr{C}_0$.

Remark 5.6. In algebraic geometry, we often encounter canonical subcategories $\mathscr{C}_0 \subset \mathscr{C}$ and $\mathscr{D}_0 \subset \mathscr{D}$ such that $f_! \mathscr{C}_0 \subset \mathscr{D}_0$ and $f^! \mathscr{D}_0 \subset \mathscr{C}_0$ together with a dualizing object Z for $\mathscr{D}_0$ such that $W = f^! Z$ is a dualizing object for $\mathscr{C}_0$. In such contexts, (5.3) and (5.4) express $f_!$ on $\mathscr{C}_0$ and $f^!$ on $\mathscr{D}_0$ in terms of f_* and f^*.

For any objects Y and Z of $\mathscr{D}$, the adjunct of the map

$$f_!(f^*Y \otimes f^!Z) \cong Y \otimes f_!f^!Z \xrightarrow{\mathrm{id}\otimes\sigma} Y \otimes Z$$

is a natural map

$$\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z). \tag{5.7}$$

It specializes to

$$\phi\colon f^*Y \otimes f^!I_{\mathscr{D}} \longrightarrow f^!Y, \tag{5.8}$$

which of course compares a right adjoint to a shift of a left adjoint. A *Verdier–Grothendieck isomorphism theorem* asserts that the map ϕ is an isomorphism; in the context of sheaves over spaces such a result was announced by Verdier in [28, §5]. The following observation abstracts a result of Neeman [25, 5.4]. In it, we only assume the projection formula for dualizable Y.

Proposition 5.9. *The map $\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$ is an isomorphism for all objects Z and all dualizable objects Y.*

Proof. Using Proposition 3.10, the projection formula, duality adjunctions (3.9), and the $(f_!, f^!)$ adjunction, we obtain isomorphisms

$$\begin{aligned}\mathscr{C}(X, f^*Y \otimes f^!Z) &\cong \mathscr{C}(f^*DY \otimes X, f^!Z) \cong \mathscr{D}(f_!(f^*DY \otimes X), Z)\\ &\cong \mathscr{D}(DY \otimes f_!X, Z) \cong \mathscr{D}(f_!X, Y \otimes Z) \cong \mathscr{C}(X, f^!(Y \otimes Z)).\end{aligned}$$

Diagram chasing shows that the composite isomorphism is induced by ϕ. □

It is natural to ask when ϕ is an isomorphism in general, and we shall return to that question in the context of triangulated categories. Of course, this discussion specializes and remains interesting in the Grothendieck context $f_! = f_*$.

We give some elementary examples of the Verdier–Grothendieck context.

Example 5.10. An example of the Verdier–Grothendieck context is already available with $\mathscr{C} = \mathscr{D}$ and $f^* = f_* = \mathrm{Id}$. Fix an object C of $\mathscr{C}$ and set

$$f_!X = X \otimes C \quad\text{and}\quad f^!(Y) = \mathrm{Hom}(C, Y).$$

The projection formula $f_!(f^*Y \otimes Z) \cong Y \otimes f_!Z$ is the associativity isomorphism

$$(Y \otimes Z) \otimes C \cong Y \otimes (Z \otimes C).$$

The map $\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$ is the canonical map

$$\nu\colon Y \otimes \mathrm{Hom}(C, Z) \longrightarrow \mathrm{Hom}(C, Y \otimes Z).$$

It is an isomorphism if Y is dualizable, and it is an isomorphism for all Y if and only if C is dualizable. Variants of this example are important in local duality theory; see for example [1, 2.1, p. 10].

The shift of an adjunction by an object of $\mathscr{C}$ used in the previous example generalizes to give a shift of any Verdier-Grothendieck context by an object of $\mathscr{C}$.

Definition 5.11. For an adjoint pair $(f_!, f^!)$ and an object $C \in \mathscr{C}$, define the twisted adjoint pair $(f_!^C, f^!_C)$ by

$$f_!^C(X) = f_!(X \otimes C) \quad\text{and}\quad f^!_C Y = \mathrm{Hom}(C, f^!Y). \tag{5.12}$$

Proposition 5.13. *If (f^*, f_*) and $(f_!, f!)$ are in the Verdier-Grothendieck context, then so are (f^*, f_*) and $(f_!^C, f!_C)$.*

Proof. The isomorphism $\hat{\pi}$ of (4.3) shifts to a corresponding isomorphism $\hat{\pi}_C$. □

We also give a simple example of the context of Definition 5.5. Recall that dualizable objects are $I_{\mathscr{C}}$-reflexive, but not conversely in general. The following observation parallels part of a standard characterization of "dualizing complexes" [11, V.2.1]. Let $d\mathscr{C}$ denote the full subcategory of dualizable objects of $\mathscr{C}$.

Proposition 5.14. *$I_{\mathscr{C}}$ is W-reflexive if and only if all $X \in d\mathscr{C}$ are W-reflexive.*

Proof. Since $I_{\mathscr{C}}$ is dualizable, the backwards implication is trivial. Assume that $I_{\mathscr{C}}$ is W-reflexive. Since $W \cong D_W I_{\mathscr{C}}$, $\mathrm{Hom}(W,W) = D_W W \cong D_W D_W I_{\mathscr{C}}$. In any closed symmetric monoidal category, such as $\mathscr{C}$, we have a natural isomorphism

$$\mathrm{Hom}(X \otimes X', X'') \cong \mathrm{Hom}(X, \mathrm{Hom}(X', X'')),$$

where X, X', and X'' are arbitrary objects. When X is dualizable,

$$\nu\colon DX \otimes X' \longrightarrow \mathrm{Hom}(X, X')$$

is an isomorphism for any object X'. Therefore

$$D_W D_W X \cong \mathrm{Hom}(DX \otimes W, W) \cong \mathrm{Hom}(DX, \mathrm{Hom}(W,W)) \cong DDX \otimes D_W D_W I_{\mathscr{C}}.$$

Identifying X with $X \otimes I_{\mathscr{C}}$, is easy to check that ρ_W corresponds under this isomorphism to $\rho_{I_{\mathscr{C}}} \otimes \rho_W$. The conclusion follows. □

Corollary 5.15. *Let W be dualizable. Then the following are equivalent.*

(i) *W is a dualizing object for $d\mathscr{C}$.*
(ii) *$I_{\mathscr{C}}$ is W-reflexive.*
(iii) *W is invertible.*
(iv) *$D_W\colon d\mathscr{C}^{op} \longrightarrow d\mathscr{C}$ is an auto–duality of $d\mathscr{C}$.*

Proof. If X is dualizable, then $D_W X \cong DX \otimes W$ is dualizable. Proposition 5.14 shows that (i) and (ii) are equivalent, and it is clear that (iii) and (iv) are equivalent. Since W is dualizable, $D_W D_W I_{\mathscr{C}} \cong \mathrm{Hom}(W,W) \cong W \otimes DW$, with ρ_W corresponding to the coevaluation map $coev\colon I_{\mathscr{C}} \longrightarrow W \otimes DW$. By [20, 2.9], W is invertible if and only if $coev$ is an isomorphism. Therefore (ii) and (iii) are equivalent. □

Finally, we have a shift comparison of Grothendieck and Wirthmüller contexts.

Remark 5.16. Start in the Grothendieck context, so that $f_! = f_*$, and assume that the map $\phi\colon f^*Y \otimes f^! I_{\mathscr{D}} \longrightarrow f^! Y$ of (5.8) is an isomorphism. Assume further that $f^! I_{\mathscr{D}}$ is invertible and let $C = Df^! I_{\mathscr{D}}$. Define a new functor $f_!$ by $f_! X = f_*(X \otimes f^! I_{\mathscr{D}})$. Then $f_!$ is *left* adjoint to f^*. Replacing X by $X \otimes C$, we see that

$$f_* X \cong f_!(X \otimes C).$$

In the next section, we shall consider isomorphisms of this general form in the Wirthmüller context. Conversely, start in the Wirthmüller context, so that $f^! = f^*$, and assume given a C such that $f_* I_{\mathscr{D}} \cong f_! C$ and the map $\omega\colon f_* X \longrightarrow f_!(X \otimes C)$ of (6.7) below is an isomorphism. Define a new functor $f^!$ by $f^! Y = \mathrm{Hom}(C, f^* Y)$ and note that $f^! I_{\mathscr{C}} \cong DC$. Then $f^!$ is *right* adjoint to f_*. If either C or Y is dualizable, then $\mathrm{Hom}(C, f^*Y) \cong f^*Y \otimes DC$ and thus $f^*Y \otimes f^! I_{\mathscr{D}} \cong f^! Y$, which is an isomorphism of the same form as in the Grothendieck context.

6. The Wirthmüller isomorphism

We place ourselves in the Wirthmüller context in this section, with $f^! = f^*$. Here the specialization of the Verdier–Grothendieck isomorphism is of no interest. In fact, ϕ reduces to the originally assumed isomorphism (3.1). However, there is now a candidate for an isomorphism between the right adjoint f_* of f^* and a shift of the left adjoint $f_!$. This is not motivated by duality questions, and it can already fail on dualizable objects. We assume in addition to the isomorphisms $\alpha = \bar{\bar{\delta}}$, hence $\bar{\pi}$ and $\bar{\gamma}$, that we are given an object $C \in \mathscr{C}$ together with an isomorphism

$$f_* I_{\mathscr{C}} \cong f_! C. \tag{6.1}$$

Observe that the isomorphism $\bar{\gamma}$ specializes to an isomorphism

$$D f_! X \cong f_* D X. \tag{6.2}$$

Taking $X = I_{\mathscr{C}}$ in (6.2) and using that $DI_{\mathscr{C}} \cong I_{\mathscr{C}}$, we see that (6.1) is equivalent to

$$D f_! I_{\mathscr{C}} \cong f_! C. \tag{6.3}$$

This version is the one most naturally encountered in applications, since it makes no reference to the right adjoint f_* that we seek to understand. In practice, $f_! I_{\mathscr{C}}$ is dualizable and C is dualizable or even invertible. It is a curious feature of our discussion that it does not require such hypotheses.

Replacing C by $I_{\mathscr{C}} \otimes C$ in (6.1), it is reasonable to hope that it continues to hold with $I_{\mathscr{C}}$ replaced by a general X. That is, we can hope for a natural isomorphism

$$f_* X \cong f_\sharp X, \quad \text{where} \quad f_\sharp X \equiv f_!(X \otimes C). \tag{6.4}$$

Note that we twist by C before applying $f_!$. We shall shortly define a particular natural map $\omega\colon f_* X \longrightarrow f_\sharp X$. A *Wirthmüller isomorphism theorem* asserts that ω is an isomorphism. We shall show that if $f_! I_{\mathscr{C}}$ is dualizable and X is a retract of some $f^* Y$, then ω is an isomorphism. However, even for dualizable X, ω need *not* be an isomorphism in general. A counterexample is given in the sequel [22]. We shall also give a categorical criterion for ω to be an isomorphism for a particular object X. An application is also given in [22].

Using the map $I_{\mathscr{D}} \longrightarrow f_* I_{\mathscr{C}}$ of (3.2), the assumed isomorphism $f_* I_{\mathscr{D}} \cong f_! C$ gives rise to maps

$$\tau\colon I_{\mathscr{D}} \longrightarrow f_* I_{\mathscr{C}} \cong f_! C$$

and

$$\xi\colon f^* f_! C \cong f^* f_* I_{\mathscr{C}} \xrightarrow{\varepsilon} I_{\mathscr{C}}$$

such that

$$\xi \circ f^* \tau = \mathrm{id}\colon I_{\mathscr{C}} \longrightarrow I_{\mathscr{C}}.$$

Using the alternative defining property (6.3) of C, we can obtain alternative descriptions of these maps that avoid reference to the functor f_* we seek to understand.

Lemma 6.5. *The maps τ and ξ coincide with the maps*

$$I_{\mathscr{D}} \cong D I_{\mathscr{D}} \xrightarrow{D\sigma} D f_! f^* I_{\mathscr{D}} \cong D f_! I_{\mathscr{C}} \cong f_! C$$

and

$$f^* f_! C \cong f^* D f_! I_{\mathscr{C}} \cong D f^* f_! I_{\mathscr{C}} \xrightarrow{D\zeta} D I_{\mathscr{C}} \cong I_{\mathscr{C}}.$$

Proof. The proofs are diagram chases that use, in addition to (6.3), the naturality of η and ε, the triangular identities for the $(f_!, f^*)$ adjunction, and the description of $\bar{\gamma}$ in Proposition 4.12. $\square$

Using the isomorphism (4.8), we extend τ to the natural map

$$(6.6)\qquad \tau\colon Y \cong Y \otimes I_{\mathscr{D}} \xrightarrow{\mathrm{id}\otimes\tau} Y \otimes f_!C \cong f_!(f^*Y \otimes C) = f_\sharp f^*Y.$$

Specializing to $Y = f_*X$, we obtain the desired comparison map ω as the composite

$$(6.7)\qquad \omega\colon f_*X \xrightarrow{\tau} f_\sharp f^* f_* X \xrightarrow{f_\sharp\varepsilon} f_\sharp X.$$

An easy diagram chase using the triangular identity $\varepsilon\circ f^*\eta = \mathrm{id}$ shows that

$$(6.8)\qquad \omega\circ\eta = \tau\colon Y \longrightarrow f_\sharp f^*Y.$$

If ω is an isomorphism, then τ must be the unit of the resulting $(f^*, f_\sharp)$ adjunction.

Similarly, using (3.1) and (4.8), we extend ξ to the natural map

$$(6.9)\qquad \xi\colon f^*f_\sharp f^*Y = f^*f_!(f^*Y\otimes C) \cong f^*Y \otimes f^*f_!C \xrightarrow{\mathrm{id}\otimes\xi} f^*Y\otimes I_{\mathscr{C}} \cong f^*Y.$$

We view ξ as a partial counit, defined not for all X but only for $X = f^*Y$. Since $\xi\circ f^*\tau = \mathrm{id}\colon I_{\mathscr{C}} \longrightarrow I_{\mathscr{C}}$, it is immediate that

$$(6.10)\qquad \xi\circ f^*\tau = \mathrm{id}\colon f^*Y \longrightarrow f^*Y,$$

which is one of the triangular identities for the desired $(f^*, f_\sharp)$ adjunction. Define

$$(6.11)\qquad \psi\colon f_\sharp f^*Y \longrightarrow f_*f^*Y$$

to be the adjunct of ξ. The adjunct of the relation (6.10) is the analogue of (6.8):

$$(6.12)\qquad \psi\circ\tau = \eta\colon Y \longrightarrow f_*f^*Y.$$

Proposition 6.13. *If Y or $f_!I_{\mathscr{C}}$ is dualizable, then $\omega\colon f_*f^*Y \longrightarrow f_\sharp f^*Y$ is an isomorphism with inverse ψ. If ψ is an isomorphism for all Y, then $f_!I_{\mathscr{C}}$ is dualizable. If X is a retract of some f^*Y, where Y or $f_!I_{\mathscr{C}}$ is dualizable, then $\omega\colon f_*X \longrightarrow f_\sharp X$ is an isomorphism.*

Proof. With $X = f^*Y$, the first part of the proof of the following result gives that $\psi\circ\omega = \mathrm{id}$, so that $\omega = \psi^{-1}$ when ψ is an isomorphism. We claim that ψ coincides with the following composite:

$$f_\sharp f^*Y = f_!(f^*Y\otimes C) \cong Y\otimes D(f_!I_{\mathscr{C}}) \xrightarrow{\nu} \mathrm{Hom}(f_!I_{\mathscr{C}}, Y) \cong f_*\,\mathrm{Hom}(I_{\mathscr{C}}, f^*Y) = f_*f^*Y.$$

Here the isomorphisms are given by (4.8) and (6.3) and by (4.6). Since ν is an isomorphism if Y or $f_!I_{\mathscr{C}}$ is dualizable, the claim implies the first statement. Note that $\psi = f_*\xi\circ\eta$ and that the isomorphism $\bar{\gamma}$ of (4.6) is $f_*\,\mathrm{Hom}(\zeta, \mathrm{id})\circ f_*\alpha\circ\eta$. Using the naturality of η and the description of ξ in Lemma 6.5, an easy, if lengthy, diagram chase shows that the diagram (3.8) gives just what is needed to check the claim. The second statement is now clear by the definition of dualizability; indeed, it suffices to consider $Y = f_!I_{\mathscr{C}}$. The last statement follows from the first since a retract of an isomorphism is an isomorphism. $\square$

We extract a criterion for ω to be an isomorphism for a general object X from the usual proof of the uniqueness of adjoint functors [19, p. 85].

Proposition 6.14. *If there is a map $\xi\colon f^*f_\sharp X = f^*f_!(X\otimes C) \longrightarrow X$ such that*

$$f_\sharp\xi\circ\tau = \mathrm{id}\colon f_\sharp X \longrightarrow f_\sharp X \tag{6.15}$$

and the following (partial naturality) diagram commutes, then $\omega\colon f_*X \longrightarrow f_\sharp X$ is an isomorphism with inverse the adjunct ψ of ξ.

$$\begin{tikzcd}
f^*f_\sharp f^*f_*X \arrow[r, "\xi"] \arrow[d, "f^*f_\sharp\varepsilon"'] & f^*f_*X \arrow[d, "\varepsilon"] \\
f^*f_\sharp X \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{6.16}$$

Moreover, (6.15) holds if and only if the following diagram commutes.

$$\begin{tikzcd}
X\otimes C \arrow[r, "\zeta"] \arrow[d, "\zeta"'] & f^*f_!(X\otimes C) \\
f^*f_!(X\otimes C) \arrow[r, "f^*\tau"'] & f^*f_!(f^*f_!(X\otimes C)\otimes C) \arrow[u, "f^*f_!(\xi\otimes\mathrm{id})"']
\end{tikzcd} \tag{6.17}$$

Proof. In the diagram (6.16), the top map ξ is given by (6.9). The diagram and the relation $\xi\circ f^*\tau = \mathrm{id}$ of (6.10) easily imply the relation $\xi\circ f^*\omega = \varepsilon$, which is complementary to the defining relation $\varepsilon\circ f^*\psi = \xi$ for the adjunct ψ. Passage to adjuncts gives that $\psi\circ\omega = \mathrm{id}$. The following diagram commutes by (6.8), the triangular identity $f_*\varepsilon\circ\eta = \mathrm{id}$, the naturality of η and ω, and the fact that ψ is adjunct to ξ. It gives that $\omega\circ\psi = f_\sharp\xi\circ\tau = \mathrm{id}$.

$$\begin{tikzcd}
f_\sharp X \arrow[rrr, "\tau"] \arrow[dr, "\eta"] \arrow[ddd, "\psi"'] & & & f_\sharp f^*f_\sharp X \arrow[dl, equals] \arrow[ddd, "f_\sharp\xi"] \\
& f_*f^*f_\sharp X \arrow[r, "\omega"] \arrow[d, "f_*f^*\psi"'] & f_\sharp f^*f_\sharp X \arrow[d, "f_\sharp f^*\psi"] & \\
& f_*f^*f_*X \arrow[r, "\omega"] \arrow[d, "f_*\varepsilon"] & f_\sharp f^*f_*X \arrow[dr, "f_\sharp\varepsilon"] & \\
f_*X \arrow[r, equals] \arrow[ur, "\eta"] & f_*X \arrow[rr, "\omega"'] & & f_\sharp X.
\end{tikzcd}$$

The last statement is clear by adjunction. □

Remark 6.18. The map ω can be generalized to the Verdier–Grothendieck context. For that, we assume given an object W of $\mathscr{C}$ such that

$$f_!C \cong Df_!f^!I_{\mathscr{D}};$$

compare (6.3). As in Lemma 6.5, we then have the map

$$\tau\colon I_{\mathscr{D}} \cong DI_{\mathscr{D}} \xrightarrow{D\sigma} Df_!f^!I_{\mathscr{D}} \cong f_!C.$$

This allows us to define the comparison map

$$\omega\colon f_*X \cong f_*X\otimes I_{\mathscr{D}} \xrightarrow{\mathrm{id}\otimes\tau} f_*X\otimes f_!W \cong f_!(f^*f_*X\otimes C) \xrightarrow{f_!(\varepsilon\otimes\mathrm{id})} f_!(X\otimes C).$$

A study of when this map ω is an isomorphism might be of interest, but we have no applications in mind. We illustrate the idea in the context of Example 5.10.

Example 6.19. Returning to Example 5.10, we seek an object C' of $\mathscr{C}$ such that $f_!C' \cong D(f_!f^!I_{\mathscr{C}})$, which is

$$C' \otimes C \cong D(DC \otimes C).$$

If C is dualizable, then the right side is isomorphic to $C \otimes DC \cong DC \otimes C$ and we can take $C' = DC$. Here the map

$$\omega\colon X = f_*X \longrightarrow f_!(X \otimes DC) = X \otimes DC \otimes C$$

turns out to be $\mathrm{id} \otimes (\gamma \circ coev)$, where $coev\colon I_{\mathscr{C}} \longrightarrow C \otimes DC$ is the coevaluation map of the duality adjunction (3.9) and γ is the symmetry isomorphism for $\otimes$. We conclude (e.g., by [20, 2.9]) that ω is an isomorphism if and only if C is invertible.

7. Preliminaries on triangulated categories

We now go beyond the hypotheses of §§3–6 to the triangulated category situations that arise in practice. We assume that $\mathscr{C}$ and $\mathscr{D}$ are triangulated and that the functors $(-) \otimes X$ and f^* are exact (or triangulated). This means that they are additive, commute with Σ up to natural isomorphism, and preserve distinguished triangles. For $(-) \otimes X$, this is a small part of the appropriate compatibility conditions that relate distinguished triangles to $\otimes$ and Hom in well-behaved triangulated closed symmetric monoidal categories; see [21] for a discussion of this, as well as for basic observations about what triangulated categories really are: the standard axiom system is redundant and unnecessarily obscure. We record the following easily proven observation relating adjoints to exactness (see for example [24, 3.9]).

Lemma 7.1. *Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be left and right adjoint functors between triangulated categories. Then F is exact if and only if G is exact.*

We also record the following definitions (see for example [12, 25]).

Definition 7.2. A full subcategory $\mathscr{B}$ of a triangulated category $\mathscr{C}$ is *thick* if any retract of an object of $\mathscr{B}$ is in $\mathscr{B}$ and if the third object of a distinguished triangle with two objects in $\mathscr{B}$ is also in $\mathscr{B}$. The category $\mathscr{B}$ is *localizing* if it is thick and closed under coproducts. The smallest thick (respectively, localizing) subcategory of $\mathscr{C}$ that contains a set of objects $\mathscr{G}$ is called the thick (respectively, localizing) subcategory generated by $\mathscr{G}$.

Definition 7.3. An object X of an additive category $\mathscr{A}$ is *compact*, or *small*, if the functor $\mathscr{A}(X,-)$ converts coproducts to direct sums. The category $\mathscr{A}$ is *compactly detected* if it has arbitrary coproducts and has a set $\mathscr{G}$ of compact objects that detects isomorphisms, in the sense that a map f in $\mathscr{A}$ is an isomorphism if and only if $\mathscr{A}(X,f)$ is an isomorphism for all $X \in \mathscr{G}$. When $\mathscr{A}$ is symmetric monoidal, we require its unit object to be compact and we include it in the detecting set $\mathscr{G}$.

When $\mathscr{A}$ is triangulated, this is equivalent to a definition given by Neeman [25, 1.7] (who used the term "generated" instead of "detected"), and we have the following generalization of a result of his [25, 5.1].

Lemma 7.4. *Let $\mathscr{A}$ be a compactly detected additive category with detecting set $\mathscr{G}$ and let $\mathscr{B}$ be any additive category. Let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor* *with right adjoint G. If G preserves coproducts, then F preserves compact objects. Conversely, if $F(X)$ is compact for $X \in \mathscr{G}$, then G preserves coproducts.*

Proof. Let $X \in \mathscr{A}$ and let $\{Y_i\}$ be a set of objects of $\mathscr{B}$. Then the evident map $f\colon \amalg G(Y_i) \longrightarrow G(\amalg Y_i)$ induces a map

$$f_*\colon \mathscr{A}(X, \amalg G(Y_i)) \longrightarrow \mathscr{A}(X, G(\amalg Y_i)).$$

If X is compact and f_* is an isomorphism, then, by adjunction and compactness, it induces an isomorphism

$$\amalg \mathscr{B}(F(X), Y_i) \longrightarrow \mathscr{B}(F(X), \amalg Y_i),$$

which shows that $F(X)$ is compact. Conversely, if X and $F(X)$ are both compact, then f_* corresponds under adjunction to the identity map of $\amalg \mathscr{B}(F(X), Y_i)$ and is therefore an isomorphism. Restricting to $X \in \mathscr{G}$, it follows from Definition 7.3 that f is an isomorphism. □

While this result is elementary, it is fundamental to the applications. We generally have much better understanding of left adjoints, so that the compactness criterion is verifiable, but it is the preservation of coproducts by right adjoints that is required in all of the formal proofs.

Returning to triangulated categories, we connect the notion of a generating set of objects from Definition 7.2 with the notion of a detecting set of objects from Definition 7.3. Its first part is [25, 3.2] (and is also given by the proof of [12, 2.3.2]). Its second part is [12, 2.1.3(d)].

Proposition 7.5. *Let $\mathscr{A}$ be a compactly detected triangulated category with detecting set $\mathscr{G}$. Then the localizing subcategory generated by $\mathscr{G}$ is $\mathscr{A}$ itself. If the objects of $\mathscr{G}$ are dualizable, then the thick subcategory generated by $\mathscr{G}$ is the full subcategory of dualizable objects in $\mathscr{A}$, and an object is dualizable if and only if it is compact.*

The following standard observation works in tandem with the previous result.

Proposition 7.6. *Let $F, F'\colon \mathscr{A} \longrightarrow \mathscr{B}$ be exact functors between triangulated categories and let $\phi\colon F \longrightarrow F'$ be a natural transformation that commutes with Σ. Then the full subcategory of $\mathscr{A}$ whose objects are those X for which ϕ is an isomorphism is thick, and it is localizing if F and F' preserve coproducts.*

Proof. Since a retract of an isomorphism is an isomorphism, closure under retracts is clear. Closure under triangles is immediate from the five lemma. A coproduct of isomorphisms is an isomorphism, so closure under coproducts holds when F and F' preserve coproducts. □

8. The formal isomorphism theorems

We assume throughout this section that $\mathscr{C}$ and $\mathscr{D}$ are closed symmetric monoidal categories with compatible triangulations and that (f^*, f_*) is an adjoint pair of functors with f^* strong symmetric monoidal and exact.

For the Wirthmüller context, we assume in addition that f^* has a left adjoint $f_!$. The maps (4.6)–(4.8) are then given by (3.3) and Proposition 4.12. When

$$\bar{\pi}\colon f_!(f^*Y \otimes X) \longrightarrow Y \otimes f_!X$$

is an isomorphism, the map

$$\omega\colon f_*X \longrightarrow f_!(X \otimes C)$$

is defined. Observe that $\bar{\pi}$ is a map between exact left adjoints and that $\bar{\pi}$ and ω commute with Σ. The results of the previous section give the following conclusion.

Theorem 8.1 (Formal Wirthmüller isomorphism)**.** *Let $\mathscr{C}$ be compactly detected with a detecting set $\mathscr{G}$ such that $\bar{\pi}$ and ω are isomorphisms for $X \in \mathscr{G}$. Then $\bar{\pi}$ is an isomorphism for all $X \in \mathscr{C}$. If all $X \in \mathscr{G}$ are dualizable, then ω is an isomorphism for all dualizable X. If f_* preserves coproducts, for example if $\mathscr{D}$ has a detecting set of compact objects $\mathscr{H}$ such that f^*Y is compact for all $Y \in \mathscr{H}$, then ω is an isomorphism for all $X \in \mathscr{C}$.*

The force of the theorem is that no construction of an inverse to ω is required: we need only check that ω is an isomorphism one detecting object at a time. Proposition 6.14 explains what is needed for that verification.

For the Grothendieck context, we can use the following basic results of Neeman [25, 3.1, 4.1] to construct the required right adjoint $f^!$ to f_* in favorable cases. A main point of Neeman's later monograph [26] and of Franke's paper [9] is to replace compact detection by a weaker notion that makes use of cardinality considerations familiar from the theory of Bousfield localization in algebraic topology.

Theorem 8.2 (Triangulated Brown representability theorem)**.** *Let $\mathscr{A}$ be a compactly detected triangulated category. A functor $H\colon \mathscr{A}^{op} \longrightarrow \mathscr{A}b$ that takes distinguished triangles to long exact sequences and converts coproducts to products is representable.*

Theorem 8.3 (Triangulated adjoint functor theorem)**.** *Let $\mathscr{A}$ be a compactly detected triangulated category and $\mathscr{B}$ be any triangulated category. An exact functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ that preserves coproducts has a right adjoint G.*

Proof. Take $G(Y)$ to be the object that represents the functor $\mathscr{B}(F(-), Y)$. $\square$

The map

$$\pi\colon Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X)$$

of (3.6) commutes with Σ. When π is an isomorphism,

$$\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$$

is defined and commutes with Σ. We obtain the following conclusion.

Theorem 8.4 (Formal Grothendieck isomorphism)**.** *Let $\mathscr{D}$ be compactly detected with a detecting set $\mathscr{G}$ such that f^*Y is compact and π is an isomorphism for $Y \in \mathscr{G}$. Then f_* has a right adjoint $f^!$, π is an isomorphism for all $Y \in \mathscr{D}$, and ϕ is an isomorphism for all dualizable Y. If the objects of $\mathscr{G}$ are dualizable and the functor $f^!$ preserves coproducts, then ϕ is an isomorphism for all $Y \in \mathscr{D}$.*

Proof. As a right adjoint of an exact functor, f_* is exact by Lemma 7.1, and it preserves coproducts by Lemma 7.4. Thus $f^!$ exists by Theorem 8.3. Now π is an isomorphism for all Y by Proposition 7.6, ϕ is an isomorphism for dualizable Y by Proposition 5.9, and the last statement holds by Propositions 7.5 and 7.6. $\square$

When $f^!$ is obtained abstractly from Brown representability, the only sensible way to check that it preserves coproducts is to appeal to Lemma 7.4, requiring $\mathscr{C}$ to be compactly detected and f_*X to be compact when X is in the detecting set. With this assumption on $\mathscr{C}$, π is an isomorphism for all X and Y if it is an isomorphism for all X in a detecting set for $\mathscr{C}$.

For the Verdier-Grothendieck context, we assume that we have a second adjunction $(f_!, f^!)$, with $f_!$ exact. We also assume given a map

$$\hat{\pi}\colon Y \otimes f_! X \longrightarrow f_!(f^*Y \otimes X)$$

that commutes with Σ. When $\hat{\pi}$ is an isomorphism, the map

$$\phi\colon f^*Y \otimes f^!Z \longrightarrow f^!(Y \otimes Z)$$

is defined and commutes with Σ. Since f^* and $f_!$ are both left adjoints and thus preserve coproducts, Propositions 7.6 and 5.9 give the following conclusion.

Theorem 8.5 (Formal Verdier isomorphism)**.** *Let $\mathscr{D}$ be compactly detected with a detecting set $\mathscr{G}$ such that $\hat{\pi}$ is an isomorphism for $Y \in \mathscr{G}$. Then $\hat{\pi}$ is an isomorphism for all $Y \in \mathscr{D}$, and ϕ is an isomorphism for all dualizable Y. If the objects of $\mathscr{G}$ are dualizable and the functor $f^!$ preserves coproducts, then ϕ is an isomorphism for all $Y \in \mathscr{D}$.*

Here again, $f^!$ preserves coproducts if and only if the objects $f_!Y$ are compact for all $Y \in \mathscr{G}$, by Lemma 7.4.

Remark 8.6. In many cases, one can construct a more explicit right adjoint $f_0^!$ from some subcategory $\mathscr{D}_0$ of $\mathscr{D}$ to some subcategory $\mathscr{C}_0$ of $\mathscr{C}$, as in Remark 5.6. In such cases we can combine approaches. Indeed, assume that we have an adjoint pair $(f_!, f_0^!)$ on full subcategories $\mathscr{C}_0$ and $\mathscr{D}_0$ such that objects isomorphic to objects in $\mathscr{C}_0$ (or $\mathscr{D}_0$) are in $\mathscr{C}_0$ (or $\mathscr{D}_0$). Then, by the uniqueness of adjoints, the right adjoint $f^!$ to $f_!$ given by Brown representability restricts on $\mathscr{D}_0$ to a functor with values in $\mathscr{C}_0$ that is isomorphic to the explicitly constructed functor $f_0^!$. That is, the right adjoint given by Brown representability can be viewed as an extension of the functor $f_0^!$ to all of $\mathscr{D}$. This allows quotation of Proposition 4.4 or 4.9 for the construction and comparison of the natural maps (4.1)–(4.3) or (4.6)–(4.8).

We give an elementary example and then some remarks on the proofs of the results that we have quoted from the literature, none of which are difficult.

Example 8.7. Return to Example 5.10, but assume further that $\mathscr{C}$ is a compactly detected triangulated category with a detecting set of dualizable objects. Here the formal Verdier duality theorem says that $\phi = \nu\colon Y \otimes \mathrm{Hom}(C, Z) \longrightarrow \mathrm{Hom}(C, Y \otimes Z)$ is an isomorphism if and only if the functor $\mathrm{Hom}(C, -)$ preserves coproducts. That is, an object C is dualizable if and only if $\mathrm{Hom}(C, -)$ preserves coproducts.

Remark 8.8. Clearly Theorem 8.3 is a direct consequence of Theorem 8.2. In turn, Theorem 8.2 is essentially a special case of Brown's original categorical representation theorem [3]. Neeman's self-contained proof closely parallels Brown's argument. The first statement of Proposition 7.5 is used as a lemma in the proof, but it is also a special case. To see this, let $\mathscr{B}$ be the localizing subcategory of $\mathscr{A}$ generated by $\mathscr{G}$. For $X \in \mathscr{A}$, application of the representability theorem to the functor $\mathscr{A}(-, X)$ on $\mathscr{B}$ gives an object $Y \in \mathscr{B}$ together with a natural isomorphism $\phi : \mathscr{A}(B, X) \longrightarrow \mathscr{B}(B, Y)$ on objects $B \in \mathscr{B}$. The map $f : Y \longrightarrow X$ such that $\phi(f) = \mathrm{id}_Y$ is an isomorphism since it induces an isomorphism $\mathscr{A}(B, f)$ for all objects $B \in \mathscr{G}$. The second part of Proposition 7.5 is intuitively clear, since objects in $\mathscr{A}$ not in the thick subcategory generated by $\mathscr{G}$ must involve infinite coproducts, and these will be neither dualizable nor compact. The formal proof in [12] starts from Example 8.7, which effectively ties together dualizability and compactness.

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