MODELS OF G-SPECTRA AS PRESHEAVES OF SPECTRA

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Abstract. Let G be a finite group. We give Quillen equivalent models for the category of G-spectra as categories of spectrally enriched functors from explicitly described domain categories to nonequivariant spectra. Our preferred model is based on equivariant infinite loop space theory applied to elementary categorical data. It recasts equivariant stable homotopy theory in terms of point-set level categories of G-spans and nonequivariant spectra. We also give a more topologically grounded model based on equivariant Atiyah duality.

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Introduction

The equivariant stable homotopy category is of fundamental importance in algebraic topology. It is the natural home in which to study equivariant stable homotopy theory, a subject that has powerful and unexpected nonequivariant applications and is also of great intrinsic interest. The foundations were well established by the mid-1980's, and by then the importance of working with equivariant spectra had already become abundantly clear, especially with Carlsson's proof of the Segal conjecture [C1]. The following decade saw much further progress; Mackey functor and $RO(G)$-graded cohomology theories came of age, the Tate square and norm maps were introduced and given their first applications [GrM2, GrM3], and THH, TC, and their applications to algebraic K-theory had made their appearance [BHM]. Summary accounts of where the subject stood in the mid-1990's are given in [C2, GrM1, M1]. While there was continued work in the following decade, the subject really took hold in the mainstream of algebraic topology with its unexpected role in the 2009 solution of the Kervaire invariant problem by Hill, Hopkins, and Ravenel [HHR]. For example, on a foundational level, understanding norms as maps of equivariant spectra plays a key role.

The first draft of this paper appeared in 2011, and the subject has truly blossomed in the decade since. Formally, just as the category of G-spaces is Quillen equivalent to the presheaf category of contravariant functors from the orbit category of G to spaces, the category of G-spectra is Quillen equivalent to the presheaf category of spectrally enriched contravariant functors from its full subcategory of suspension spectra of orbits to spectra. We shall say more about that shortly. The purpose of this paper is to replace the target presheaf category by one that is Quillen equivalent and yet is accessible to concrete constructions on the level of related presheaf categories of spaces and categories.

Setting up the equivariant stable homotopy category with its attendant model structures takes a fair amount of work. The first version was due to Lewis and May [LMSM], and more modern versions that we shall start from are given in Mandell and May [MaM] and, even more recently, [HHR]. A result of Schwede and Shipley [SchSh] (reworked in [GM1] to give the starting point of this paper) asserts that any stable model category $\mathscr{M}$ is equivalent to a category $\mathbf{Pre}(\mathscr{D}, \mathscr{S})$ of spectrally enriched presheaves with values in a chosen category $\mathscr{S}$ of spectra. However, the domain $\mathscr{S}$-category $\mathscr{D}$ is a *full* $\mathscr{S}$-subcategory of $\mathscr{M}$ and typically is as inexplicit and mysterious as $\mathscr{M}$ itself. From the point of view of applications and calculations, this is therefore only a starting point. One wants a more concrete understanding of the category $\mathscr{D}$. We shall give explicit equivalents to the domain category $\mathscr{D}$ in the case when $\mathscr{M} = G\mathscr{S}$ is the category of G-spectra for a finite group G, and we fix a finite group G throughout.

We shall define an $\mathscr{S}$-category (or spectral category) $G\mathscr{A}$ by applying a suitable infinite loop space machine to simply defined categories of finite G-sets. The spectral category $G\mathscr{A}$ is a spectrally enriched version of the Burnside category of G. We shall prove the following result.

Theorem 0.1 (Main theorem)**.** *There is a zig-zag of Quillen equivalences*

$$G\mathscr{S} \simeq \mathbf{Pre}(G\mathscr{A}, \mathscr{S})$$

relating the category of G-spectra to the category of spectrally enriched contravariant functors $G\mathscr{A} \longrightarrow \mathscr{S}$.

Such functors are often called presheaves. We reemphasize the simplicity of our spectral category $G\mathscr{A}$: no prior knowledge of G-spectra is required to define it.

We give a precise description of the relevant categorical input and restate the main theorem more precisely in Section 1. The central point of the proof is to use equivariant infinite loop space theory to construct the spectral category $G\mathscr{A}$ from elementary categories of finite G-sets. We prove our main theorem in Section 2, using the equivariant Barratt-Priddy-Quillen (BPQ) theorem to compare $G\mathscr{A}$ to the spectral category $G\mathscr{D}_{\mathrm{All}}$ given by the suspension G-spectra $\Sigma^\infty_G(A_+)$ of based finite G-sets A_+, which is a standard choice for application of the theorem of Schwede and Shipley to $G\mathscr{S}$. The classical Burnside category of isomorphism classes of spans of finite G-sets leads to a calculation of the homotopy category $\mathrm{Ho}G\mathscr{D}_{\mathrm{All}}$ (see Theorem 1.12 below), and $G\mathscr{A}$ starts from the bicategory of such spans, in which isomorphisms of spans give the 2-cells.

Intuitively, (algebraic) Mackey functors can be viewed as functors from $\mathrm{Ho}G\mathscr{D}_{\mathrm{All}}$ to abelian groups, and the result of Schwede and Shipley says that G-spectra can be viewed as functors from $G\mathscr{D}_{\mathrm{All}}$ to spectra. We are lifting the standard purely algebraic understanding of Mackey functors to obtain an analogous algebraic understanding of G-spectra as functors from $G\mathscr{A}$ to spectra. Thus the slogan is that G-spectra are spectral Mackey functors.

It is crucial to our work that the G-spectra $\Sigma^\infty_G(A_+)$ are self-dual. Our original proof took this as a special case of equivariant Atiyah duality (Section 4.2), thinking of A as a trivial example of a smooth closed G-manifold. We later found a direct categorical proof (Section 2.3) of this duality based on equivariant infinite loop space theory and the equivariant BPQ theorem. This allows us to give an illuminating new proof of the required self-duality as we go along. We give presheaf versions of a few standard constructions on G-spectra in Section 3. Switching gears, we give an alternative presheaf model for the category of G-spectra in terms of classical Atiyah duality in Section 4. An appendix, Section 5, provides some background on the two model categories of G-spectra used here, equivariant orthogonal spectra and equivariant S-modules, and describes and compares the specialization of [GM1] to those categories that provides the starting point for our work.

We take what we need from equivariant infinite loop space theory as a black box in this paper. The additive and multiplicative space level theories are worked out in [MMO] and [GMMO1], respectively. The generalization from space level to category level input is based on general (and not necessarily equivariant) categorical coherence theory that is worked out in [GMMO3]. What is needed for this paper is a small part of the full story there.

We thank a diligent referee for demanding a reorganization of our original paper. We also thank Angélica Osorno and Inna Zakharevich for very helpful comments, and we especially thank Osorno and Anna Marie Bohmann for catching an error in the handling of pairings in earlier versions of this work. That error is one reason for the very long delay in the publication of this paper, which was first posted on ArXiv on August 21, 2011. The delay is no fault of this journal.

In the interim, we teamed with Osorno and Mona Merling to fully work out the relevant infinite loop space theory, which turned out to be both surprisingly demanding and unexpectedly interesting. Also in the interim, Bohmann and Osorno [BO] introduced categorical Mackey functors and used these, together with our main result, to produce a functorial construction of equivariant Eilenberg-Mac Lane spectra for Mackey functors. The prospect of applications like theirs was a major motivation for our variant of the Schwede and Shipley model for the homotopy category of G-spectra. A small error[1] in [BO] is corrected in the short appendix, Section 6, of this paper. Further applications to the concrete construction of genuine G-spectra are in development in their work and in work of Cary Malkievich and Merling [MM1, MM2]. During the delay, Jonathan Rubin combed through our draft and caught a great many errors of detail and infelicities. Needless to say, we are responsible for all that remain.

Comparison with alternative approaches. We also note that since this article first appeared online in 2011, several alternative approaches have been given by other authors. First among these was the work of Barwick [B]. A notable difference is that our spectral Burnside category $G\mathscr{A}$ is a group completion of Barwick's effective Burnside category. A second difference is that Barwick is working in the ∞-categorical setting, so that questions of strictness, such as those necessitating our Section 6, do not arise. Moreover, Barwick's work provides a conceptual generalization that applies to handle the case of profinite groups, as well as other applications. Later, streamlined alternative approaches were given in [N] and [CMNN, Appendix A]. The version described in [CMNN] has the advantage of providing a monoidal equivalence (see also [BGS, Section 11]). See Remark 3.9 for further discussion.

1. The bicategory $G\mathscr{E}$ and $\mathscr{S}$-category $G\mathscr{A}$

In this paper, $\mathscr{S}$ denotes the category of (nonequivariant) orthogonal spectra. See Section 5 for some discussion of the comparison between models of G-spectra. We first define the $\mathscr{S}$-category $G\mathscr{A}$ (Definition 1.13) and restate our main theorem. Conceptually $G\mathscr{A}$ can be viewed as obtained by applying a nonequivariant infinite loop space machine $\mathbb{K}$ to a category $G\mathscr{E}$ "enriched in permutative categories".[2] The term in quotes can be made categorically precise [G, HP, Sch], but we shall use it just as an informal slogan since no real categorical background is necessary to our work here: we shall give direct elementary definitions of the examples we use, and they do satisfy the axioms specified in the cited sources. We then define (Definition 1.29) a G-category[3] $\mathscr{E}_G$ "enriched in permutative G-categories", from which $G\mathscr{E}$ is obtain by passage to G-fixed subcategories. Section 1.5 contains a discussion of duality that will be needed in Section 2 for the proof of our main theorem.

1.1. **The bicategory $G\mathscr{E}$ of G-spans.** In any category $\mathscr{C}$ with pullbacks, the bicategory of spans in $\mathscr{C}$ has 0-cells the objects of $\mathscr{C}$. The 1-cells and 2-cells

[1] We are grateful to Angélica Osorno for helping us discover and fix this error.

[2] A permutative category is a symmetric strict monoidal category.

[3] In general, we understand a G-category to be a category internal and not just enriched in G-sets, meaning that G can act on both objects and morphisms.

$A \longrightarrow B$ are the diagrams

$$(1.1) \qquad \begin{tikzcd} B & D \arrow[l] \arrow[r] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ B & & A. \\ & E. \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Composites of 1-cells are given by (chosen) pullbacks

$$(1.2) \qquad \begin{tikzcd} & & F \arrow[dl] \arrow[dr] & & \\ & E \arrow[dl] \arrow[dr] & & D \arrow[dl] \arrow[dr] & \\ C & & B & & A. \end{tikzcd}$$

The identity 1-cells are the diagrams $A \xleftarrow{=} A \xrightarrow{=} A$. The associativity and unit constraints are determined by the universal property of pullbacks. Observe that the 1-cells $A \longrightarrow B$ can just as well be viewed as objects over $B \times A$. Viewed this way, the identity 1-cells are given by the diagonal maps $A \longrightarrow A \times A$.

Our starting point is the bicategory of spans of (unbased) finite G-sets. Here the disjoint union of G-sets over $B \times A$ gives us a symmetric monoidal structure on the category of 1-cells and 2-cells $A \longrightarrow B$ for each pair (A, B). We can think of the bicategory of spans as a category "enriched in the category of symmetric monoidal categories". Again, the notion in quotes does not make obvious mathematical sense since there is no obvious monoidal structure on the category of symmetric monoidal categories, but category theory due to the first author [G] (see also [HP, Sch]) explains what these objects are and how to rigidify them to categories enriched in permutative categories.

We repeat that we have no need to go into such categorical detail. Rather than apply such category theory, we give a direct elementary construction of a strict structure that is equivalent to the intuitive notion of the category "enriched in symmetric monoidal categories" of spans of finite G-sets. We first define a bipermutative category $G\mathscr{E}(1)$ that is equivalent to the symmmetric bimonoidal groupoid of finite G-sets.

Definition 1.3. Any finite G-set is isomorphic to one of the form $A = \underline{n}^{\alpha}$, where $\underline{n} = \{1, \cdots, n\}$, α is a homomorphism $G \longrightarrow \Sigma_n$, and G acts on $\underline{n}$ by $g \cdot i = \alpha(g)(i)$ for $1 \leq i \leq n$. *We understand finite G-sets to be of this restricted form from now on.* A G-map $f \colon \underline{m}^{\alpha} \longrightarrow \underline{n}^{\beta}$ is a function $f \colon \underline{m} \longrightarrow \underline{n}$ such that $f \circ \alpha(g) = \beta(g) \circ f$ for $g \in G$. The morphisms of $G\mathscr{E}(1)$ are the isomorphisms $\underline{n}^{\alpha} \longrightarrow \underline{n}^{\beta}$ of G-sets.

The disjoint union $D \amalg E$ of finite G-sets $D = \underline{s}^{\sigma}$ and $E = \underline{t}^{\tau}$ is $\underline{s+t}^{\sigma \oplus \tau}$, with $\sigma \oplus \tau$ being the evident block sum $G \longrightarrow \Sigma_{s+t}$. With the evident commutativity isomorphism, this gives the permutative groupoid[4] $G\mathscr{E}(1)$ of finite G-sets; the empty finite G-set is the unit for $\amalg$. To define the cartesian product, for each s and t let $\lambda_{s,t} \colon \underline{st} \longrightarrow \underline{s} \times \underline{t}$ denote the lexicographic ordering. Then $D \times E$ is $\underline{st}^{\sigma \otimes \tau}$ where $\sigma \otimes \tau$ is the permutation

$$\underline{st} \xrightarrow{\lambda_{s,t}} \underline{s} \times \underline{t} \xrightarrow{\sigma \times \tau} \underline{s} \times \underline{t} \xrightarrow{\lambda_{s,t}^{-1}} \underline{st}$$

[4] Though the terminology "permutative category" is more prevalent than "permutative groupoid", we find it useful to remind the reader that we are only considering isomorphisms.

as in [GMMO3, (3.6)]. There is again an evident commutativity isomorphism, and $\amalg$ and $\times$ give $G\mathscr{E}(1)$ a structure of bipermutative category in the sense of [M6]; the multiplicative unit is the trivial G-set $1 = (\underline{1}, \varepsilon)$, where $\varepsilon(g) = 1$ for $g \in G$.

As we will need it later, we also introduce the reordering permutation $\tau_{s,t} \in \Sigma_{st}$, defined as the composition

$$\underline{st} \xrightarrow{\lambda_{s,t}} \underline{s} \times \underline{t} \xrightarrow{\cong} \underline{t} \times \underline{s} \xrightarrow{\lambda_{t,s}^{-1}} \underline{ts} = \underline{st}.$$

as in [GMMO3, Definition 3.8].

We may view $G\mathscr{E}(1)$ as the category of finite G-sets over the one point G-set 1, and we generalize the definition as follows.

Definition 1.4. For a finite G-set A, we define a permutative groupoid $G\mathscr{E}(A)$ of finite G-sets over A. The objects of $G\mathscr{E}(A)$ are the G-maps $p\colon D \longrightarrow A$. The morphisms $p \longrightarrow q$, $q\colon E \longrightarrow A$, are the G-isomorphisms $f\colon D \longrightarrow E$ such that $q \circ f = p$. Disjoint union of G-sets over A gives $G\mathscr{E}(A)$ a structure of permutative category; its unit is the empty set over A. When $A = 1$, $G\mathscr{E}(A)$ is the ("additive") permutative category of the previous definition.

Remark 1.5. There is also a product $\times\colon G\mathscr{E}(A) \times G\mathscr{E}(B) \longrightarrow G\mathscr{E}(A \times B)$. It takes (D, E) to $D \times E$, where D and E are finite G-sets over A and B, respectively. This product is also strictly associative and unital, with unit the unit of $G\mathscr{E}(1)$, and it has an evident commutativity isomorphism. Restriction to the object 1 gives the "multiplicative" permutative category of Definition 1.3. This product distributes over $\amalg$ and *almost* makes the enriched category $G\mathscr{E}$ of the next definition into a "category enriched in permutative categories", in the sense defined in [G]. The "almost" refers to the fact that the category we define does not have a strict unit, a problem that was encountered in [BO] and is fixed in Section 6 below.

Definition 1.6. We define a bicategory $G\mathscr{E}$ with a permutative hom groupoid for each pair of objects as follows. The 0-cells of $G\mathscr{E}$ are the finite G-sets, which may be thought of as the categories $G\mathscr{E}(A)$. The permutative groupoid $G\mathscr{E}(A, B)$ of 1-cells and 2-cells $A \longrightarrow B$ is $G\mathscr{E}(B \times A)$, as defined in Definition 1.4. The 1-cells are thought of as spans and the 2-cells as isomorphisms of spans. The composition

$$\circ\colon G\mathscr{E}(B, C) \times G\mathscr{E}(A, B) \longrightarrow G\mathscr{E}(A, C)$$

is defined via pullbacks, as in the diagram (1.2). The diagonal map $\Delta_A : A \longrightarrow A \times A$ serves as a unit 1-cell. Precisely, following [BO, 7.2], we choose the pullback F in (1.2) to be the sub G-set of $E \times D$, ordered lexicographically, consisting of the elements (e, d) such that d and e map to the same element of B.

Remark 1.7. This bicategory is almost a 2-category. The composition of spans is strictly associative, but if $|A| \geq 2$ then $\Delta_A : A \longrightarrow A \times A$ acts as a strict unit only on the right and so should be called a pseudo-unit 1-cell. The point is that with our chosen model for the pullback, the left map in the span composition

$$\begin{tikzcd}[column sep=small]
 & & \Delta_B \circ E \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & B \arrow[dl, equal] \arrow[dr, equal] & & E \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B & & B & & A
\end{tikzcd}$$

must be order-preserving. Therefore, if f is not order-preserving, then $\Delta_B \circ E \neq E$. However, in view of the evident commutative diagram

$$\begin{tikzcd} & \Delta_B \circ E \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "g \circ p_2"] & \\ B & E \arrow[l, "f"] \arrow[r, "g"'] & A, \end{tikzcd}$$

the function p_2 specifies a reordering isomorphism of spans

$$\Delta_B \circ E \xrightarrow{\ell_{B,E}} E \tag{1.8}$$

In Section 6, we show how to whisker the pseudo-unit 1-cells to obtain an equivalent construction $G\mathscr{E}'$ that still has a strictly associative composition but now has strict two-sided unit 1-cells. The construction is closely analogous to the usual whiskering of a degenerate basepoint in a space to obtain a nondegenerate basepoint.

Remark 1.9. We are suppressing some categorical details that are irrelevant to our work. The composition distributes over coproducts, and it should be defined on a "tensor product" rather than a cartesian product of permutative categories. Such a tensor product does in fact exist [HP], but we shall not use the relevant category theory. Rather we will change notation to $\wedge$ since the composition is a pairing that gives rise to a pairing defined on the smash product of the spectra constructed from $G\mathscr{E}(B, C)$ and $G\mathscr{E}(A, B)$. The passage from pairings of permutative categories to pairings of spectra has a checkered history even nonequivariantly,[5] and it is here that a mistake occurred in earlier versions of this paper. As explained in [GMMO3], categorical strictification and the full development of multiplicative equivariant infinite loop space theory resolve the relevant issues.

Remark 1.10. It is helpful to observe that the composition just defined can be viewed as a composite of maps of finite G-sets induced contravariantly and covariantly by the maps of finite G-sets

$$C \times B \times B \times A \xleftarrow{\mathrm{id} \times \Delta \times \mathrm{id}} C \times B \times A \xrightarrow{\quad \pi \quad} C \times A,$$

where $\pi : C \times B \times A \longrightarrow C \times A$ is the projection.

Before beginning work, we recall an old result that motivated this paper. The category $[G\mathscr{E}]$ of isomorphism classes of G-spans is obtained from the bicategory $G\mathscr{E}$ of G-spans by identifying spans from A to B if there is an isomorphism between them. Composition is again by pullbacks. We add spans from A to B by taking disjoint unions, and that gives the morphism set $[G\mathscr{E}](A, B)$ a structure of abelian monoid. We apply the Grothendieck construction to obtain an abelian group of morphisms $A \longrightarrow B$. This gives an additive category $\mathscr{A}b[G\mathscr{E}]$.

Definition 1.11. Define $G\mathscr{D}_{\mathrm{All}}$ to be the full subcategory of $G\mathscr{S}$ whose objects are fibrant replacements of the G-spectra $\Sigma^\infty_G(A_+)$ in the stable model structure [MaM], where A runs over the finite G-sets and let $\mathrm{Ho}G\mathscr{D}_{\mathrm{All}} \subset \mathrm{Ho}G\mathscr{S}$ denote its homotopy category.

[5]That starts from [M3], which is modernized, corrected, and generalized in [GMMO3].

Theorem 1.12 ([LMSM, V.9.6][6]). *The categories $HoG\mathscr{D}_{\text{All}}$ and $\mathscr{A}b[G\mathscr{E}]$ are isomorphic.*

1.2. **The precise statement of the main theorem.** Infinite loop space theory associates a spectrum $\mathbb{K}\mathscr{A}$ to a permutative category $\mathscr{A}$. There are several machines available and all are equivalent [M5]. Since it is especially convenient for the equivariant generalization, we require $\mathbb{K}$ to take values in the category $\mathscr{S}$ of orthogonal spectra [MMSS], but symmetric spectra would also work. Slightly modifying the axiomatization of [M5], we require $\mathbb{K}$ to take values in positive[7] Ω-spectra and we require a natural map $\eta\colon B\mathscr{A} \longrightarrow (\mathbb{K}\mathscr{A})_0$ whose composition with $(\mathbb{K}\mathscr{A})_0 \longrightarrow \Omega(\mathbb{K}\mathscr{A})_1$ gives a group completion.

Since $\mathscr{S}$ is closed symmetric monoidal under the smash product, it makes sense to enrich categories in $\mathscr{S}$. Our preferred version of spectral categories is categories enriched in $\mathscr{S}$, abbreviated $\mathscr{S}$-categories. Model theoretically, $\mathscr{S}$ is a particularly nice enriching category since its unit S is cofibrant in the stable model structure and $\mathscr{S}$ satisfies the monoid axiom [MMSS, 12.5].

When a spectral category $\mathscr{D}$ is used as the domain category of a presheaf category, the objects and maps of the underlying category are unimportant. The important data are the morphism spectra $\mathscr{D}(A,B)$, the unit maps $S \longrightarrow \mathscr{D}(A,A)$, and the composition maps

$$\mathscr{D}(B,C) \wedge \mathscr{D}(A,B) \longrightarrow \mathscr{D}(A,C).$$

The presheaves $\mathscr{D}^{op} \longrightarrow \mathscr{S}$ can be thought of as (right) $\mathscr{D}$-modules.

Recall that an object a in a permutative category $\mathscr{A}$ determines a point of $B\mathscr{A}$ hence, via η, a point of $(\mathbb{K}\mathscr{A})_0$. Therefore each $a \in \mathscr{A}$ determines a map $S \longrightarrow \mathbb{K}\mathscr{A}$. We will use this to specify unit maps for spectral categories.

Definition 1.13. We define a spectral category $G\mathscr{A}$. Its objects are the finite G-sets A, which may be viewed as the spectra $\mathbb{K}G\mathscr{E}(A)$. Its morphism spectra are defined by $G\mathscr{A}(A,B) = \mathbb{K}G\mathscr{E}'(A,B)$. Its unit maps $S \longrightarrow G\mathscr{A}(A,A)$ are induced by the identity 1-cells in $G\mathscr{E}'(A,A)$ and its composition

$$G\mathscr{A}(B,C) \wedge G\mathscr{A}(A,B) \longrightarrow G\mathscr{A}(A,C)$$

is induced by composition in $G\mathscr{E}'$.

As written, the definition makes little sense: to make the word "induced" meaningful requires a suitably behaved machine $\mathbb{K}$, as we will spell out in Section 2.2. For the purpose of Definition 1.13, the machine of [EM] would be sufficient, although it takes values in symmetric rather than orthogonal spectra. However, the proof of our main theorem, given in Section 2.4, will use the equivariant machine of [GMMO3], and we will therefore use the same machine to make sense of Definition 1.13. Once this is done, we will have the presheaf category $\mathbf{Pre}(G\mathscr{A},\mathscr{S})$ of $\mathscr{S}$-functors $(G\mathscr{A})^{op} \longrightarrow \mathscr{S}$ and $\mathscr{S}$-natural transformations. As shown for example in [GM1], it is a cofibrantly generated model category enriched in $\mathscr{S}$, or an $\mathscr{S}$-model category for short. As shown in [MaM], the category $G\mathscr{S}$ of (genuine) orthogonal G-spectra is also an $\mathscr{S}$-model category. Our main theorem can be restated as follows.

[6] All G-spectra in [LMSM] are fibrant, but we are using orthogonal G-spectra in this paper. The homotopy categories are equivalent.

[7] This means that $E_0 \longrightarrow \Omega E_1$ need not be an equivalence

Theorem 1.14 (Main theorem)**.** *There is a zigzag of enriched Quillen equivalences connecting the $\mathscr{S}$-model categories $G\mathscr{S}$ and $\mathbf{Pre}(G\mathscr{A}, \mathscr{S})$.*

Therefore G-spectra can be thought of as constructed from the very elementary category $G\mathscr{E}$ enriched in permutative categories, ordinary nonequivariant spectra, and the black box of infinite loop space theory.

We have chosen to take all finite G-sets A as the objects of $G\mathscr{A}$. As we discuss in Remark 5.4, Theorem 1.14 holds just as well if we allow A to instead range only over the orbits G/H for subgroups $H \subset G$ (or even over one H in each conjugacy class). Intuitively, a G-spectrum is then described by its fixed point spectra X^H, together with enriched restriction and transfer data. A bit more precisely, let $\mathscr{O}_G$ denote the category of orbits G/H and G-maps between them. For a G-spectrum X, passage to fixed point spectra specifies a contravariant functor $X^{(-)}\colon \mathscr{O}_G \longrightarrow \mathscr{S}$. The following reassuring result falls out of the proof of Theorem 1.14. We shall be more precise about this in Corollary 3.7.

Corollary 1.15. *The zigzag of equivalences induces a natural zigzag of equivalences between the fixed point orbit functor, $X \mapsto \{G/H \mapsto X^H\}$, on G-spectra and the functor given by restricting presheaves $G\mathscr{A} \longrightarrow \mathscr{S}$ to the (unenriched) orbit category.*

Thus, if X is a fibrant G-spectrum that corresponds to the presheaf Y, then X^H is equivalent to $Y(G/H)$.

Remark 1.16. For any n, the homotopy groups $\pi_n(X^H)$ define a Mackey functor, and so do the homotopy groups $\pi_n(Y(G/H))$. The corollary implies an isomorphism between these Mackey functors.

We view Theorem 1.14 as a G-spectrum analog of the standard equivalence between G-spaces and space-valued presheaves on $\mathscr{O}_G$ (e.g. [M1, Chapter VI]). As there, we do not in any sense view the theorem as giving a *replacement* for the category of G-spectra. We regard G-spectra as natural objects of intrinsic interest, and their presheaf descriptions as an illuminating perspective. We give some comparisons of functors to illustrate this in the brief Section 3.

1.3. **The G-bicategory $\mathscr{E}_G$ of spans: intuitive definition.** Everything we do depends on first working equivariantly and then passing to fixed points. We fix some generic notations. For a category $\mathscr{C}$, let $G\mathscr{C}$ be the category of G-objects in $\mathscr{C}$ and G-maps between them. Let $\mathscr{C}_G$ be the G-category of G-objects and nonequivariant maps, with G acting on morphisms by conjugation. The two categories are related conceptually by $G\mathscr{C} = (\mathscr{C}_G)^G$. The objects, being G-objects, are already G-fixed; we apply the G-fixed point functor to hom sets. More generally, we can start with a category $\mathscr{C}$ with actions by G on its objects and again define a category $G\mathscr{C}$ of G-maps and a G-category $\mathscr{C}_G$ with G-fixed category $G\mathscr{C}$. The reader may prefer to think of $G\mathscr{C}$ as a category enriched in G-categories, with enriched hom objects the G-categories $\mathscr{C}_G(A, B)$ for G-objects A and B.

We apply this framework to the category of finite G-sets. We have already defined the G-fixed bicategory $G\mathscr{E}$, and we shall give two definitions of G-bicategories $\mathscr{E}_G$ with fixed point bicategories equivalent to $G\mathscr{E}$. The first, given in this section, is more intuitive, but the second is more convenient for the proof of our main theorem.

Let U be a countable G-set that contains all orbit types G/H infinitely many times. Again let A, B, and C denote finite G-sets, but now think of the D, E, and F of (1.1) and (1.2) as finite subsets of the G-set U; these subsets need *not* be G-subsets. The action of G on U gives rise to an action of G on the finite subsets of U: for a finite subset D of U and $g \in G$, gD is another finite subset of U.

Definition 1.17. We define a G-groupoid $\mathscr{E}^U_G(A)$. The objects of $\mathscr{E}^U_G(A)$ are the nonequivariant maps $p\colon D \longrightarrow A$, where A is a finite G-set and D is a finite subset of U. The morphisms $f\colon p \longrightarrow q$, $q\colon E \longrightarrow A$, are the bijections $f\colon D \longrightarrow E$ such that $q \circ f = p$. The group G acts on objects and morphisms by sending D to gD and sending a bijection $f\colon D \longrightarrow E$ over A to the bijection $gf\colon gD \longrightarrow gE$ over A given by $(gf)(gd) = g(f(d))$.

Definition 1.18. We define a bicategory $\mathscr{E}^U_G$ with objects the finite G-sets and with G-groupoids of morphisms between objects given by $\mathscr{E}^U_G(A,B) = \mathscr{E}^U_G(B \times A)$. Thinking of the objects of $\mathscr{E}^U_G(A,B)$ as nonequivariant spans $B \longleftarrow D \longrightarrow A$, composition and units are defined as in Definition 1.6.

Observe that taking disjoint unions of finite sets over A will not keep us in U and is thus not well-defined. Therefore the $\mathscr{E}^U_G(A)$ are not even symmetric monoidal (let alone permutative) G-categories in the naive sense of symmetric monoidal categories with G acting compatibly on all data.

1.4. The G-bicategory $\mathscr{E}_G$ of spans: working definition.

We shall work with a less intuitive definition of $\mathscr{E}_G$, one that solves the problem of disjoint unions by avoiding any explicit use of them. It uses an especially convenient E_∞ operad of G-categories, denoted $\mathscr{P}_G$. We recall it from [GM2], where we define a genuine permutative G-category to be an algebra over $\mathscr{P}_G$. More generally, in [GMMO2] we define a genuine symmetric monoidal G-category to be a pseudoalgebra over $\mathscr{P}_G$, but we will not need that notion here. Such pseudoalgebras provide input for an equivariant infinite loop space machine.

To define $\mathscr{P}_G$, we apply our general point of view on equivariant categories to the category $\mathscr{C}at$ of small categories. Thus, for G-categories $\mathscr{A}$ and $\mathscr{B}$, let $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$ be the G-category of functors $\mathscr{A} \longrightarrow \mathscr{B}$ and natural transformations, with G acting by conjugation, and let $G\mathscr{C}at(\mathscr{A},\mathscr{B}) = Cat_G(\mathscr{A},\mathscr{B})^G$ be the category of G-functors and G-natural transformations.

Definition 1.19. Let $\mathcal{E}G$ be the groupoid[8] with object set G and a unique morphism, denoted (h,k), from k to h for each pair of objects. Let G act from the right on $\mathcal{E}G$ by $h \cdot g = hg$ on objects and $(h,k)\cdot g = (hg,kg)$ on morphisms. Define $\mathscr{P}(j) = \mathcal{E}\Sigma_j$; this is the jth category of an E_∞ operad of categories whose algebras are the permutative categories [D, M4]. Define $\mathscr{P}_G(j)$ to be the G-category

$$\mathscr{P}_G(j) = \mathscr{C}at_G(\mathcal{E}G, \mathcal{E}\Sigma_j) = \mathscr{C}at_G(\mathcal{E}G, \mathscr{P}(j)).$$

Here G acts trivially on $\mathcal{E}\Sigma_j$. The left action of G on $\mathscr{P}_G(j)$ is induced by the right action of G on $\mathcal{E}G$, and the right action of Σ_j is induced by the right action of Σ_j on $\mathcal{E}\Sigma_j$. The functor $\mathscr{C}at_G(\mathcal{E}G,-)$ is product preserving and the operad structure maps are induced from those of $\mathscr{P}$. We interpret $\mathscr{P}(0)$ and $\mathscr{P}_G(0)$ to be trivial categories; $\mathscr{P}_G(1)$ is also trivial, with unique object denoted $\mathbb{1}$.

[8] While $\mathcal{E}G$ is isomorphic as a G-category to the translation category of G, the action of G on that category is defined differently, as is explained in [GMM, Lemma 1.7]. Our $\mathcal{E}G$ is the chaotic category of G, sometimes denoted $\tilde{G}$.

Definition 1.20. Regard a finite G-set A as a discrete G-groupoid (identity morphisms only). Define the G-groupoid $\mathscr{E}_G(A)$ by

$$\mathscr{E}_G(A) = \coprod_{n\geq 0} \mathscr{P}_G(n) \times_{\Sigma_n} A^n = (\coprod_{n\geq 1} \mathscr{P}_G(n) \times_{\Sigma_n} A^n)_+. \tag{1.21}$$

We interpret the term with $n = 0$ to be a trivial base category $*$, which explains the second equality, and we identify the term with $n = 1$ with A.

The following result is neither obvious nor difficult. It is proven in [GM2], where it is one ingredient in a categorical proof of the tom Dieck splitting theorem.

Theorem 1.22 ([GM2, Theorem 5.5])**.** *The G-fixed permutative groupoid $\mathscr{E}_G(A)^G$ is naturally isomorphic to the permutative groupoid $G\mathscr{E}(A)$ of Definition 1.4.*

The starting point of the proof is the observation that a functor $\mathcal{E}G \longrightarrow \mathcal{E}\Sigma_n$ is uniquely determined by its object function $G \longrightarrow \Sigma_n$. In particular, for a finite G-set $B = \underline{n}^\beta$ we may view the group homomorphism $\beta\colon G \longrightarrow \Sigma_n$ as an object of the category $\mathscr{P}_G(n)$. With a little care, we see that a G-fixed object $(\beta; a_1, \cdots, a_n)$ of $\mathscr{P}_G(n) \times_{\Sigma_n} A^n$ can be interpreted as a G-map $B \longrightarrow A$ and that all finite G-sets over A are of this form.

Remark 1.23. Conceptually, Definition 1.20 hides an important identification and extension of functoriality. A priori, $\mathscr{E}_G(A)$ appears to be a functor on *unbased* finite G-sets, but an alternative reformulation is

$$\mathscr{E}_G(A) = \mathbb{P}_G(A_+) \tag{1.24}$$

which exhibits $\mathscr{E}_G$ as a functor on *based* finite G-sets A_+. Here $\mathbb{P}_G$ is the monad in the category of based G-categories whose algebras are the same as the $\mathscr{P}_G$-algebras. Thus $\mathbb{P}_G(A_+)$ is the free $\mathscr{P}_G$-algebra (= genuine permutative G-category) generated by A_+, with unit given by the disjoint trivial base category added to A.

We need to be more precise about this identification and extended functoriality.

Definition 1.25. Let Λ be the category of finite based sets $\mathbf{n}$ and injections.[9] For a finite based G-set $\mathcal{A}$, define $\mathbb{P}_G(\mathcal{A})$ to be the categorical tensor product

$$\mathscr{P}_G(\bullet) \otimes_\Lambda \mathcal{A}^\bullet.$$

Since any based injection $\sigma \in \Lambda(\mathbf{m}, \mathbf{n})$ can be written uniquely as the composition of a permutation of $\mathbf{m}$ followed by an order-preserving injection, the contravariant functoriality of $\mathscr{P}_G(\bullet)$ on based injections is given by combining the right Σ_m-action on $\mathscr{P}_G(m)$ with the contravariant functoriality with regards to ordered injections described in [M2, 2.3]. Insertion of basepoints makes the powers $\mathscr{A}^n$ into a covariant functor $\mathcal{A}^\bullet$ on Λ, and

$$\mathbb{P}_G(\mathcal{A}) = \Big(\coprod_{n\geq 0} \mathscr{P}_G(n) \times \mathcal{A}^n\Big)/(\sim), \tag{1.26}$$

where

$$(\sigma^*\mu; a) \sim (\mu; \sigma_* a) \quad \text{for} \quad \mu \in \mathscr{P}_G(n), \quad \sigma \in \Lambda(\mathbf{m}, \mathbf{n}), \quad \text{and} \quad a \in \mathcal{A}^m.$$

[9]The category Λ of finite based sets and injections is isomorphic to the category of finite sets and injections. We use based here both for historical reasons and because it fits better into the machinery of infinite loop space theory.

As in [M2, 2.3], we can first pass to orbits using the permutations in Λ and then use the equivalence relation induced by the proper injections to rewrite this as

$$\mathbb{P}_G(\mathcal{A}) = \Big(\coprod_{n\geq 0} \mathscr{P}_G(n) \times_{\Sigma_n} \mathcal{A}^n\Big)/(\sim), \tag{1.27}$$

thus highlighting the comparison with (1.21).

Definition 1.28. For a based G-map $f\colon A_+ \longrightarrow B_+$, define a functor

$$f_!\colon \mathscr{E}_G(A) \longrightarrow \mathscr{E}_G(B)$$

using the identification (1.24) and the functoriality of $\mathbb{P}_G$ on based maps.[10] In the case that $f^{-1}(*) = *$, so that f is in the image of the disjoint basepoint functor $X \mapsto X_+$, then $f_!$ is given by the disjoint union over n of the functors $\mathrm{id} \times_{\Sigma_n} f^n$. If $i\colon A \longrightarrow B$ is an inclusion of unbased finite G-sets, define an associated retraction $r\colon B_+ \longrightarrow A_+$ of based finite G-sets by setting $ri(a) = a$ and $r(b) = *$ if $b \notin \mathrm{im}(A)$. Then define[11]

$$i^* = r_!\colon \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A).$$

By Remark 2.18 below, we may think of i^* as the dual of i.

The following definition gives the G-category analogue of Definition 1.6. It specifies a G-category (almost) "enriched in permutative G-categories".

Definition 1.29. We define a G-bicategory[12] $\mathscr{E}_G$ with a permutative G-groupoid of hom objects for each pair of objects as follows. The 0-cells of $\mathscr{E}_G$ are the finite G-sets A, which may be thought of as the G-categories $\mathscr{E}_G(A)$. The permutative G-groupoid $\mathscr{E}_G(A,B)$ of 1-cells and 2-cells $A \longrightarrow B$ is $\mathscr{E}_G(B\times A)$, as defined in Definition 1.20. The composition

$$\circ\colon \mathscr{E}_G(B,C) \times \mathscr{E}_G(A,B) \longrightarrow \mathscr{E}_G(A,C)$$

is given by the diagram (1.30). Its first map ω is a pairing of free $\mathscr{P}_G$-algebras that will be made precise in Definition 1.34. Its second and third maps are specializations of the contravariant functoriality of $\mathscr{E}_G$ on inclusions and its covariant functoriality on surjections, as is made precise in Definition 1.28.

$$\begin{tikzcd}
\mathscr{E}_G(C\times B)\wedge \mathscr{E}_G(B\times A) \arrow[r, dashed, "\circ"] \arrow[d, "\omega"'] & \mathscr{E}_G(C\times A). \\
\mathscr{E}_G(C\times B\times B\times A) \arrow[r, "(\mathrm{id}\times\Delta\times\mathrm{id})^*"] & \mathscr{E}_G(C\times B\times A) \arrow[u, "\pi_!"']
\end{tikzcd} \tag{1.30}$$

This composition is strictly associative. With $A = \underline{n}^\alpha$, $\mathscr{E}_G(A,A)$ has a pseudo-unit 1 cell

$$\Delta_A = (\alpha; \Delta_A) \in \mathscr{E}_G(A\times A) = \mathscr{P}_G(n) \times_{\Sigma_n} (A\times A)^n \tag{1.31}$$

where

$$\Delta_A = \big((1,1), \cdots, (n,n)\big) \in (A\times A)^n.$$

It is a strict right unit, but it is not a strict left unit (see Remark 1.35 below).

[10] With the intuitive version of $\mathscr{E}_G$, $f_!\colon \mathscr{E}_G(A) \longrightarrow \mathscr{E}_G(B)$ is then just the pushforward functor obtained by composing maps over A with f.

[11] With the intuitive version of $\mathscr{E}_G$, $i^*\colon \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A)$ is just the functor obtained by using i to pull back maps over B to maps over A.

[12] As in Remark 1.7, it is almost a 2-category, just missing strict units as we shall explain.

To rectify to obtain a strict unit, we need whiskered G-categories $\mathscr{E}'_G$ analogous to the whiskered categories $G\mathscr{E}'$, and we define them in Section 6. They are defined in such a way that Theorem 1.22 has the following corollary by direct comparison of definitions.

Corollary 1.32. *The G-fixed category $(\mathscr{E}'_G)^G$ enriched in permutative categories is isomorphic to the category $G\mathscr{E}'$ enriched in permutative categories.*

We place the following ad hoc definition of the pairing ω required in Definition 1.29 in a general multicategorical context in [GMMO3, Definition 3.20]. We first comment on its domain; compare Remark 1.9.

Remark 1.33. We can define the smash product of based G-categories in the same way as the smash product of based G-spaces (see [EM, Lemma 4.20]). We are most interested in examples of the form $\mathscr{A}_+$ and $\mathscr{B}_+$ for unbased G-categories $\mathscr{A}$ and $\mathscr{B}$, and then $\mathscr{A}_+ \wedge \mathscr{B}_+$ can be identified with $(\mathscr{A} \times \mathscr{B})_+$. In particular,

$$\mathscr{E}_G(A) \wedge \mathscr{E}_G(B) = \Big(\coprod_{m\geq 1} \mathscr{P}_G(m) \times_{\Sigma_m} A^m\Big)_+ \wedge \Big(\coprod_{n\geq 1} \mathscr{P}_G(n) \times_{\Sigma_n} B^n\Big)_+$$

is isomorphic to

$$\Big(\coprod_{m\geq 1, n\geq 1} \mathscr{P}_G(m) \times \mathscr{P}_G(n) \times_{\Sigma_m \times \Sigma_n} A^m \times B^n\Big)_+.$$

We note that this smash product is not a $\mathscr{P}_G$-algebra.

Definition 1.34. The homomorphism $\otimes : \Sigma_m \times \Sigma_n \longrightarrow \Sigma_{mn}$ defined using lexicographic ordering in Definition 1.3 is the object function of a functor

$$\mathcal{E}\Sigma_m \times \mathcal{E}\Sigma_n \longrightarrow \mathcal{E}\Sigma_{mn}.$$

Applying the functor $\mathscr{C}at_G(\mathcal{E}G, -)$, we obtain pairings

$$\otimes \colon \mathscr{P}_G(m) \times \mathscr{P}_G(n) \longrightarrow \mathscr{P}_G(mn);$$

on objects of $\mathcal{E}G$, $(\mu \otimes \nu)(g) = \mu(g) \otimes \nu(g)$. For G-sets A and B, we have the injection

$$\boxtimes : A^m \times B^n \longrightarrow (A \times B)^{mn}$$

that sends $(a_1, \cdots, a_m) \times (b_1, \cdots, b_n)$ to the set of pairs (a_i, b_j), ordered lexicographically. Combining, there result functors

$$\omega_{m,n} \colon (\mathscr{P}_G(m) \times_{\Sigma_m} A^m) \times (\mathscr{P}_G(n) \times_{\Sigma_n} B^n) \longrightarrow \mathscr{P}_G(mn) \times_{\Sigma_{mn}} (A \times B)^{mn},$$

$$\omega_{m,n}\big((\mu, a), (\nu, b)\big) = (\mu \otimes \nu, a \boxtimes b).$$

Distributing products over disjoint unions, these specify pairings of G-categories

$$\omega \colon \mathscr{E}_G(A) \wedge \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A \times B).$$

While $\mathscr{E}_G(A) \wedge \mathscr{E}_G(B)$ is not a $\mathscr{P}_G$-algebra, we show in [GMMO3, Proposition 3.25] that ω is an example of a bilinear, or 2-ary, morphism in the multicategory of $\mathscr{P}_G$-algebras. The machine of [GMMO3] then produces from this bilinear map a pairing of G-spectra, as we will discuss in Section 2.2 below.

Remark 1.35. The associativity of the composition $\circ$ defined in Definition 1.29 is an easy diagram chase, starting from the associativity of the pairing on $\mathscr{P}_G$. We illustrate how Definition 1.28 works by considering composites with the pseudo-unit objects Δ_A. Let E be a 1-cell in $\mathscr{E}_G(A,B)$ and choose an object

$$(\mu;(b_1,a_1),\cdots,(b_m,a_m)) \in \mathscr{P}_G(m) \times_{\Sigma_m} (B\times A)^m$$

in the orbit E.

We first prove that $E \circ \Delta_A = E$. Take $A = \underline{n}^\alpha$. Then the object

$$\big(\mu\otimes\alpha;((b_i,a_i,j,j))\big) \in \mathscr{P}_G(mn)\times_{\Sigma_{mn}} (B\times A\times A\times A)^{mn}$$

is in the orbit $\omega(E,\Delta_A)$. The ordering of the four-tuples is lexicographic on i and j. The four-tuple (b_i,a_i,j,j) is in the image of $\mathrm{id}\times\Delta\times\mathrm{id}$ if and only if $a_i = j$. The retraction corresponding to this inclusion maps all other (b_i,a_i,j,j) to the basepoint. Applying $\pi_!$ we arrive at

$$\sigma_*((b_1,a_1),\cdots,(b_m,a_m)) \in (B\times A)^{mn},$$

where $\sigma\colon \mathbf{m}\longrightarrow\mathbf{mn}$ is the ordered injection that sends i to $\lambda^{-1}_{m,n}(i,a_i)$. Therefore

$$E\circ\Delta_A = (\mu\otimes\alpha;\sigma_*((b_1,a_1),\cdots,(b_m,a_m))) = (\sigma^*(\mu\otimes\alpha);(b_1,a_1),\cdots,(b_m,a_m)).$$

Since σ^* reverses the lexicographic ordering used to define $\mu\otimes\alpha$, we have the relation $\sigma^*(\mu\otimes\alpha)=\mu$.

Now take $B=\underline{p}^\beta$ and consider $\Delta_B\circ E$. Then the object

$$\big(\beta\otimes\mu;(k,k,b_i,a_i))\big) \in \mathscr{P}_G(pm)\times_{\Sigma_{pm}} (B\times B\times B\times A)^{pm}$$

is in the orbit $\omega(\Delta_B,E)$. The ordering of the four-tuples is lexicographic on k and i. The four-tuple (k,k,b_i,a_i) is in the image of $\mathrm{id}\times\Delta\times\mathrm{id}$ if and only if $k=b_i$. The retraction corresponding to this inclusion maps all other (k,k,b_i,a_i) to the basepoint. Applying $\pi_!$ we arrive at

$$\tau_*((b_1,a_1),\cdots,(b_m,a_m)) \in (B\times A)^{pm},$$

where $\tau\colon\mathbf{m}\longrightarrow\mathbf{pm}$ is the injection that sends i to $\lambda^{-1}_{p,m}(b_i,i)$. But now the injection τ is not ordered, although it becomes so after composition with some $\rho\in\Sigma_m$. We have

$$\Delta_B\circ E = (\beta\otimes\mu,\tau_*((b_1,a_1),\cdots,(b_m,a_m)) = (\tau^*(\beta\otimes\mu);(b_1,a_1),\cdots,(b_m,a_m)),$$

but $\tau^*(\beta\otimes\mu)$ is not equal to μ. We define

$$\ell_{B,E}:\Delta_B\circ E\longrightarrow E \tag{1.36}$$

to be the 2-cell induced by the (unique) morphism $\tau^*(\beta\otimes\mu)\longrightarrow\mu$ in $\mathscr{P}_G(m)$. The structure $\mathscr{E}_G$ is only a bicategory, while $\mathscr{E}'_G$ defined in Section 6 is a strict 2-category. The inclusion $\mathscr{E}_G\longrightarrow\mathscr{E}'_G$ is a pseudofunctor with unit constraint given by ζ of Definition 6.1. In [GMMO3], the category of $\mathscr{P}_G$-algebras is given the structure of a multicategory. The composition functors in both $\mathscr{E}_G$ and $\mathscr{E}'_G$ are examples of bilinear maps in the multicategorical sense.

1.5. **The categorical duality maps.** Since various specializations are central to our work, we briefly recall how duality works categorically, following [LMSM, III§1] for example. We then define maps of $\mathscr{P}_G$-algebras that will lead in Section 2.3 to the proof that the objects of $G\mathscr{A}$ are self-dual.

Let $\mathscr{V}$ be a closed symmetric monoidal category with product $\wedge$, unit S, and hom objects $F(X,Y)$; write $DX = F(X,S)$. A pair of objects (X,Y) in $\mathscr{V}$ is a dual pair if there are maps $\eta\colon S \longrightarrow X\wedge Y$ and $\varepsilon\colon Y\wedge X \longrightarrow S$ such that the composites

$$X \cong S\wedge X \xrightarrow{\eta\wedge\mathrm{id}} X\wedge Y\wedge X \xrightarrow{\mathrm{id}\wedge\varepsilon} X\wedge S \cong X$$

$$Y \cong Y\wedge S \xrightarrow{\mathrm{id}\wedge\eta} Y\wedge X\wedge Y \xrightarrow{\varepsilon\wedge\mathrm{id}} S\wedge Y \cong Y$$

are identity maps. For any such pair, the adjoint $\tilde{\varepsilon}\colon Y \longrightarrow DX$ of ε is an isomorphism. When (X,Y) and (X',Y') are dual pairs, the dual of a map $f\colon X \longrightarrow X'$ is the composite

$$Y' \cong Y'\wedge S_G \xrightarrow{\mathrm{id}\wedge\eta} Y'\wedge X\wedge Y \xrightarrow{\mathrm{id}\wedge f\wedge\mathrm{id}} Y'\wedge X'\wedge Y \xrightarrow{\varepsilon'\wedge\mathrm{id}} S_G\wedge Y \cong Y. \tag{1.37}$$

For any pair of objects X and Y, we have a natural map

$$\zeta\colon Y\wedge DX = Y\wedge F(X,S) \longrightarrow F(X,Y) \tag{1.38}$$

in $\mathscr{V}$, namely the adjoint of

$$\mathrm{id}\wedge\varepsilon\colon Y\wedge DX\wedge X \longrightarrow Y\wedge S \cong Y,$$

where ε is the evident evaluation map. The map ζ is an isomorphism when either X or Y is dualizable [LMSM, III.1.3]. When X is self-dual and Y is arbitrary, we have the composite isomorphism

$$\delta = \zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})\colon Y\wedge X \longrightarrow Y\wedge DX \longrightarrow F(X,Y). \tag{1.39}$$

This map in various categories will play an important role in our work.

There are two maps of $\mathscr{P}_G$-algebras that are central to duality and therefore to everything we do. Let $S^0 = \{*, 1\}$, where $*$ is the basepoint and 1 is not. We think of S^0 as 1_+, where 1 is the one-point G-set. In line with this convention, we also think of 1 as a trivial category with object 1. Remember that $\mathscr{E}_G(A) = \mathbb{P}_G(A_+)$ is the free $\mathscr{P}_G$-algebra generated by A_+, where we view finite G-sets as categories with only identity morphisms. We have already seen the first map implicitly.

Definition 1.40. For a finite G-set $A = \underline{n}^\alpha$, define based G-maps

$$\varepsilon\colon (A\times A)_+ \longrightarrow S^0, \quad r\colon (A\times A)_+ \longrightarrow A_+ \quad\text{and}\quad \pi\colon A_+ \longrightarrow S^0$$

by $r(a,b) = *$ if $a\neq b$ and $r(a,a) = a$, $\pi(a) = 1$, and $\varepsilon = \pi\circ r$, so that $\varepsilon(a,b) = *$ if $a\neq b$ and $\varepsilon(a,a) = 1$. Note that $r\circ\Delta = \mathrm{id}$ and that ε is just an example of a Kronecker δ-function. We agree to again write ε for the induced map of $\mathscr{P}_G$-algebras

$$\varepsilon = \mathscr{E}_G\varepsilon\colon \mathscr{E}_G(A\times A) \longrightarrow \mathscr{E}_G(1).$$

Definition 1.41. For a finite G-set $A = \underline{n}^\alpha$, regard the object $\Delta_A \in \mathscr{E}_G(A\times A)$ as the map of G-categories $i_A\colon 1 \longrightarrow \mathscr{E}_G(A\times A)$ that sends the object 1 of the trivial category to the object Δ_A. By freeness, there results a map of $\mathscr{P}_G$-algebras

$$\eta\colon \mathscr{E}_G(1) \longrightarrow \mathscr{E}_G(A\times A).$$

Explicitly,[13] η is the disjoint union over m of the maps

$$\mathscr{P}_G(m) \times_{\Sigma_m} 1^m \longrightarrow \mathscr{P}_G(mn) \times_{\Sigma_{mn}} (A \times A)^{mn}$$

given by

$$\eta(\mu, 1^m) = \big(\mu \otimes \alpha; (\Delta_A)^m\big).$$

The following categorical observation will lead to our proof in Section 2.3 that the G-spectra $\Sigma_G^\infty(A_+)$ are self-dual. Since care of basepoints is crucial, we use the alternative notation $\mathbb{P}_G(A_+)$. Remember that $(A \times A)_+$ can be identified with $A_+ \wedge A_+$. We identify $1_+ \wedge A_+$ and $A_+ \wedge 1_+$ with A_+ at the bottom center of our diagrams.

Proposition 1.42. *In the diagrams below, square (1) commutes up to isomorphism, and the other three squares commute on the nose.*

$$\begin{tikzcd}
\mathbb{P}_G(A_+ \wedge A_+) \wedge \mathbb{P}_G(A_+) \arrow[r, "\omega"] & \mathbb{P}_G(A_+ \wedge A_+ \wedge A_+) \arrow[d, "\mathbb{P}_G(\mathrm{id}\wedge\varepsilon)"] & \mathbb{P}_G(A_+) \wedge \mathbb{P}_G(A_+ \wedge A_+) \arrow[l, "\omega"'] \arrow[d, "\mathrm{id}\wedge\varepsilon"] \\
\mathbb{P}_G(1_+) \wedge \mathbb{P}_G(A_+) \arrow[u, "\eta\wedge\mathrm{id}"] \arrow[r, "\omega"'] \arrow[ur, phantom, "\Downarrow(1)"] & \mathbb{P}_G(A_+) & \mathbb{P}_G(A_+) \wedge \mathbb{P}_G(1_+) \arrow[l, "\omega"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}_G(A_+) \wedge \mathbb{P}_G(A_+ \wedge A_+) \arrow[r, "\omega"] & \mathbb{P}_G(A_+ \wedge A_+ \wedge A_+) \arrow[d, "\mathbb{P}_G(\varepsilon\wedge\mathrm{id})"] & \mathbb{P}_G(A_+ \wedge A_+) \wedge \mathbb{P}_G(A_+) \arrow[l, "\omega"'] \arrow[d, "\varepsilon\wedge\mathrm{id}"] \\
\mathbb{P}_G(A_+) \wedge \mathbb{P}_G(1_+) \arrow[u, "\mathrm{id}\wedge\eta"] \arrow[r, "\omega"'] \arrow[ur, phantom, "(2)"] & \mathbb{P}_G(A_+) & \mathbb{P}_G(1_+) \wedge \mathbb{P}_G(A_+) \arrow[l, "\omega"]
\end{tikzcd}$$

Proof. In the right vertical arrows, ε means $\mathbb{P}_G(\varepsilon)$. Both right squares are naturality diagrams, so it remains to consider the squares on the left. The difference between squares (1) and (2) is closely analogous to the difference between left and right composition with Δ_A explained in Remark 1.35. Let $A = \underline{n}^\alpha$ and let $(\mu, 1^m) \in \mathscr{P}(m) \times_{\Sigma_m} 1^m$ and $(\nu, a) \in \mathscr{P}(q) \times_{\Sigma_q} A^q$. We consider square (2) first, paying close attention to the order in which variables appear.

By Definitions 1.34 and 1.41,

$$\omega\big((\nu, a), (\mu, 1^m)\big) = (\nu \otimes \mu, a \boxtimes 1^m) \in \mathscr{P}(qm) \times A^{qm}$$

and

$$\omega \circ (\mathrm{id} \wedge \eta)\big((\nu, a), (\mu, 1^m)\big) = \big(\nu \otimes \mu \otimes \alpha; a \boxtimes (\Delta_A)^m\big) \in \mathscr{P}_G(qmn) \times_{\Sigma_{qmn}} (A^3)^{qmn}.$$

Identifying $\underline{qm}$ with $\underline{q} \times \underline{m}$ lexicographically, the (k, i)th coordinate of $a \boxtimes 1^m$ is a_k. Identifying $\underline{qmn}$ with $\underline{q} \times \underline{m} \times \underline{n}$ lexicographically, the (k, j, i)th coordinate of $a \boxtimes (\Delta_A)^m$ is (a_k, i, i). By Definition 1.40, $\varepsilon \wedge \mathrm{id}$ sends this coordinate to the basepoint unless $a_k = i$, when it sends it to i. Noticing the agreement of lexicographic orderings, we see as in Remark 1.35 that the injection $\sigma\colon \mathbf{qm} \longrightarrow \mathbf{qmn}$ such that

$$\sigma_*(a \boxtimes 1^m) = (\varepsilon \wedge \mathrm{id})_*(a \boxtimes (\Delta_A)^m)$$

is ordered and satisfies $\sigma^*(\nu \otimes \mu \otimes \alpha) = \nu \otimes \mu$.

Now consider square (1). By Definition 1.34 and Definition 1.41,

$$\omega\big((\mu, 1^m), (\nu, a)\big) = \big(\mu \otimes \nu, 1^m \boxtimes a\big) \in \mathscr{P}(mq) \times_{\Sigma_{mq}} A^{mq}$$

[13]This uses that $\gamma(\mu; \alpha^n) = \mu \otimes \alpha$, where $\gamma\colon \mathscr{P}_G(m) \times \mathscr{P}_G(n)^m \longrightarrow \mathscr{P}_G(mn)$, as explained in [GMMO3, §3.1].

and

$$\omega\circ(\eta\wedge\mathrm{id})\big((\mu,1^m),(\nu,a)\big)=\big(\gamma(\mu;\alpha^n)\otimes\nu;(\Delta_A)^m\boxtimes a\big)\in\mathscr{P}_G(mnq)\times_{\Sigma_{mnq}}(A^3)^{mnq}.$$

Identifying $\underline{mq}$ with $\underline{m}\times\underline{q}$ lexicographically, the (i,k)th coordinate of $1^m\boxtimes a$ is a_k. Identifying $\underline{mnq}$ with $\underline{m}\times\underline{n}\times\underline{q}$ lexicographically, the (i,j,k)th coordinate of $(\Delta_A)^m\boxtimes a$ is (j,j,a_k). By Definition 1.40, $\mathrm{id}\wedge\varepsilon$ sends this coordinate to the basepoint unless $j=a_k$, when it sends it to j. Here the injection $\tau\colon \mathbf{mq}\longrightarrow\mathbf{mnq}$ such that

$$\tau(1^m\boxtimes a)=(\mathrm{id}\wedge\varepsilon)_*((\Delta_A)^m\boxtimes a)$$

is not ordered, although it becomes so after composition with some $\rho\in\Sigma_{mq}$, and $\tau^*(\mu\otimes\alpha\otimes\nu)$ is not equal to $\mu\otimes\nu$ in $\mathscr{P}_G(mq)$. As in Remark 1.35, there is a unique 2-cell, necessarily an isomorphism,

$$\vartheta\colon(\mu\otimes\nu)\Longrightarrow\tau^*(\mu\otimes\alpha\otimes\nu)$$

in $\mathscr{P}_G(mq)$. As the input varies, the 2-cells

$$(\vartheta,\mathrm{id})\colon\big(\mu\otimes\nu;1^m\boxtimes a\big)\Longrightarrow\big(\tau^*(\mu\otimes\alpha\otimes\nu),1^m\boxtimes a\big)$$

specify the 2-natural isomorphism $\Rightarrow$ in the square (1). □

2. The proof of the main theorem

2.1. **The equivariant approach to Theorem 1.14.** As we explain in [GMMO3], following [GM2], equivariant infinite loop space theory associates an orthogonal G-spectrum $\mathbb{K}_G\mathscr{C}_G$ to a genuine permutative (or more generally genuine symmetric monoidal) G-category $\mathscr{C}_G$. The map $B\mathscr{C}_G=(\mathbb{K}_G\mathscr{C}_G)_0\longrightarrow\Omega(\mathbb{K}_G\mathscr{C}_G)_1$ is an equivariant group completion.[14] The category $G\mathscr{S}$ of orthogonal G-spectra is the G-fixed category of a G-category $\mathscr{S}_G$ of G-spectra and non-equivariant maps with the same objects as $\mathscr{S}_G$ and with G acting by conjugation on morphisms. Applying the functor $\mathbb{K}_G$ to $\mathscr{E}_G$, we obtain the following equivariant analogue of Definition 1.13.

Definition 2.1. We define a G-spectral category, or $\mathscr{S}_G$-category, $\mathscr{A}_G$. Its objects are the finite G-sets A, which may be viewed as the G-spectra $\mathbb{K}_G\mathscr{E}_G(A)$. Its morphism G-spectra $\mathscr{A}_G(A,B)$ are the $\mathbb{K}_G\mathscr{E}'_G(B\times A)$. Its unit G-maps $S_G\longrightarrow\mathscr{A}_G(A,A)$ are induced by the points $I_A\in G\mathscr{E}'(A,A)$ (see Section 6) and its composition G-maps

$$\mathscr{A}_G(B,C)\wedge\mathscr{A}_G(A,B)\longrightarrow\mathscr{A}_G(A,C)$$

are induced by composition in $\mathscr{E}'_G$.

Again, as written, the definition makes little sense: to make the word "induced" meaningful requires properties of the equivariant infinite loop space machine $\mathbb{K}_G$ that we will spell out in Section 2.2. This depends on having a functor that takes pairings (alias bilinear maps) of free $\mathscr{P}_G$-algebras to pairings of G-spectra.

The equivariant and non-equivariant infinite loop space functors are related by the following result.

[14] The papers from around 1990, such as [CW, Sh] are not adequate, in part because genuine permutative G-categories were not explicitly defined and the group completion property was not worked out rigorously, but more substantially because a symmetric monoidal category of G-spectra had not yet been discovered. A key feature of the version of the Segal machine [GMMO1] used in our proofs is that it is given by a symmetric monoidal functor, a claim that would not have made sense in 1990.

Theorem 2.2 ([GM2]). *There is a natural equivalence of spectra*

$$\iota\colon \mathbb{K}(G\mathscr{C}) \longrightarrow (\mathbb{K}_G\mathscr{C}_G)^G$$

for permutative G-categories $\mathscr{C}_G$ with G-fixed permutative categories $G\mathscr{C}$.

In view of Corollary 1.32, there results an equivalence of $\mathscr{S}$-categories

$$G\mathscr{A} \xrightarrow{\simeq} (\mathscr{A}_G)^G.$$

The proof of Theorem 1.14 goes as follows. We start with the following Theorem 2.3, which is a specialization of [GM1, Lemma 1.35]; it is discussed in Section 5.1. The essential point is that the collection $\{\Sigma^\infty_G A_+\}$ is a set of generators for $\mathrm{Ho}G\mathscr{S}$.

Theorem 2.3. *There is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S},$$

and it is a Quillen equivalence.

Remark 2.4. Instead of using $G\mathscr{D}_{\mathrm{All}}$, we can use its full subcategory $G\mathscr{D}_{\mathrm{Orb}}$ obtained by restricting the A to be orbits G/H, and then the result generalizes to compact Lie groups G; see Theorem 5.1. We define $G\mathscr{D}_{\mathrm{Orb}}$ as we defined $G\mathscr{D}_{\mathrm{All}}$ in Definition 1.11, again using fibrant replacements. Then $G\mathscr{D}_{\mathrm{All}} = (\mathscr{D}_{\mathrm{All}})^G$ and $G\mathscr{D}_{\mathrm{Orb}} = (\mathscr{D}_{\mathrm{Orb}})^G$, where we now regard our G-spectra $\Sigma^\infty_G A_+$ as objects of full sub $\mathscr{S}_G$-categories $\mathscr{D}_{All}$ and $\mathscr{D}_{Orb}$ of $\mathscr{S}_G$, so that the morphisms between them are given by G-spectra.

We will prove the following result in Section 2.4.

Theorem 2.5 (Equivariant version of the main theorem). *There is a zigzag of weak equivalences connecting the $\mathscr{S}_G$-categories $\mathscr{A}_G$ and $\mathscr{D}_{All}$.*

A weak equivalence between $\mathscr{S}_G$-categories with the same object sets is just an $\mathscr{S}_G$-functor that induces weak equivalences on morphism G-spectra.[15] On passage to G-fixed categories, this equivariant zigzag induces a zigzag of weak $\mathscr{S}$-equivalences connecting the $\mathscr{S}$-categories $G\mathscr{A}$ and $G\mathscr{D}_{\mathrm{All}}$. In turn, by [GM1, Proposition 2.4], this zigzag induces a zigzag of Quillen equivalences between $\mathbf{Pre}(G\mathscr{A}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S})$. Since $\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S})$ is Quillen equivalent to $G\mathscr{S}$, it follows that Theorem 2.5 implies Theorem 1.14.

Remark 2.6. For a G-spectrum X, the functor $\mathbb{U}(X)$ sends an orbit G/H to $F_G(\Sigma^\infty_G G/H_+, X)^G \cong X^H$. Keeping that fact in mind shows why Corollary 1.15 follows from the proof of Theorem 1.14.

To understand $G\mathscr{S}$ as an $\mathscr{S}$-category, we must first understand $\mathscr{S}_G$ as an $\mathscr{S}_G$-category. That is, to understand the G-fixed spectra $F_G(X,Y)^G$, we must first understand the function G-spectra $F_G(X,Y)$. Using infinite loop space theory to model function spectra implicitly raises a conceptual issue: there is no known infinite loop space machine that knows about function spectra. That is, given input data X and Y (permutative G-categories, E_∞-G-spaces, Γ-G-spaces, etc) for an infinite loop space machine $\mathbb{K}_G$, we do not know what input data will have as output the function G-spectra $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$. The problem does not even make sense

[15] A more general definition is given in [GM1, Definition 2.3].

as just stated because the output G-spectra $\mathbb{K}_G X$ are always connective, whereas $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$ is generally not. The most that one could hope for in general is to detect the connective cover of $F(\mathbb{K}_G X, \mathbb{K}_G Y)$. In our case, the relevant function G-spectra are connective since the suspension G-spectra $\Sigma^\infty_G(A_+)$ are self-dual, as we shall reprove in Section 2.3.

2.2. **Results from equivariant infinite loop space theory.** The proof of Theorem 2.5 is the heart of this paper, and of course it depends on equivariant infinite loop space theory and in particular on the relationship between the G-spectra $\mathscr{A}_G(A) = \mathbb{K}_G \mathscr{E}_G(A)$ and the suspension G-spectra $\Sigma^\infty_G(A_+)$. We collect the results that we need from [GMMO3] in this section. We warn the skeptical reader that the results of this paper depend fundamentally on Theorems 2.7 and 2.9. However, the proofs of those results require work far afield from the applications in this paper.

In fact, Theorem 2.5 is an application of a categorical version of the equivariant Barratt-Priddy-Quillen (BPQ) theorem for the identification of suspension G-spectra.[16] We state the theorem in full generality before restricting attention to finite G-sets. We shall find use for the full generality in Section 2.5.

Recall from Remark 1.23 that $\mathscr{E}_G(A)$ can be identified with the category $\mathbb{P}_G(A_+)$, where $\mathbb{P}_G$ is the free $\mathscr{P}_G$-category functor on based G-categories. The functor $\mathbb{P}_G$ applies equally well to based topological G-categories.[17] We view a based G-space X as a topological G-category that is discrete in the categorical sense: its morphism and object G-spaces are both X, and its source, target, identity, and composition maps are all its identity map. Thus we have the topological $\mathscr{P}_G$-category $\mathbb{P}_G(X)$. The geometric realization of its nerve is the free E_∞ G-space generated by X.

Henceforward, we use the term stable equivalence, rather than weak equivalence, for the weak equivalences in our model categories of spectra and G-spectra. In previous work, we established [GM2, Theorem 6.2] an equivariant version of the Barratt-Priddy-Quillen theorem, giving a natural equivalence between $\Sigma^\infty_G X_+$ and $\mathbb{K}_G \mathbb{P}_G(X)$. However, in order to produce our spectral category $\mathscr{A}_G$, we require a more structured version of that result.

First, it is essential that we have a machine with good multiplicative properties. The following result, which is proven in [GMMO3], gives far more than we need. As explained in [GMMO3, §3], we have a multicategory $\mathbf{Mult}(\mathscr{P}_G)$ of (strict) $\mathscr{P}_G$-algebras and pseudomorphisms between them; it is a submulticategory of a multicategory $\mathbf{Mult}(\mathscr{P}_G\textbf{-PsAlg})$ of $\mathscr{P}_G$-pseudoalgebras. Its multilinear maps require $\mathscr{P}_G$-pseudomaps despite the restriction to strict $\mathscr{P}_G$-algebras as objects. We also have the multicategory $\mathbf{Mult}(\mathscr{S}_G)$ associated to the symmetric monoidal category of orthogonal G-spectra under the smash product.

Theorem 2.7. [GMMO3] *$\mathbb{K}_G$ extends to a multifunctor*

$$\mathbb{K}_G : \mathbf{Mult}(\mathscr{P}_G) \longrightarrow \mathbf{Mult}(\mathscr{S}_G).$$

Remark 2.8. At one place in the duality proof of Section 2.3 below, we use from [GMMO3, Proposition 9.24] that $\mathbb{K}_G$ converts 2-cells, such as ϑ in Proposition 1.42, to homotopies between maps of G-spectra.

[16] For $A = *$, Carlsson [C2, p.6] mentions a space level version of the BPQ theorem. Shimakawa [Sh, p. 242] states and gives a sketch proof of a G-spectrum level version.

[17] We understand a topological G-category to mean an internal category in the category of G-spaces.

Theorem 2.7 yields an important consequence. Observe that the pairing ω of Definition 1.34 generalizes from G-sets A and B to G-spaces X and Y, giving a natural pairing

$$\omega\colon \mathbb{P}_G(X_+) \wedge \mathbb{P}_G(Y_+) \longrightarrow \mathbb{P}_G(X_+ \wedge Y_+).$$

Then Theorem 2.7 produces a map of G-spectra

$$\wedge\colon \mathbb{K}_G\mathbb{P}_G(X_+) \wedge \mathbb{K}_G\mathbb{P}_G(Y_+) \longrightarrow \mathbb{K}_G\mathbb{P}_G(X_+ \wedge Y_+).$$

With this multiplicative machine in hand, it now makes sense to ask for a Barratt-Priddy-Quillen comparison that is also compatible with the multiplicative structure. That is another main result of [GMMO3].

Theorem 2.9 ([GMMO3])**.** *For G-CW-complexes X, there is a monoidal natural stable equivalence*

$$\alpha\colon \Sigma^\infty_G(X_+) \longrightarrow \mathbb{K}_G\mathbb{P}_G(X_+).$$

For the remainder of this section, we restrict our attention to the case when X is a finite G-set A, although we will return to the generality of G-spaces X in Section 2.5. We therefore use the identification (1.24) to rewrite $\mathbb{P}_G(A_+)$ as $\mathscr{E}_G(A)$.

That the transformation of Theorem 2.9 is monoidal means that we have a commutative diagram

$$\begin{tikzcd} \Sigma^\infty_G(A_+) \wedge \Sigma^\infty_G(B_+) \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathscr{E}_G(A) \wedge \mathbb{K}_G\mathscr{E}_G(B) \arrow[d, "\wedge"] \\ \Sigma^\infty_G(A\times B)_+ \arrow[r, "\alpha"'] & \mathbb{K}_G\mathscr{E}_G(A\times B). \end{tikzcd} \tag{2.10}$$

We restate the naturality of α with respect to G-maps $f\colon A \longrightarrow B$ in the diagram

$$\begin{tikzcd} \Sigma^\infty_G(A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G f_+"'] & \mathbb{K}_G\mathscr{E}_G(A) \arrow[d, "\mathbb{K}_G f_!"] \\ \Sigma^\infty_G(B_+) \arrow[r, "\alpha"'] & \mathbb{K}_G\mathscr{E}_G(B). \end{tikzcd} \tag{2.11}$$

If $i\colon A \longrightarrow B$ is an inclusion with retraction $r\colon B_+ \longrightarrow A_+$, we have the induced map of G-spectra

$$\mathbb{K}_G i^* = \mathbb{K}_G r_!\colon \mathbb{K}_G\mathscr{E}_G(B) \longrightarrow \mathbb{K}_G\mathscr{E}_G(A),$$

and (2.11) specializes to

$$\begin{tikzcd} \Sigma^\infty_G(B_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathscr{E}_G(B) \arrow[d, "\mathbb{K}_G i^*"] \\ \Sigma^\infty_G(A_+) \arrow[r, "\alpha"'] & \mathbb{K}_G\mathscr{E}_G(A) \end{tikzcd} \tag{2.12}$$

By Remark 2.18 below, we may identify $\mathbb{K}_G i^*$ as the dual of $\mathbb{K}_G i$ and thus $\Sigma^\infty_G r$ as the dual of $\Sigma^\infty_G i_+$.

We combine these diagrams to construct those that we need to prove Theorem 2.5. Let A, B, and C be finite G-sets and recall Definition 1.29.

Proposition 2.13. *The following diagram of G-spectra commutes.*

$$\begin{tikzcd}
\Sigma^\infty_G(C\times B)_+\wedge\Sigma^\infty_G(B\times A)_+ \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathscr{E}_G(C\times B)\wedge\mathbb{K}_G\mathscr{E}_G(B\times A) \arrow[d, "\wedge"] \\
\Sigma^\infty(C\times B\times B\times A)_+ \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathscr{E}_G(C\times B\times B\times A) \arrow[d, "\mathbb{K}_G(\mathrm{id}\times\Delta\times\mathrm{id})^*"] \\
\Sigma^\infty(C\times B\times A)_+ \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty\pi"'] & \mathbb{K}_G\mathscr{E}_G(C\times B\times A) \arrow[d, "\mathbb{K}_G\pi_!"] \\
\Sigma^\infty_G(C\times A)_+ \arrow[r, "\alpha"'] & \mathbb{K}_G\mathscr{E}_G(C\times A)
\end{tikzcd} \tag{2.14}$$

Here r is the retraction which sends the complement of the image of $\mathrm{id}\times\Delta\times\mathrm{id}$ *to the basepoint.*

The diagram (2.14) relates the composition pairing of the $\mathscr{S}_G$-category $\mathscr{A}_G$ to remarkably simple and explicit maps between suspension G-spectra. In fact, recalling Definition 1.40 and again writing $\varepsilon = \Sigma^\infty_G\varepsilon$, we see that the left vertical composite in (2.14) can be identified with $\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}$. We have proven the following result, where we abuse notation by writing α for the composite

$$\Sigma^\infty_G(B\times A)_+ \longrightarrow \mathbb{K}_G\mathscr{E}_G(B\times A) \longrightarrow \mathbb{K}_G\mathscr{E}'_G(B\times A).$$

Theorem 2.15. *The following diagram of G-spectra commutes in $HoG\mathscr{S}$.*

$$\begin{tikzcd}
\Sigma^\infty_G(C\times B)_+\wedge\Sigma^\infty_G(B\times A)_+ \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\cong"'] & \mathscr{A}_G(B,C)\wedge\mathscr{A}_G(A,B) \arrow[ddd, "\circ"] \\
\Sigma^\infty_G(C_+)\wedge\Sigma^\infty_G(B\times B)_+\wedge\Sigma^\infty_G(A_+) \arrow[d, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"'] & \\
\Sigma^\infty_G(C_+)\wedge S_G\wedge\Sigma^\infty_G(A_+) \arrow[d, "\cong"'] & \\
\Sigma^\infty_G(C\times A)_+ \arrow[r, "\alpha"'] & \mathscr{A}_G(A,C)
\end{tikzcd}$$

2.3. **The self-duality of $\Sigma^\infty_G(A_+)$.** Let A be a finite G-set and write $\mathbb{A} = \Sigma^\infty_G(A_+)$ for brevity of notation. As recalled in Section 1.5, we must define maps $\eta\colon S_G \longrightarrow \mathbb{A}\wedge\mathbb{A}$ and $\varepsilon\colon \mathbb{A}\wedge\mathbb{A}\longrightarrow S_G$ in the stable homotopy category $HoG\mathscr{S}$ such that the composites

$$\mathbb{A}\xrightarrow{\eta\wedge\mathrm{id}}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\mathrm{id}\wedge\varepsilon}\mathbb{A} \quad\text{and}\quad \mathbb{A}\xrightarrow{\mathrm{id}\wedge\eta}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\varepsilon\wedge\mathrm{id}}\mathbb{A} \tag{2.16}$$

are the identity map in $HoG\mathscr{S}$. Using the stable equivalence α and the definitions of η and ε from Definition 1.40 and Definition 1.41, we let η and ε be the composites

$$S_G\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\mathbb{K}_G\eta}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\alpha^{-1}}\Sigma^\infty_G(A\times A)_+\cong\mathbb{A}\wedge\mathbb{A}$$

and

$$\mathbb{A}\wedge\mathbb{A}\cong\Sigma^\infty_G(A\times A)_+\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\mathbb{K}_G\varepsilon}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\alpha^{-1}}S_G.$$

The following commutative diagram proves that the first composite in (2.16) is the identity map in $HoG\mathscr{S}$; the second is dealt with similarly. We abbreviate notation by setting $\mathscr{A}_G A = \mathbb{K}_G\mathscr{E}_G(A)$. Remember that $\mathscr{E}_G(A) = \mathbb{P}_G(A_+)$. The center two squares are derived by use of the diagrams from Proposition 1.42.

$$\begin{tikzcd}
\mathscr{A}_G(A^2)\wedge\mathbb{A} \arrow[dr, "\mathrm{id}\wedge\alpha"] & & (\mathbb{A}^2)\wedge\mathbb{A}\cong\mathbb{A}^3\cong\mathbb{A}\wedge(\mathbb{A}^2) \arrow[ll, "\alpha\wedge\mathrm{id}"'] \arrow[rr, "\mathrm{id}\wedge\alpha"] \arrow[d, "\alpha"] & & \mathbb{A}\wedge\mathscr{A}_G(A^2) \arrow[dl, "\alpha\wedge\mathrm{id}"'] \arrow[dd, "\mathrm{id}\wedge\varepsilon"] \\
& \mathscr{A}_G(A^2)\wedge\mathscr{A}_G A \arrow[r, "\wedge"] & \mathscr{A}_G(A^3) \arrow[dd, "\mathrm{id}\times\varepsilon"] & \mathscr{A}_G A\wedge\mathscr{A}_G(A^2) \arrow[l, "\wedge"'] \arrow[dd, "\mathrm{id}\wedge\varepsilon"] & \\
\mathscr{A}_G 1\wedge\mathbb{A} \arrow[uu, "\eta\wedge\mathrm{id}"] \arrow[ur, "\eta\wedge\alpha"'] & & & & \mathbb{A}\wedge\mathscr{A}_G 1 \arrow[dl, "\alpha\wedge\mathrm{id}"'] \\
& \mathscr{A}_G 1\wedge\mathscr{A}_G A \arrow[uu, "\eta\wedge\mathrm{id}"] \arrow[r, "\wedge"] & \mathscr{A}_G A & \mathscr{A}_G A\wedge\mathscr{A}_G 1 \arrow[l, "\wedge"'] & \\
S_G\wedge\mathbb{A} \arrow[uu, "\alpha\wedge\mathrm{id}"] \arrow[ur, "\alpha\wedge\alpha"'] \arrow[rr, "\cong"'] & & \mathbb{A} \arrow[u, "\alpha"] & & \mathbb{A}\wedge S_G \arrow[ll, "\cong"] \arrow[ul, "\alpha\wedge\alpha"] \arrow[uu, "\mathrm{id}\wedge\alpha"']
\end{tikzcd}$$

Given Theorem 2.9, it is trivial that the outer parts of the diagram commute. The right central diagram is just a naturality diagram, as in Proposition 1.42. The left central diagram commutes up to homotopy by that result and Remark 2.8.

Specializing general observations about duality recalled in Section 1.5, we have the following corollary. This homotopical input is the crux of the proof of Theorem 2.5.

Corollary 2.17. *For finite G-sets A and B, the canonical map*

$$\delta = \zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})\colon \mathbb{B}\wedge\mathbb{A}\longrightarrow\mathbb{B}\wedge D\mathbb{A}\longrightarrow F_G(\mathbb{A},\mathbb{B})$$

of (1.39) is a stable equivalence.

We insert a mild digression concerning the identification of some of our maps.

Remark 2.18. For an inclusion $i\colon A\longrightarrow B$ of finite G-sets, (1.37) and the precise constructions of η and ε starting from Definition 1.40 and Definition 1.41 imply that the dual of i is the map $\mathbb{B}\longrightarrow\mathbb{A}$ induced by the evident retraction $r\colon B_+\longrightarrow A_+$. A G-map $\pi\colon G/H\longrightarrow G/K$ is a bundle, and the dual of $\Sigma^\infty\pi_+$ is the associated transfer map (see e.g. [LMSM, IV.pp 182 and 192]). It can be identified explicitly by a similar (but not especially illuminating) inspection of definitions.

2.4. **The proof that $\mathscr{A}_G$ is equivalent to $\mathscr{D}_{\mathbf{All}}$.** We will have to chase large diagrams, and we again abbreviate notations by writing

$$\mathbb{A} = \Sigma^\infty_G(A_+),\quad \mathbb{B} = \Sigma^\infty_G(B_+),\quad\text{and}\quad \mathbb{C} = \Sigma^\infty_G(C_+)$$

for finite G-sets A, B, and C. We also abbreviate notation by writing

$$\mathscr{A}_G(A) = \mathscr{A}_G(*, A).$$

This is the G-spectrum $\mathscr{A}_G(A) = \mathbb{K}_G\mathscr{E}_G(A)$, which is equivalent to $\mathbb{A}$ by Theorem 2.9. Remember that we are free to choose any bifibrant equivalents of the G-spectra $\mathbb{A}$ as the objects of $\mathscr{D}_{\mathrm{All}}$.

Proof of Theorem 2.5. We use model categorical arguments, and we work with the stable model structure on $G\mathscr{S}$. We use [GM1, §2.4] to obtain a model structure on the category $G\mathscr{S}\mathbb{O}$-$\mathscr{C}at$ of $G\mathscr{S}$-categories with the same object set $\mathbb{O}$ as $G\mathscr{E}$. We emphasize that this is a model structure on a category of categories. Maps are weak equivalences or fibrations if they induce weak equivalences or fibrations on hom objects in $G\mathscr{S}$. Here the nature of the objects is irrelevant; we are concerned with $G\mathscr{S}$-categories with one object for each finite G-set A.

Let $\lambda\colon Q\mathscr{A}_G \longrightarrow \mathscr{A}_G$ be a cofibrant approximation of $\mathscr{A}_G$. By [GM1, Theorem 2.16], since S_G is cofibrant in the stable model structure each morphism G-spectrum $Q\mathscr{A}_G(A,B)$ is cofibrant in $G\mathscr{S}$. The maps $\lambda\colon Q\mathscr{A}_G(A,B) \longrightarrow \mathscr{A}_G(A,B)$ are stable acyclic fibrations. Digressively, since the $\mathscr{A}_G(A,B)$ are fibrant in the positive stable model structure, that is also true of the $Q\mathscr{A}_G(A,B)$; we will use this fact later, in Section 2.5.

Let $\rho\colon Q\mathscr{A}_G \longrightarrow RQ\mathscr{A}_G$ be a fibrant approximation of $Q\mathscr{A}_G$. The morphism G-spectra $RQ\mathscr{A}_G(A,B)$ are then bifibrant in the stable model structure. Therefore $RQ\mathscr{A}_G(A)$ is bifibrant for each A, and it is stably equivalent to $\mathbb{A}$. We take the $RQ\mathscr{A}_G(A)$ as the bifibrant approximations of the $\mathbb{A}$ that we use to define the full $G\mathscr{S}$-subcategory $\mathscr{D}_{\text{All}}$ of $G\mathscr{S}$.

We define $\mathscr{C}_G$ to be the full $G\mathscr{S}$-subcategory of $G\mathscr{S}$ with objects the $Q\mathscr{A}_G(A)$. To abbreviate notation, we agree to write

$$Q\mathscr{A}_G(*,A) = Q\mathscr{A}_G A \quad \text{and} \quad RQ\mathscr{A}_G(*,A) = RQ\mathscr{A}_G A.$$

With our notational conventions, it is consistent to write $Q\mathscr{A}_G(B\times A) = Q\mathscr{A}_G(A,B)$.

For finite G-sets A and B, let

$$\beta\colon Q\mathscr{A}_G(A,B) \longrightarrow \mathscr{C}_G(A,B) = F_G(Q\mathscr{A}_G A, Q\mathscr{A}_G B)$$

be the adjoint of the composition map

$$\circ\colon Q\mathscr{A}_G(A,B) \wedge Q\mathscr{A}_G A \longrightarrow Q\mathscr{A}_G B$$

and let

$$\gamma\colon RQ\mathscr{A}_G(A,B) \longrightarrow \mathscr{D}_{\text{All}}(A,B) = F_G(RQ\mathscr{A}_G A, RQ\mathscr{A}_G B)$$

be the adjoint of the composition map

$$\circ\colon RQ\mathscr{A}_G(A,B) \wedge RQ\mathscr{A}_G A \longrightarrow RQ\mathscr{A}_G B.$$

By [GM1, Construction 5.6], these define $G\mathscr{S}$-functors

$$\beta\colon Q\mathscr{A}_G \longrightarrow \mathscr{C}_G \qquad \text{and} \qquad \gamma\colon RQ\mathscr{A}_G \longrightarrow \mathscr{D}_{\text{All}}.$$

It suffices to prove that each of the maps γ is a stable equivalence. For each finite G-set A, $\mathbb{A}$ is cofibrant and $\lambda\colon Q\mathscr{A}_G A \longrightarrow \mathscr{A}_G A$ is an acyclic fibration in the stable model structure on $G\mathscr{S}$. Therefore there is a map $\mu\colon \mathbb{A} \longrightarrow Q\mathscr{A}_G A$ such that the diagram

$$\begin{tikzcd} & Q\mathscr{A}_G A \arrow[d, "\lambda"] \\ \mathbb{A} \arrow[ur, "\mu"] \arrow[r, "\alpha"'] & \mathscr{A}_G A \end{tikzcd}$$

commutes. Since α and λ are stable equivalences, so is μ. In the same way, we get a stable equivalence $\mu : \mathbb{B} \wedge \mathbb{A} \longrightarrow Q\mathscr{A}_G(A,B)$.

For the remainder of the proof, we work in the homotopy category $HoG\mathscr{S}$. In particular, the distinction between $\mathbb{K}_G\mathscr{E}_G$ and $\mathbb{K}_G\mathscr{E}'_G$ vanishes. We claim that the following diagram of G-spectra commutes in $HoG\mathscr{S}$. Indeed, modulo inversion of maps which are stable equivalences, it commutes on the nose. As before, we identify $\mathbb{B} \wedge \mathbb{A} = \Sigma^\infty_G B_+ \wedge \Sigma^\infty_G A_+$ with $\Sigma^\infty_G (B \times A)_+$.

$$\begin{tikzcd}
RQ\mathscr{A}_G(A,B) \arrow[r, "\gamma"] & F_G(RQ\mathscr{A}_G A, RQ\mathscr{A}_G B) \arrow[r, "F_G(\rho,\mathrm{id})", "\simeq"'] & F_G(Q\mathscr{A}_G A, RQ\mathscr{A}_G B) \arrow[d, "F_G(\mu,\mathrm{id})", "\simeq"'] \\
Q\mathscr{A}_G(A,B) \arrow[u, "\rho"', "\simeq"] \arrow[r, "\beta"] & F_G(Q\mathscr{A}_G A, Q\mathscr{A}_G B) \arrow[ur, "F_G(\mathrm{id},\rho)"'] \arrow[dr, "F_G(\mu,\mathrm{id})"] & F_G(\mathbb{A}, RQ\mathscr{A}_G B) \\
\mathbb{B} \wedge \mathbb{A} \arrow[u, "\mu"', "\simeq"] \arrow[r, "\simeq", "\delta"'] & F_G(\mathbb{A},\mathbb{B}) \arrow[r, "\simeq", "F_G(\mathrm{id},\mu)"'] & F_G(\mathbb{A}, Q\mathscr{A}_G B) \arrow[u, "F_G(\mathrm{id},\rho)"', "\simeq"]
\end{tikzcd}$$

The map δ is the stable equivalence of Corollary 2.17. The maps μ and ρ are also stable equivalences. The maps $F_G(\rho, \mathrm{id})$ and $F_G(\mu, \mathrm{id})$ that are labeled $\simeq$ are stable equivalences by [GM1, Lemma 1.22] since ρ and μ are maps between cofibrant objects and $RQ\mathscr{A}_G B$ is fibrant. The maps $F_G(\mathrm{id}, \mu)$ and $F_G(\mathrm{id}, \rho)$ that are labeled $\simeq$ are stable equivalences by [MaM, III.3.9], which shows that the functor $F_G(\mathbb{A}, -)$ preserves stable equivalences. Provided that the diagram commutes, it follows that γ is a stable equivalence since all of the other outer arrows of the diagram are stable equivalences.

The top pentagon commutes since ρ is a map of spectral categories, and both composites on the right give $F_G(\mu, \rho)$. It therefore remains to consider the lower pentagon. To prove that the diagram commutes in $HoG\mathscr{S}$, we consider its adjoint, which is displayed as the outer rectangle of the diagram below. Here we have inserted the map $\circ\colon \mathscr{A}_G(A,B) \wedge \mathscr{A}_G A \longrightarrow \mathscr{A}_G B$ and wrong way arrows into its source and target for purposes of proof.

$$\begin{tikzcd}
Q\mathscr{A}_G(A,B) \wedge Q\mathscr{A}_G A \arrow[rrr, "\circ"] \arrow[dr, "\lambda\wedge\lambda"] & & & Q\mathscr{A}_G B \arrow[dl, "\lambda"'] \\
& \mathscr{A}_G(A,B) \wedge \mathscr{A}_G A \arrow[r, "\circ"] & \mathscr{A}_G B & \\
\mathbb{B} \wedge \mathbb{A} \wedge \mathbb{A} \arrow[uu, "\mu\wedge\mu"] \arrow[ur, "\alpha\wedge\alpha"] \arrow[rrr, "\mathrm{id}\wedge\Sigma^\infty_G \varepsilon"'] & & & \mathbb{B} \arrow[uu, "\mu"'] \arrow[ul, "\alpha"']
\end{tikzcd}$$

Since λ is a map of $G\mathscr{S}$-categories, it is apparent that all parts of the diagram commute except for the bottom trapezoid. Taking $(A,B,C) = (*,A,B)$ in Theorem 2.15, we see that the trapezoid commutes. Since the wrong way map λ is a stable equivalence and can be inverted upon passage to the homotopy category, this diagram and its adjoint commute there. $\square$

2.5. **The identification of suspension G-spectra.** We expand the adjoint $\mathscr{S}$-equivalences in Theorem 1.14 more explicitly as follows.

$$\begin{tikzcd} G\mathscr{S} \arrow[r, shift left, "\mathbb{T}"] & \mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S}) \arrow[l, shift left, "\mathbb{U}"] \arrow[r, shift right, "\gamma^*"'] & \mathbf{Pre}((RQ\mathscr{A}_G)^G, \mathscr{S}) \arrow[l, shift right, "\gamma_!"'] \arrow[d, shift right, "\rho^*"'] \\ \mathbf{Pre}(G\mathscr{A}, \mathscr{S}) \arrow[r, shift left, "\iota_!"] & \mathbf{Pre}((\mathscr{A}_G)^G, \mathscr{S}) \arrow[l, shift left, "\iota^*"] \arrow[r, shift right, "\lambda^*"'] & \mathbf{Pre}((Q\mathscr{A}_G)^G, \mathscr{S}) \arrow[l, shift right, "\lambda_!"'] \arrow[u, shift right, "\rho_!"'] \end{tikzcd} \tag{2.19}$$

The map $\iota : G\mathscr{A} \longrightarrow (\mathscr{A}_G)^G$ is the equivalence of Theorem 2.2. Before passage to G-fixed points, the proof in Section 2.4 gives stable equivalences of $\mathscr{S}_G$-categories

$$\rho\colon Q\mathscr{A}_G \longrightarrow RQ\mathscr{A}_G, \quad \gamma : RQ\mathscr{A}_G \longrightarrow \mathscr{D}_{\mathrm{All}}, \text{ and } \lambda\colon Q\mathscr{A}_G \longrightarrow \mathscr{A}_G,$$

and these maps give stable equivalences of $\mathscr{S}$-categories after passage to fixed points.

For a finite G-set B, $\Sigma^\infty_G B_+$ corresponds under this zigzag to the presheaf $\mathbf{B}$ that sends A to $G\mathscr{A}(A,B)$. This is almost a tautology since, for $E \in G\mathscr{S}$, $\mathbb{U}(E)$ is the presheaf represented by E, while $G\mathscr{E}(-,B)$ is the functor represented by B. In the proof of Theorem 2.5, we chose the bifibrant approximation of $\Sigma^\infty_G B_+$ in $\mathscr{D}_{\mathrm{All}}$ to be $RQ\mathscr{A}_G(B)$. With B fixed, that proof shows that γ gives an equivalence of presheaves

$$RQ\mathscr{A}_G(-,B) \longrightarrow \gamma^*\mathbb{U}RQ\mathscr{A}_G(B)$$

(before passage to G-fixed points). The functors ρ^* and $\lambda_!$ and the isomorphism ι^* preserve representable functors, and therefore $\iota^*\lambda_!\rho^*RQ\mathscr{A}_G(-,B) \simeq \mathbb{K}_G\mathscr{E}_G(-,B)$.

This observation can be generalized from finite based G-sets B_+ to arbitrary based G-spaces X. To see this, we mix general based G-spaces X with finite based G-sets A_+ to obtain a functorial construction of a presheaf $\mathbf{Pr}_G(X)$.

Definition 2.20. Define a presheaf $\mathbf{Pr}_G(X)\colon (\mathscr{A}_G)^{op} \longrightarrow \mathscr{S}_G$ by letting

$$\mathbf{Pr}_G(X)(A) = \mathbb{K}_G\mathbb{P}_G(X \wedge A_+).$$

The contravariant functoriality map

$$\mathbf{Pr}_G(X)\colon \mathscr{A}_G(A,B) \longrightarrow F_G(\mathbf{Pr}_G(X)(B), \mathbf{Pr}_G(X)(A))$$

is the composite of the retraction $\mathscr{A}_G(A,B) = \mathbb{K}_G\mathscr{E}'_G(A,B) \longrightarrow \mathbb{K}_G(\mathscr{E}_G(B\times A))$ with the adjoint of the right vertical composite in the commutative diagram

$$\begin{tikzcd} \Sigma^\infty_G(X\wedge B_+)\wedge\Sigma^\infty_G(B_+\wedge A_+) \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+)\wedge\mathbb{K}_G\mathbb{P}_G(B_+\wedge A_+) \arrow[d, "\wedge"] \\ \Sigma^\infty(X\wedge B_+\wedge B_+\wedge A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+\wedge B_+\wedge A_+) \arrow[d, "\mathbb{K}_G\mathbb{P}_G(r)"] \\ \Sigma^\infty(X\wedge B_+\wedge A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty\pi"'] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+\wedge A_+) \arrow[d, "\mathbb{K}_G\mathbb{P}_G\pi"] \\ \Sigma^\infty_G(X\wedge A_+) \arrow[r, "\alpha"'] & \mathbb{K}_G\mathbb{P}_G(X\wedge A_+). \end{tikzcd} \tag{2.21}$$

Here r is the evident left inverse of $\mathrm{id} \wedge \Delta \wedge \mathrm{id}$ and π is the projection. The diagram commutes by the same concatenation of commutative diagrams as in Proposition 2.13. Note that there is no need to whisker the G-categories $\mathbb{P}_G(X \wedge A_+)$ in order to get a strict functor. The spans in $\mathbb{P}_G(X \wedge A_+)$ are only composed on the right with spans in $\mathscr{A}_G$ in this construction, and the Δ_B were already strict units on the right. Therefore use of the retraction does not destroy functoriality.

Theorem 2.22. *Let X be a based G-space. Under our zigzag of equivalences, $\Sigma_G^\infty X$ corresponds naturally to the presheaf $(\mathbf{Pr}_G(X))^G$ that sends A to $\mathbb{K}\big(\mathbb{P}_G(X \wedge A_+)^G\big)$.*

Proof. Note that $\mathbb{K}_G\mathbb{P}_G(X \wedge -_+)$ is no longer a representable presheaf. We again work with G-spectra and obtain the conclusion after passage to G-fixed spectra. According to Theorem 2.9, we may replace $\Sigma_G^\infty X$ by the positive fibrant G-spectrum $\mathbb{K}_G\mathbb{P}_G(X)$, which we abbreviate to $\mathscr{A}_G(X)$ by a slight abuse of notation. After this replacement, the presheaf $\mathbb{U}(\Sigma_G^\infty X)$ may be computed as

$$\mathbb{U}(\Sigma_G^\infty X)(A) = F_G(RQ\mathscr{A}_G(A), \mathscr{A}_G(X)).$$

Therefore, following the chain of (2.19), we may compute $\rho^*\gamma^*\mathbb{U}(\Sigma_G^\infty X)$ as

$$\rho^*\gamma^*\mathbb{U}(\Sigma_G^\infty X) \simeq F_G(Q\mathscr{A}_G(-), \mathscr{A}_G(X)).$$

Replacing (B, A) by $(A, 1)$ in (2.21) and recalling that $1_+ = S^0$, the right column gives the second map in the composite

$$\mathbf{Pr}_G(X)(A) \wedge Q\mathscr{A}_G(A) \xrightarrow{\mathrm{id}\wedge\lambda} \mathbf{Pr}_G(X)(A) \wedge \mathscr{A}_G(A) \xrightarrow{\circ} \mathbf{Pr}_G(X)(1). \tag{2.23}$$

Its target is the G-spectrum $\mathscr{A}_G(X)$, and its adjoint gives a map of presheaves

$$\lambda^*\mathbf{Pr}_G(X) \longrightarrow F_G(Q\mathscr{A}_G(-), \mathscr{A}_G(X)) \tag{2.24}$$

with domain $Q\mathscr{A}_G$. It remains to show that this map is an equivalence. To compute the adjoint (2.24), observe that (2.23) is the top horizontal composite in the diagram

$$\begin{tikzcd}
\mathbf{Pr}_G(X)(A) \wedge Q\mathscr{A}_G(A) \arrow[r, "\mathrm{id}\wedge\lambda"] & \mathbf{Pr}_G(X)(A) \wedge \mathscr{A}_G(A) \arrow[r, "\circ"] & \mathbf{Pr}_G(X)(1) \\
\Sigma_G^\infty(X \wedge A_+) \wedge Q\mathscr{A}_G(A) \arrow[u, "\alpha\wedge\mathrm{id}"] & \mathbf{Pr}_G(X)(A) \wedge \Sigma_G^\infty A_+ \arrow[u, "\mathrm{id}\wedge\alpha"'] & \\
\Sigma_G^\infty(X \wedge A_+) \wedge \Sigma_G^\infty A_+ \arrow[u, "\mathrm{id}\wedge\mu"] \arrow[ur, "\alpha\wedge\mathrm{id}"'] \arrow[r, "\cong"'] & \Sigma_G^\infty X \wedge \Sigma_G^\infty(A_+ \wedge A_+) \arrow[r, "\mathrm{id}\wedge\varepsilon"'] & \Sigma_G^\infty X. \arrow[uu, "\alpha"']
\end{tikzcd}$$

The left pentagon commutes since $\lambda \circ \mu = \alpha$ and the right pentagon is a special case of (2.21). Therefore the map (2.24) is the top horizontal composite in the diagram

$$\begin{tikzcd}
\mathbf{Pr}_G(X)(A) \arrow[r] & F_G(\mathscr{A}_G(A), \mathscr{A}_G(X)) \arrow[r, "F_G(\lambda,\mathrm{id})"] & F_G(Q\mathscr{A}_G(A), \mathscr{A}_G(X)) \arrow[d, "F_G(\mu,\mathrm{id})"] \\
\Sigma_G^\infty(X \wedge A_+) \arrow[u, "\alpha"] \arrow[r, "\delta"'] & F_G(\Sigma_G^\infty A_+, \Sigma_G^\infty X) \arrow[r, "F_G(\mathrm{id},\alpha)"'] & F_G(\Sigma_G^\infty A_+, \mathscr{A}_G(X)).
\end{tikzcd}$$

The map α is a stable equivalence by Theorem 2.9. The map δ is the stable equivalence of (1.39). The map $F_G(\mathrm{id}, \alpha)$ is a stable equivalence by [MaM, III.3.9]. Finally, the map $F_G(\mu, \mathrm{id})$ is a stable equivalence by [GM1, Lemma 1.22]. $\square$

3. Some comparisons of functors

3.1. **Change of groups and fixed point functors.** We discuss several constructions on G-spectra from the point of view of Theorem 1.14. Categorical fixed points are already built into the setup: for any subgroup $H \subset G$, the functor of H-fixed points is given by evaluating presheaves at the orbit G/H. We will return to this in Construction 3.5.

Construction 3.1 (Restriction to subgroups)**.** Let $H \subset G$ be a subgroup. Then induction of G-sets provides a strong monoidal (in other words, coproduct-preserving) bifunctor $G \times_H (-): H\mathscr{E} \longrightarrow G\mathscr{E}$. Using our models for $H\mathscr{E}$ and $G\mathscr{E}$, we must declare a preferred ordering for an induced G-set $G \times_H A$, given an ordering of the H-set A. For this, we choose an ordering of G/H as well as a set of coset representatives for H in G. The choice of coset representatives gives a bijection of sets $G \times_H A \cong G/H \times A$, and we use the lexicographic ordering of $G/H \times A$ to order the induced G-set $G \times_H A$.

This extends to a (strict) 2-functor $G \times_H -: H\mathscr{E}' \longrightarrow G\mathscr{E}'$ if, recalling that the 1-cell $I_A \in H\mathscr{E}'(A, A)$ is the identity of A as in Definition 6.1, we then define $G \times_H I_A = I_{G \times_H A}$ for all H-sets A. For finite H-sets A and B, there is a unique G-equivariant isomorphism $G \times_H (A \amalg B) \cong (G \times_H A) \amalg (G \times_H B)$, though it is not order-preserving in general. It follows that the induction functor gives rise to a spectral functor $\mathbb{K}(G \times_H -): H\mathscr{A} \longrightarrow G\mathscr{A}$. Then

$$\mathbb{K}(G \times_H -)^*: \mathbf{Pre}(G\mathscr{A}, \mathscr{S}) \longrightarrow \mathbf{Pre}(H\mathscr{A}, \mathscr{S})$$

gives a model for the restriction $G\mathscr{S} \longrightarrow H\mathscr{S}$.

Construction 3.2 (Induction)**.** Let $H \subset G$ be a subgroup. The spectrum-level induction functor $G_+ \wedge_H -: H\mathscr{S} \longrightarrow G\mathscr{S}$ is left adjoint to restriction. Given the description of restriction provided in Construction 3.1, it follows that induction can be described as the enriched Kan extension (as in [GM1, Lemma 2.2])

$$\mathbb{K}(G \times_H -)_!: \mathbf{Pre}(H\mathscr{A}, \mathscr{S}) \longrightarrow \mathbf{Pre}(G\mathscr{A}, \mathscr{S})$$

along the spectral functor $\mathbb{K}(G \times_H -): H\mathscr{A} \longrightarrow G\mathscr{A}$.

Construction 3.3 (Inflation along a quotient)**.** Let $N \trianglelefteq G$ be a normal subgroup. Then passage to N-fixed points defines a functor $\mathrm{Fix}^N: G\mathscr{E} \longrightarrow G/N\mathscr{E}$. Note that since $\mathrm{Fix}^N(A)$ is a subset of A, the G/N-set $\mathrm{Fix}^N(A)$ inherits an ordering from that of A. Moreover, Fix^N preserves pullbacks and coproducts. It follows that Fix^N gives rise to a spectral functor $\mathbb{K}(\mathrm{Fix}^N): G\mathscr{A} \longrightarrow G/N\mathscr{A}$. Then

$$\mathbb{K}(\mathrm{Fix}^N)^*: \mathbf{Pre}(G/N\mathscr{A}, \mathscr{S}) \longrightarrow \mathbf{Pre}(G\mathscr{A}, \mathscr{S})$$

gives a model for the inflation functor, whose image consists of G-spectra "concentrated over N". In the language of [MaM, Section VI.5], this is the functor $X \mapsto \widetilde{E\mathcal{F}}[N] \wedge \varepsilon^{\#} X$.

Construction 3.4 (Geometric fixed points)**.** Let $N \trianglelefteq G$ be a normal subgroup. Then the geometric N-fixed points functor is left adjoint to inflation. Given the description of inflation provided in Construction 3.3, the enriched Kan extension (as in [GM1, Lemma 2.2])

$$\mathbb{K}(\mathrm{Fix}^N)_!: \mathbf{Pre}(G\mathscr{A}, \mathscr{S}) \longrightarrow \mathbf{Pre}(G/N\mathscr{A}, \mathscr{S})$$

gives a model for the geometric N-fixed points functor $\Phi^N: G\mathscr{S} \longrightarrow G/N\mathscr{S}$.

This construction extends to arbitrary subgroups as follows. For a subgroup $H \subset G$, the H-fixed points functor $\mathrm{Fix}^H \colon G\mathscr{E} \longrightarrow \mathscr{E}$ gives rise to a spectral functor $\mathbb{K}(\mathrm{Fix}^H)\colon G\mathscr{A} \longrightarrow \mathscr{A}$, and the enriched Kan extension

$$\mathbb{K}(\mathrm{Fix}^H)_!\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \longrightarrow \mathbf{Pre}(\mathscr{A},\mathscr{S})$$

gives a model for the geometric H-fixed points functor $\Phi^H \colon G\mathscr{S} \longrightarrow \mathscr{S}$. We leave it to the reader to verify that, in the case of a normal subgroup, the two versions agree after restricting from G/N-spectra to underlying spectra.

Construction 3.5 (Categorical fixed points)**.** There is an inclusion $\iota\colon \mathscr{E} \hookrightarrow G\mathscr{E}$ of the finite sets as the G-trivial finite G-sets. This functor preserves pullbacks and coproducts and therefore induces a spectral functor $\mathbb{K}(\iota)\colon \mathscr{A} \hookrightarrow G\mathscr{A}$. As generalized equivariantly in Remark 5.4, spectrally enriched presheaves on finite sets are determined by their value at a one-point set, and

$$\mathbb{K}(\iota)^*\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \longrightarrow \mathbf{Pre}(\mathscr{A},\mathscr{S}) \simeq \mathscr{S}$$

gives a model for the (categorical) G-fixed points functor $(-)^G\colon G\mathscr{S} \longrightarrow \mathscr{S}$. For a subgroup $H \subset G$, the H-fixed points functor is given by first using the restriction functor of Construction 3.1 and then passing to fixed points.

Construction 3.6 (G-trivial G-spectra)**.** Left adjoint to the G-fixed points functor is the trivial G-action functor. Given the description of G-fixed points provided in Construction 3.5, the enriched Kan extension (as in [GM1, Lemma 2.2])

$$\mathbb{K}(\iota)_!\colon \mathscr{S} \simeq \mathbf{Pre}(\mathscr{A},\mathscr{S}) \longrightarrow \mathbf{Pre}(G\mathscr{A},\mathscr{S})$$

gives a model for the trivial G-spectrum functor $\varepsilon^{\#}\colon \mathscr{S} \longrightarrow G\mathscr{S}$ (using the notation of [MaM, Section VI.3]). This functor describes the tensoring of G-spectra over nonequivariant spectra. We return to this in Section 3.3.

3.2. **Fixed point orbit functors.** We return to Corollary 1.15 and give a more precise formulation. We know from Construction 3.5 how to pass to H-fixed points for each H, but a more functorial perspective may be illuminating. Again let $\mathscr{O}_G$ denote the orbit category of G. For a G-spectrum X, passage to H-fixed point spectra for $H \subset G$ gives a functor $X^\bullet\colon \mathscr{O}_G^{op} \longrightarrow \mathscr{S}$. Recall Remark 2.4. By definition, $G\mathscr{D}_{\mathrm{Orb}}$ is the image of the composition j of $\Sigma^\infty_{G,+}\colon \mathscr{O}_G \longrightarrow G\mathscr{S}$ with our bifibrant replacement functor. Pulling back along j defines a functor

$$G\mathscr{S} \xrightarrow{\mathbb{U}} \mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}},\mathscr{S}) \xrightarrow{j^*} \mathrm{Pre}(\mathscr{O}_G,\mathscr{S}),$$

where the target denotes ordinary (i.e. unenriched) presheaves. On the other hand, we have the functor $k\colon \mathscr{O}_G \longrightarrow G\mathscr{E}$ that associates to a map of finite G-sets its graph, considered as a span. This gives rise to a functor $\mathscr{O}_G \longrightarrow G\mathscr{A}$, which we also denote by k. Now pullback along k gives a functor

$$\mathbf{Pre}(G\mathscr{A},\mathscr{S}) \xrightarrow{k^*} \mathrm{Pre}(\mathscr{O}_G,\mathscr{S}).$$

Corollary 3.7. *The zigzag of equivalences of Theorem 1.14 identifies the composition $j^* \circ \mathbb{U}$ with k^* up to equivalence.*

3.3. **Tensors with spectra and smash products.** There is another visible identification. The category $G\mathscr{S}$ and our presheaf categories are $\mathscr{S}$-complete, so that they have tensors and cotensors over $\mathscr{S}$ (see [GM1, §5.1]). It is formal that the left adjoint of an $\mathscr{S}$-adjunction preserves tensors and the right adjoint preserves cotensors. A quick chase of our zigzag of Quillen $\mathscr{S}$-equivalences gives the following conclusion.

Proposition 3.8. *For G-spectra Y and spectra X, if Y corresponds to a presheaf $\mathscr{P}Y$ under our zigzag of weak equivalences, then the tensor $Y \odot X$ corresponds to the tensor $\mathscr{P}Y \odot X$.*

Remark 3.9 (Smash products)**.** We have not described the behavior of smash products under the equivalences of Theorem 1.14. On the presheaf side, one would expect to use Day convolution to describe the smash product, starting from the cartesian product of finite G-sets. Indeed, this is the approach taken in [CMNN], where a symmetric monoidal version of Theorem 1.14 is given. We warn the reader, however, of two notable differences in their approach. First, in the approach of [CMNN], the functor from G-spectra to presheaves is a *left* adjoint, so that their right adjoint plays the role of our $\mathbb{T}$ in Theorem 2.5. Secondly, they produce a monoidal functor on the category of G-spectra by using [CMNN, Theorem A.2] that the category of G-spectra can be obtained as a monoidal category from the category of based G-spaces by inverting smash products with representation spheres.

Remark 3.10. We here give a sketch of an approach to a monoidal version of Theorem 1.14. Starting from an enriched symmetric monoidal structure on $G\mathscr{D}_{\mathrm{All}}$, Day convolution provides a symmetric monoidal structure on our category of spectral presheaves, and Theorem 2.2 can be promoted to a monoidal Quillen equivalence, as in [ABS, Theorem 4.3]. It then remains to equip the spectral category $G\mathscr{A}$ with an enriched monoidal structure and promote Theorem 2.5 to a zig-zag of *monoidal* weak equivalences.

However, there are several difficulties with this approach. First, starting with the enriched monoidal structure on $G\mathscr{D}_{\mathrm{All}}$, it is clear what to do on objects, since they are in bijective correspondence with finite G-sets. Namely, employing again the notation of Section 2.4, the objects are of the form $R\mathbb{A} = R\Sigma^\infty_G A_+$, and we set $R\mathbb{A} \wedge R\mathbb{B} = R\Sigma^\infty_G (A \times B)_+$. At the level of morphism spectra, we would need maps of spectra

$$F_G(R\mathbb{A}, R\mathbb{B}) \wedge F_G(R\mathbb{C}, R\mathbb{D}) \longrightarrow F_G(R\Sigma^\infty_G(A \times C)_+, R\Sigma^\infty_G(B \times D)_+)$$

Without a monoidal fibrant replacement functor (see Section 5.3), there is no obvious candidate for such a map.

It is also not straightforward to equip $G\mathscr{A}$ with an enriched monoidal structure. Again, it is clear what to do on objects. The machine developed in [GMMO3] does convert the product functors

(3.11)

$$G\mathscr{E}(B \times A) \times G\mathscr{E}(D \times C) \xrightarrow{\times} G\mathscr{E}(B \times A \times D \times C) \xrightarrow{\cong} G\mathscr{E}(B \times D \times A \times C)$$

of Remark 1.5 to morphisms of spectra

$$\mathbb{K}G\mathscr{E}(B \times A) \wedge \mathbb{K}G\mathscr{E}(D \times C) \longrightarrow \mathbb{K}G\mathscr{E}(B \times D \times A \times C)$$

However, recall from Definition 1.13 that the morphism spectra of $G\mathscr{A}$ are defined using $G\mathscr{E}'$ rather than $G\mathscr{E}$, so some care is required to handle that change. A little more seriously, the functors (3.11) do not give a strict functor $G\mathscr{A} \times G\mathscr{A} \longrightarrow G\mathscr{A}$ since the evident diagram relating products to composition only commutes up to isomorphism. We have not pursued this idea further.

4. Atiyah duality for finite G-sets

It is illuminating to see that we can come very close to constructing an alternative model for the spectrally enriched category $G\mathscr{D}_{\mathrm{All}}$ just by applying the suspension G-spectrum functor Σ_G^∞ to the category of based finite G-sets and G-maps and then passing to G-fixed points. This is based on a close inspection of classical Atiyah duality specialized to finite G-sets. However, it depends on working in the alternative category $G\mathscr{Z}$ of S_G-modules [EKMM, MaM] rather than in the category $G\mathscr{S}$ of orthogonal G-spectra. Because every object of $G\mathscr{Z}$ is fibrant and its suspension G-spectra are easily understood, it is considerably more convenient than $G\mathscr{S}$ for comparison with space level constructions. This leads us to a variant, Theorem 4.20, of Theorem 0.1 that does not invoke infinite loop space theory. It is more topological and less categorical, and it best captures the geometric intuition behind our results. It is also more elementary.

4.1. **The categories $G\mathscr{Z}$, $G\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$, and $\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$.** Relevant background about $G\mathscr{Z}$ appears in Section 5.4, and we just give a minimum of notation here. In analogy with Theorem 2.3, we have the following specialization of the same general result about stable model categories. It is discussed in Section 5.1.

Theorem 4.1. *Let $G\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$ be the full $\mathscr{Z}$-subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the suspension G-spectra $\Sigma_G^\infty(A_+)$, where A runs through the finite G-sets. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}, \mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence.

We must be explicit about cofibrant approximation here. The construction of $G\mathscr{Z}$ starts from the Lewis-May category $G\mathscr{S}p$ of G-spectra, and S_G-modules are G-spectra with additional structure. We have an elementary suspension G-spectrum functor $\Sigma_G^\infty\colon G\mathscr{T} \longrightarrow G\mathscr{S}p$. Viewing Σ_G^∞ as a functor $G\mathscr{T} \longrightarrow G\mathscr{Z}$, it is strong symmetric monoidal. However, the $\Sigma_G^\infty X$ are not cofibrant. As explained in section 5.4 below, there is a left Quillen equivalence $\mathbb{F}\colon G\mathscr{S}p \longrightarrow G\mathscr{Z}$ such that the composite $\mathbf{\Sigma_G^\infty} = \mathbb{F} \circ \Sigma_G^\infty$ takes based G-CW complexes X, such as A_+ for a finite G-set A, to cofibrant S_G-modules. Therefore $\mathbf{\Sigma_G^\infty}$ may be viewed as a cofibrant replacement functor for Σ_G^∞. In particular, we write $\mathbf{S_G} = \mathbf{\Sigma_G^\infty} S^0$ and have a cofibrant approximation $\gamma\colon \mathbf{S_G} \longrightarrow S_G$ of the unit object S_G. Moreover, the cofibrant approximation $\mathbf{\Sigma_G^\infty}(A_+)$ is isomorphic over $\Sigma_G^\infty(A_+)$ to $\mathbf{S_G} \wedge \Sigma_G^\infty(A_+)$.

As before, we consider finite G-sets A, B, and C, but we now agree to write

$$\mathbb{A} = \mathbf{\Sigma_G^\infty} A_+, \quad \mathbb{B} = \mathbf{\Sigma_G^\infty} B_+, \quad \text{and} \quad \mathbb{C} = \mathbf{\Sigma_G^\infty} C_+.$$

The $\mathbb{A}$ are bifibrant objects of $G\mathscr{Z}$ and we let $G\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$ and $\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$ be the full subcategories of $G\mathscr{Z}$ and $\mathscr{Z}_G$ whose objects are the S_G-modules $\mathbb{A}$, where A runs over the finite G-sets. Then $\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}}$ is enriched in $G\mathscr{Z}$ and $G\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}} = (\mathscr{D}_{\mathrm{All}}^{\mathscr{Z}})^G$ is enriched in the category $\mathscr{Z}$ of S-modules. The functor $\mathbf{\Sigma^\infty_G}$ is almost strong symmetric monoidal. Precisely, by Proposition 5.10 below, there is a natural isomorphism

$$\mathbb{A} \wedge \mathbb{B} \cong \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times B)_+ \tag{4.2}$$

with appropriate coherence properties with respect to associativity and commutativity. Since S_G is the unit for the smash product, we can compose with

$$\gamma \wedge \mathrm{id}\colon \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times B)_+ \longrightarrow \mathbf{\Sigma^\infty_G}(A \wedge B)_+$$

to give a pairing as if $\mathbf{\Sigma^\infty_G}$ were a lax symmetric monoidal functor. However, the map $\gamma\colon \mathbf{S_G} \longrightarrow S_G$ points the wrong way for the unit map of such a functor.

4.2. **Space level Atiyah duality for finite G-sets.** To lift the self-duality of $Ho\mathscr{D}_{\mathrm{All}}$ to obtain a new model for $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$, we need representatives in $G\mathscr{Z}$ for the maps

$$\eta\colon S_G \longrightarrow \mathbb{A} \wedge \mathbb{A} \quad \text{and} \quad \varepsilon\colon \mathbb{A} \wedge \mathbb{A} \longrightarrow S_G$$

in $\mathrm{Ho}G\mathscr{Z}$ that express the duality there. The map ε is induced from the elementary map ε of Definition 1.40. The observation that it plays a key role in Atiyah duality seems to be new. The definition of η requires desuspension by representation spheres.

Let A be a finite G-set and let $V = \mathbb{R}[A]$ be the real representation generated by A, with its standard inner product, so that $|a| = 1$ for $a \in A$. Since we are working on the space level, we may view $A_+ \wedge S^V$ as the wedge over $a \in A$ of the spaces (not G-spaces) $\{a\}_+ \wedge S^V$, with G acting by $g(a,v) = (ga, gv)$. There is no such wedge decomposition after passage to G-spectra.

Definition 4.3. Recall that $\varepsilon\colon (A\times A)_+ \longrightarrow S^0$ is the G-map defined by $\varepsilon(a,b) = *$ if $a \neq b$ and $\varepsilon(a,a) = 1$. Recall too that $(A \times B)_+$ can be identified with $A_+ \wedge B_+$ and that the functor $\mathbf{\Sigma^\infty_G}$ is almost strong symmetric monoidal. We shall also write ε for the composite map of S_G-modules

$$\mathbb{A} \wedge \mathbb{A} \cong \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times A)_+ \xrightarrow{\mathrm{id}\wedge\mathbf{\Sigma^\infty_G}\varepsilon} \mathbf{S_G} \wedge \mathbf{S_G} \xrightarrow{\gamma\wedge\gamma} S_G \wedge S_G \cong S_G, \tag{4.4}$$

where the unlabeled isomorphisms are two instances of (4.2).

Definition 4.5. Embed A as the basis of the real representation $V = \mathbb{R}[A]$. The normal bundle of the embedding is just $A \times V$, and its Thom complex is $A_+ \wedge S^V$. We obtain an explicit tubular embedding $\nu\colon A \times V \longrightarrow V$ by setting

$$\nu(a,v) = a + (\rho(|v|)/|v|)v,$$

where $\rho\colon [0,\infty) \longrightarrow [0,d)$ is a homeomorphism for some $d < 1/2$; ν is a G-map since $|gv| = |v|$ for all g and v. Applying the Pontryagin-Thom construction, we obtain a G-map $t\colon S^V \longrightarrow A_+ \wedge S^V$, which is an equivariant pinch map

$$S^V \longrightarrow \vee_{a\in A} S^V \cong A_+ \wedge S^V.$$

To be more precise, after collapsing the complement of the tubular embedding to a point, we use ν^{-1} to expand each small homeomorphic copy of S^V to the canonical full-sized one; explicitly, if $|w| < d$, then

$$\nu^{-1}(a+w) = (a, (\rho^{-1}(|w|)/|w|)w).$$

The diagonal map on A induces the Thom diagonal $\Delta\colon A_+ \wedge S^V \longrightarrow A_+ \wedge A_+ \wedge S^V$, and we let

$$\eta = \eta_A\colon S^V \longrightarrow A_+ \wedge A_+ \wedge S^V \tag{4.6}$$

be the composite $\Delta \circ t$. Explicitly,

$$\eta(v) = \begin{cases} (a, a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a + w \text{ where } a \in A \text{ and } |w| < d \\ * & \text{otherwise.} \end{cases} \tag{4.7}$$

The negative sphere G-spectrum S^{-V} in $G\mathscr{S}p$ is obtained by applying the left adjoint of the V^{th}-space functor to S^0, and S_G is isomorphic to $S^V \odot S^{-V}$ (see [LMSM, I.4.2] and [MaM, IV.2.2]). Taking the tensor of η with S^{-V} we obtain a map of G-spectra

$$S_G \cong S^V \odot S^{-V} \longrightarrow (A_+ \wedge A_+ \wedge S^V) \odot S^{-V} \cong (A_+ \wedge A_+) \odot S_G \cong \Sigma^\infty_G (A_+ \wedge A_+).$$

Applying the functor $\mathbb{F}$ to this map and smashing with $\mathbf{S_G}$ we obtain the second map in the diagram

$$S_G \cong S_G \wedge S_G \xleftarrow{\gamma\wedge\gamma} \mathbf{S_G} \wedge \mathbf{S_G} \xrightarrow{\eta} \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times A)_+ \cong \mathbb{A} \wedge \mathbb{A}. \tag{4.8}$$

The following result is a reminder about space level Atiyah duality. The notion of a V-duality was defined and explained for smooth G-manifolds in [LMSM, §III.5].

Proposition 4.9. *The maps*

$$\eta\colon S^V \longrightarrow A_+ \wedge A_+ \wedge S^V \quad \textit{and} \quad \varepsilon \wedge \mathrm{id}\colon A_+ \wedge A_+ \wedge S^V \longrightarrow S^V$$

specify a V-duality between A_+ and itself.

Proof. This could be proven from scratch by proving the required triangle identities, but in fact it is a special case of equivariant Atiyah duality for smooth G-manifolds, A being a 0-dimensional example. Our specification of η is a specialization of the description of η for a general smooth G-manifold M given in [LMSM, p. 152]. We claim that our $\varepsilon \wedge \mathrm{id}$ is a specialization of the definition of ε for a general smooth G-manifold given there. Indeed, letting s be the zero section of the normal bundle ν of the embedding $A \subset \mathbb{R}[A] = V$, we have the composite embedding

$$A \xrightarrow{\Delta} A \times A \xrightarrow{s\times \mathrm{id}} (A \times V) \times A \cong A \times A \times V.$$

The normal bundle of this embedding is $A \times V$, and we may view

$$\Delta \times \mathrm{id}\colon A \times V \longrightarrow A \times A \times V$$

as giving a big tubular neighborhood. The Pontryagin-Thom map here is obtained by smashing the map $r\colon (A \times A)_+ \longrightarrow A_+$ that sends (a, b) to a if $a = b$ and to $*$ if $a \neq b$ with the identity map of S^V. Composing with the map induced by the projection $\pi\colon A_+ \longrightarrow S^0$ that sends a to 1, this gives $\varepsilon \wedge \mathrm{id}$. We observed this factorization of ε in Definition 1.40 and we have used it before, in the proof of Theorem 2.15. □

Tensoring with S^{-V}, applying the functor $\mathbf{S_G} \wedge \mathbb{F}$, and composing with γ, we obtain the explicit duality maps in $G\mathscr{Z}$ displayed in (4.4) and (4.8).

4.3. **The weakly unital categories $G\mathscr{B}$ and $\mathscr{B}_G$.** Since the G-spectra $\mathbb{A}$ are self-dual, $F_G(\mathbb{A}, \mathbb{B})$ is naturally isomorphic to $\mathbb{B} \wedge \mathbb{A}$ in $\mathrm{Ho}G\mathscr{Z}$, and the composition and unit

$$F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) \longrightarrow F_G(\mathbb{A}, \mathbb{C}) \quad \text{and} \quad S_G \longrightarrow F_G(\mathbb{B}, \mathbb{B}) \tag{4.10}$$

can be expressed as maps

$$\mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{C} \wedge \mathbb{A} \quad \text{and} \quad S_G \longrightarrow \mathbb{A} \wedge \mathbb{A} \tag{4.11}$$

in Ho$G\mathscr{Z}$. We want to understand these maps in terms of duality in $G\mathscr{Z}$, without use of infinite loop space theory. However, since we are working in $G\mathscr{Z}$, we must take the isomorphisms (4.2) and the cofibrant approximation $\gamma\colon \mathbf{S}_G \longrightarrow S_G$ into account, and we cannot expect to have strict units. The notion of a weakly unital enriched category was introduced in [GM1, §3.5] to formalize what we see here.

Thus we shall construct a weakly unital $G\mathscr{Z}$-category $\mathscr{B}_G$, analogous to $\mathscr{A}_G$, and compare it with $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$. The G-fixed category $G\mathscr{B}$ will be a weakly unital $\mathscr{Z}$-category. The objects of $\mathscr{B}_G$ and $G\mathscr{B}$ are the S_G-modules $\mathbb{A}$ for finite G-sets A. The morphism S_G-modules of $\mathscr{B}_G$ are $\mathscr{B}_G(\mathbb{A},\mathbb{B}) = \mathbb{B}\wedge\mathbb{A}$. Composition is given by the maps

$$\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}\colon \mathbb{C}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A}\longrightarrow \mathbb{C}\wedge\mathbb{A}, \tag{4.12}$$

where ε is the map of (4.4); compare Theorem 2.15.

As recalled in Section 1.5, the adjoint $\tilde{\varepsilon}\colon \mathbb{A}\longrightarrow D\mathbb{A} = F_G(\mathbb{A}, S_G)$ of ε is a stable equivalence, and we have the composite stable equivalence

$$\delta = \zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})\colon \mathbb{B}\wedge\mathbb{A}\longrightarrow \mathbb{B}\wedge D\mathbb{A}\longrightarrow F_G(\mathbb{A},\mathbb{B}). \tag{4.13}$$

Formal properties of the adjunction $(\wedge, F_G)$ give the following commutative diagram in $G\mathscr{Z}$, which uses δ to compare composition in $\mathscr{B}_G$ with composition in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$.

$$\begin{tikzcd}
\mathbb{C}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} \arrow[r, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\tilde{\varepsilon}\wedge\mathrm{id}\wedge\tilde{\varepsilon}"'] & \mathbb{C}\wedge\mathbb{A} \arrow[d, "\mathrm{id}\wedge\tilde{\varepsilon}"] \\
\mathbb{C}\wedge D\mathbb{B}\wedge\mathbb{B}\wedge D\mathbb{A} \arrow[r, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \arrow[d, "\zeta\wedge\zeta"'] & \mathbb{C}\wedge D\mathbb{A} \arrow[d, "\zeta"] \\
F_G(\mathbb{B},\mathbb{C})\wedge F_G(\mathbb{A},\mathbb{B}) \arrow[r, "\circ"'] & F_G(\mathbb{A},\mathbb{C})
\end{tikzcd} \tag{4.14}$$

At the bottom, we do not know that the function S_G-modules or their smash product are cofibrant, but all objects at the top are cofibrant and thus bifibrant. In general, to compute the smash product of G-spectra X and Y in the homotopy category, we should take the smash product of cofibrant approximations $\mathbb{Q}X$ and $\mathbb{Q}Y$ of X and Y. Since all objects of $G\mathscr{Z}$ are fibrant, to compute a map $X\wedge Y\longrightarrow Z$ in the homotopy category, we should represent it by a map $\mathbb{Q}X\wedge\mathbb{Q}Y\longrightarrow\mathbb{Q}Z$ and take its homotopy class. The diagram displays such a cofibrant approximation of the composition in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$.

The unit $S_G\longrightarrow F_G(\mathbb{A},\mathbb{A})$ of $\mathscr{B}_G$ is represented by the (formal) composite

$$S_G\xrightarrow{\ \eta\ }\mathbb{A}\wedge\mathbb{A}\xrightarrow{\mathrm{id}\wedge\tilde{\varepsilon}}\mathbb{A}\wedge D\mathbb{A}\xrightarrow{\ \zeta\ }F_G(\mathbb{A},\mathbb{A}) \tag{4.15}$$

that is obtained by inverting the map $\gamma\wedge\gamma$ in (4.8) to obtain the map denoted η. The weak unital property is a way of expressing the unital property by maps in $\mathscr{Z}_G$, without use of inverses in $Ho\mathscr{Z}_G$. This is a bit tedious. Here are the details.

Definition 4.16. Let $V=\mathbb{R}[A]$. For $a\in A$, define $\xi_a\colon \{a\}_+\wedge S^V\longrightarrow\{a\}_+\wedge S^V$ by

$$\xi_a(a,v) = \begin{cases} (a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a+w \text{ and } |w|<d \\ * & \text{otherwise,}\end{cases} \tag{4.17}$$

where ρ is as in Definition 4.5. Then the wedge of the ξ_a is a G-map

$$\xi = \xi_A\colon A_+\wedge S^V\longrightarrow A_+\wedge S^V; \tag{4.18}$$

ξ is G-homotopic to the identity map of $A_+ \wedge S^V$ via the explicit G-homotopy

$$h(a,v,t) = \begin{cases} (a,v) & \text{if } t=0 \text{ or } v=a \\ (a,(1-t)v + t(\rho^{-1}(t|w|)/|w|)w) & \text{if } v = a+w \text{ and } t|w| < d \\ * & \text{otherwise.} \end{cases}$$

Tensoring with S^{-V} and using the natural isomorphisms

$$(X \wedge S^V) \odot S^{-V} \cong X \odot S_G \cong \Sigma^\infty_G X$$

for based G-spaces X, we see that the space level G-equivalence ξ induces a spectrum level G-equivalence $\xi \colon \mathbb{A} \longrightarrow \mathbb{A}$.

With η as specified in (4.6), easy and perhaps illuminating inspections show that the following unit diagrams already commute in $G\mathscr{T}$, before passage to homotopy. In both, A and B are finite G-sets. In the first, $V = \mathbb{R}[A]$. In the second, $W = \mathbb{R}[B]$ and we move S^W from the right to the left for clarity.

$$\begin{tikzcd} B_+ \wedge A_+ \wedge S^V \arrow[r, "\mathrm{id}\wedge\eta_A"] \arrow[d, "\mathrm{id}\wedge\xi_A"'] & B_+ \wedge A^3_+ \wedge S^V \arrow[dl, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \\ B_+ \wedge A_+ \wedge S^V & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S^W \wedge B_+ \wedge A_+ \arrow[r, "\eta_B\wedge\mathrm{id}"] \arrow[d, "\xi_B\wedge\mathrm{id}_A"'] & S^W \wedge B^3_+ \wedge A_+ \arrow[dl, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \\ S^W \wedge B_+ \wedge A_+ & \end{tikzcd}$$

Tensoring with S^{-V} and S^{-W} and using (4.2) to pass to smash products of S_G-modules, a little diagram chase shows that the previous pair of diagrams in $G\mathscr{T}$ gives rise to the following pair of commutative diagrams in $G\mathscr{Z}$. These express the unit laws for a weakly unital $G\mathscr{Z}$-category $\mathscr{B}_G$ [GM1, §3.5] with objects the $\mathbb{A}$ and composition as specified in (4.12). The cited unit laws allow us to start with any chosen cofibrant approximation $\gamma \colon \mathbb{Q}S_G \longrightarrow S_G$ of the unit S_G, and we are led by (4.8) to choose our cofibrant approximation to be $\gamma\wedge\gamma \colon \mathbf{S_G}\wedge\mathbf{S_G} \longrightarrow S_G \wedge S_G \cong S_G$. Using the notation $\gamma \colon \mathbb{Q}S_G \longrightarrow S_G$ for this map, we obtain the required diagrams

$$\begin{tikzcd} \mathbb{B}\wedge\mathbb{A}\wedge\mathbb{Q}S_G \arrow[r, "\mathrm{id}\wedge\eta"] \arrow[d, "\mathrm{id}\wedge\xi\wedge\gamma"'] & \mathbb{B}\wedge\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \arrow[d, "\circ"] \\ \mathbb{B}\wedge\mathbb{A}\wedge S_G \arrow[r, "\cong"'] & \mathbb{B}\wedge\mathbb{A} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{Q}S_G\wedge\mathbb{B}\wedge\mathbb{A} \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[d, "\gamma\wedge\xi\wedge\mathrm{id}"'] & \mathbb{B}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} \arrow[d, "\circ"] \\ S_G\wedge\mathbb{B}\wedge\mathbb{A} \arrow[r, "\cong"'] & \mathbb{B}\wedge\mathbb{A}. \end{tikzcd}$$

Taking $A = S^0$ in our second space level diagram and changing B to A, we also obtain the following commutative diagrams in $G\mathscr{Z}$, where the second diagram is adjoint to the first.

$$(4.19) \quad \begin{tikzcd} \mathbb{Q}S_G\wedge\mathbb{A} \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[d, "\gamma\wedge\xi"'] & \mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \arrow[d, "\mathrm{id}\wedge\varepsilon"] \\ S_G\wedge\mathbb{A} \arrow[r, "\cong"'] & \mathbb{A} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{Q}S_G \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \mathbb{A}\wedge\mathbb{A} \arrow[r, "\mathrm{id}\wedge\tilde{\varepsilon}"] & \mathbb{A}\wedge D\mathbb{A} \arrow[d, "\zeta"] \\ S_G \arrow[r, "\eta"'] & F_G(\mathbb{A},\mathbb{A}) \arrow[r, "F_G(\xi,\mathrm{id})"'] & F_G(\mathbb{A},\mathbb{A}) \end{tikzcd}$$

Here η at the bottom right is adjoint to the identity map of $\mathbb{A}$. In effect, this uses $\delta = \zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})$ to compare the actual unit η in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ at the top with the weak unit in $\mathscr{B}_G$, which is given by the interrelated maps η, γ, and ξ.

4.4. **The category of presheaves with domain $G\mathscr{B}$.** The diagrams (4.14) and (4.19) show that the maps $\delta\colon \mathbb{A}\wedge\mathbb{B}\longrightarrow F_G(\mathbb{A},\mathbb{B})$ specify a map of weakly unital $\mathscr{Z}_G$-categories from the weakly unital $\mathscr{Z}_G$-category $\mathscr{B}_G$ to the (unital) $\mathscr{Z}_G$-category $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$. Passing to G-fixed points, we obtain a weakly unital $\mathscr{Z}$-category $G\mathscr{B}$ and a map $\delta\colon G\mathscr{B}\longrightarrow G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ of weakly unital $\mathscr{Z}$-categories. Weakly unital presheaves and presheaf categories are defined in [GM1, Definition 3.25]. By [GM1, Remark 3.26], we obtain the same category of presheaves $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}},\mathscr{Z})$ using unital or weakly unital presheaves. Since δ is an equivalence, we can adapt the methodology of [GM1, §2] to complete the proof of the following theorem, using the details relating the functor $\mathbf{\Sigma^\infty_G}$ to smash products from Section 5.4. Since we find the use of weakly unital categories unpleasant and our main result Theorem 1.14 more satisfactory, we shall leave the details to the interested reader.

Theorem 4.20. *The categories $\mathbf{Pre}(G\mathscr{B},\mathscr{Z})$ and $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}},\mathscr{Z})$ are Quillen equivalent.*

5. Appendix: Enriched model categories of G-spectra

The results in this section show how to model categories of G-spectra as categories of presheaves of spectra, where G is any compact Lie group. We specialize results of [GM1] to prove and compare two such models. More precisely, in Section 5.1 we establish Theorems 2.3 and 4.1, which state that G-spectra can be modeled as presheaves of spectra in both the orthogonal and S-module contexts. In Section 5.2, we compare these two presheaf models. In sections 5.3 and 5.4 we discuss suspension spectra for orthogonal spectra and S-modules, respectively, in order to be precise about the domain categories for our presheaves. We shall rely on [EKMM, LMSM, MaM, MMSS] for definitions of the relevant categories.

5.1. **Presheaf models for categories of G-spectra.** We focus on two categories of G-spectra treated in detail in [MaM]. We have the closed symmetric monoidal category $\mathscr{S}$ of nonequivariant orthogonal spectra [MMSS]. Its function spectra are denoted $F(X,Y)$. We also have the closed symmetric monoidal category $G\mathscr{S}$ of orthogonal G-spectra for a fixed G-universe U [MaM]. Its function G-spectra are denoted $F_G(X,Y)$. Note that, in contrast to the previous sections, here we allow G-spectra to be indexed over any G-universe. The homotopy type of $F_G(X,Y)$ very much depends on the choice of universe. Then $G\mathscr{S}$ is enriched over $\mathscr{S}$ via the G-fixed point spectra $F_G(X,Y)^G$.[18] We have stable model structures on $\mathscr{S}$ and $G\mathscr{S}$ [MaM, MMSS]. Then [GM1, Lemma 1.35] specializes to give Theorem 2.3. It also gives the following result, in which G can be a compact Lie group (see also [SchSh, Example 3.4(i)].

Theorem 5.1. *Let $G\mathscr{D}_{\mathrm{Orb}}$ be the full $\mathscr{S}$-subcategory of $G\mathscr{S}$ whose objects are fibrant approximations of the suspension G-spectra $\Sigma^\infty_G(G/H_+)$, where H runs over the closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}},\mathscr{S})\underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}}G\mathscr{S},$$

and it is a Quillen equivalence.

[18] In terms of the general context of [GM1], we are taking $\mathscr{V}=\mathscr{S}$ and $\mathscr{M}=G\mathscr{S}$.

Remark 5.2. Adapting our work for finite groups to incomplete universes would requre us to use incomplete Mackey functors and to reconcile the conflict between needing to use all orbits G/H to obtain generators for $\mathrm{Ho}G\mathscr{S}$ and needing to use only those orbits G/H that embed in the given universe to have self-duality of orbits, which is vital to our theory but irrelevant to Theorem 5.1.

We have a second specialization of [GM1, Lemma 1.35]. We have the closed symmetric monoidal category $\mathscr{Z}$ of nonequivariant S-modules [EKMM].[19] Its function spectra are again denoted $F(X,Y)$. We also have the closed symmetric monoidal category $G\mathscr{Z}$ of S_G-modules (for a fixed G-universe U as above) [MaM]. Its function G-spectra are denoted $F_G(X,Y)$. Then $G\mathscr{Z}$ is enriched over $\mathscr{Z}$ via the G-fixed point spectra $F_G(X,Y)^G$. We are taking $\mathscr{V} = \mathscr{Z}$ and $\mathscr{M} = G\mathscr{Z}$. We have stable model structures on $\mathscr{Z}$ and $G\mathscr{Z}$ [EKMM, MaM]. The following specialization of [GM1, Lemma 1.35] is Theorem 4.1.

Theorem 5.3. *Let $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ be the full $\mathscr{Z}$-subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the orbit suspension G-spectra (= S_G-modules) $\Sigma^\infty_G(G/H_+)$, where H runs over the closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence.

Remark 5.4. We stated Theorems 5.1 and 5.3 in terms of orbits G/H. We could equally well shrink the category $G\mathscr{D}_{\mathrm{Orb}}$ by choosing one H in each conjugacy class.

When G is finite, we can instead expand $G\mathscr{D}_{\mathrm{Orb}}$ to the full subcategory $G\mathscr{D}_{\mathrm{All}}$ of $G\mathscr{S}$ or $G\mathscr{Z}$ whose objects are bifibrant approximations of the suspension G-spectra $\Sigma^\infty_G(A_+)$, where A runs over the finite G-sets. By [GM1, Remark 2.5], [GM1, Lemma 1.35] applies to *any* set of compact generators, hence Theorems 5.1 and 5.3 remain true for these expanded versions of the categories $G\mathscr{D}$.

Alternatively, still defining $G\mathscr{D}_{\mathrm{All}}$ using finite G-sets, we can restrict attention to additive presheaves, namely those that take finite wedges in $G\mathscr{D}_{\mathrm{All}}$ to finite products (which are weakly equivalent to finite wedges). The original categories $\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z})$ are Quillen equivalent to the respective categories of additive presheaves defined using finite G-sets. The point is that since our functors are spectrally enriched, the enrichment builds in additivity, just as functors enriched over abelian groups automatically preserve coproducts.

Either way, when G is finite Theorems 5.1 and 5.3 remain valid with $G\mathscr{D}$ reinterpreted to allow general finite G-sets rather than just orbits.

Homotopically, Theorems 5.1 and 5.3 are essentially the same result since $G\mathscr{S}$ and $G\mathscr{Z}$ are Quillen equivalent. On the point set level they are quite different, and they have different virtues and defects.

We say just a bit about the proofs of these theorems. By [GM1, Theorem 4.32], the presheaf categories used in them are well-behaved model categories. The acyclicity condition there holds in Theorem 5.1 because $\mathscr{S}$ satisfies the monoid axiom, by [MMSS, 12.5]. It holds in Theorem 5.3 by use of the "Cofibration Hypothesis" of [EKMM, p. 146], which also holds equivariantly. The orbit G-spectra give compact

[19] The notation $\mathscr{S}$ is short for $\mathscr{I}\mathscr{S}$ and the notation $\mathscr{Z}$ is short for $\mathscr{M}_S$ in the original sources; as a silly mnemonic device, $\mathscr{Z}$ stands for the Z in the middle of Elmendorf-KriZ-Mandell-May.

generating sets in both $\mathrm{Ho}(G\mathscr{S})$ and $\mathrm{Ho}(G\mathscr{Z})$. We require bifibrant representatives. In Theorem 5.1, the orbit G-spectra are cofibrant, and fibrant approximation makes them bifibrant. We say more about the relevant functors in Section 5.3.

By contrast, in Theorem 5.3, all S_G-modules are fibrant, and cofibrant approximation makes them bifibrant. Here cofibrant approximation is given by a well understood left adjoint that very nearly preserves smash products, as we shall explain in Section 5.4.

Technically, [GM1, Lemma 1.35] requires *either* that the unit object of the enriching category $\mathscr{V}$ be cofibrant *or* that every object in $\mathscr{V}$ be fibrant. The first hypothesis holds in $\mathscr{S}$ and the second holds in $\mathscr{Z}$. It is impossible to have both of these conditions in the same symmetric monoidal model category for the stable homotopy category [L, M7]. That is a key reason that both of these results are of interest.

5.2. **Comparison of presheaf models of G-spectra.** Theorems 5.1 and 5.3 are related by the following result, which is [MaM, IV.1.1]; the nonequivariant special case is [MaM, I.1.1]. In this result, $G\mathscr{S}$ is given its positive stable model structure from [MaM] and is denoted $G\mathscr{S}_{pos}$ to indicate the distinction; in that model structure, the sphere G-spectrum in $G\mathscr{S}$, like the sphere G-spectrum in $G\mathscr{Z}$ is not cofibrant. In [MaM], the result is proven for genuine G-spectra for compact Lie groups G, but the same proof applies to naive G-spectra for any topological group G.

Theorem 5.5. *There is a Quillen equivalence*

$$G\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}.$$

The functor $\mathbb{N}$ is strong symmetric monoidal, hence $\mathbb{N}^{\#}$ is lax symmetric monoidal.

The identity functor is a left Quillen equivalence $G\mathscr{S}_{pos} \longrightarrow G\mathscr{S}$. Therefore Theorems 5.1, 5.3, and 5.5, have the following immediate consequence.

Corollary 5.6. *The categories $\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z})$ are Quillen equivalent. More precisely, there are left Quillen equivalences*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}}, \mathscr{S}) \longrightarrow G\mathscr{S} \longleftarrow G\mathscr{S}_{pos} \longrightarrow G\mathscr{Z} \longleftarrow \mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z}).$$

In fact, we can compare the $\mathscr{S}$-category $G\mathscr{D}_{\mathrm{Orb}}$ with the $\mathscr{Z}$-category $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ via the right adjoint $\mathbb{N}^{\#}$. The adjunction

$$G\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}$$

is tensored over the adjunction

$$\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{Z}$$

in the sense of [GM1, Definition 3.20]. Indeed, since $G\mathscr{S}$ is a bicomplete $\mathscr{S}$-category, it is tensored over $\mathscr{S}$. While a more explicit definition is easy enough, we can define $Y \odot X$ to be $Y \wedge i_*\varepsilon^* X$, where $i_*\varepsilon^* \colon \mathscr{S} \longrightarrow G\mathscr{S}$ is the change of group and universe functor associated to $\varepsilon \colon G \longrightarrow e$ that assigns a genuine G-spectrum to a nonequivariant spectrum. The same is true with $\mathscr{S}$ replaced by $\mathscr{Z}$. These functors are discussed in both contexts and compared in [MaM]. Results there (see [MaM, IV.1.1]) imply that

$$\mathbb{N}Y \odot \mathbb{N}X \cong \mathbb{N}(Y \odot X),$$

which is the defining condition for a tensored adjunction. Now [GM1, Corollary 3.24] gives that the $\mathscr{S}$-category $\mathbb{N}^{\#}G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ is quasi-equivalent to $G\mathscr{D}_{\mathrm{Orb}}$. Using [GM1, Remark 2.15 and Theorem 3.17], this implies a direct proof of the Quillen equivalence of Corollary 5.6. Therefore Theorems 5.1 and 5.3 are equivalent: each implies the other.

We reiterate the generality: the results above do *not* require G to be finite. In that generality, we do not know how to simplify the description of the domain category $G\mathscr{D}_{\mathrm{Orb}}$ to transform it into a weakly equivalent $\mathscr{S}$-category or $\mathscr{Z}$-category that is intuitive and perhaps even familiar, something accessible to study independent of knowledge of the category of G-spectra that we seek to understand. Our main theorem shows how to do just that when G is finite.

5.3. **Suspension spectra and fibrant replacement functors in $G\mathscr{S}$.** We here give some observations relevant to understanding the category $G\mathscr{D}_{\mathrm{Orb}}$ of Theorem 5.1. We start with a parenthetical observation about fibrant approximations that is immediate from Theorem 5.5 but does not appear in the literature.

Proposition 5.7. *The unit $\eta\colon E \longrightarrow \mathbb{N}^{\#}\mathbb{N}E$ of the adjunction between $G\mathscr{S}$ and $G\mathscr{Z}$ specifies a lax monoidal fibrant replacement functor on cofibrant objects for the positive stable model structure on $G\mathscr{S}$.*

Remark 5.8. Nonequivariantly, Kro [K] has given a different lax monoidal positive fibrant replacement functor for orthogonal spectra. His construction does not restrict to cofibrant objects, but as he notes, it does not apply to symmetric spectra. However, by [MMSS, 3.3], the unit $E \longrightarrow \mathbb{N}^{\sharp}\mathbb{U}\mathbb{P}\mathbb{N}E$ of the composite of the adjunction $(\mathbb{P}, \mathbb{U})$ between symmetric and orthogonal spectra and the adjunction $(\mathbb{N}, \mathbb{N}^{\sharp})$ gives a lax monoidal positive fibrant replacement functor for symmetric spectra.

Unfortunately the restriction to the positive model structure is necessary, and the only fibrant approximation functor we know of for use in Theorem 5.1 is that given by the small object argument. The point is that the suspension G-spectra $\Sigma^{\infty}_{G}(G/H_{+})$ are cofibrant but not positive cofibrant. For an inner product space V and a based G-space X, the V^{th} space of $\Sigma^{\infty}_{G}X$ is $X \wedge S^{V}$. The functor Σ^{∞}_{G}, also denoted F_0, is left adjoint to the zeroth space $(-)_0\colon G\mathscr{S} \longrightarrow G\mathscr{T}$. Nonequivariantly, it is part of [MMSS, 1.8] that for based spaces X and Y, $F_0X \wedge F_0Y$ is naturally isomorphic to $F_0(X \wedge Y)$. The categorical proof of that result in [MMSS, §21] applies equally well equivariantly to give the following complement to Proposition 5.7.

Proposition 5.9. *The functor $\Sigma^{\infty}_{G}\colon G\mathscr{T} \longrightarrow G\mathscr{S}$ is strong symmetric monoidal.*

Therefore the zeroth space functor is lax symmetric monoidal, but of course that functor is not homotopically meaningful except on objects that are fibrant in the stable model structure. There is no known fibrant replacement functor in that model structure that is well-behaved with respect to smash products.

Nonequivariantly, a homotopically meaningful version of the adjunction $(\Sigma^{\infty}, \Omega^{\infty})$ has been worked out for symmetric spectra by Sagave and Schlichtkrull [SaSc] and for symmetric and orthogonal spectra by Lind [Li], who compares his constructions with the adjunction $(\Sigma^{\infty}, \Omega^{\infty})$ in $\mathscr{S}p$ (see below) and with its analogue for $\mathscr{Z}$.

This generalizes to the equivariant context, although details have not been written down.

5.4. **Suspension spectra and smash products in $G\mathscr{Z}$.** We here give some observations relevant to understanding the category $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ of Theorem 5.3. In particular, we give properties of cofibrant approximations of suspension spectra that are used in Section 4. For more information, see [M1, XXIV], [MaM, §IV.2], and the nonequivariant precursor [EKMM].

We have a category $G\mathscr{P}$ of (coordinate-free)-prespectra. Its objects Y are based G-spaces $Y(V)$ and based G-maps $Y(V)\wedge S^W \longrightarrow Y(W-V)$ for $V\subset W$. Here V and W are sub inner product spaces of a G-universe U. A G-spectrum is a G-prespectrum Y whose adjoint G-maps $Y(V)\longrightarrow \Omega^{W-V}Y(W)$ are homeomorphisms. The (Lewis-May) category $G\mathscr{S}p$ of G-spectra is the full subcategory of G-spectra in $G\mathscr{P}$. The suspension G-prespectrum functor Π sends a based G-space X to $\{X\wedge S^V\}$. There is a left adjoint spectrification functor $L\colon G\mathscr{P}\longrightarrow G\mathscr{S}p$, and the suspension G-spectrum functor $\Sigma^\infty_G\colon G\mathscr{T}\longrightarrow G\mathscr{S}p$ is $L\circ\Pi$. Explicitly, let

$$Q_G X = \operatorname{colim} \Omega^V\Sigma^V X,$$

where V runs over the finite dimensional subspaces of a complete G-universe U. Then the V^{th} G-space of $\Sigma^\infty_G X$ is $Q_G\Sigma^V X$.

All objects of $G\mathscr{S}p$ are fibrant, and the zeroth space functor $\Omega^\infty_G\colon G\mathscr{S}p\longrightarrow G\mathscr{T}$ is now homotopically meaningful. For a based G-CW complex X (with based attaching maps), $\Sigma^\infty_G X$ is cofibrant in $G\mathscr{S}p$. In particular, the sphere G-spectrum $S_G=\Sigma^\infty_G S^0$ is cofibrant. Since G is a compact Lie group, the orbits G/H are G-CW complexes, hence the $\Sigma^\infty_G(G/H_+)$ are cofibrant. However, $G\mathscr{S}p$ is not symmetric monoidal under the smash product. The implicit trade off here is intrinsic to the mathematics, as was explained by Lewis [L]; see [M7] for a more recent discussion.

We summarize some constructions in [EKMM] that work in exactly the same fashion equivariantly as nonequivariantly. We have the G-space $\mathscr{L}(j)$ of linear isometries $U^j\longrightarrow U$, with G acting by conjugation. These spaces form an E_∞ G-operad when U is complete. The G-monoid $\mathscr{L}(1)$ gives rise to a monad $\mathbb{L}$ on $G\mathscr{S}p$. Its algebras are called $\mathbb{L}$-spectra, and we have the category $G\mathscr{S}p[\mathbb{L}]$ of $\mathbb{L}$-spectra. It has a smash product $\wedge_{\mathscr{L}}$ which is associative and commutative but not unital. The action map $\xi\colon \mathbb{L}Y\longrightarrow Y$ of an $\mathbb{L}$-spectrum Y is a stable equivalence.

Suspension G-spectra are naturally $\mathbb{L}$-spectra. In particular, the sphere G-spectrum S_G is an $\mathbb{L}$-spectrum. There is a natural stable equivalence $\lambda\colon S_G\wedge_{\mathscr{L}} Y\longrightarrow Y$ for $\mathbb{L}$-spectra Y. The S_G-modules are those Y for which λ is an isomorphism, and they are the objects of $G\mathscr{Z}$. All suspension G-spectra are S_G-modules, and so are all $\mathbb{L}$-spectra of the form $S_G\wedge_{\mathscr{L}} Y$. The smash product $\wedge$ on S_G-modules is just the restriction of the smash product $\wedge_{\mathscr{L}}$, and it gives $G\mathscr{Z}$ its symmetric monoidal structure.

We have a sequence of Quillen left adjoints

$$G\mathscr{T}\xrightarrow{\Sigma^\infty_G} G\mathscr{S}p\xrightarrow{\mathbb{L}} G\mathscr{S}p[\mathbb{L}]\xrightarrow{\mathbb{J}} G\mathscr{Z},$$

where $\mathbb{L}X$ is the free $\mathbb{L}$-spectrum generated by a G-spectrum X and $\mathbb{J}Y=S_G\wedge_{\mathscr{L}}Y$ is the S_G-module generated by an $\mathbb{L}$-spectrum Y. We let $\mathbb{F}=\mathbb{J}\mathbb{L}$; then $\mathbb{L}$, $\mathbb{J}$, and $\mathbb{F}$ are Quillen equivalences. The composite $\gamma=\xi\circ\lambda\colon \mathbb{F}Y\longrightarrow Y$ is a stable equivalence for any $\mathbb{L}$-spectrum Y. We have defined $\mathbf{\Sigma^\infty_G}$ to be the composite functor $\mathbb{F}\Sigma^\infty_G$, and we have the natural stable equivalence of S_G-modules $\gamma\colon \mathbf{\Sigma^\infty_G}X\longrightarrow \Sigma^\infty_G X$.

The tensor $Y \odot X$ of a G-prespectrum and a based G-space X has V^{th} G-space $Y(V) \wedge X$. When Y is a G-spectrum, the G-spectrum $Y \odot X$ is $L(\ell Y \odot X)$, where ℓY is the underlying G-prespectrum of Y [LMSM, I.3.1]. Tensors in $G\mathscr{S}p[\mathbb{L}]$ and $G\mathscr{Z}$ are inherited from those in $G\mathscr{S}p$. All of our left adjoints are enriched in $\mathscr{T}$ and preserve tensors. This leads to the following relationship between $\wedge$ and $\mathbf{\Sigma^\infty_G}$.

Proposition 5.10. *For based G-spaces X and Y, there are natural isomorphisms*

$$\mathbf{\Sigma^\infty_G} X \wedge \mathbf{\Sigma^\infty_G} Y \cong (\mathbf{S_G} \wedge \mathbf{S_G}) \odot (X \wedge Y) \cong \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(X \wedge Y).$$

Proof. We have $\Sigma^\infty_G X \cong S_G \odot X$ and therefore

$$\mathbf{\Sigma^\infty_G} X = \mathbb{F}\Sigma^\infty_G X \cong \mathbb{F}(S_G \odot X) \cong (\mathbb{F}S_G) \odot X = \mathbf{S_G} \odot X.$$

We also have

$$(\mathbf{S_G} \odot X) \wedge (\mathbf{S_G} \odot Y) \cong (\mathbf{S_G} \wedge \mathbf{S_G}) \odot (X \wedge Y)$$

and the conclusion follows. □

6. Appendix: Whiskering $G\mathscr{E}$ to obtain strict unit 1-cells

The bicategory $G\mathscr{E}$ of Definition 1.6 narrowly misses being a strict 2-category, and we whisker the unit 1-cells to obtain a strict 2-category $G\mathscr{E}'$.[20] Before focusing on specifics we give an elementary general definition.

Definition 6.1. For a category $\mathscr{D}$ with a privileged object Δ, define the whiskering $\mathscr{D}'$ of $\mathscr{D}$ at Δ by adjoining a new object I and an isomorphism $\zeta\colon I \longrightarrow \Delta$. We have the inclusion $i\colon \mathscr{D} \longrightarrow \mathscr{D}'$, and we define a retraction functor $r\colon \mathscr{D}' \longrightarrow \mathscr{D}$ by $r(I) = \Delta$ and $r(\zeta) = \mathrm{id}_\Delta$. Thus $r \circ i = \mathrm{Id}_{\mathscr{D}}$ and the isomorphism ζ on the object I together with the identify map on all other objects of $\mathscr{D}'$ defines a natural isomorphism $\mathrm{Id}_{\mathscr{D}'} \longrightarrow i \circ r$. If $\mathscr{D}$ is a G-category and Δ is G-fixed, then $\mathscr{D}'$ is a G-category with I and ζ fixed by G, and then $\mathscr{D}$ and $\mathscr{D}'$ are G-equivalent.

The whiskered category $G\mathscr{E}'$ "enriched in permutative categories" and the whiskered G-category $\mathscr{E}'_G$ "enriched in permutative G-categories" are defined to have the same objects, or 0-cells, as $G\mathscr{E}$ and $\mathscr{E}_G$, namely the finite G-sets A in both cases.

Definition 6.2. If $A \neq B$ or if $|A| \leq 1$ and $A = B$, we define $G\mathscr{E}'(A, B)$ to be the permutative category $G\mathscr{E}(A, B)$. For each A of cardinality at least 2, we define

$$G\mathscr{E}'(A, A) = G\mathscr{E}(A, A)'.$$

We denote the adjoined 1-cell by I_A and the adjoined isomorphism 2-cell by ζ_A. We specify a permutative structure on $G\mathscr{E}'(A, A)$ by setting

$$E \amalg F = \begin{cases} I_A & \text{if } (E, F) = (I_A, \emptyset) \text{ or } (\emptyset, I_A) \\ i\big(r(E) \amalg r(F)\big) & \text{otherwise.} \end{cases}$$

We have denoted the monoidal product as $\amalg$ since the product in $G\mathscr{E}(A \times A)$ is given by the disjoint union of spans. As the only 2-cell in $G\mathscr{E}'(A, A)$ with source or target $\emptyset$ is $\mathrm{id}_\emptyset$, this product extends uniquely to a functor. Since the retraction

$$r\colon G\mathscr{E}'(A, A) \longrightarrow G\mathscr{E}(A, A)$$

is strict monoidal (even though the unit of $G\mathscr{E}(A, A)$ is not strict) and an equivalence of categories, the symmetry isomorphism $\gamma\colon \amalg \cong \amalg\tau$ on $G\mathscr{E}(A, A)$ lifts uniquely to a symmetry isomorphism $\gamma\colon \amalg \cong \amalg\tau$ on $G\mathscr{E}'(A, A)$.

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[20] We thank Angélica Osorno for help with the material in this section.

To extend composition to functors

$$G\mathscr{E}'(B,C)\times G\mathscr{E}'(A,B)\xrightarrow{\circ}G\mathscr{E}'(A,C)$$

we declare I_A to be a strict 2-sided unit. It remains to define composition with a 2-cell with source or target I_A. Since every such 2-cell factors through ζ_A and composition with Δ_A is already defined, it suffices to define composition with ζ_A. Since Δ_A is a strict right unit, for a span $B \longleftarrow E \longrightarrow A$, abbreviated E, we may define $E\circ\zeta_A : E\circ I_A \longrightarrow E\circ\Delta_A$ to be the identity 2-cell id_E. We define $\zeta_B\circ E : I_B\circ E \longrightarrow \Delta_B\circ E$ to be $\ell_{B,E}^{-1}$, where $\ell_{B,E}$ is the 2-cell defined in (1.8).

Remark 6.3. In [BO], and also in a previous version of this article, a different strictification of $G\mathscr{E}$ was proposed, namely just redefining composition with Δ_A to force this to be a unit 1-cell. Unfortunately, this breaks associativity, since the 1-cell Δ_A is decomposable under composition if $|A|\geq 2$.

We have a precisely analogous definition on the level of G-categories, obtaining a strict 2-category $\mathscr{E}'_G$ from $\mathscr{E}_G$.

Definition 6.4. If $A\neq B$ or if $|A|\leq 1$ and $A=B$, we define $\mathscr{E}'_G(A,B)$ to be the permutative G-category $\mathscr{E}_G(A,B)$. For each A of cardinality at least 2, we define

$$\mathscr{E}'_G(A,A)=\mathscr{E}_G(A,A)'.$$

We denote the adjoined 1-cell by I_A and the adjoined isomorphism 2-cell by ζ_A. We specify a G-permutative structure on $\mathscr{E}'_G(A,A)$ by setting

$$\theta(\mu;E_1,\dots,E_n)=\begin{cases} I_A & \text{if } E_i=I_A \text{ and } E_j=\emptyset \text{ for } j\neq i\\ \theta(\mu;r(E_1),\dots,r(E_n)) & \text{otherwise.}\end{cases}$$

To extend composition to a functor

$$\mathscr{E}'_G(B,C)\times\mathscr{E}'_G(A,B)\xrightarrow{\circ}\mathscr{E}'_G(A,C),$$

we declare the object $I_A\in\mathscr{E}'_G(A,A)$ to be a strict 2-sided unit. We define composition with a 2-cell whose source or target is of the form I_A exactly as in Definition 6.2, except that to define $\zeta_B\circ E$ we now use the $\ell_{B,E}$ defined in (1.36).

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