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Categorical models for equivariant classifying spaces

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Starting categorically, we give simple and precise models for classifying spaces of equivariant principal bundles. We need these models for work in progress in equivariant infinite loop space theory and equivariant algebraic K-theory, but the models are of independent interest in equivariant bundle theory and especially equivariant covering space theory.

Introduction

Let Π and G be topological groups and let G act on Π, so that we have a semi-direct product $\Gamma = \Pi \rtimes G$ and a split extension

$$(0\text{--}1) \qquad 1 \longrightarrow \Pi \overset{\subset}{\longrightarrow} \Gamma \overset{q}{\longrightarrow} G \longrightarrow 1.$$

The underlying space of Γ is $\Pi \times G$, and the product is given by

$$(\sigma, g)(\tau, h) = (\sigma(g \cdot \tau), gh).$$

There is a general theory of (G, Π_G)-bundles [2, 8, 9, 15] corresponding to such extensions. Here Π_G denotes Π together with its given action of G. We shall only be interested in principal (G, Π_G)-bundles $p \colon E \longrightarrow B$.

Definition 0.2 Let $p \colon E \longrightarrow B$ be a principal Π-bundle where B is a G-space. Then p is a principal (G, Π_G)-bundle if the (free) action of Π on E extends to an action of Γ and p is a Γ-map, where Γ acts on B through the quotient map $\Gamma \longrightarrow G$.

The more general theory of $(\Pi; \Gamma)$-bundles applicable to non-split extensions Γ is included in [9, 14, 15]. The theory is especially familiar when G acts trivially on Π, so that $\Gamma = G \times \Pi$. With $\Pi = O(n)$ or $U(n)$, the trivial action case gives classical equivariant bundle theory and equivariant topological K-theory.

Definition 0.3 A principal (G, Π_G)-bundle $p\colon E \longrightarrow B$ is universal if for all G-spaces X of the homotopy types of G-CW complexes, pullback of p along G-maps $f\colon X \longrightarrow B$ induces a natural bijection from the set of homotopy classes of G-maps $X \longrightarrow B$ to the set of equivalence classes of (G, Π_G)-bundles over X.

For applications in equivariant infinite loop space theory and equivariant algebraic K-theory, we need to understand classifying G-spaces for (G, Π_G)-bundles as classifying spaces of categories. Nonequivariantly, it was already emphasized in Segal's classical paper [21, §3] that the universal principal Π-bundle of a topological group Π can be constructed on the level of topological categories, and the intuition is that we are giving the equivariant generalization of his classical construction.

One motivation is to give new constructions of E_∞ operads of G-categories and G-spaces. This much only requires trivial actions of G on Π. By definition, the jth-space of an E_∞ operad of G-spaces is a universal principal (G, Σ_j)-bundle. Having various category level models for such classifying spaces allows us to construct examples of E_∞ G-spaces from E_∞ categories, and these feed into equivariant infinite loop space machines to construct interesting G-spectra [4, 5, 6].

The examples relevant to the equivariant algebraic K-theory of G-rings, namely rings with G-action by automorphisms, require more general split extensions. If R is a G-ring, then G acts entrywise on $GL(n, R)$. The classifying spaces of $(G, GL(n, R)_G)$-bundles are central to the definition of the genuine equivariant algebraic K-theory spectrum $\mathbb{K}_G(R)$ of R [4, 16]. Our treatment of the fixed point spaces of the classifying spaces of equivariant bundles is crucial to determining the fixed point spectra of the $\mathbb{K}_G(R)$. The paradigmatic example is a finite Galois extension E/F with Galois group G. As explained in [4], it is an immediate application of examples in this paper, which demonstrate the relevance of Hilbert's theorem 90, that the fixed point spectrum $\mathbb{K}_G(E)^H$ is the classical nonequivariant K-theory spectrum of the fixed field E^H. The use of genuine G-spectra in algebraic K-theory is new and is explored in [16].

The results we need are close to those of [8, 9, 14] and those stated by Murayama and Shimakawa [18][1], but we require a more precise and rigorous categorical and topological understanding than the literature affords. This is intended as a service paper that displays the relevant constructions in their fullblown simplicity.

We start with the topologized equivariant version of the elementary theory of chaotic categories in §1. We analyze a general construction that specializes to give our classifying G-spaces in §2. We show how it gives universal equivariant bundles in §3.

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[1]But see Scholium 3.12.

Our explicit description of the classifying spaces of (G, Π_G)-bundles as classifying spaces of categories allows us to compute their fixed point spaces categorically in §4. This gives precise information already on the category level, before passage to classifying spaces, and that is essential to our applications.

The main results of the paper are summarized in the following two theorems: the first gives a categorical model for equivariant universal bundles and their classifying spaces, and the second gives a description of the fixed points of the classifying spaces of equivariant bundles. Details of the first are in Theorems 3.10 and 3.11 and details of the second are in Theorems 4.18, 4.23, and 4.24. We need some preliminary definitions and notations to state these results.

Let G be discrete and let $\mathcal{E}G$ denote the unique contractible groupoid with object set G. It is a (right) G-category, meaning that G acts on both objects and morphisms, and it has a unique morphism between any two objects. We agree to identify the topological group Π with the topological groupoid with a single object and with morphism space Π. Then the action of G on Π makes it a G-groupoid.

For small topological categories $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{C}at(\mathcal{A}, \mathcal{B})$ denote the category of all continuous functors $\mathcal{A} \longrightarrow \mathcal{B}$ and all natural transformations. When $\mathcal{A}$ and $\mathcal{B}$ are G-categories, $\mathcal{C}at(\mathcal{A}, \mathcal{B})$ inherits an action of G given by conjugation. We shall give more details in §1.1.

We assume that the reader is familiar with the classifying space functor B from categories to spaces, or more generally from topological categories to spaces. It works equally well to construct G-spaces from topological G-categories. It is the composite of the nerve functor N from topological categories to simplicial spaces (e.g. [13, §7]) and geometric realization $|-|$ from simplicial spaces to spaces (e.g. [12, §11]), both of which are product-preserving functors.

Theorem 0.4 *If G is discrete and Π is either discrete or a compact Lie group, then the canonical map*

$$B\mathcal{C}at(\mathcal{E}G, \mathcal{E}\Pi) \longrightarrow B\mathcal{C}at(\mathcal{E}G, \Pi)$$

is a universal principal (G, Π_G)-bundle.

Thus the classifying space of the G-category $\mathcal{C}at(\mathcal{E}G, \Pi)$ is a G-space that classifies (G, Π_G)-bundles.

Crossed homomorphisms, their automorphism groups, and the non-Abelian cohomology group $H^1(G; \Pi_G)$ are defined in Definitions 4.1, 4.11, and 4.17.

Theorem 0.5 *The fixed point category $\mathscr{C}at(\mathcal{E}G, \Pi)^G$ is the disjoint union of the groups $\operatorname{Aut}\alpha$, where α runs over crossed homomorphisms representing the elements of $H^1(G; \Pi_G)$. Equivalently, $\mathscr{C}at(\mathcal{E}G, \Pi)^G$ is the disjoint union of the groups $\Pi \cap N_\Gamma \Lambda$, where Λ runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$. Therefore $B\mathscr{C}at(\mathcal{E}G, \Pi)^G$ is the disjoint union of the classifying spaces $B(\Pi \cap N_\Gamma \Lambda)$.*

With more work, our hypotheses on G and Π could surely be weakened. We should admit that we are especially interested in discrete groups in many of our current applications. Since Π is the relevant structural group, we are then studying equivariant covering spaces. However, it is important for some applications to allow Π to have a topology. For example, in [16], equivariant algebraic K-theory is related to equivariant topological K-theory and to Atiyah's Real K-theory. There it is crucial that Π be allowed to be compact Lie in Theorem 0.4.

There is an earlier topological analogue of our categorical construction in terms of mapping spaces rather than mapping categories [14]. It applies in considerably greater topological generality, but it does not generally start categorically. We compare the categorical and topological constructions in §5.

The choices of Π relevant to equivariant infinite loop space theory and equivariant algebraic K-theory, namely symmetric groups and the general linear groups of G-rings, have alternative categorical models, which play a key role. These alternative categorical models are given in §6, which is entirely algebraic, with all groups discrete. We call special attention to §6.2, where we relate crossed homomorphisms to skew group rings and their skew modules. The algebraic ideas here may not be as well-known as they should be and deserve further study.

The letter B for the classifying space functor from categories to spaces would sometimes be awkward in our context, since the classifying space functor will also be used to construct universal bundles rather than classifying spaces for bundles, hence we agree to write out $|N-|$ rather than B whenever B seems likely to confuse.

This notation also displays a key technical problem that is sometimes overlooked in the literature. The functor $|-|$ is a left adjoint and therefore preserves all colimits, such as passage to orbits in the equivariant setting. The functor N is a right adjoint and it generally does not preserve colimits or passage to orbits, as we illustrate with elementary examples. This problem is the subject of the paper [1] by Babson and Kozlov. For topological categories, there is no discussion in the literature. Exceptionally, N does commute with passage to orbits in the key examples that appear in equivariant bundle theory. Clear understanding of passage to orbits is essential to our calculations of fixed point spaces.

Remark 0.6 The functor $\mathscr{C}at(\mathcal{E}G, -)$ from G-categories to G-categories plays a central role in our work. Its G-fixed category was introduced by Thomason [24, (2.1)], who called it the lax limit of the action of G on $\mathscr{C}$ and denoted it by $\mathscr{C}at_G(\underline{EG}, \underline{C})$. The relevance to equivariant bundle theory of the equivariant precursor $\mathscr{C}at(\mathcal{E}G, \mathscr{B})$ was first noticed by Shimakawa [18, 23].

Acknowledgements

1 Preliminaries on chaotic and translation categories

The definitions we start with are familiar and elementary. However, to keep track of categorical data and group actions later, we shall be pedantically precise.

1.1 Preliminaries on topological G-categories

Let $\mathscr{C}at$ be the category of categories and functors. We may also view it as the 2-category of categories, with 0-cells, 1-cells, and 2-cells the categories, functors, and natural transformations. From that point of view, $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is the internal hom category whose objects are the functors $\mathscr{A} \longrightarrow \mathscr{B}$ and whose morphisms are the natural transformations between them; they enrich $\mathscr{C}at$ over itself.

For a group G, a G-category $\mathscr{A}$ is a category with an action of G specified by a homomorphism from G to the automorphism group of $\mathscr{A}$. Regarding G as a groupoid with one object, the action is specified by a functor $G \longrightarrow \mathscr{C}at$. We have the 2-category $G\mathscr{C}at$ of G-categories, G-functors, and G-natural transformations, where the latter notions are defined in the evident way: everything must be equivariant.

We may view $G\mathscr{C}at$ as the underlying 2-category of a category enriched over $G\mathscr{C}at$. The 0-cells are still G-categories, but now we have the G-category $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ as the internal hom between them. Its underlying category is $\mathscr{C}at(\mathscr{A}, \mathscr{B})$, and G acts by conjugation on functors and natural transformations. Thus, for $F \colon \mathscr{A} \longrightarrow \mathscr{B}$, $g \in G$,

and A either an object or a morphism of $\mathscr{A}$, $(gF)(A) = gF(g^{-1}A)$. Similarly, for a natural transformation $\eta\colon E \longrightarrow F$ and an object A of $\mathscr{A}$,

$$(g\eta)_A = g\eta_{g^{-1}A}\colon gE(g^{-1}A) \longrightarrow gF(g^{-1}A).$$

The category $G\mathscr{C}at(\mathscr{A},\mathscr{B})$ is the same as the G-fixed category $\mathscr{C}at(\mathscr{A},\mathscr{B})^G$, and we sometimes vary the choice of notation.

We can topologize the definitions so far, starting with the 2-category of categories internal to the category $\mathscr{U}$ of (compactly generated) spaces, together with continuous functors and continuous natural transformations. Recall that a category $\mathscr{A}$ internal to a cartesian monoidal category $\mathscr{V}$ has object and morphism objects in $\mathscr{V}$ and structure maps Source, Target, Identity and Composition in $\mathscr{V}$. These maps are denoted S, T, I, and C, and the usual category axioms must hold. When $\mathscr{V} = \mathscr{U}$, we refer to internal categories as topological categories; we refer to them as topological G-categories when $\mathscr{V} = G\mathscr{U}$. These are more general than (small) topologically enriched categories, which have discrete sets of objects. We can now allow G to be a topological group in the equivariant picture. We continue to use the notations already given in the more general topological situation.

1.2 Chaotic topological G-categories

Definition 1.1 A small category $\mathscr{C}$ is *chaotic* if there is exactly one morphism from b to a for each pair of objects a and b. The unique morphism from a to b must then be inverse to the unique morphism from b to a. Thus $\mathscr{C}$ is a groupoid, and its classifying space is contractible since every object is initial and terminal; in fact, it is the unique contractible groupoid with the given object set. A topological category $\mathscr{C}$ is chaotic if its underlying category is chaotic. Its classifying space is again contractible (see Remark 2.11), but there are other topological groupoids with the given object space and contractible classifying spaces. Similarly, a topological G-category is chaotic if its underlying category is chaotic. It is then contractible but not usually G-contractible.

The senior author remembers hearing the name "chaotic" long ago, but we do not know its source. The idea is that everything is the same as everything else, which does seem rather chaotic.[2]

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[2]Some category theorists suggest the name "indiscrete category", by formal analogy with indiscrete spaces in topology. The key difference is that indiscrete spaces are of no interest, whereas we hope to convince the reader that chaotic categories are of considerable interest.

Lemma 1.2 *If $\mathscr{A}$ is any category and $\mathscr{B}$ is a chaotic category, then the category $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is again chaotic.*

Proof The unique natural map $\zeta\colon E \longrightarrow F$ between functors $E, F\colon \mathscr{A} \longrightarrow \mathscr{B}$ is given on an object A of $\mathscr{A}$ by the unique map $\zeta_A\colon E(A) \longrightarrow F(A)$ in $\mathscr{B}$. □

Lemma 1.3 *If $\mathscr{A}$ is any topological G-category and $\mathscr{B}$ is a chaotic topological G-category, then the topological G-category $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ and its G-fixed category $G\mathscr{C}at(\mathscr{A}, \mathscr{B})$ are again chaotic.*

Proof Since $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is just the category $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ with its conjugation action by G, [Lemma 1.2](#) implies the conclusion for $\mathscr{C}at(\mathscr{A}, \mathscr{B})$. The conclusion is inherited by $G\mathscr{C}at(\mathscr{A}, \mathscr{B}) = \mathscr{C}at(\mathscr{A}, \mathscr{B})^G$ since the unique natural transformation between G-functors E and F is necessarily a G-natural transformation. □

Definition 1.4 The chaotic topological category $\mathcal{E}X$ generated by a space X is the topological category with object space X and morphism space $X \times X$. The source, target, identity, and composition maps are defined by

$$S = \pi_2\colon X \times X \longrightarrow X, \; T = \pi_1\colon X \times X \longrightarrow X, \; I = \Delta\colon X \longrightarrow X \times X, \; \text{and}$$

$$C = \mathrm{id} \times \varepsilon \times \mathrm{id}\colon (X \times X) \times_X (X \times X) \cong X \times X \times X \longrightarrow X \times X,$$

where $\varepsilon\colon X \longrightarrow *$ is the trivial map. On elements, $S(y, x) = x$, $T(y, x) = y$, $I(x) = (x, x)$, and $C(z, y, x) = (z, x)$. Forgetting the topology, the element (y, x) is the unique morphism $x \longrightarrow y$. Reversing the order of source and target in the notation this way, so that $(z, y) \circ (y, x) = (z, x)$, will turn out to be helpful later.

A map $f\colon X \longrightarrow Y$ induces the functor $\tilde{f}\colon \mathcal{E}X \longrightarrow \tilde{Y}$ given by f on objects and $f \times f$ on morphisms. When X is a (left or right) G-space, we give $\mathcal{E}X$ the action specified by the given action on the object space X and the diagonal action on the morphism space $X \times X$; $\mathcal{E}X$ is then a chaotic topological G-category. Sending X to $\mathcal{E}X$ specifies a functor from the category $G\mathscr{U}$ of G-spaces to the category $G\mathscr{G}pd$ of topological G-groupoids (a full subcategory of $G\mathscr{C}at$).

1.3 The adjunction between G-spaces and topological G-groupoids

Sending a category to its set of objects restricts to an object functor $\mathscr{O}b\colon G\mathscr{G}pd \longrightarrow G\mathscr{U}$.

Lemma 1.5 *The chaotic category functor is right adjoint to the object functor, so that*

$$G\mathscr{C}at(\mathscr{C}, \mathcal{E}X) \cong G\mathrm{Map}(\mathscr{O}b\mathscr{C}, X)$$

for a topological G-category $\mathscr{C}$ with object space $\mathscr{O}b\mathscr{C}$ and a topological G-space X. If $\mathscr{C}$ is chaotic with object G-space X, then the unit of the adjunction is an isomorphism of topological G-groupoids $\eta\colon \mathscr{C} \longrightarrow \mathcal{E}X$.

Proof Let $\mathscr{M}or\mathscr{C}$ be the morphism G-space of $\mathscr{C}$. The functor $\mathscr{C} \longrightarrow \mathcal{E}X$ determined by a continuous G-map $f\colon \mathscr{O}b\mathscr{C} \longrightarrow X$ is f on object G-spaces and the composite

$$\mathscr{M}or\mathscr{C} \xrightarrow{(T,S)} \mathscr{O}b\mathscr{C} \times \mathscr{O}b\mathscr{C} \xrightarrow{f \times f} X \times X$$

on morphism G-spaces. The last statement rephrases the meaning of chaotic. $\qquad \square$

1.4 Translation categories and chaotic categories

We use another simple definition to relate chaotic categories to other familiar categories. Let G be a topological group and Y be a left G-space. Generalizing how we think of G as a one object category, we can think of Y together with its action by G as the functor $Y\colon G \longrightarrow \mathscr{U}$ that sends the single object $*$ to Y and is given on morphism spaces by the map $G \longrightarrow \mathrm{Map}(Y, Y)$ adjoint to the action map $G \times Y \longrightarrow Y$.

Definition 1.6 Let Y be a left G-space. Define the translation category $T(G, Y)$ as follows. The object space is Y and the morphism space is $G \times Y$. We think of (g, y) as a morphism $g\colon y \longrightarrow gy$. The map $I\colon Y \longrightarrow G \times Y$ sends y to (e, y). The maps S and T send (g, y) to y and gy, respectively. The domain of composition, $(G \times Y) \times_Y (G \times Y)$, can be identified with $(G \times G) \times Y$, and composition sends (h, g, y) to (hg, y). The construction is functorial in Y, for fixed G, and in the pair (G, Y) in general. If Y has a right action by G that commutes with the left action, then $T(Y, G)$ is a right G-category via the given right action on the object space Y and on the second coordinate of the morphism space $G \times Y$.

Remark 1.7 The definition makes sense when G is only a monoid, not necessarily a group. When Y is a point, $T(Y, G)$ is G regarded as a one object category. When G is a group, $T(Y, G)$ is the standard groupoid associated to a G-space, but it is not generally chaotic.

Proposition 1.8 *For left G-spaces Y, there is a natural comparison functor $\mu\colon T(G, Y) \longrightarrow \tilde{Y}$. If Y has a right action that commutes with its left action, then μ is a map of right G-categories. The functor $\mu\colon T(G, G) \longrightarrow \mathcal{E}G$ is an isomorphism of right G-categories.*

Proof Define μ to be the identity map on object spaces and the map that sends (g, y) to (gy, y) on morphism spaces. Since $\tilde{Y}$ is chaotic, this is the unique functor that is the identity on objects, and it is easy to check equivariance when Y has a right G-action. When $Y = G$ with left action and right action given by its product, μ is an isomorphism with $\mu^{-1}(h, g) = (hg^{-1}, g)$ on morphism spaces. □

In view of the differing group actions on the morphism spaces $G \times G$, namely action on the right coordinate in $T(G, G)$ and diagonal action in $\mathcal{E}G$, the isomorphism between $T(G, G)$ and $\mathcal{E}G$ must not be viewed as a tautology.

Remark 1.9 When we return to the split extension (0–1), the group Π there will play a role close to that of the group denoted G in Definition 1.6 and Proposition 1.8. When $G = e$, we would then specialize to $Y = \Pi$ with its natural left Π action and see the usual universal principal Π-bundle. When $G \neq e$, the relevant specialization is a little less obvious; see Lemma 3.4, which is a follow up of Proposition 1.8.

2 The category $\mathscr{C}at(\mathcal{E}X, \Pi)$

We let X be a space and Π be a topological group in this section. We regard Π as a category with one object without change of notation; it should be clear from the context when we mean the group Π and when we mean the category Π. From now on, functors and natural transformations are to be continuous (in the topological sense), even when we neglect to say so. We are especially interested in the functor categories $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$, which are chaotic by Lemma 1.2, and in the functor categories $\mathscr{C}at(\mathcal{E}X, \Pi)$, which are not. The right action of Π on $\mathcal{E}\Pi$ induces a right action of Π on $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$.

This section and the next give a pedantically explicit description of $\mathscr{C}at(\mathcal{E}X, \Pi)$ and of the induced map

$$\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi) \longrightarrow \mathscr{C}at(\mathcal{E}X, \Pi),$$

showing in particular that it is obtained by passage to orbits over Π. When $X = G$, this elementary analysis will be at the heart of all our proofs. We defer adding in the second group G that appears in the bundle theory until after we have this description in place since a group defined solely in terms of the diagonal on X and the product on Π plays a central role in the description.

2.1   An explicit description of $\mathscr{C}at(\mathcal{E}X, \Pi)$

By the adjunction given in Lemma 1.5 (with $G = e$), the object space of the chaotic category $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$ can be identified with the space $\mathrm{Map}(X, \Pi)$ of maps $X \longrightarrow \Pi$ with its standard (compactly generated) function space topology. Therefore $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$ can be identified with the chaotic category $\mathcal{E}\mathrm{Map}(X, \Pi)$.

Definition 2.1   Define the pointwise product $*$ on $\mathrm{Map}(X, \Pi)$ by

$$(\alpha * \beta)(x) = \alpha(x)\beta(x)$$

for $\alpha, \beta \colon X \longrightarrow \Pi$. The unit element ε is given by $\varepsilon(x) = e$ and inverses are given by $\alpha^{-1}(x) = \alpha(x)^{-1}$. The topological group $\mathrm{Map}(X, \Pi)$ contains Π as a (closed) subgroup, where we regard an element $\sigma \in \Pi$ as the constant map $\sigma \colon X \longrightarrow \Pi$ at σ. The inclusion of Π in $\mathrm{Map}(X, \Pi)$ and composition give $\mathrm{Map}(X, \Pi)$ its right Π-action.

Definition 2.2   Choose a basepoint $x_0 \in X$. There is a unique representative map α such that $\alpha(x_0) = e$ in each orbit of $\mathrm{Map}(X, \Pi)$ under the right action by Π. Let $\mathscr{O}(X, \Pi) \subset \mathrm{Map}(X, \Pi)$ denote the subspace of such representative maps. It is a subgroup of $\mathrm{Map}(X, \Pi)$. The Π-action and the product $*$ on $\mathrm{Map}(X, \Pi)$ are related by $\alpha\sigma = \alpha * \sigma$ for $\sigma \in \Pi$, and $*$ restricts to a homeomorphism of Π-spaces $\mathscr{O}(X, \Pi) \times \Pi \longrightarrow \mathrm{Map}(X, \Pi)$. Write elements of $\mathrm{Map}(X, \Pi)$ in the form $\alpha\sigma$, where $\alpha(x_0) = e$. Passage to orbits restricts to a homeomorphism $\mathscr{O}(X, \Pi) \cong \mathrm{Map}(X, \Pi)/\Pi$. Observe that the product $*$ on $\mathrm{Map}(X, \Pi)$ induces a left action of $\mathrm{Map}(X, \Pi)$ on $\mathscr{O}(X, \Pi)$ by sending (β, α) to the orbit representative of $\beta * \alpha$.

The proofs of the follow three lemmas are simple exercises from the fact that there is a unique morphism (y, x) from x to y in $\mathcal{E}X$; compare Lemma 1.2.

Lemma 2.3   *A functor $E \colon \mathcal{E}X \longrightarrow \Pi$ is given by the trivial map $X \longrightarrow *$ of object spaces and a map $E \colon X \times X \longrightarrow \Pi$ of morphism spaces such that $E(x, x) = e$ and $E(z, y)E(y, x) = E(z, x)$. Define $\alpha \in \mathscr{O}(X, \Pi)$ by $\alpha(x) = E(x, x_0)$. Then α determines E by the formula*

$$E(y, x) = E(y, x_0)E(x_0, x) = \alpha(y)\alpha(x)^{-1}.$$

Writing $E = E_\alpha$, sending E_α to α specifies a homeomorphism from the space of functors $\mathcal{E}X \longrightarrow \Pi$ to $\mathscr{O}(X, \Pi)$.

Lemma 2.4   *For $E_\alpha, E_\beta \colon \mathcal{E}X \longrightarrow \Pi$, a natural transformation $\eta \colon E_\alpha \longrightarrow E_\beta$ is given by a map $\eta \colon X \longrightarrow \Pi$ such that $\eta(y)E_\alpha(y, x) = E_\beta(y, x)\eta(x)$ for $x, y \in X$. If $\sigma \in \Pi$ is defined by $\sigma = \eta(x_0)$, then the pair $(\beta\sigma, \alpha)$ determines η by the formula*

$$\eta(x) = E_\beta(x, x_0)\eta(x_0)E_\alpha(x, x_0)^{-1} = (\beta\sigma * \alpha^{-1})(x).$$

Writing $\eta = \eta_\sigma$, sending η_σ to $(\beta\sigma, \alpha)$ specifies a homeomorphism from the space of morphisms of $\mathscr{C}at(\mathcal{E}X, \Pi)$ to the space $\mathrm{Map}(X, \Pi) \times \mathscr{O}(X, \Pi)$.

Lemma 2.5 *Identify the object and morphism spaces of $\mathscr{C}at(\mathcal{E}X, \Pi)$ with*

$$\mathscr{O}(X, \Pi) \text{ and } \mathscr{M}(X, \Pi) \equiv \mathrm{Map}(X, \Pi) \times \mathscr{O}(X, \Pi)$$

via the homeomorphisms of the previous two lemmas. Then the identity map I sends α to $(\alpha e, \alpha)$ and the source and target maps S and T send $(\beta\sigma, \alpha)$ to α and to β. The $S = T$ pullback

$$\mathscr{M}(X, \Pi) \times_{\mathscr{O}(X,\Pi)} \mathscr{M}(X, \Pi)$$

can be identified with $\mathrm{Map}(X, \Pi) \times \mathrm{Map}(X, \Pi) \times \mathscr{O}(X, \Pi)$ via

$$((\gamma\tau, \beta), (\beta\sigma, \alpha)) \leftrightarrow (\gamma\tau, \beta\sigma, \alpha)$$

and the composition map C sends $(\gamma\tau, \beta\sigma, \alpha)$ to $(\gamma\tau\sigma, \alpha)$.

Proof If we compose $\eta_\tau \colon E_\beta \longrightarrow E_\gamma$ with $\eta_\sigma \colon E_\alpha \longrightarrow E_\beta$, we obtain

$$\eta_\tau * \eta_\sigma = \gamma^{-1}\tau * \beta * \beta^{-1}\sigma * \alpha = \gamma^{-1}\tau\sigma * \alpha,$$

which corresponds to the given description. $\qquad\qquad\square$

2.2 Two identifications of $\mathscr{C}at(\mathcal{E}X, \Pi)$

We show here that [Proposition 1.8](#) leads to one identification of $\mathscr{C}at(\mathcal{E}X, \Pi)$, and the lemmas of the previous section lead to a closely related one. These elementary identifications commute passage to orbits with the functor $\mathscr{C}at(\mathcal{E}X, -)$, and that will be crucial to understanding $B\mathscr{C}at(\mathcal{E}G, \Pi)$ as an equivariant classifying space.

Notation 2.6 The category Π is isomorphic to the orbit category $\mathcal{E}\Pi/\Pi$. The quotient functor $p \colon \mathcal{E}\Pi \longrightarrow \Pi$ is the trivial map $\Pi \longrightarrow *$ on object spaces and is given on morphism spaces by the map $p \colon \Pi \times \Pi \longrightarrow (\Pi \times \Pi)/\Pi \cong \Pi$ specified by $p(\tau, \sigma) = \tau\sigma^{-1}$. Let q denote the functor

$$\mathscr{C}at(\mathrm{id}, p) \colon \mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi) \longrightarrow \mathscr{C}at(\mathcal{E}X, \Pi).$$

We also let q denote the functor between translation categories

$$T(\mathrm{Map}(X, \Pi), \mathrm{Map}(X, \Pi)) \longrightarrow T(\mathrm{Map}(X, \Pi), \mathscr{O}(X, \Pi))$$

that is induced by the quotient map $p \colon \mathrm{Map}(X, \Pi) \longrightarrow \mathrm{Map}(X, \Pi)/\Pi \cong \mathscr{O}(X, \Pi)$.

Theorem 2.7 *There is a commutative diagram of topological categories in which μ, ν, and ξ are isomorphisms.*

$$
\begin{array}{ccc}
T(\mathrm{Map}(X,\Pi),\mathrm{Map}(X,\Pi)) & \xrightarrow{\quad\quad\mu\quad\quad} & \mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi) \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\
T(\mathrm{Map}(X,\Pi),\mathscr{O}(X,\Pi)) \xrightarrow{\ \nu\ } \mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi)/\Pi & \xrightarrow{\ \xi\ } & \mathscr{C}at(\mathcal{E}X,\Pi)
\end{array}
$$

Proof The map p is the quotient map given by passage to orbits over Π. Since q on the right is a Π-map with Π acting trivially on $\mathscr{C}at(\mathcal{E}X,\Pi)$, q factors through a map ξ that makes the triangle commute. Since $\mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi)$ is the chaotic category whose object space is the topological group $\mathrm{Map}(X,\Pi)$, Proposition 1.8 gives the isomorphism μ. Since q on the left is obtained by passage to orbits from the relevant action of Π, it is clear that μ induces an isomorphism ν making the left trapezoid commute.

All that remains is to prove that ξ is an isomorphism, and that follows from the results of §2.1. For a functor $E_\alpha\colon \mathcal{E}X \longrightarrow \Pi$, $\alpha\colon X \longrightarrow \Pi$ and $\alpha\times\alpha\colon X\times X \longrightarrow \Pi\times\Pi$ define the object and morphism maps of a functor $F\colon \mathcal{E}X \longrightarrow \mathcal{E}\Pi$. The functoriality properties of E_α show that $p\circ F = E_\alpha$, so that q is surjective on objects. If we also have $p\circ F' = E_\alpha$, then a quick check shows that $F(x)^{-1}F'(x) = F(y)^{-1}F'(y)$ for all $x,y\in X$. If the common value is denoted by σ, then $F'(x) = F(x)\sigma$ for all x. In view of the specification of p and q in Notation 2.6, this implies that ξ is a homeomorphism on object spaces.

Now let $E_\alpha, E_\beta\colon \mathcal{E}X \longrightarrow \Pi$ be any two functors. For any choices of functors $F,F'\colon \mathcal{E}X \longrightarrow \mathcal{E}\Pi$ such that $q\circ F = E_\alpha$ and $q\circ F' = E_\beta$, define $\zeta\colon X \longrightarrow \Pi\times\Pi$ by $\zeta(x) = (F(x),F'(x))$. Then ζ is a map from the object space of $\mathcal{E}X$ to the morphism space of $\mathcal{E}\Pi$. A quick check shows that ζ is a natural transformation $F \longrightarrow F'$ such that $\eta = q\circ\zeta$ is a natural transformation $E_\alpha \longrightarrow E_\beta$ with $\eta_{x_0} = F'(x_0)F(x_0)^{-1}$. Via our enumeration of the possible choices, this implies that q restricted to the inverse image of the space of natural transformations $E_\alpha \longrightarrow E_\beta$ can be identified with the quotient map $p\colon \Pi\times\Pi \longrightarrow \Pi$ of Notation 2.6. It follows that ξ is a homeomorphism on morphism spaces. $\square$

2.3 The nerve functor and classifying spaces

We recall the definition of the nerve functor N in more detail than might be thought warranted at this late date since, in the presence of the left-right action dichotomy of multiple group actions, the original definitions in category theory can cause real

problems arising from categorical dyslexia. There are two standard conventions in the literature, and we must choose. Let $\mathscr{C}$ be a topological category with object space $\mathscr{O}$ and morphism space $\mathscr{M}$. Then $N_0\mathscr{C} = \mathscr{O}$ and, for $q > 0$,

$$N_q\mathscr{C} = \mathscr{M} \times_{\mathscr{O}} \cdots \times_{\mathscr{O}} \mathscr{M},$$

with q factors $\mathscr{M}$. The pullbacks are over pairs of maps (S, T). To avoid dyslexia, we remember that $g \circ f$ means first f and then g, and choose to forget the picture

$$\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \longrightarrow \cdots \longrightarrow \bullet \xrightarrow{f_{q-1}} \bullet \xrightarrow{f_q} \bullet$$

of q composable arrows and instead remember that the picture

$$(2\text{--}8) \qquad x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \xleftarrow{\quad} \cdots \xleftarrow{\quad} x_{q-2} \xleftarrow{f_{q-1}} x_{q-1} \xleftarrow{f_q} x_q$$

corresponds to an element $[f_1, \cdots, f_q]$ of $N_q\mathscr{C}$, so that $S(f_i) = T(f_{i+1})$. For $x \in \mathscr{O}$, we write $\mathrm{id} = I(x)$ generically. Then

$$d_0[f] = T(f), \quad d_1[f] = S(f), \quad \text{and} \quad s_0(x) = [\mathrm{id}_x].$$

For $q \geq 2$,

$$d_i[f_1, \cdots, f_q] = \begin{cases} [f_2, \cdots, f_q] & \text{if } i = 0 \\ [f_1, \cdots, f_{i-1}, f_i \circ f_{i+1}, f_{i+2}, \cdots, f_q] & \text{if } 0 < i < q \\ [f_1, \cdots, f_{q-1}] & \text{if } i = q \end{cases}$$

and, for $q \geq 1$,

$$s_i[f_1, \ldots, f_q] = [f_1, \cdots, f_i, \mathrm{id}, f_{i+1}, \cdots, f_q].$$

Of course, these can and should be expressed in terms of the maps S, T, I, and C so as to remember the topology and check continuity.

Recall that a (right) action of a group G on a simplicial space Y_* is specified by levelwise group actions such that the d_i and s_i are G-maps; formally, Y_* is a simplicial object in the category of (right) G-spaces. Orbit and fixed point simplicial spaces are constructed levelwise, $(Y_*/G)_q = Y_q/G$ and $(Y_*)_q^G = Y_q^G$. For a G-category $\mathscr{C}$, $N(\mathscr{C}^G) \cong (N\mathscr{C})^G$ since N is a right adjoint, but it is rarely the case that $N(\mathscr{C}/G) \cong (N\mathscr{C})/G$, as the following counterexample should make clear.

Example 2.9 Let G be a group and let G act on itself by conjugation. Let A be the abelianization of G. Regarding G and A as categories with a single object, $G/G \cong A$, and NA is generally much smaller than NG/G. Here $[g_1, \ldots, g_q]$ and $[h_1, \cdots, h_q]$ are in the same orbit under the conjugation action if and only if there is a single g such that $gg_ig^{-1} = gh_ig^{-1}$ for all i. For example if G is a finite simple group of order n, then A is trivial but N_qG/G has at least n^{q-1} elements.

In this example, NG is the simplicial space, often denoted B_*G, whose geometric realization is the classifying space BG. Parametrizing with a left G-space Y gives a familiar simplicial space $B_*(*, G, Y)$ (e.g. [13, §7]). Write $q: E_*G \longrightarrow B_*G$ for the map

$$B_*(*, G, G) \longrightarrow B_*(*, G, *) \cong B_*(*, G, G)/G$$

induced by $G \longrightarrow *$. The isomorphism on the right is obvious, but it is in fact an example of an isomorphism of the form $N(\mathscr{C}/G) \cong (N\mathscr{C})/G$, as the following observations make clear. Recall the translation category from Definition 1.6.

Lemma 2.10 *The simplicial space $NT(G, Y)$ is isomorphic to $B_*(*, G, Y)$.*

Proof A typical q-tuple (2–8) in $N_qT(G, Y)$ has i^{th} term

$$f_i = (g_i, g_{i+1} \cdots g_q y): g_{i+1} \cdots g_q y \longrightarrow g_i g_{i+1} \cdots g_q y$$

for elements $g_i \in G$ and $y \in Y$. It corresponds to $[g_1, \cdots, g_q]y$ in $B_q(*, G, Y)$. $\square$

Remark 2.11 For any space X, $N\mathcal{E}X$ is the simplicial space denoted D_*X in [12, p. 97]. Our choice of S and T on $\mathcal{E}X$ is consistent with (2–8) and the usual notation $(x_0, \cdots, x_q)$ for q-simplices. The claim in Definition 1.1 that $|N\mathcal{E}X|$ is contractible is immediate from [12, 10.4], which says that D_*X is simplicially contractible. The isomorphism $N\mu: NT(G, G) \longrightarrow N\mathcal{E}G$ implied by Proposition 1.8 coincides with the isomorphism $\alpha_*: E_*G \longrightarrow D_*G$ of [12, 10.4].

Applying geometric realization, write $B(*, G, Y) = |B_*(*, G, Y)|$, and similarly for EG and BG. Then $B(*, G, Y) \cong B(*, G, G) \times_G Y = EG \times_G Y$. By Lemma 2.10,

$$BT(Y, G) = EG \times_G Y.$$

A relevant example is $Y = G/H$ for a (closed) subgroup H of G. The space

$$BT(G, G/H) = EG \times_G (G/H) \cong (EG)/H$$

is a classifying space BH since EG is a free contractible H-space.

In particular, take $G = \text{Map}(X, \Pi)$ and $H = \Pi$ for a space X and group Π, remembering that $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$ is the chaotic category with object space the group $\text{Map}(X, \Pi)$. Applying the classifying space functor to the diagram of Theorem 2.7 and using Lemma 2.10, we obtain the following commutative diagram, in which the horizontal

maps are homeomorphisms and, up to canonical homeomorphisms, the vertical maps are obtained by passage to orbits over Π.

$$
\begin{CD}
E(\mathrm{Map}(X,\Pi)) @>\cong>> B\mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi) @>=>> B\mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi) \\
@VVV @VVV @VVV \\
(E\,\mathrm{Map}(X,\Pi))/\Pi @>\cong>> B(\mathscr{C}at(\mathcal{E}X,\mathcal{E}\Pi)/\Pi) @>\cong>> B\mathscr{C}at(\mathcal{E}X,\Pi)
\end{CD}
$$

Ignoring minor topological niceness conditions[3], for any space X the diagram gives isomorphic categorical models for the universal principal Π-bundle $E\Pi \longrightarrow B\Pi$.

3 Categorical universal equivariant principal bundles

3.1 Preliminaries on actions by the semi-direct product Γ

Now return to the split extension (0–1) of the introduction. For a Γ-category or Γ-space, passage to orbits with respect to Π gives a G-category or a G-space. It is standard in equivariant bundle theory to let G act from the left and Π act from the right. Thus suppose that X is a left G and right Π object in any category. Using elementwise notation, turn the right action of Π into a left action by setting $\sigma x = x\sigma^{-1}$.

By an action of Γ on X, we understand a left action that coincides with the given actions when restricted to the subgroups $G = e \times G$ and $\Pi = \Pi \times e$ of Γ. Since $(\sigma, g) = (\sigma, e)(e, g)$, the action must be defined by

$$(3\text{–}1) \qquad (\sigma, g)x = (\sigma, e)(e, g)x = (\sigma, e)gx = \sigma gx = (gx)\sigma^{-1}.$$

For now, we will denote the action of G on Π by $\cdot$, but we just use juxtaposition for the prescribed actions of G and Π on X. Since the action by g on Π is a group homomorphism, $g \cdot (\sigma\tau) = (g \cdot \sigma)(g \cdot \tau)$ and $g \cdot \sigma^{-1} = (g \cdot \sigma)^{-1}$. The interaction of Π and G in Γ is given by the twisted commutation relation

$$(e, g)(\sigma, e) = (g \cdot \sigma, g) = (g \cdot \sigma, e)(e, g),$$

or the same relation with σ replaced by σ^{-1}. Therefore (3–1) gives an action of Γ if and only if the given actions of Π and G satisfy the twisted commutation relation

$$(3\text{–}2) \qquad g(x\sigma) = (gx)(g \cdot \sigma).$$

---

[3]The identity element of the group $\mathrm{Map}(X,\Pi)$ should be a nondegenerate basepoint and the space $\mathrm{Map}(X,\Pi)$ should be paracompact; see [13, 9.10].

The placement of parentheses is crucial: we are taking group actions in different orders. When the action of G on Π is trivial, $g \cdot \sigma = \sigma$, this is the familiar statement that commuting left and right actions define an action by the product $G \times \Pi$.

Lemma 3.3 *For a G-category $\mathscr{A}$, the left G and right Π-actions on $\mathscr{C}at(\mathscr{A}, \mathcal{E}\Pi)$ extend naturally to a Γ-action.*

Proof We must verify that $g(F\sigma) = (gF)(g \cdot \sigma)$ for $g \in G$, $\sigma \in \Pi$ and a functor $F \colon \mathscr{A} \longrightarrow \Pi$. The unique natural transformation $E \longrightarrow F$ between a pair of functors E and F will then necessarily be given by Γ-maps. The verification is formal from the fact that G acts by conjugation, so that the action of G on Π is part of the prescription of the action of G on F. Recall that the left action of G on $\mathscr{C}at(\mathscr{A}, \mathcal{E}\Pi)$ is given by conjugation, $(gF)(a) = g \cdot F(g^{-1}a)$ for $g \in G$ and an object or morphism $a \in \mathscr{A}$. The right action of Π is given by $(F\sigma)(a) = F(a)\sigma$. Then

$$
\begin{aligned}
(g(F\sigma))(a) &= g \cdot (F\sigma)(g^{-1}a) \\
&= g \cdot (F(g^{-1}a)\sigma) \\
&= (g \cdot F(g^{-1}a))(g \cdot \sigma) \\
&= ((gF)(a))(g \cdot \sigma) \\
&= ((gF)(g \cdot \sigma))(a). \qquad \square
\end{aligned}
$$

In particular, let $\mathscr{A} = \mathcal{E}X$ for a left G-space X. Then the given action of G on the object space X and the diagonal action of G on the morphism space $X \times X$ give a left G-action on the category $\mathcal{E}X$. Lemma 3.3 shows that the left G and right Π-action on $\mathscr{C}at(\mathcal{E}X, \mathcal{E}\Pi)$ give it an action by Γ. Explicitly, the conjugation left action by G and the evident right action by Π on the object space $\operatorname{Map}(X, \Pi)$ induce diagonal actions on the morphism space $\operatorname{Map}(X, \Pi) \times \operatorname{Map}(X, \Pi)$, and these specify left G and right Π-actions on $\mathscr{C}at(\mathcal{E}X, \Pi)$ that satisfy the commutation relation required for a Γ-action.

Specializing further to $X = G$, we have the following equivariant elaboration of Proposition 1.8. We change the group G there to the group $\operatorname{Map}(G, \Pi)$ here and remember that the product on $\operatorname{Map}(G, \Pi)$ is just the pointwise product induced by the product on Π, with no dependence on the product of G. Ignoring the group action, we may identify the chaotic right $\operatorname{Map}(G, \Pi)$-category with object space $\operatorname{Map}(G, \Pi)$ with the category $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)$. The following lemma identifies group actions. Remember that Π is a subgroup of $\operatorname{Map}(G, \Pi)$.

Lemma 3.4 *The isomorphism of right $\operatorname{Map}(G, \Pi)$-categories*

$$
\mu \colon T\big(\operatorname{Map}(G, \Pi), \operatorname{Map}(G, \Pi)\big) \longrightarrow \mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)
$$

is an isomorphism of Γ-categories, where the G-action on both source and target categories is given by the conjugation action on the object space $\mathrm{Map}(G, \Pi)$ and the resulting diagonal action on the morphism space $\mathrm{Map}(G, \Pi) \times \mathrm{Map}(G, \Pi)$.

Proof Since μ is an isomorphism and a Π-map, we can and must give the source category the unique G-action such that μ is a G-map. Since μ is the identity map on object spaces, the action must be the conjugation action on the object space. On an element (β, α) of the morphism space, we must define

$$g(\beta, \alpha) = \mu^{-1}(g\mu(\beta, \alpha)) = \mu^{-1}(g(\beta\alpha), g\alpha) = \mu^{-1}((g\beta)(g\alpha), g\alpha) = (g\beta, g\alpha). \qquad \square$$

Lemma 3.5 *With $X = G$, the diagram of Theorem 2.7 is a commutative diagram of Γ-categories and maps of Γ-categories, where Γ acts through the quotient homomorphism $\Gamma \longrightarrow G$ on the three categories on the bottom row.*

Proof Since the trapezoid is obtained by passing to orbits under the action of Π, it is a diagram of Γ-categories by Lemma 3.4. The functor $p \colon \mathcal{E}\Pi \longrightarrow \Pi$ of Notation 2.6 is a G-map since

$$g \cdot (\tau, \sigma) = g \cdot (\tau\sigma^{-1}) = (g \cdot \tau)(g \cdot \sigma)^{-1} = p(g \cdot \tau, g \cdot \sigma).$$

It follows that the right vertical arrow $q = \mathcal{C}at(\mathcal{E}G, p)$ is a map of Γ-categories. Letting $[F]$ denote the orbit of a functor $F \colon \mathcal{E}G \longrightarrow \mathcal{E}\Pi$ under the right action of Π, the functor ξ is specified by $\xi[F] = p \circ F$, and it follows that ξ is Γ-equivariant. $\qquad \square$

3.2 Universal principal (G, Π_G)-bundles

Observe that for any G-category $\mathscr{A}$, the corepresented functor $\mathcal{C}at(\mathscr{A}, -)$ from G-categories to G-categories is a right adjoint and therefore preserves all limits. We take $\mathscr{A}$ to be the G-category $\mathcal{E}G$ from now on, and we use the functor $\mathcal{C}at(\mathcal{E}G, -)$ to obtain a convenient categorical description of universal principal (G, Π_G)-bundles. Variants of the construction are given in [14, 18].

Definition 3.6 Let G and Π be topological groups and let G act on Π. Define $E(G, \Pi_G)$ to be the Γ-space $B\mathcal{C}at(\mathcal{E}G, \mathcal{E}\Pi) = |N\mathcal{C}at(\mathcal{E}G, \mathcal{E}\Pi)|$ and define $B(G, \Pi_G)$ to be the orbit G-space $E(G, \Pi_G)/\Pi$. Let $p \colon E(G, \Pi_G) \longrightarrow B(G, \Pi_G)$ be the quotient map.

We need a lemma in order to prove that p is a universal (G, Π_G)-bundle in favorable cases. We defer the proof to the next section. We believe that the result is true more generally, but there are point-set topological issues obstructing a proof. We shall not obscure the simplicity of our work by seeking maximum generality. As usual in equivariant bundle theory, we assume that all given subgroups are closed.

Lemma 3.7 *Let Λ be a subgroup of Γ. If $\Lambda \cap \Pi \neq e$, then the fixed point category $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)^\Lambda$ is empty. At least if G is discrete, if $\Lambda \cap \Pi = e$, then $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)^\Lambda$ is non-empty and chaotic.*

The following result is [9, Thm. 9], but the details of the proof are in [8, §2]. A principal (G, Π_G)-bundle is numerable if it is trivial over the subspaces of B in a numerable open cover.

Theorem 3.8 *A numerable principal (G, Π_G)-bundle $p \colon E \longrightarrow B$ is universal if and only if E^Λ is contractible for all (closed) subgroups Λ of Γ such that $\Lambda \cap \Pi = \{e\}$.*

We comment on the hypotheses. Recall from point-set topology that a space X is completely regular if for every closed subspace C and every point x not in C, there is a continuous function $f \colon X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$. This is a weak condition that is satisfied by reasonable spaces, such as CW complexes.

Remark 3.9 Specializing [9, Propositions 4 and 5], a principal (G, Π_G)-bundle with completely regular total space is locally trivial, and a locally trivial principal (G, Π_G)-bundle over a paracompact base space (such as a CW complex) is numerable. Therefore, modulo weak point-set topological conditions, the fixed point condition in Theorem 3.8 is the essential criterion for a universal bundle.

Therefore Lemma 3.7 has the following consequence. Its condition on Π serves only to ensure that p is a numerable principal (G, Π_G)-bundle.

Theorem 3.10 *If G is discrete and Π is either discrete or a compact Lie group, the map*

$$p \colon E(G, \Pi_G) \longrightarrow B(G, \Pi_G)$$

obtained by passage to orbits over Π is a universal principal (G, Π_G)-bundle.

The classifying space $B(G, \Pi_G) = |N\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)|/\Pi$ is obtained by first applying the classifying space functor and then passing to orbits. On the other hand, the space $B\mathscr{C}at(\mathcal{E}G, \Pi) = |N\mathscr{C}at(\mathcal{E}G, \Pi)|$ is obtained by first passing to orbits on the categorical

level and then applying the classifying space functor. The category $\mathscr{C}at(\mathcal{E}G, \Pi)$ is thoroughly understood, as explained in §2. The key virtue of our model for $B(G, \Pi_G)$ is that these two G-spaces can be identified, by Theorem 2.7.

Theorem 3.11 *The canonical map*

$$B(G, \Pi_G) = |N\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)|/\Pi \longrightarrow |N\mathscr{C}at(\mathcal{E}G, \Pi)| = B\mathscr{C}at(\mathcal{E}G, \Pi)$$

is a homeomorphism of G-spaces. Therefore, if G is discrete and Π is either discrete or a compact Lie group, the map

$$Bq \colon B\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi) \longrightarrow B\mathscr{C}at(\mathcal{E}G, \Pi)$$

is a universal principal (G, Π_G)-bundle.

Scholium 3.12 For finite groups G, this result is claimed in [18, p. 1294]. For more general groups G, [18, 3.1] states an analogous result, but with $\mathcal{E}\Pi \longrightarrow \Pi$ replaced by a functor defined in terms of the nonequivariant universal bundle $E\Pi \longrightarrow B\Pi$, resulting in a much larger construction. The replacement is needed for the proof of their analogue [18, 3.3] of our Lemma 3.7. A commutation relation of the form $N(\mathscr{C}/\Pi) = (N\mathscr{C})/\Pi$ for their larger construction is stated (five lines above [18, 3.1]), but there is no hint of a proof or of the need for one. It is not altogether clear to us that the commutation relation stated there is true, and we view the commutation relation Theorem 2.7 as the main point of the proof of Theorem 3.11. Nevertheless, [18] had the insightful right idea that led to our work.

4 Determination of fixed points

4.1 The fixed point spaces of $E(G, \Pi_G)$

We must prove Lemma 3.7, but we place no restrictions on G and Π until they are needed. Since Π acts freely on $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)$, it is clear that $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)^\Lambda$ is empty if $\Lambda \cap \Pi \neq e$. Thus assume that $\Lambda \cap \Pi = e$. By Lemma 1.3, the fixed point category $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Pi)^\Lambda$ is chaotic. It remains to prove that it is non-empty, and Lemma 1.5 implies that this is so if and only if the space $\mathrm{Map}(G, \Pi)^\Lambda$ is non-empty. Thus it suffices to show that $\mathrm{Map}(G, \Pi)$ has a Λ-fixed point, which means that there is a Λ-map $f \colon G \longrightarrow \Pi$. We prove this using the following standard generalization of a homomorphism and a variant needed later.

Definition 4.1   A function $\alpha\colon G \longrightarrow \Pi$ is a crossed homomorphism if

(4–2) $$\alpha(gh) = \alpha(g)(g \cdot \alpha(h))$$

for all $g, h \in G$. In particular,

(4–3) $$\alpha(e) = e, \ \alpha(g)^{-1} = g \cdot \alpha(g^{-1}) \ \text{ and } \ \alpha(g^{-1})^{-1} = g^{-1} \cdot \alpha(g).$$

A map $\alpha\colon G \longrightarrow \Pi$ is a crossed anti-homomorphism if

(4–4) $$\alpha(gh) = (g \cdot \alpha(h))\alpha(g).$$

Note that we should require the function α to be continuous in our general topological context. However, the continuity is sometimes automatic, as indicated in the following lemma. Remember that we understand subgroups to be closed.

Lemma 4.5   *All subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ are of the form*

$$\Lambda_\alpha = \{(\alpha(h), h)|h \in H\},$$

where H is a subgroup of G and $\alpha\colon H \longrightarrow \Pi$ is a crossed homomorphism. At least if G is discrete or Γ is compact, α is continuous.

Proof   Clearly Λ_α is a subgroup of Γ such that $\Lambda_\alpha \cap \Pi = e$. Conversely, let $\Lambda \cap \Pi = e$. Define H to be the image of the composite of the inclusion $\iota\colon \Lambda \subset \Gamma$ and the projection $\pi\colon \Gamma \longrightarrow G$. Since $\Lambda \cap \Pi = e$, the composite $\pi \circ \iota$ is injective and so restricts to a continuous isomorphism $\nu\colon \Lambda \longrightarrow H$. For $h \in H$, define $\alpha(h) = \sigma$, where σ is the unique element of Π such that $(\sigma, h) \in \Lambda$. Thus α is the composite of $\iota \circ \nu^{-1}\colon H \longrightarrow \Gamma$ and the projection $\rho\colon \Gamma \longrightarrow \Pi$. If G is discrete or if Γ and therefore Λ is compact, then ν is a homeomorphism and α is continuous. For $h, k \in H$,

$$(\alpha(h), h)(\alpha(k), k) = (\alpha(h)(h \cdot \alpha(k)), hk) \in \Lambda,$$

so $\alpha(hk) = \alpha(h)(h \cdot \alpha(k))$. Thus α is a crossed homomorphism and $\Lambda = \Lambda_\alpha$. $\square$

Proof of [Lemma 3.7]   We must obtain a Λ-map $f\colon G \to \Pi$, where $\Lambda = \Lambda_\alpha$ for a crossed homomorphism α. By the definition of the action by Λ, this means that

$$f(g) = (h \cdot f(h^{-1}g))\alpha(h)^{-1}$$

or equivalently

$$h \cdot f(h^{-1}g) = f(g)\alpha(h)$$

for all $h \in H$ and $g \in G$. We choose right coset representatives $\{g_i\}$ to write G as a disjoint union of cosets Hg_i. We then define $f\colon G \longrightarrow \Pi$ by

$$f(kg_i) = \alpha(k)^{-1}$$

for $k \in H$. By using (4–2), writing out the inverse of a product as the product of inverses, using that $h^{-1} \cdot$ and $h \cdot$ are group homomorphisms and that $\cdot$ is a group action, and finally using (4–3) and, again, that $\cdot$ is a group action, we see that

$$
\begin{aligned}
h \cdot f(h^{-1} k g_i) &= h \cdot \alpha(h^{-1} k)^{-1} \\
&= h \cdot (\alpha(h^{-1})(h^{-1} \cdot \alpha(k))^{-1} \\
&= h \cdot ((h^{-1} \cdot \alpha(k))^{-1}(\alpha(h^{-1}))^{-1}) \\
&= (h \cdot (h^{-1} \cdot \alpha(k)^{-1})(h \cdot (\alpha(h^{-1})^{-1}) \\
&= \alpha(k)^{-1}(h \cdot (h^{-1} \cdot \alpha(h))) \\
&= f(k g_i) \alpha(h).
\end{aligned}
$$

for all $h \in H$. Thus f is a Λ-map. We have assumed that G is discrete in order to ensure that f is continuous. □

Remark 4.6 If we relax the condition that G is discrete, we do not see how to prove that f is continuous, as would be needed for a more general result.

4.2 The fixed point categories of $\mathscr{C}at(\mathcal{E}G, \Pi)$

For $H \subset G$, the structure of the fixed point space $B(G, \Pi_G)^H$ is known (up to homotopy), for example by specialization of more general results in [9]. We show here how to see that structure on the category level. In fact, we identify the fixed point categories $\mathscr{C}at(\mathcal{E}G, \Pi)^H$, with no restrictions on Π and G. However, the reader may prefer to assume that G is discrete for the rest of Section 4.

Since the functor B commutes with fixed points, this gives a categorically precise interpretation of the fixed point space $B(G, \Pi_G)^H$.

We return to §2, taking $X = G$ there. The H-fixed functors and H-natural transformations in $\mathscr{C}at(\mathcal{E}G, \Pi)$ are the H-equivariant functors and natural transformations, in accord with our notational convention $\mathscr{C}at(\mathcal{E}G, \Pi)^H = H\mathscr{C}at(\mathcal{E}G, \Pi)$. Since $\mathcal{E}G$ and $\tilde{H}$ are both H-free contractible categories, they are equivalent as H-categories. Therefore

$$(4\text{–}7) \qquad \mathscr{C}at(\mathcal{E}G, \Pi)^H \simeq \mathscr{C}at(\tilde{H}, \Pi)^H = H\mathscr{C}at(\tilde{H}, \Pi).$$

This implies that we may restrict to the case $G = H$ and deduce conclusions in general. The objects and morphisms of $G\mathscr{C}at(\mathcal{E}G, \Pi)$ are the G-equivariant functors $E : \mathcal{E}G \to \Pi$ and the G-equivariant natural transformations η. In Lemma 2.3, we described a functor E in terms of the map $\alpha : G \longrightarrow \Pi$ defined by $\alpha(h) = E(h, e)$.

Lemma 4.8 *The G-action on functors $E\colon \mathcal{E}G \longrightarrow \Pi$ induces the G-action on maps $\alpha\colon G \longrightarrow \Pi$ specified by*

$$(g\alpha)(h) = (g \cdot (\alpha(g^{-1}h))(g \cdot \alpha(g^{-1})^{-1})).$$

Proof

$$(gE)(h, e) = g \cdot E(g^{-1}h, g^{-1}) = g \cdot (E(g^{-1}h, e)E(e, g^{-1})). \qquad \square$$

Lemma 4.9 *The space of objects of $G\mathcal{C}at(\mathcal{E}G, \Pi)$ can be identified with the subspace of $\mathrm{Map}(G, \Pi)$ consisting of the crossed anti-homomorphisms $\alpha\colon G \longrightarrow \Pi$.*

Proof Setting $g\alpha = \alpha$ and applying $g^{-1} \cdot (-)$ to the formula for the action of G on α, we obtain

$$g^{-1} \cdot \alpha(h) = \alpha(g^{-1}h)\alpha(g^{-1})^{-1}.$$

Replacing g^{-1} by g and multiplying on the right by $\alpha(g)$, this gives

$$\alpha(gh) = (g \cdot \alpha(h))\alpha(g)$$

for all $g, h \in G$, which says that α is a crossed anti-homomorphism. $\qquad \square$

Similarly, as in Lemma 2.4, a natural transformation $\eta\colon E_\alpha \longrightarrow E_\beta$ is determined by $\sigma = \eta(e)$. Explicitly,

$$\eta(g) = E_\beta(g, e)\eta(e)E_\alpha(g, e)^{-1} = \beta(g)\sigma\alpha(g)^{-1}$$

for $g \in G$. Now a G-fixed natural transformation η satisfies $\eta(gh) = g \cdot \eta(h)$ for $g, h \in G$ and thus $\eta(g) = \eta(ge) = g \cdot \eta(e) = g \cdot \sigma$. Therefore the naturality square for G-fixed natural transformations translates into

$$g \cdot \sigma = \beta(g)\sigma\alpha(g)^{-1}$$

or equivalently

(4–10) $$\beta(g)\sigma = (g \cdot \sigma)\alpha(g).$$

We use the following definitions and lemma to put things together.

Definition 4.11 Let G act on Π. Define the crossed functor category $\mathcal{C}at_\times(G, \Pi)$ to be the category whose objects are the crossed homomorphisms $G \longrightarrow \Pi$ and whose morphisms $\sigma\colon \alpha \longrightarrow \beta$ are the elements $\sigma \in \Pi$ such that $\beta(g)(g \cdot \sigma) = \sigma\alpha(g)$; they are are called isomorphisms of crossed homomorphisms. The composite $\tau \circ \sigma$, $\tau\colon \beta \longrightarrow \gamma$, is given by $\tau\sigma$. Define the centralizer Π^α of a crossed homomorphism $\alpha\colon G \longrightarrow \Pi$ to be the subgroup

$$\Pi^\alpha = \{\sigma \in \Pi \,|\, \alpha(g)(g \cdot \sigma) = \sigma\alpha(g) \text{ for all } g \in G\}$$

of Π. It is the automorphism group $\mathrm{Aut}(\alpha)$ of the object α in $\mathcal{C}at_\times(G, \Pi)$.

Definition 4.12 Define the anti-crossed functor category $\mathscr{C}at_{\times}^{-}(G, \Pi)$ to have objects the crossed anti-homomorphisms $\alpha\colon G \longrightarrow \Pi$ and morphisms $\sigma\colon \alpha \longrightarrow \beta$ the elements $\sigma \in \Pi$ such that $\beta(g)\sigma = (g \cdot \sigma)\alpha(g)$, with $\tau \circ \sigma = \tau\sigma$. The centralizer Π^{α} of a crossed anti-homomorphism $\alpha\colon G \longrightarrow \Pi$ is

$$\Pi^{\alpha} = \{\sigma \in \Pi | \alpha(g)\sigma = (g \cdot \sigma)\alpha(g) \text{ for all } g \in G\}.$$

Again, $\Pi^{\alpha} = \mathrm{Aut}(\alpha)$ in $\mathscr{C}at_{\times}^{-}(G, \Pi)$.

If the action of G on Π is trivial, then the crossed functor category is just the functor category $\mathscr{C}at(G, \Pi)$ since homomorphisms $\alpha\colon G \to \Pi$ correspond to functors $\alpha\colon G \longrightarrow \Pi$ and elements $\sigma \in \Pi$ such that $\beta(g)\sigma = \sigma\alpha(g)$ for $g \in G$ correspond to natural transformations $\alpha \longrightarrow \beta$. In that case,

$$\Pi^{\alpha} = \{\sigma \in \Pi | \sigma^{-1}\alpha(g)\sigma = \alpha(g) \text{ for all } g \in G\}$$

is the usual centralizer of α in Π, and then the following identification is obvious.

Lemma 4.13 *The categories* $\mathscr{C}at_{\times}(G, \Pi)$ *and* $\mathscr{C}at_{\times}^{-}(G, \Pi)$ *of crossed homomorphisms and crossed anti-homomorphisms are canonically isomorphic.*

Proof For a crossed homomorphism $\alpha\colon G \longrightarrow \Pi$, define $\bar{\alpha}\colon G \longrightarrow \Pi$ by

$$\bar{\alpha}(g) = g \cdot \alpha(g^{-1}).$$

Then

$$\bar{\alpha}(gh) = (gh) \cdot \alpha(h^{-1}g^{-1}) = g \cdot h \cdot (\alpha(h^{-1})(h^{-1} \cdot \alpha(g^{-1})) = (g \cdot \bar{\alpha}(h))(\bar{\alpha}(g)),$$

so that $\bar{\alpha}$ is a crossed anti-homomorphism. If σ is a morphism $\alpha \longrightarrow \beta$ in $\mathscr{C}at_{\times}(G, \Pi)$, then $\beta(g)(g \cdot \sigma) = \sigma\alpha(g)$. It follows that

$$\bar{\beta}(g)\sigma = (g \cdot \beta(g^{-1}))\sigma = g \cdot (\beta(g^{-1})(g^{-1} \cdot \sigma)) = g \cdot (\sigma\alpha(g^{-1})) = (g \cdot \sigma)\bar{\alpha}(g),$$

so that σ is also a morphism $\bar{\alpha} \longrightarrow \bar{\beta}$ in $\mathscr{C}at_{\times}^{-}(G, \Pi)$. The construction of the inverse isomorphism is similar. $\square$

Returning to the G-fixed category of interest, we summarize our discussion in terms of these definitions and results.

Theorem 4.14 *The fixed point category* $G\mathscr{C}at(\mathcal{E}G, \Pi) = \mathscr{C}at(\mathcal{E}G, \Pi)^{G}$ *is isomorphic to the anti-crossed functor category* $\mathscr{C}at_{\times}^{-}(G, \Pi)$. *Therefore it is also isomorphic to the crossed functor category* $\mathscr{C}at_{\times}(G, \Pi)$.

Corollary 4.15 *For $H \subset G$, the fixed point category $\mathscr{C}at(\mathcal{E}G, \Pi)^H$ is equivalent to the anti-crossed functor category $\mathscr{C}at_\times^-(H, \Pi)$. Therefore it is also equivalent to the crossed functor category $\mathscr{C}at_\times(H, \Pi)$.*

Remark 4.16 The appearance of anti-homomorphisms in this context is not new; see e.g. [25]. As we have seen, it is also innocuous. We have chosen not to introduce opposite groups, but the anti-isomorphism $(-)^{-1} \colon \Pi \longrightarrow \Pi^{op}$ is relevant.

4.3 Fixed point categories, $H^1(G; \Pi_G)$, and Hilbert's Theorem 90

Since $G\mathscr{C}at(\mathcal{E}G, \Pi)$ is a groupoid, it is equivalent to the coproduct of its subcategories $Aut(\alpha)$, where we choose one α from each isomorphism class of objects. The following definition is standard when Π and G are discrete but makes sense in general.

Definition 4.17 The first non-abelian cohomology group $H^1(G; \Pi_G)$ is the pointed set of isomorphism classes of (continuous) crossed homomorphisms $G \longrightarrow \Pi$. We write $[\alpha]$ for the isomorphism class of α. The basepoint of $H^1(G; \Pi_G)$ is $[\varepsilon]$, where ε is the trivial crossed homomorphism given by $\varepsilon(g) = e$ for $g \in G$.

With this language, (4–7) and Corollary 4.15 can be restated as follows.

Theorem 4.18 *For $H \subset G$, $\mathscr{C}at(\mathcal{E}G, \Pi)^H$ is equivalent to the coproduct of the categories $Aut(\alpha)$, where the coproduct runs over $[\alpha] \in H^1(H; \Pi_H)$.*

Here $Aut(\alpha)$ implicitly refers to the ambient group $\Pi \rtimes H$, not $\Gamma = \Pi \rtimes G$. By (4–7) or, more concretely, Lemma 4.22 below, we obtain the same group $Aut(\alpha)$ for α considered as an object of $\mathscr{C}at(\tilde{K}, \Pi)^H$ for any $H \subset K \subset G$.

For any G-category $\mathscr{A}$, we have a natural map of G-categories

$$\iota \colon \mathscr{A} \longrightarrow \mathscr{C}at(\mathcal{E}G, \mathscr{A}).$$

It is induced by the unique G-functor $\mathcal{E}G \longrightarrow *$, where $*$ is the trivial G-category with one object and its identity morphism. The G-fixed point functor ι^G played a central role in Thomason's paper [24]. When $\mathscr{A} = \Pi$ for a G-group Π, ι sends the unique object of Π to the basepoint $[\varepsilon] \in H^1(G; \Pi)$.

We shall describe the groups $Aut(\alpha)$ in familiar group theoretic terms in the next section. As a special case, $Aut(\varepsilon) = \Pi^G$ and ι^G restricts to the identity functor from Π^G to $Aut(\varepsilon)$. This implies the following result.

Proposition 4.19 *The functor $\iota^G \colon \Pi^G \longrightarrow \mathscr{C}\!at(\mathcal{E}G, \Pi)^G$ is an equivalence of categories if and only if $H^1(G; \Pi_G) = [\varepsilon]$.*

Example 4.20 Let E be a Galois extension of a field F with Galois group G. Then G acts on E and $E^G = F$. Let G act entrywise on $GL(n, E)$. Then Serre's general version of Hilbert's Theorem 90 [22, Ch 10, Prop. 3] gives that $H^1(G; GL(n, E)_G) = [\varepsilon]$. Since $GL(n, E)^G = GL(n, F)$, we conclude that ι^G is an equivalence of categories

$$GL(n, F) \longrightarrow \mathscr{C}\!at(\mathcal{E}G, GL(n, E))^G.$$

More generally, for $H \subset G$, ι^H is an equivalence of categories

$$GL(n, E^H) \longrightarrow \mathscr{C}\!at(\mathcal{E}G, GL(n, E)^H.$$

As explained in [4] this gives precisely the information that ensures that the algebraic K-theory fixed point spectrum $\mathbb{K}_G(E)^H$ is equivalent to $\mathbb{K}(E^H)$. We shall return to consideration of G-rings such as E in §6.

We recall the easy calculation of $H^1(G; \Pi)$ in group theoretic terms. Here we must restrict G since the proof depends on Lemma 3.7.

Lemma 4.21 *At least if G is discrete, the set $H^1(G; \Pi)$ is in bijective correspondence with the set of Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = G$.*

Proof By Lemma 3.7, the subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ are of the form

$$\Lambda_\alpha = \{(\alpha(h), h) | h \in H\}$$

for a crossed homomorphism $\alpha \colon H \to \Pi$. If $\sigma \in \Pi$, then $\sigma \Lambda_\alpha \sigma^{-1} \cap \Pi = e$ and therefore $\sigma \Lambda_\alpha \sigma^{-1} = \Lambda_\beta$ for some crossed homomorphism β. The equality forces β and α to be defined on the same subgroup H and to satisfy $\beta(g)(g \cdot \sigma) = \sigma \alpha(g)$. We are concerned only with the case $H = G$, and then this says that σ is a morphism and thus an isomorphism $\alpha \longrightarrow \beta$ in $\mathscr{C}\!at_\times(G, \Pi)$. $\qquad\square$

4.4 The fixed point spaces of $B(G, \Pi_G)$

We here identify the automorphism groups $\mathrm{Aut}(\alpha)$ group theoretically and so complete the identification of $\mathscr{C}\!at(\mathcal{E}G, \Pi)^G$.

Lemma 4.22 *Let $\alpha\colon H \longrightarrow \Pi$ be a crossed homomorphism and Π be a G-group, where $H \subset G$. Then the crossed centralizer Π^α is the intersection $\Pi \cap N_\Gamma \Lambda_\alpha$. Therefore this intersection is the same for all $\Gamma_K = \Pi \rtimes K$, $H \subset K \subset G$.*

Proof Let $(\pi, g) \in \Pi \rtimes G$ and $h \in H$. Calculating in $\Gamma = \Pi \rtimes G$, we have

$$
\begin{aligned}
(\sigma, g)^{-1}(\alpha(h), h)(\sigma, g) &= (g^{-1} \cdot \sigma^{-1}, g^{-1})(\alpha(h), h)(\sigma, g) \\
&= ((g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h)), g^{-1}h)(\sigma, g) \\
&= ((g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h))((g^{-1}h) \cdot \sigma), g^{-1}hg).
\end{aligned}
$$

Therefore (σ, g) is in $N_\Gamma \Lambda_\alpha$ if and only if g is in $N_G H$ and

$$
\alpha(g^{-1}hg) = (g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h))((g^{-1}h) \cdot \sigma)
$$

for all $h \in H$. When $g = e$, so that $\sigma = (\sigma, e)$ is a typical element of $\Pi \cap N_\Gamma \Lambda_\alpha$, this simplifies to

$$
\alpha(h) = \sigma^{-1}\alpha(h)(h \cdot \sigma). \qquad \square
$$

Passing to classifying spaces from Theorem 4.18 gives the following result.

Theorem 4.23 *For $H \subset G$,*

$$
B(G, \Pi_G)^H = B\mathscr{C}\!at(\mathcal{E}G, \Pi)^H \simeq \coprod B \operatorname{Aut}(\alpha),
$$

where the coproduct runs over $[\alpha] \in H^1(H; \Pi_H)$.

By Lemmas 4.21 and 4.22, at least when G is discrete we can restate Theorem 4.23 as follows.

Theorem 4.24 *Let $\Gamma = \Pi \rtimes G$, where G is discrete. For a subgroup H of G,*

$$
B(G, \Pi_G)^H \simeq \coprod B(\Pi \cap N_\Gamma \Lambda),
$$

where the union runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$.

Of course, we are only entitled to consider $B(G, \Pi_G)$ as a classifying space for principal Γ-bundles when Theorem 3.11 applies. The fixed point spaces $B(\Pi; \Gamma)^H$ of classifying spaces are studied more generally in [9] when Γ is given by a not necessarily split extension of compact Lie groups

$$
(4\text{–}25) \qquad\qquad 1 \longrightarrow \Pi \longrightarrow \Gamma \overset{q}{\longrightarrow} G \longrightarrow 1.
$$

For such groups Γ, [9, Theorem 10] gives an entirely different bundle theoretic proof that the conclusion of Theorem 4.24 still holds as stated, but without the restriction on G. However, when [9] was written, no particularly nice model for the homotopy type $B(\Pi; \Gamma)$ was known.

5   The comparison between $B\mathscr{C}at(\mathcal{E}G, \Pi)$ and $\mathrm{Map}(EG, B\Pi)$

A convenient model $p\colon E(\Pi; \Gamma) \longrightarrow B(\Pi; \Gamma)$ for a universal principal $(\Pi; \Gamma)$-bundle was later given in terms of mapping spaces [14]. Here we assume given an extension (4–25), with no restrictions on our topological groups.[4] Start with the classical models in §2.3 for universal principal Π, G, and Γ-bundles and let $Eq\colon E\Gamma \longrightarrow EG$ be the map induced by the quotient homomorphism $q\colon \Gamma \longrightarrow G$. Let $\mathrm{Sec}(EG, E\Gamma)$ denote the Γ-space of sections $f\colon EG \longrightarrow E\Gamma$, so that $Eq \circ f = \mathrm{id}$. The following result is part of [14, Theorem 5].

Theorem 5.1 *The quotient map $p\colon \mathrm{Sec}(EG, E\Gamma) \longrightarrow \mathrm{Sec}(EG, E\Gamma)/\Pi$ is a universal principal $(\Pi; \Gamma)$-bundle.*

Now let the extension be split, so that $\Gamma = \Pi \rtimes G$. The given action of G induces a left action of G on $E\Pi$ that, together with the free right action by Π, makes it a Γ-space. Taking EG to be a left G-space and letting Γ act through q on EG, we have the product Γ-space $E\Pi \times EG$. It is free as a Γ-space because $E\Pi$ is free as a Π-space and EG is free as a G-space. Since it is contractible, we may as well take $E\Gamma = E\Pi \times EG$. Since the second coordinate of a section $f\colon EG \longrightarrow E\Pi \times EG$ must be the identity, we then have

$$\mathrm{Sec}(EG, E\Gamma) = \mathrm{Map}(EG, E\Pi).$$

Its Γ-action is defined just as was the Γ-action on $\mathscr{C}at(\mathcal{E}G, \Pi)$ in Lemma 3.3. This gives the following specialization of Theorem 5.1, which is the space level forerunner of the categorical Theorem 3.10.

Theorem 5.2 *The quotient map $p\colon \mathrm{Map}(EG, E\Pi) \longrightarrow \mathrm{Map}(EG, E\Pi)/\Pi$ is a universal principal (G, Π_G)-bundle.*

We also have the mapping space $\mathrm{Map}(EG, B\Pi)$. The canonical map $E\Pi \longrightarrow B\Pi$ induces a map $q\colon \mathrm{Map}(EG, E\Pi) \longrightarrow \mathrm{Map}(EG, B\Pi)$. Then there is an induced map ξ that makes the following diagram commute.

$$
\begin{array}{ccc}
 & & \mathrm{Map}(EG, E\Pi) \\
 & {}^{p}\swarrow & \downarrow{}^{q} \\
\mathrm{Map}(EG, E\Pi)/\Pi & \xrightarrow{\ \xi\ } & \mathrm{Map}(EG, B\Pi).
\end{array}
$$

---

[4]We do assume their identity elements are nondegenerate basepoints.

The analogy with the triangle in Theorem 2.7 should be evident. As observed in [14, Theorem 5], elementary covering space theory gives the following space level forerunner of the categorical Theorem 3.11.

Theorem 5.3 *If Π is discrete, then $\xi\colon \mathrm{Map}(EG, E\Pi)/\Pi \longrightarrow \mathrm{Map}(EG, B\Pi)$ is a homeomorphism and therefore $q\colon \mathrm{Map}(EG, E\Pi) \longrightarrow \mathrm{Map}(EG, B\Pi)$ is a universal principal (G, Π_G)-bundle.*

Note that G but not Π is required to be discrete in Theorem 3.11, whereas Π but not G is required to be discrete in Theorem 5.3.[5] There is an obvious comparison map relating the categorical and space level constructions. For any G-categories $\mathscr{A}$ and $\mathscr{B}$, we have the evaluation G-functor

$$\varepsilon\colon \mathscr{C}at(\mathscr{A}, \mathscr{B}) \times \mathscr{A} \longrightarrow \mathscr{B}.$$

Applying the classifying space functor and taking adjoints, this gives a G-map

$$(5\text{--}4) \qquad\qquad \xi\colon B\mathscr{C}at(\mathscr{A}, \mathscr{B}) \longrightarrow \mathrm{Map}(B\mathscr{A}, B\mathscr{B}).$$

When $\mathscr{A}$ and $\mathscr{B}$ are both discrete (in the topological sense), there is a simple analysis of this map in terms of the simplicial mapping space $\mathrm{Map}^\Delta(N\mathscr{A}, N\mathscr{B})$. The following two lemmas are well-known nonequivariantly.

Lemma 5.5 *For discrete categories $\mathscr{A}$ and $\mathscr{B}$, there is a natural isomorphism*

$$\mu\colon N\mathscr{C}at(\mathscr{A}, \mathscr{B}) \cong \mathrm{Map}^\Delta(N\mathscr{A}, N\mathscr{B}),$$

and this is an isomorphism of simplicial G-sets if $\mathscr{A}$ and $\mathscr{B}$ are G-categories.

Proof Let Δ_n be the poset $\{0, 1, \cdots, n\}$, viewed as a category. The n-simplices of $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ are the functors $\Delta_n \longrightarrow \mathscr{C}at(\mathscr{A}, \mathscr{B})$. By adjunction, they are the functors $\mathscr{A} \times \Delta_n \longrightarrow \mathscr{B}$. Since N is full and faithful, these functors are the maps of simplicial sets

$$N\mathscr{A} \times N\Delta_n \cong N(\mathscr{A} \times \Delta_n) \longrightarrow N\mathscr{B}.$$

By definition, these maps are the n-simplices of $\mathrm{Map}^\Delta(N\mathscr{A}, N\mathscr{B})$. These identifications give the claimed isomorphism of simplicial sets. The compatibility with the actions of G when $\mathscr{A}$ and $\mathscr{B}$ are G-categories is clear.   □

---

[5]When G is a compact Lie group acting trivially on a compact Abelian Lie group Π, results of [10] imply that the map ξ is a weak G-equivalence; in [20], Charles Rezk proves that this remains true when Π is a finite extension of a torus (a compact Lie homotopy 1-type).

Lemma 5.6 *For simplicial sets K and L, there is a natural map*

$$\nu \colon |\operatorname{Map}^{\Delta}(K, L)| \longrightarrow \operatorname{Map}(|K|, |L|).$$

If K and L are simplicial G-sets, ν is a map of G-spaces, and it is a weak equivalence of G-spaces when L is a Kan complex.

Proof The evaluation map $\operatorname{Map}^{\Delta}(K, L) \times K \longrightarrow L$ induces a map

$$|\operatorname{Map}^{\Delta}(K, L)| \times |K| \cong |\operatorname{Map}^{\Delta}(K, L) \times K| \longrightarrow |L|$$

whose adjoint is ν. When L is a Kan complex, so is $\operatorname{Map}^{\Delta}(K, L)$ (e.g. [11, 6.9]), and the natural maps $L \longrightarrow S|L|$ and $\operatorname{Map}^{\Delta}(K, L) \longrightarrow S|\operatorname{Map}^{\Delta}(K, L)|$ are homotopy equivalences, where S is the total singular complex functor. A diagram chase shows that ξ induces a bijection on homotopy classes of maps
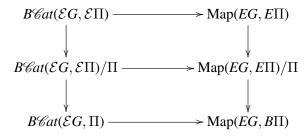
$$\xi_* \colon [|J|, |\operatorname{Map}^{\Delta}(K, L)|] \longrightarrow [|J|, \operatorname{Map}(|K|, |L|)]$$

for any simplicial set J. Letting G act trivially on J, all functors in sight commute with passage to H-fixed points, and the equivariant conclusions follow. $\square$

Now the following result is immediate from the definitions and lemmas above.

Proposition 5.7 *For discrete G-categories $\mathscr{A}$ and $\mathscr{B}$, the map ξ of (5–4) is the composite $\nu \circ \mu$, and it is a weak G-equivalence if $\mathscr{B}$ is a groupoid.*

Returning to the topological setting, take $\mathscr{A} = \mathcal{E}G$ and write $EG = |N\mathcal{E}G|$, as we may. Recalling that $E\Pi \longrightarrow B\Pi$ is obtained by applying B to the functor $\mathcal{E}\Pi \longrightarrow \Pi$, we obtain the following commutative diagram.

$$
\begin{array}{ccc}
B\mathscr{C}\!at(\mathcal{E}G, \mathcal{E}\Pi) & \longrightarrow & \operatorname{Map}(EG, E\Pi) \\
\downarrow & & \downarrow \\
B\mathscr{C}\!at(\mathcal{E}G, \mathcal{E}\Pi)/\Pi & \longrightarrow & \operatorname{Map}(EG, E\Pi)/\Pi \\
\downarrow & & \downarrow \\
B\mathscr{C}\!at(\mathcal{E}G, \Pi) & \longrightarrow & \operatorname{Map}(EG, B\Pi)
\end{array}
$$

Theorems 3.10 and 5.2 say that that the top two vertical arrows are often universal principal $(\Pi; \Gamma)$-bundles, in which case the top two horizontal arrows are equivalences. Theorems 3.11 and 5.3 say that the lower two vertical arrows and therefore also the bottom horizontal arrow are also often equivalences. When both Π and G are discrete, the equivalences are immediate from Proposition 5.7. More elaborate arguments might prove all of these results in greater topological generality.

6   Other categorical models for classifying spaces $B(G, \Pi_G)$

For particular G-groups Π, there are alternative categorical models for universal principal (G, Π_G)-bundles that are important in our applications in [4, 16]. They lead to equivalent, but more intuitive, constructions of categorical models for a number of interesting G-spectra, in particular suspension G-spectra and the equivariant K-theory spectra of rings with actions by G.

Perhaps surprisingly, the symmetric groups Σ_n with trivial G-action are of particular importance in equivariant infinite loop space theory. For a ring R with an action of a group G via ring maps, the general linear groups $GL(n, R)$ with G-action on all matrix entries are of particular importance. We give alternative models for universal principal bundles applicable to these cases. We focus on the total spaces here and explain additional structure on the resulting classifying spaces in [4]. We assume that G is finite, although some of the definitions make sense and are interesting more generally.

6.1   A model $\tilde{\mathscr{E}}_G(n)$ for $E(G, \Sigma_n)$

Definition 6.1 Let U be a countable ambient G-set that contains countably many copies of each orbit G/H. The action of G on U fixes bijections $g \colon A \longrightarrow gA$ for all finite subsets A of U, denoted $a \mapsto g \cdot a$.

Let $\mathbf{n} = \{1, \cdots, n\}$ and view elements $\sigma \in \Sigma_n$ as functions $\mathbf{n} \longrightarrow \mathbf{n}$, so that $\sigma(i) = \sigma \cdot i$ gives a left action of Σ_n on $\mathbf{n}$.

Definition 6.2 For $n \geq 0$, let $\tilde{\mathscr{E}}_G(n)$ denote the chaotic $(\Sigma_n \times G)$-category whose set $\mathscr{O}b$ of objects is the set of pairs (A, α), where A is an n-element subset of U and $\alpha \colon \mathbf{n} \longrightarrow A$ is a bijection. Let G act on $\mathscr{O}b$ on the left by postcomposition and let Σ_n act on the right by precomposition. Thus $g(A, \alpha) = (gA, g \circ \alpha)$ for $g \in G$, and $(A, \alpha)\sigma = (A, \alpha \circ \sigma)$ for $\sigma \in \Sigma$; of course

$$(g \circ \alpha) \circ \sigma = g \circ \alpha \circ \sigma = g \circ (\alpha \circ \sigma).$$

The action of $\Sigma_n \times G$ is given by $(\sigma, g)(A, \alpha) = (gA, g \circ \alpha \circ \sigma^{-1})$. Since $\tilde{\mathscr{E}}_G(n)$ is chaotic, this fixes the actions on the morphism set, which the map (S, T) identifies with $\mathscr{O}b \times \mathscr{O}b$ with $\Sigma_n \times G$ acting diagonally.

Proposition 6.3 *For each n, the classifying space $|N\tilde{\mathscr{E}}_G(n)|$ is a universal principal (G, Σ_n)-bundle.*

Proof For each A, choose a base bijection $\eta_A\colon \mathbf{n} \longrightarrow A$. The function sending σ to $(A, \eta_A \circ \sigma)$ is an isomorphism of right Σ_n-sets from Σ_n to the set of objects (A, α); its inverse sends (A, α) to $\eta_A^{-1} \circ \alpha$. Thus Σ_n acts freely on $\tilde{\mathscr{E}}_G(n)$. Since $\tilde{\mathscr{E}}_G(n)$ is chaotic, it suffices to show that the set of objects of $\tilde{\mathscr{E}}_G(n)^\Lambda$ is non-empty if $\Lambda \cap \Sigma_n = \{e\}$. As usual, $\Lambda = \{(\rho(h), h)|h \in H\}$, where H is a subgroup of G and $\rho\colon H \longrightarrow \Sigma_n$ is a homomorphism.

Let H act through ρ on $\mathbf{n}$, so that $h \cdot i = \rho(h)(i)$. Since U contains a copy of every finite G-set, there is a bijection of G-sets $\beta\colon G \times_H \mathbf{n} \longrightarrow B \subset U$. Its restriction to $\mathbf{n}$ gives a bijection of H-sets $\alpha\colon \mathbf{n} \longrightarrow A \subset B$. We claim that this (A, α) is a Λ-fixed object. Obviously $hA = A$ for $h \in H$. By Definition 6.2, we have $(\rho(h), h)(A, \alpha) = (A, h \circ \alpha \circ \rho(h)^{-1})$, where

$$
\begin{aligned}
(h \circ \alpha \circ \rho(h)^{-1})(i) &= h \cdot \alpha(\rho(h)^{-1}(i)) \\
&= h \cdot h^{-1} \cdot \alpha(i) = \alpha(i). \qquad \square
\end{aligned}
$$

Definition 6.4 Define $\mathscr{E}_G(n)$ to be the orbit G-category $\tilde{\mathscr{E}}_G(n)/\Sigma_n$.

By Proposition 6.3 and §2.3, $B\mathscr{E}_G(n)$ is a classifying space $B(G, \Sigma_n)$. Up to isomorphism, the G-category $\mathscr{E}_G(n)$ admits the following more explicit description.

Lemma 6.5 *The objects of $\mathscr{E}_G(n)$ are the n-pointed subsets A of U. The morphisms are the bijections $\alpha\colon A \longrightarrow B$, with the evident composition and identities. The group G acts by translation on objects and by conjugation on morphisms. That is, g sends A to gA and α to $g\alpha$, where $g\alpha = g \circ \alpha \circ g^{-1}$, so that $(g\alpha)(g \cdot a) = g \cdot \alpha(a)$.*

Proof The objects (A, α) are all in the same orbit, denoted A, and the bijections η_A chosen in the proof of Proposition 6.3 give orbit representatives for the objects of $\mathscr{E}_G(n)$. In $\tilde{\mathscr{E}}_G(n)$, we have a unique morphism $\iota_\beta\colon (A, \eta_A) \longrightarrow (B, \beta)$ for each bijection $\beta\colon \mathbf{n} \longrightarrow B$, and these morphisms give orbit representatives for the set of morphisms $A \longrightarrow B$ in $\mathscr{E}_G(n)$. Letting the orbit of ι_β correspond to the bijection $\alpha = \beta \circ \eta_A^{-1}\colon A \longrightarrow B$ and noting that $\alpha = \eta_B \circ \sigma \circ \eta_A^{-1}$ for a unique $\sigma \in \Sigma_n$, we obtain the claimed description of $\mathscr{E}_G(n)$. Since η_A specifies an ordering on A, η_{gA} is fixed as $g \circ \eta_A$. Then if $\alpha = \beta \circ \eta_A^{-1}$,

$$
g \circ \alpha \circ g^{-1} = g \circ (\beta \circ \eta_A^{-1}) \circ (\eta_A \circ \eta_{gA}^{-1}) = g \circ \beta \circ \eta_{gA}^{-1}\colon gA \longrightarrow gB. \qquad \square
$$

6.2 G-rings, G-ring modules, and crossed homomorphisms

By a G-ring we understand a ring R with a left action of G on R through ring automorphisms. We do not assume that R is commutative, although that is the case of greatest interest to us. Following the literature, we write $g(r) = r^g$ for the automorphism $g\colon R \longrightarrow R$ determined by $g \in G$. Then $r^{gh} = g(h(r)) = (r^h)^g$.

When R is a subquotient of $\mathbb{Q}$, the only automorphism of R is the identity and the action of G must be trivial, but non-trivial examples abound. One important example is the action of the Galois group on a Galois extension E of a field F.

In the next section we will give an analogue of $\tilde{\mathscr{E}}_G(n)$ but with $\Pi = \Sigma_n$ replaced by $\Pi = GL(n, R)$ with the entrywise action of G. We will need a tiny bit of what appears to us to be a relatively undeveloped part of representation theory.

For a G-ring R, there are standard notions of a "crossed product" ring, a "group-graded ring", and, as a special case of both, a "skew group ring", variously denoted $R \rtimes G$ or $R * G$. We shall use the notation $R_G[G]$ for the last of these notions. If the action of G on R is given by the homomorphism $\theta\colon G \longrightarrow \mathrm{Aut}(R)$, a more precise notation would be $R_\theta[G]$. Observe that R is a k-algebra, where k denotes the intersection of the center of R with R^G.

Definition 6.6 As an R-module, $R_G[G]$ is the same as the group ring $R[G]$, which is the case when G acts trivially on R. We define the product on $R_G[G]$ by k-linear (not R-linear) extension of the relation

$$(rg)(sh) = rs^g\, gh$$

for $r, s \in R$ and $g, h \in G$. Thus R and $k[G]$ are subrings of $R_G[G]$ and

$$g\, r = r^g\, g.$$

Definition 6.7 We call (left) $R_G[G]$-modules "G-ring modules" or "skew G-modules". Such an M is a left R-module and a left $k[G]$-module such that $g(rm) = r^g(gm)$ for $m \in M$. If M is R-free, we call M a skew representation of G over R.

Although special cases have appeared and there is a substantial literature on crossed products, group-graded rings, and skew group rings (for example [3, 17, 19]), we have not found a systematic study of these representations in the literature. Kawakubo's paper [7] gives a convenient starting point. The following relationship with crossed homomorphisms is his [7, 5.1].

Theorem 6.8 *Let R be a G-ring. Then the set of isomorphism classes of $R_G[G]$-module structures on the R-module R^n is in canonical bijective correspondence with $H^1(G; GL(n, R))$. In detail, let $\{e_i\}$ be the standard basis for R^n. Then the formula*

$$ge_i = \rho(g)(e_i)$$

establishes a bijection between $R_G[G]$-module structures on R^n and crossed homomorphisms $\rho\colon G \longrightarrow GL(n, R)$. Moreover, two $R_G[G]$-modules with underlying R-module R^n are isomorphic if and only if their corresponding crossed homomorphisms are isomorphic.

Proof Given an $R_G[G]$-module structure on R^n, define the matrix $\rho(g)$ in $GL(n, R)$ by letting its ith column be $(s_{i,j})$, where

$$ge_i = \sum_j s_{i,j} e_j.$$

Conversely, given ρ, write $\rho(g) = (s_{i,j})$ and define ge_i by the same formula. From either starting point, we have $ge_i = \rho(g)(e_i)$. For a second element $h \in G$, write $\rho(h) = (t_{i,j})$, where $\rho(h)$ is either determined by an $R_G[G]$-module structure or is given by a crossed homomorphism ρ. Since $g r = r^g g$ in $R_G[G]$ and $g(r_{i,j}) = (r^g_{i,j})$ in $GL(n, R)$, the relation $(gh)e_i = g(he_i)$ required of an $R_G[G]$-module is the same as the relation $\rho(g)\rho(h)(e_i) = \rho(g)(g\rho(h))(e_i)$ required of a crossed homomorphism. Indeed, $(gh)e_i = \rho(gh)(e_i)$ and

$$
\begin{aligned}
g(he_i) &= g\rho(h)(e_i) = \sum_j g(t_{i,j} e_j) \\
&= \sum_j t^g_{i,j} ge_j = \sum_j \sum_k t^g_{i,j} s_{j,k} e_k \\
&= \rho(g)(\sum_j t^g_{i,j} e_j) = \rho(g)(g\rho(h)(e_i)).
\end{aligned}
$$

The remaining compatibilities, in particular for the transitivity relation required of a module, are equally straightforward verifications, as is the verification of the statement about isomorphisms. □

The following easy observation specifies the permutation skew representations. For a set A, let $R[A]$ denote the free R-module on the basis A.

Proposition 6.9 *Let A be a G-set and define*

$$g(\sum_a r_a a) = \sum_a r^g_a ga$$

for $g \in G$, $r_a \in R$, and $a \in A$. Then $R[A]$ is an $R_G[G]$-module.

In view of Theorem 6.8, this has the following immediate consequence.

Corollary 6.10 *For a G-ring R, any n-pointed G-set A canonically gives rise to a crossed homomorphism $\rho_A\colon G \longrightarrow GL(n,R)$.*

We shall need to embed skew representations in permutation skew representations to apply these notions in equivariant bundle (or covering space) theory. Of course, in classical representation theory over $\mathbb{C}$, every representation embeds in a permutation representation. We need an analogue for skew representations.

Definition 6.11 A G-ring R is amenable if there is a monomorphism of $R_G[G]$-modules that embeds any finite dimensional skew representation of G over R into a finite dimensional permutation skew representation.

Example 6.12 Let G act trivially on $\mathbf{n} = \{1,\cdots,n\}$. The trivial permutation skew representation $R[\mathbf{n}]$ is the $R_G[G]$-module corresponding to the trivial crossed homomorphism $\varepsilon\colon G \longrightarrow GL(n,R)$. Thus, when $H^1(G;GL(n,R)) = [\varepsilon]$ for all n, every skew representation of G over R is isomorphic to a permutation skew representation and R is amenable. This holds, for example, when G is the Galois group of a Galois extension $R = K$ over a field k.

More generally, we have the following analogue of the situation in classical representation theory, which shows that amenability is not an unduly restrictive condition. It is proven in Passman [19, 4.1 in Chapter 1]. Even in this generality, he ascribes it to Maschke.

Lemma 6.13 *Let $N \subset M$ be $R_G[G]$-modules with no $|G|$-torsion. If $M = N \oplus V$ as an R-module, then there is an $R_G[G]$-submodule $P \subset M$ such that $|G|M \subset N \oplus P$.*

An irreducible skew representation is one that has no non-trivial proper skew subrepresentations.

Theorem 6.14 *Suppose that R is semisimple and $|G|^{-1} \in R$. Then every $R_G[G]$-module is completely reducible and R is amenable.*

Proof By the lemma, if $N \subset M$, then $M = N \oplus P$. That is, the complete reducibility of R-modules implies the complete reducibility of $R_G[G]$-modules. If N is an irreducible $R_G[G]$-module, then any choice of an element $n \neq 0$ determines a map of $R_G[G]$-modules $f\colon R_G[G] \longrightarrow N$ such that $f(1) = n$. The image of f is a submodule of N, and

it is all of N since N is irreducible. By complete reducibility, $\text{Ker}(f)$ has a complement in $R_G[G]$, and that complement must be isomorphic to N. Thus N is a direct summand of the permutation skew representation $R_G[G]$. Therefore, by complete reducibility, all skew representations are direct summands of permutation skew representations. $\quad\square$

6.3 A model $\widetilde{\mathcal{GL}}_G(n, R)$ for $E(G, GL(n, R)_G)$

Again let R be a G-ring, and assume that R is amenable. We have the entrywise left action of G on $GL(n, R)$, and we have the right action of $GL(n, R)$ on $GL(n, R)$ given by matrix multiplication.

Lemma 6.15 *The left action of G and the right action of $GL(n, R)$ on $GL(n, R)$ specify an action of $GL(n, R) \rtimes G$ on $GL(n, R)$ via $(\tau, g)(x) = (gx)\tau^{-1}$ for $g \in G$, $x \in GL(n, R)$, and $\tau \in GL(n, R)$.*

Proof The required relation $g \cdot (x\tau) = (g \cdot x)(g \cdot \tau)$ is immediate from the fact that $g \colon R \longrightarrow R$ is an automorphism of rings. $\quad\square$

Recall the G-set U from Definition 6.1. By Proposition 6.9, $R[U]$ is an $R_G[G]$-module with

$$(6\text{–}16) \qquad\qquad g \cdot (ru) = r^g gu \quad \text{for } g \in G, \ r \in R \text{ and } u \in U.$$

Similarly, we have the entrywise (equivalently, diagonal) left action of g on R^n, $g \cdot (re_i) = r^g e_i$, where we think of G as acting trivially on the set $\{e_i\}$. Regard elements $\tau \in GL(n, R)$ as homomorphisms $\tau \colon R^n \longrightarrow R^n$. That fixes the left action of $GL(n, R)$ on R^n given by matrix multiplication, where elements of R^n are thought of as row matrices.

Definition 6.17 We define the chaotic general linear category $\widetilde{\mathcal{GL}}_G(n, R)$. The objects of $\widetilde{\mathcal{GL}}_G(n, R)$ are the monomorphisms of left R-modules $\alpha \colon R^n \longrightarrow R[U]$. Let G act from the left on objects by $g\alpha = g \circ \alpha \circ g^{-1}$. By (6–16), we have

$$
\begin{aligned}
(g \circ \alpha \circ g^{-1})\left(\sum r_i e_i\right) &= \sum_i (g \circ \alpha)(r_i^{g^{-1}} e_i)) = \sum_i g(r_i^{g^{-1}})\alpha(e_i) \\
&= \sum_i r_i^{g^{-1}g}(g \cdot \alpha(e_i)) = \sum_i r_i(g \cdot \alpha(e_i)).
\end{aligned}
$$

In particular, $(g\alpha)(e_i) = g \cdot \alpha(e_i)$. Let $GL(n, R)$ act from the right on objects by $\alpha\tau = \alpha \circ \tau \colon R^n \longrightarrow R[U]$; this uses the left, not the right, action of $GL(n, R)$ on R^n. Since $\widetilde{\mathcal{GL}}_G(n, R)$ is chaotic, this fixes the actions on the morphism set, which the map (S, T) identifies with the product of two copies of the object set.

Proposition 6.18 *The actions of G and $GL(n, R)$ on $\widetilde{\mathcal{GL}}_G(n, R)$ determine a left action of $GL(n, R) \rtimes G$ via*

$$(\tau, g)\alpha = (g\alpha)\tau^{-1}.$$

The classifying space $|N\widetilde{\mathcal{GL}}_G(n, R)|$ is a universal principal $(G, GL(n, R)_G)$-bundle.

Proof For the first claim, we must show that $g(\alpha\tau) = (g\alpha)(g \cdot \tau) \colon R^n \longrightarrow R[U]$ for $\alpha \colon R^n \longrightarrow R[U]$, $g \in G$, and $\tau = (t_{i,j}) \in GL(n, R)$. On elements e_i,

$$
\begin{aligned}
g(\alpha\tau)(e_i) &= g \cdot (\alpha\tau)(e_i) = g \cdot (\alpha(\sum_j t_{i,j} e_j) \\
&= g \cdot \sum_j (t_{i,j}\alpha(e_j)) = \sum_j t_{i,j}^g (g \cdot \alpha(e_j)) \\
&= (g\alpha)(\sum_j t_{i,j}^g e_j) = (g\alpha)(g \cdot \tau)(e_i).
\end{aligned}
$$

For each free R-module $M \subset R[U]$, choose an R-linear isomorphism $\eta_M \colon R^n \longrightarrow M$. Sending $\alpha \colon R^n \longrightarrow M$ to $\eta_M^{-1} \circ \alpha$ specifies an isomorphism of right $GL(n, R)$-sets from the set of objects α with image M to $GL(n, R)$; the inverse sends $\tau \in GL(n, R)$ to $\eta_M \circ \tau$. Therefore $GL(n, R)$ acts freely on $\widetilde{\mathcal{GL}}_G(n, R)$. Since $\widetilde{\mathcal{GL}}_G(n, R)$ is chaotic, it only remains to show that the set of objects of $\widetilde{\mathcal{GL}}_G(n, R)^\Lambda$ is non-empty if $\Lambda \cap GL(n, R) = \{e\}$. By [Lemma 4.5]{.color}, $\Lambda = \{(\rho(h), h) | h \in H\}$, where H is a subgroup of G and $\rho \colon H \longrightarrow GL(n, R)$ is a crossed homomorphism.

By [Theorem 6.8]{.color}, we may use ρ to endow R^n with a structure of left $R_H[H]$-module. By the assumed amenability of R, there is a monomorphism of left $R_H[H]$-modules $R^n \longrightarrow R[A]$ for some finite H-set A. We can embed A in the finite G-set $B = G \times_H A$ and then B is isomorphic to a sub G-set of U. This fixes a monomorphism $\alpha \colon R^n \longrightarrow R[U]$ of left $R_H[H]$-modules. Writing $\rho(h) = (s_{i,j})$ and $\rho(h)^{-1} = (t_{i,j})$, we have

$$h\alpha(e_j) = \alpha(\rho(h)(e_j)) = \alpha(\sum_k s_{j,k} e_k) = \sum_k s_{j,k}\alpha(e_k)$$

and therefore, using the display in [Definition 6.17]{.color},

$$((h\alpha)\rho(h)^{-1})(e_i) = (h\alpha)(\sum_j t_{i,j} e_j) = \sum_j t_{i,j} h \cdot \alpha(e_j) = \sum_j \sum_k t_{i,j} s_{j,k}\alpha(e_k) = \alpha(e_i). \qquad \square$$

Definition 6.19 Define $\mathcal{GL}_G(n, R)$ to be the orbit G-category $\widetilde{\mathcal{GL}}_G(n, R)/GL(n, R)$.

The classifying space $|N\mathcal{GL}_G(n, R)|$ is a model for $B(G, GL(n, R)_G)$. Up to isomorphism, the G-category $\mathcal{GL}_G(n, R)$ admits the following explicit description.

Lemma 6.20 *The objects of $\mathscr{GL}_G(n, R)$ are the n-dimensional free R-submodules M of $R[U]$. The morphisms $\alpha\colon M \longrightarrow N$ are the isomorphisms of R-modules. The group G acts by translation on objects, so that $gM = \{gm \,|\, m \in M\}$, and by conjugation on morphisms, so that $(g\alpha)(gm) = \alpha(m)$ for $m \in M$ and $g \in G$.*

Proof The objects α of $\widetilde{\mathscr{GL}}_G(n, R)$ with a fixed image M are all in the same orbit. Choose $\eta_M\colon R^n \longrightarrow M$ to fix an orbit representative. In $\widetilde{\mathscr{GL}}_G(n, R)$, we have a unique morphism $\iota\colon \eta \longrightarrow \beta$ for each object $\beta\colon R^n \longrightarrow N$. We define $\alpha\colon M \longrightarrow N$ to be the composite $\beta \circ \eta_M^{-1}$. The α are isomorphisms of R-modules that give orbit representatives specifying the morphisms of $\mathscr{GL}_G(n, R)$. As in the proof of Lemma 6.5, the description of the action of G follows. $\qquad\square$

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