Equivariant orthogonal spectra and S-modules

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ABSTRACT. The last few years have seen a revolution in our understanding of the foundations of stable homotopy theory. Many symmetric monoidal model categories of spectra whose homotopy categories are equivalent to the stable homotopy category are now known, whereas no such categories were known before 1993. The most well-known examples are the category of S-modules and the category of symmetric spectra. We focus on the category of orthogonal spectra, which enjoys some of the best features of S-modules and symmetric spectra and which is particularly well-suited to equivariant generalization. We first complete the nonequivariant theory by comparing orthogonal spectra to S-modules. We then develop the equivariant theory. For a compact Lie group G, we construct a symmetric monoidal model category of orthogonal G-spectra whose homotopy category is equivalent to the classical stable homotopy category of G-spectra. We also complete the theory of S_G-modules and compare the categories of orthogonal G-spectra and S_G-modules. A key feature is the analysis of change of universe, change of group, fixed point, and orbit functors in these two highly structured categories for the study of equivariant stable homotopy theory.

Contents

Introduction

There are two general approaches to the construction of symmetric monoidal categories of spectra, one based on an encoding of operadic structure in the definition of the smash product and the other based on the categorical observation [**4**] that categories of diagrams with symmetric monoidal domain are symmetric monoidal. The first had its origins in the Lewis-May theory of coordinate free spectra [**19**] and was worked out by Elmendorf, Kriz, and the authors in the theory of "S-modules" [**6**]. The second started with Smith's introduction of symmetric spectra of simplicial sets, the details of which were worked out by Hovey, Shipley, and Smith [**16**]. The diagrammatic approach was later worked out by Schwede, Shipley, and the authors in a general topological setting that also includes orthogonal spectra, among other variants.

Philosophically, orthogonal spectra are intermediate between S-modules and symmetric spectra, enjoying some of the best features of both. They are defined in the same diagrammatic fashion as symmetric spectra, but with orthogonal groups rather than symmetric groups building in the symmetries required to define an associative and commutative smash product. They were first introduced by the second author in [**24**], although he failed to notice their internal smash product.

We prove in Chapter I that the categories of orthogonal spectra and S-modules are Quillen equivalent and that this equivalence induces Quillen equivalences between the respective categories of ring spectra, of modules over ring spectra, and of commutative ring spectra. Combined with the analogous comparison between symmetric spectra and orthogonal spectra of [**20**], this reproves and improves Schwede's comparison between symmetric spectra and S-modules [**31**]. We refer the reader to I§1 for further discussion. We reinterpret the second author's approach to infinite loop space theory in terms of symmetric and orthogonal spectra in I§7, where we recall the purposes for which orthogonal spectra were first introduced [**24**].

With this understanding of the nonequivariant foundations of stable homotopy theory in place, we develop new foundations for equivariant stable homotopy theory in the rest of this monograph. We let G be a compact Lie group throughout, and we understand subgroups of G to be closed. There is no treatment of diagram G-spectra in the literature, and we shall provide one. Just as in the nonequivariant case, a major advantage of such a treatment is the simplicity of the resulting definitional framework.

Orthogonal spectra are far more suitable than symmetric spectra for this purpose. They are defined just as simply as symmetric spectra but, unlike symmetric spectra, they share two of the essential features of the spectra of [**19**] that facilitate equivariant generalization. First, they are defined in a coordinate-free fashion. This makes it simple and natural to build in spheres associated to representations, which play a central role in the theory. Second, their weak equivalences are just the maps that induce isomorphisms of homotopy groups. This simplifies the equivariant generalization of the relevant homotopical analysis.

We define orthogonal G-spectra and show that the category of orthogonal G-spectra is a closed symmetric monoidal category in Chapter II. We prove that this category has a proper Quillen model structure whose homotopy category is equivalent to the classical homotopy category of G-spectra in Chapter III. Moreover, we show that the various categories of orthogonal ring and module G-spectra have induced model structures.

The original construction of the equivariant stable homotopy category, due to Lewis and the second author, was in terms of G-spectra, which are equivariant versions of coordinate-free spectra. These are much more highly structured and much less elementary objects than orthogonal G-spectra. The Lewis-May construction was modernized to a symmetric monoidal category of structured G-spectra, called S_G-modules, by Elmendorf, Kriz, and the authors [**6, 27**]. Those monographs did not consider model structures on S_G-spectra and S_G-modules, and we rectify that omission in Chapter IV.

In fact there are two stable model structures on the categories of G-spectra and S_G-modules, and the difference between them is fundamental to the understanding of equivariant stable homotopy theory. One has cofibrant objects defined in terms of spheres of representations and is essential to the comparison with orthogonal G-spectra. The other has cofibrant objects defined in terms of integer spheres and is essential for the equivariant versions of classical arguments in terms of CW spectra. We refer the reader to IV§1 for further discussion of this vital point.

Generalizing our nonequivariant comparison between orthogonal spectra and S-modules, we prove in Chapter IV that the categories of orthogonal G-spectra and S_G-modules are Quillen equivalent and that this equivalence induces Quillen equivalences between the respective categories of ring G-spectra, of modules over ring G-spectra, and of commutative ring G-spectra. We also generalize the model theoretic framework to deal with families and cofamilies of subgroups of G.

We discuss change of universe functors, change of group functors, orbit functors, and categorical and geometric fixed point functors on orthogonal G-spectra in Chapter V. We discuss the analogous functors on S_G-modules in Chapter VI, and we prove there that the equivalences among G-spectra, S_G-modules, and orthogonal G-spectra are compatible with all of these functors interrelating equivariant and nonequivariant stable homotopy categories. We conclude that all homotopical results proven in the original stable homotopy category of G-spectra apply verbatim to the new stable homotopy categories of S_G-modules and orthogonal G-spectra.

Implicitly, equivariant orthogonal spectra have already been applied. A global form of the definition, with orthogonal G-spectra varying functorially in G, was exploited in the proof of the completion theorem for complex cobordism of Greenlees and May [**13**]. In retrospect, orthogonal S^1-spectra are intrinsic to the construction of topological cyclic homology given by Hesselholt and Madsen [**14**], as is apparent from a glance at their definitions; we plan to give a conceptual rationale for their construction elsewhere.

CHAPTER I

Orthogonal spectra and S-modules

1. Introduction and statements of results

We assume that the reader is familiar with the notion of a Quillen equivalence of model categories (see for example [**20**, A.1]). This is the most structured kind of equivalence that ensures an adjoint equivalence of the associated homotopy categories. With Schwede and Shipley, we proved in [**20**] that the category $\Sigma\mathscr{S}$ of symmetric spectra is Quillen equivalent to the category $\mathscr{I}\mathscr{S}$ of orthogonal spectra. In [**31**], Schwede proved that $\Sigma\mathscr{S}$ is also Quillen equivalent to the category $\mathscr{M}$ of S-modules. However, these comparisons do not give a satisfactory Quillen equivalence between the categories of orthogonal spectra and S-modules since the resulting functor $\mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$ is the composite of the right adjoint $\mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ and the left adjoint $\Sigma\mathscr{S} \longrightarrow \mathscr{M}$ and therefore fails to preserve either q-cofibrations or q-fibrations.

We shall construct a Quillen equivalence between $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$ such that Schwede's left adjoint $\Sigma\mathscr{S} \longrightarrow \mathscr{M}$ is the composite of the left adjoint $\Sigma\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$ of [**20**] and our new left adjoint $\mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$. This shows that orthogonal spectra are mathematically as well as philosophically intermediate between symmetric spectra and S-modules. The force of our work is the construction of the Quillen adjunction relating $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$. That it must be a Quillen equivalence follows from the Quillen equivalences of [**20**] and [**31**]. However, it is simpler to argue the other way around, deducing Schwede's Quillen equivalence of [**31**] from the Quillen equivalence between $\Sigma\mathscr{S}$ and $\mathscr{I}\mathscr{S}$ of [**20**] and our Quillen equivalence between $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$. The point is that the weak equivalences in $\mathscr{I}\mathscr{S}$, unlike those in $\Sigma\mathscr{S}$ and like those in $\mathscr{M}$, are just the π_*-isomorphisms. For this reason, our proof that $\mathscr{I}\mathscr{S}$ is Quillen equivalent to $\mathscr{M}$ is significantly simpler than Schwede's proof that $\Sigma\mathscr{S}$ is Quillen equivalent to $\mathscr{M}$. Moreover, our construction of the adjunction gives a concrete Thom space level understanding of the relationship between orthogonal spectra and S-modules. To complete the picture, we also point out Quillen equivalences relating coordinatized prespectra, coordinate-free prespectra, and spectra to S-modules and orthogonal spectra, in §4.

To separate formalities from substance, we begin in §2 by establishing a formal framework for constructing symmetric monoidal left adjoint functors whose domain is a category of diagram spaces. In fact, this elementary category theory sheds new light on the basic constructions that are studied in all work on diagram spectra. In §3, we explain in outline how this formal theory combines with model theory to prove the following comparison results. We recall the relevant model structures and give the homotopical parts of the proofs in §§4 and 5 but we note right away that, since the sphere S-module is not cofibrant whereas the sphere orthogonal spectrum is cofibrant in the usual stable model structure, we must use the positive stable model structure on orthogonal spectra [**20**, §14] to have any hope of obtaining Quillen equivalences. We defer the basic construction that gives substance to the theory to §6.

THEOREM 1.1. *There is a strong symmetric monoidal functor* $\mathbb{N} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$ *and a lax symmetric monoidal functor* $\mathbb{N}^{\#} : \mathscr{M} \longrightarrow \mathscr{I}\mathscr{S}$ *such that* $(\mathbb{N}, \mathbb{N}^{\#})$ *is a Quillen equivalence between* $\mathscr{I}\mathscr{S}$ *and* $\mathscr{M}$. *The induced equivalence of homotopy categories preserves smash products.*

THEOREM 1.2. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of orthogonal ring spectra and of* S*-algebras.*

THEOREM 1.3. *For a cofibrant orthogonal ring spectrum* R*, the pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of* R*-modules and of* $\mathbb{N}R$*-modules.*

By [**20**, 12.1(iv)], the assumption that R is cofibrant results in no loss of generality. As in [**20**, §13], this result implies the following one.

COROLLARY 1.4. *For an* S*-algebra* R*, the categories of* R*-modules and of* $\mathbb{N}^{\#}R$*-modules are Quillen equivalent.*

THEOREM 1.5. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of commutative orthogonal ring spectra and of commutative* S*-algebras.*

THEOREM 1.6. *Let* R *be a cofibrant commutative orthogonal ring spectrum. The categories of* R*-modules,* R*-algebras, and commutative* R*-algebras are Quillen equivalent to the categories of* $\mathbb{N}R$*-modules,* $\mathbb{N}R$*-algebras, and commutative* $\mathbb{N}R$*-algebras.*

By [**20**, 12.1(iv) and 15.2(ii)], the assumption that R is cofibrant results in no loss of generality. As we shall see, arguments like those in [**20**, §§13 16] show that this result implies the following one.

COROLLARY 1.7. *Let* R *be a commutative* S*-algebra. The categories of* R*-modules,* R*-algebras, and, if* R *is cofibrant, commutative* R*-algebras are Quillen equivalent to the categories of* $\mathbb{N}^{\#}R$*-modules,* $\mathbb{N}^{\#}R$*-algebras, and commutative* $\mathbb{N}^{\#}R$*-algebras.*

These last results are the crucial comparison theorems since most of the deepest applications of structured ring and module spectra concern E_∞ ring spectra or, equivalently by [**6**], commutative S-algebras. By [**20**, 22.4], commutative orthogonal ring spectra are the same objects as commutative orthogonal FSP's. Under the name "$\mathscr{I}_*$-prefunctor", these were defined and shown to give rise to E_∞ ring spectra in [**23**]. Theorem 1.5 shows that, up to equivalence, all E_∞ ring spectra arise this way. The second author has wondered since 1973 whether or not that is the case.

The analogues of the results above with orthogonal spectra and S-modules replaced by symmetric spectra and orthogonal spectra are proven in [**20**]. This has the following immediate consequence, which reproves all of the results of [**31**].

THEOREM 1.8. *The analogues of the results above with orthogonal spectra replaced by symmetric spectra are also true.*

The functor $\mathbb{N}$ that occurs in the results above has all of the formal and homotopical properties that one might desire. However, a quite different and considerably more intuitive functor $\mathbb{M}$ from orthogonal spectra to S-modules is implicit in [**23**]. The functor $\mathbb{M}$ gives the most natural way to construct Thom spectra as commutative S-algebras, and its equivariant version was used in an essential way in the proof of the localization and completion theorem for complex cobordism given in [**13**]. We define $\mathbb{M}$ and compare it with $\mathbb{N}$ in §7.

Orthogonal spectra were first introduced in [**24**], where they were called "$\mathscr{I}_*$-prespectra" and were used as intermediaries in the passage from pairings of spaces with operad actions and pairings of permutative categories to pairings of spectra. What was missing then was Jeff Smith's crucial insight that the evident external smash products of diagram spectra can be internalized by use of left Kan extension. As we explain in §8, symmetric spectra could have been used for the same purposes for which orthogonal spectra were used in [**24**]. The theory of this paper sharpens the conclusions of [**24**] by showing how to obtain point-set level rather than homotopy category level pairings of spectra from the given input data: in retrospect, the weaker conclusions were an artifact of the passage from orthogonal spectra to Lewis-May spectra that was used there.

It is a pleasure to thank our collaborators Brooke Shipley and Stefan Schwede. Like Schwede's paper [**31**], which gives a blueprint for some of §3 here, this chapter is an outgrowth of our joint work in [**20**].

2. Right exact functors on categories of diagram spaces

To clarify our arguments, we first give the formal structure of our construction of the adjoint pair $(\mathbb{N}, \mathbb{N}^{\#})$ in a suitably general framework. We consider categories $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-shaped diagrams of based spaces for some domain category $\mathscr{D}$, and we show that, to construct left adjoint functors from $\mathscr{D}\mathscr{T}$ to suitable categories $\mathscr{C}$, we need only construct contravariant functors $\mathscr{D} \longrightarrow \mathscr{C}$. The proof is an exercise in the use of representable functors and must be standard category theory, but we do not know a convenient reference.

Let $\mathscr{T}$ be the category of based spaces, where spaces are understood to be compactly generated (= weak Hausdorff k-spaces). Let $\mathscr{D}$ be any based topological category with a small skeleton $sk\mathscr{D}$. A $\mathscr{D}$-space is a continuous based functor $\mathscr{D} \longrightarrow \mathscr{T}$. Let $\mathscr{D}\mathscr{T}$ be the category of $\mathscr{D}$-spaces. As observed in [**20**, §1], the evident levelwise constructions define limits, colimits, smash products with spaces, and function $\mathscr{D}$-spaces that give $\mathscr{D}\mathscr{T}$ a structure of complete and cocomplete, tensored and cotensored, topological category. We call such a category *topologically bicomplete*. We fix a topologically bicomplete category $\mathscr{C}$ for the rest of this section. We write $C \wedge A$ for the tensor of an object C of $\mathscr{C}$ and a based space A. All functors are assumed to be continuous.

Definition 2.1. A functor between topologically cocomplete categories is *right exact* if it commutes with colimits and tensors. For example, any functor that is a continuous left adjoint is right exact.

For a contravariant functor $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$ and a $\mathscr{D}$-space X, we have the coend

$$\mathbb{E} \otimes_{\mathscr{D}} X = \int^{d} \mathbb{E}(d) \wedge X(d) \tag{2.2}$$

in $\mathscr{C}$. Explicitly, $\mathbb{E} \otimes_{\mathscr{D}} X$ is the coequalizer in $\mathscr{C}$ of the diagram

$$\bigvee_{d,e} \mathbb{E}(e) \wedge \mathscr{D}(d,e) \wedge X(d) \underset{\mathrm{id}\wedge\varepsilon}{\overset{\varepsilon\wedge\mathrm{id}}{\rightrightarrows}} \bigvee_d \mathbb{E}(d) \wedge X(d),$$

where the wedges run over pairs of objects and objects of $sk\mathscr{D}$ and the parallel arrows are wedges of smash products of identity and evaluation maps of $\mathbb{E}$ and X.

For an object $d \in \mathscr{D}$, we have a left adjoint $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ to the functor given by evaluation at d. If d^* is defined by $d^*(e) = \mathscr{D}(d,e)$, then $F_d A = d^* \wedge A$. In particular, $F_d S^0 = d^*$.

Definition 2.3. Let $\mathbb{D} = \mathbb{D}_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{D}\mathscr{T}$ be the evident contravariant functor that sends d to d^*.

The following observation is [**20**, 1.6].

Lemma 2.4. *The evaluation maps $\mathscr{D}(d,e) \wedge X(d) \longrightarrow X(e)$ of $\mathscr{D}$-spaces X induce a natural isomorphism of $\mathscr{D}$-spaces $\mathbb{D} \otimes_{\mathscr{D}} X \longrightarrow X$.*

Together with elementary categorical observations, this has the following immediate implication. It shows that (covariant) right exact functors $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ determine and are determined by contravariant functors $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$.

Theorem 2.5. *If $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ is a right exact functor, then $(\mathbb{F} \circ \mathbb{D}) \otimes_{\mathscr{D}} X \cong \mathbb{F}X$. Conversely, if $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$ is a contravariant functor, then the functor $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ specified by $\mathbb{F}X = \mathbb{E} \otimes_{\mathscr{D}} X$ is right exact and $\mathbb{F} \circ \mathbb{D} \cong \mathbb{E}$.*

Notation 2.6. Write $\mathbb{F} \leftrightarrow \mathbb{F}^*$ for the correspondence between right exact functors $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ and contravariant functors $\mathbb{F}^* : \mathscr{D} \longrightarrow \mathscr{C}$. Thus, given $\mathbb{F}$, $\mathbb{F}^* = \mathbb{F} \circ \mathbb{D}$, and, given $\mathbb{F}^*$, $\mathbb{F} = \mathbb{F}^* \otimes_{\mathscr{D}} (-)$. In particular, on representable $\mathscr{D}$-spaces, $\mathbb{F}d^* \cong \mathbb{F}^* d$.

Corollary 2.7. *Via $\xi^* = \mathbb{F}\eta \circ \mathbb{D}$ and $\xi = \xi^* \otimes_{\mathscr{D}} (-)$, natural transformations $\xi : \mathbb{F} \longrightarrow \mathbb{G}$ between right exact functors $\mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ determine and are determined by natural transformations $\xi^* : \mathbb{F}^* \longrightarrow \mathbb{G}^*$ between the corresponding contravariant functors $\mathscr{D} \longrightarrow \mathscr{C}$.*

Proposition 2.8. *Any right exact functor $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ has the right adjoint $\mathbb{F}^{\#}$ specified by*

$$(\mathbb{F}^{\#} C)(d) = \mathscr{C}(\mathbb{F}^* d, C)$$

for $C \in \mathscr{C}$ and $d \in \mathscr{D}$. The evaluation maps

$$\mathscr{D}(d,e) \wedge \mathscr{C}(\mathbb{F}^* d, C) \longrightarrow \mathscr{C}(\mathbb{F}^* e, C)$$

of the functor $\mathbb{F}^{\#}$ are the adjoints of the composites

$$\mathbb{F}^* e \wedge \mathscr{D}(d,e) \wedge \mathscr{C}(\mathbb{F}^* d, C) \xrightarrow{\varepsilon\wedge\mathrm{id}} \mathbb{F}^* d \wedge \mathscr{C}(\mathbb{F}^* d, C) \xrightarrow{\zeta} C,$$

where ε is an evaluation map of the functor $\mathbb{F}^*$ and ζ is an evaluation map of the category $\mathscr{C}$.

Proof. We must show that

$$\mathscr{C}(\mathbb{F}X, C) \cong \mathscr{D}\mathscr{T}(X, \mathbb{F}^{\#} C). \tag{2.9}$$

The description of $\mathbb{F}X$ as a coend implies a description of $\mathscr{C}(\mathbb{F}X, C)$ as an end constructed out of the spaces $\mathscr{C}(\mathbb{F}^* d \wedge X(d), C)$. Under the adjunction isomorphisms

$$\mathscr{C}(\mathbb{F}^* d \wedge X(d), C) \cong \mathscr{T}(X(d), \mathscr{C}(\mathbb{F}^* d, C)),$$

this end transforms to the end that specifies $\mathscr{D}\mathscr{T}(X, \mathbb{F}^{\#}C)$. □

As an illustration of the definitions, we show how the prolongation and forgetful functors studied in [**20**] fit into the present framework.

EXAMPLE 2.10. A (covariant) functor $\iota : \mathscr{D} \longrightarrow \mathscr{D}'$ induces the forgetful functor $\mathbb{U} : \mathscr{D}'\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ that sends Y to $Y \circ \iota$. It also induces the contravariant functor $\mathbb{D}_{\mathscr{D}'} \circ \iota : \mathscr{D} \longrightarrow \mathscr{D}'\mathscr{T}$. Let $\mathbb{P}X = (\mathbb{D}_{\mathscr{D}'} \circ \iota) \otimes_{\mathscr{D}} X$. Then $\mathbb{P}$ is the prolongation functor left adjoint to $\mathbb{U}$.

Now let $\mathscr{D}$ be symmetric monoidal with product $\oplus$ and unit $u_{\mathscr{D}}$. By [**20**, §21], $\mathscr{D}\mathscr{T}$ is symmetric monoidal with unit $u^*_{\mathscr{D}}$. We denote the smash product of $\mathscr{D}\mathscr{T}$ by $\wedge_{\mathscr{D}}$. Actually, the construction of the smash product is another simple direct application of the present framework.

EXAMPLE 2.11. We have the external smash product $\bar{\wedge} : \mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T} \longrightarrow (\mathscr{D} \times \mathscr{D})\mathscr{T}$ specified by $(X \bar{\wedge} Y)(d, e) = X(d) \wedge Y(e)$ [**20**, 21.1]. We also have the contravariant functor $\mathbb{D}_{\mathscr{D}} \circ \oplus : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}\mathscr{T}$. The internal smash product is given by

$$X \wedge_{\mathscr{D}} Y = (\mathbb{D}_{\mathscr{D}} \circ \oplus) \otimes_{\mathscr{D}\times\mathscr{D}} (X \bar{\wedge} Y). \tag{2.12}$$

It is an exercise to rederive the universal property

$$\mathscr{D}\mathscr{T}(X \wedge_{\mathscr{D}} Y, Z) \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(X \bar{\wedge} Y, Z \circ \oplus) \tag{2.13}$$

that characterizes $\wedge_{\mathscr{D}}$ from this definition.

PROPOSITION 2.14. *Let $\mathbb{F}^* : \mathscr{D} \longrightarrow \mathscr{C}$ be a strong symmetric monoidal contravariant functor. Then $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ is a strong symmetric monoidal functor and $\mathbb{F}^{\#} : \mathscr{C} \longrightarrow \mathscr{D}\mathscr{T}$ is a lax symmetric monoidal functor.*

PROOF. We are given an isomorphism $\lambda : \mathbb{F}^*u_{\mathscr{D}} \longrightarrow u_{\mathscr{C}}$ and a natural isomorphism

$$\phi : \mathbb{F}^*d \wedge_{\mathscr{C}} \mathbb{F}^*e \longrightarrow \mathbb{F}^*(d \oplus e).$$

Since $\mathbb{F}^*u_{\mathscr{D}} \cong \mathbb{F}u^*_{\mathscr{D}}$, we may view λ as an isomorphism $\mathbb{F}u^*_{\mathscr{D}} \longrightarrow u_{\mathscr{C}}$. By (2.9) and (2.13), we have

$$\mathscr{C}(\mathbb{F}(X \wedge_{\mathscr{D}} Y), C) \cong (\mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T})(X \bar{\wedge} Y, \mathbb{F}^{\#}C \circ \oplus).$$

Commuting coends past smash products and using isomorphisms

$$(\mathbb{F}^*d \wedge X(d)) \wedge (\mathbb{F}^*e \wedge Y(e)) \cong \mathbb{F}^*(d \oplus e) \wedge X(d) \wedge Y(e)$$

induced by ϕ, we obtain the first of the following two isomorphisms. We obtain the second by using the tensor adjunction of $\mathscr{C}$ and applying the defining universal property of coends.

$$\begin{aligned} \mathscr{C}(\mathbb{F}X \wedge_{\mathscr{C}} \mathbb{F}Y, C) &\cong \mathscr{C}\Big(\int^{(d,e)} \mathbb{F}^{\#}(d \oplus e) \wedge X(d) \wedge Y(e), C\Big) \\ &\cong (\mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T})(X \bar{\wedge} Y, \mathbb{F}^{\#}C \circ \oplus). \end{aligned}$$

There results a natural isomorphism $\mathbb{F}X \wedge_{\mathscr{C}} \mathbb{F}Y \cong \mathbb{F}(X \wedge_{\mathscr{D}} Y)$, and coherence is easily checked.

The second statement follows formally from the first, but we can describe the relevant maps for $\mathbb{F}^{\#}$ concretely. The adjoint $u^*_{\mathscr{D}} \longrightarrow \mathbb{F}^{\#}u_{\mathscr{C}}$ of λ gives the unit map. Taking the smash products of maps in $\mathscr{C}$ and applying isomorphisms ϕ, we obtain maps

$$\mathscr{C}(\mathbb{F}^*(d), C) \wedge \mathscr{C}(\mathbb{F}^*(e), C') \longrightarrow \mathscr{C}(\mathbb{F}^*(d \oplus e), C \wedge_{\mathscr{C}} C')$$

that together define a map

$$\mathbb{F}^{\#}C \,\bar{\wedge}\, \mathbb{F}^{\#}C' \longrightarrow \mathbb{F}^{\#}(C \wedge_{\mathscr{C}} C') \circ \oplus.$$

Using (2.13), there results a natural map

$$\mathbb{F}^{\#}C \wedge_{\mathscr{D}} \mathbb{F}^{\#}C' \longrightarrow \mathbb{F}^{\#}(C \wedge_{\mathscr{C}} C'),$$

and coherence is again easily checked. □

3. The proofs of the comparison theorems

We refer to [**20**] for details of the category $\mathscr{I}\mathscr{S}$ of orthogonal spectra and to [**6**] for details of the category $\mathscr{M} = \mathscr{M}_S$ of S-modules. Much of our work depends only on basic formal properties. Both of these categories are closed symmetric monoidal and topologically bicomplete. They are Quillen model categories, and their model structures are compatible with their smash products. Actually, in [**20**], the category of orthogonal spectra is given two model structures with the same (stable) weak equivalences. In one of them, the sphere spectrum is cofibrant, in the other, the "positive stable model structure", it is not. In [**20**], use of the positive stable model structure was essential to obtain an induced model structure on the category of commutative orthogonal ring spectra. It is also essential here, since the sphere S-module S is not cofibrant. We will review the model structures in §5.

We begin by giving a quick summary of definitions from [**20**], recalling how orthogonal spectra fit into the framework of the previous section. Let $\mathscr{I}$ be the symmetric monoidal category of finite dimensional real inner product spaces and linear isometric isomorphisms. We call an $\mathscr{I}$-space an *orthogonal space*. The category $\mathscr{I}\mathscr{T}$ of orthogonal spaces is closed symmetric monoidal under its smash products $X \wedge Y$ and function objects $F(X,Y)$.

The sphere orthogonal space $S_{\mathscr{I}}$ has Vth space the one-point compactification S^V of V; $S_{\mathscr{I}}$ is a commutative monoid in $\mathscr{I}\mathscr{T}$. An *orthogonal spectrum*, or *$\mathscr{I}$-spectrum*, is a (right) $S_{\mathscr{I}}$-module. The category $\mathscr{I}\mathscr{S}$ of orthogonal spectra is closed symmetric monoidal. We denote its smash products and function spectra by $X \wedge_{\mathscr{I}} Y$ and $F_{\mathscr{I}}(X,Y)$ (although this is not consistent with the previous section).

There is a symmetric monoidal category $\mathscr{J}$ with the same objects as $\mathscr{I}$ such that the category of *$\mathscr{J}$-spaces* is isomorphic to the category of *$\mathscr{I}$-spectra*; $\mathscr{J}$ contains $\mathscr{I}$ as a subcategory. The construction of $\mathscr{J}$ is given in [**20**, 2.1], where it is denoted $\mathscr{I}_S$. Its space of morphisms $\mathscr{J}(V,W)$ is $(V^* \wedge S_{\mathscr{I}})(W)$, where $V^*(W) = \mathscr{I}(V,W)_+$. In §6, we shall give a concrete alternative description of $\mathscr{J}$ in terms of Thom spaces, and we shall use it to construct a coherent family of cofibrant $(-V)$-sphere S-modules $\mathbb{N}^*(V)$ that give us a contravariant "negative spheres" functor $\mathbb{N}^*$ to which we can apply the constructions of the previous section. Precisely, we shall prove the following theorem. Note that the unit of $\mathscr{I}$ is 0, the unit of $\mathscr{J}$ is $S_{\mathscr{I}}$, and, as required for consistency, $\mathscr{J}(0,W) = S^W$.

THEOREM 3.1. *There is a strong symmetric monoidal contravariant functor $\mathbb{N}^* : \mathscr{J} \longrightarrow \mathscr{M}$. If $V \neq 0$, then $\mathbb{N}^*(V)$ is a cofibrant S-module and the evaluation map*

$$\varepsilon : \mathbb{N}^*(V) \wedge S^V = \mathbb{N}^*(V) \wedge \mathscr{J}(0,V) \longrightarrow \mathbb{N}^*(0) \cong S$$

of the functor is a weak equivalence.

Here $\mathbb{N}^*(0) \cong S$ since $\mathbb{N}^*$ is strong symmetric monoidal. Propositions 2.8 and 2.14 give the following immediate consequence.

THEOREM 3.2. *Define functors* $\mathbb{N} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$ *and* $\mathbb{N}^{\#} : \mathscr{M} \longrightarrow \mathscr{I}\mathscr{S}$ *by letting* $\mathbb{N}(X) = \mathbb{N}^* \otimes_{\mathscr{J}} X$ *and* $(\mathbb{N}^{\#}M)(V) = \mathscr{M}(\mathbb{N}^*(V), M)$. *Then* $(\mathbb{N}, \mathbb{N}^{\#})$ *is an adjoint pair such that* $\mathbb{N}$ *is strong symmetric monoidal and* $\mathbb{N}^{\#}$ *is lax symmetric monoidal.*

This gives the formal properties of $\mathbb{N}$ and $\mathbb{N}^{\#}$, and we turn to their homotopical properties. According to [**20**, A.2], to show that these functors give a Quillen equivalence between $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$, it suffices to prove the following three results. Thus, since its last statement is formal [**15**, 4.3.3], these results will prove Theorem 1.1. We give the proofs in §5. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between model categories is said to create the weak equivalences in $\mathscr{A}$ if the weak equivalences in $\mathscr{A}$ are exactly the maps f such that Ff is a weak equivalence in $\mathscr{B}$, and similarly for other classes of maps.

LEMMA 3.3. *The functor* $\mathbb{N}^{\#}$ *preserves homotopy groups and creates the weak equivalences in* $\mathscr{M}$.

LEMMA 3.4. *The functor* $\mathbb{N}^{\#}$ *preserves* q*-fibrations.*

PROPOSITION 3.5. *The unit* $\eta : X \longrightarrow \mathbb{N}^{\#}\mathbb{N}X$ *of the adjunction is a weak equivalence for all cofibrant orthogonal spectra* X.

In Lemma 3.4, we are concerned with q-fibrations of orthogonal spectra in the positive stable model structure. To prove Theorem 1.1, we only need Proposition 3.5 for orthogonal spectra that are cofibrant in the positive stable model structure, but we shall prove it more generally for orthogonal spectra that are cofibrant in the stable model structure. We refer to *positive cofibrant* and *cofibrant* orthogonal spectra to distinguish these classes.

In the rest of this section, we show that these results imply their multiplicatively enriched versions needed to prove Theorems 1.2, 1.3, 1.5, and 1.6 and Corollaries 1.4 and 1.7. That is, in all cases, we have an adjoint pair $(\mathbb{N}, \mathbb{N}^{\#})$ such that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations and the unit of the adjunction is a weak equivalence on cofibrant objects. The subtlety is that, to apply Proposition 3.5, we must relate cofibrancy of multiplicatively structured orthogonal spectra and S-modules with cofibrancy of their underlying orthogonal spectra or S-modules.

THE PROOF OF THEOREM 1.2. The category of orthogonal ring spectra has two model structures. The respective weak equivalences and q-fibrations are created in the category of orthogonal spectra with its stable model structure or its positive stable model structure. The category of S-algebras is a model category with weak equivalences and q-fibrations created in the category of S-modules. Our claim is that $(\mathbb{N}, \mathbb{N}^{\#})$ restricts to a Quillen equivalence relating the category of orthogonal ring spectra with its positive stable model structure to the category of S-algebras. It is clear from Lemmas 3.3 and 3.4 that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations. We must show that $\eta : R \longrightarrow \mathbb{N}^{\#}\mathbb{N}R$ is a weak equivalence when R is a positive cofibrant orthogonal ring spectrum. More generally, if R is a cofibrant orthogonal ring spectrum, then the underlying orthogonal spectrum of R is cofibrant (although not positive cofibrant) by [**20**, 12.1]. The conclusion follows from Proposition 3.5. □

THE PROOF OF THEOREM 1.3. The category of R-modules is a model category with weak equivalences and q-fibrations created in the category of orthogonal spectra with its positive stable model structure. The category of $\mathbb{N}R$-modules is a model category with weak equivalences and q-fibrations created in the category of S-modules. Again, it is clear that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations. We must show that $\eta : Y \longrightarrow \mathbb{N}^{\#}\mathbb{N}Y$ is a weak equivalence when Y is a positive cofibrant R-module. We are assuming that R is positive cofibrant as an orthogonal ring spectrum, and it follows from [**20**, 12.1] that the underlying orthogonal spectrum of a cofibrant R-module is cofibrant (although not necessarily positive cofibrant). The conclusion follows from Proposition 3.5. □

THE PROOF OF THEOREM 1.5. The category of commutative orthogonal ring spectra has a model structure with weak equivalences and q-fibrations created in the category of orthogonal spectra with its positive stable model structure [**20**, 15.1]. The category of commutative S-algebras has a model structure with weak equivalences and q-fibrations created in the category of S-modules [**6**, VII.4.8]. Again, it is clear that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations, and we must prove that $\eta : R \longrightarrow \mathbb{N}^{\#}\mathbb{N}R$ is a weak equivalence when R is a cofibrant commutative orthogonal ring spectrum. Since the underlying orthogonal spectrum of R is not cofibrant, we must work harder here. We use the notations and results of [**20**, §§15, 16], where the structure of cofibrant commutative orthogonal ring spectra is analyzed and the precisely analogous proof comparing commutative symmetric ring spectra and commutative orthogonal ring spectra is given.

We may assume that R is a $\mathbb{C}F^+I$-cell complex (see [**20**, 15.1]), where $\mathbb{C}$ is the free commutative orthogonal ring spectrum functor, and we claim first that η is a weak equivalence when $R = \mathbb{C}X$ for a positive cofibrant orthogonal spectrum X. Here $\mathbb{C}X$ is the wedge over $i \geq 0$ of the $X^{(i)}/\Sigma_i$, the term for $i = 0$ being S. Therefore, by Lemma 3.3 and Theorems 4.10(ii) and 4.11 below, it suffices to prove that $\eta : X^{(i)}/\Sigma_i \longrightarrow \mathbb{N}^{\#}\mathbb{N}(X^{(i)}/\Sigma_i)$ is a weak equivalence for $i \geq 1$. On the right, $\mathbb{N}(X^{(i)}/\Sigma_i) \cong (\mathbb{N}X)^{(i)}/\Sigma_i$, and $\mathbb{N}X$ is a cofibrant S-module. Consider the commutative diagram

$$\begin{array}{ccccc}
E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} & \xrightarrow{\eta} & \mathbb{N}^{\#}\mathbb{N}(E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)}) & \cong & \mathbb{N}^{\#}(E\Sigma_{i+} \wedge_{\Sigma_i} (\mathbb{N}X)^{(i)}) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathbb{N}^{\#}\mathbb{N}q} & & \downarrow{\scriptstyle \mathbb{N}^{\#}q} \\
X^{(i)}/\Sigma_i & \xrightarrow[\eta]{} & \mathbb{N}^{\#}\mathbb{N}(X^{(i)}/\Sigma_i) & \cong & \mathbb{N}^{\#}((\mathbb{N}X)^{(i)}/\Sigma_i).
\end{array}$$

The q are the evident quotient maps, and the left and right arrows q are weak equivalences by [**20**, 15.5] and [**6**, III.5.1]. The top map η is a weak equivalence by Proposition 3.5 since an induction up the cellular filtration of $E\Sigma_i$, the successive subquotients of which are wedges of copies of $\Sigma_{i+} \wedge S^n$, shows that $E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)}$ is positive cofibrant since $X^{(i)}$ is positive cofibrant.

By passage to colimits, as in the analogous proof in [**20**, §16], the result for general R follows from the result for a $\mathbb{C}F^+I$-cell complex that is constructed in finitely many stages. We have proven the result when R requires only a single stage, and we assume the result when R is constructed in n stages. Thus suppose that R is constructed in $n+1$ stages. Then R is a pushout (in the category of commutative orthogonal ring spectra) of the form $R_n \wedge_{\mathbb{C}X} \mathbb{C}Y$, where R_n is constructed in n-stages and $X \longrightarrow Y$ is a wedge of maps in F^+I. As in the proof of [**20**, 15.9], $R \cong B(R_n, \mathbb{C}X, \mathbb{C}T)$, where T is a suitable wedge of orthogonal spectra $F_r S^0$. The bar construction here is the geometric realization of a proper simplicial orthogonal spectrum and $\mathbb{N}$ commutes with geometric realization. Tracing through the cofibration sequences used in the proof of the invariance of bar constructions in [**6**, X.4], we see that it suffices to show that η is a weak equivalence on the commutative orthogonal ring spectrum

$$R_n \wedge (\mathbb{C}X)^{(q)} \wedge \mathbb{C}T \cong R_n \wedge \mathbb{C}(X \vee \cdots \vee X \vee T)$$

of q-simplices for each q. By the definition of $\mathbb{C}F^+I$-cell complexes, we see that this smash product can be constructed in n-stages, hence the conclusion follows from the induction hypothesis. □

THE PROOF OF THEOREM 1.6. For a cofibrant commutative orthogonal ring spectrum R, we must prove that the unit $\eta : X \longrightarrow \mathbb{N}^{\#}\mathbb{N}X$ of the adjunction is a weak equivalence when X is a cofibrant R-module, R-algebra, or commutative R-algebra. For R-modules X, this reduces as in [**20**, 10.3 and §16] to the case when $X = R \wedge_S F_V S^V$, where F_V is a shift desuspension functor [**20**, 1.3]. In turn, this case reduces as in [**20**, §16] to an application of [**6**, III.3.8], which gives that the functor $(\mathbb{N}R) \wedge (-)$ preserves weak equivalences. The case of R-algebras follows since a cofibrant R-algebra is cofibrant as an R-module [**20**, 12.1]. The case of commutative R-algebras follows from the previous proof since a cofibrant commutative R-algebra is cofibrant as a commutative orthogonal ring spectrum. □

THE PROOFS OF COROLLARIES 1.4 AND 1.7. Let R be a commutative S-algebra, let $\gamma : Q \longrightarrow \mathbb{N}^{\#}R$ be a cofibrant approximation of the commutative orthogonal ring spectrum $\mathbb{N}^{\#}R$, and let $\tilde{\gamma} : \mathbb{N}Q \longrightarrow R$ be its adjoint. Writing $\mathscr{I}\mathscr{S}_Q$ and $\mathscr{M}_R$ for the respective categories of modules, we have the following commutative diagram of right adjoints in Quillen adjoint pairs:

$$\begin{array}{ccc} \mathscr{M}_{\mathbb{N}Q} & \xleftarrow{\tilde{\gamma}^*} & \mathscr{M}_R \\ {\scriptstyle \mathbb{N}^{\#}}\downarrow & & \downarrow{\scriptstyle \mathbb{N}^{\#}} \\ \mathscr{I}\mathscr{S}_Q & \xleftarrow[\gamma^*]{} & \mathscr{I}\mathscr{S}_{\mathbb{N}^{\#}}. \end{array}$$

The left adjoints to γ^* and $\tilde{\gamma}^*$ are given by extension of scalars. We have similar diagrams with modules replaced by algebras or by commutative algebras, and we have a similar diagram of modules in the non-commutative case. Since Q is cofibrant, the left vertical arrow is the right adjoint of a Quillen equivalence in all cases, by Theorems 1.3 and 1.6. Thus, to show that the right vertical arrow is the right adjoint of a Quillen equivalence, it suffices to prove that both extension of scalars adjunctions are Quillen equivalences in all cases. For the bottom arrow, this is given by [**20**, 12.1 and 15.2]. For the top arrow, we use the following result. □

THEOREM 3.6. *Let $f : Q \longrightarrow R$ be a map of S-algebras or of commutative S-algebras. Define $f_* : \mathscr{M}_Q \longrightarrow \mathscr{M}_R$ by $f_*M = R \wedge_Q M$. Then (f_*, f^*) is a Quillen adjoint pair. If f is a weak equivalence, then (f_*, f^*) is a Quillen equivalence. Moreover, in the commutative case, (f_*, f^*) then induces a Quillen equivalence* *between the categories of Q-algebras and R-algebras and, if Q and R are cofibrant, between the categories of commutative Q-algebras and commutative R-algebras.*

PROOF. Since weak equivalences and q-fibrations are created in the underlying category of S-modules, it is immediate that all adjoint pairs here are Quillen adjoint pairs. We need only check that the units of the adjunctions are weak equivalences on cofibrant objects. For the cases of modules, this is proven in [**6**, III.4.2]. Thus restrict to the case when Q and R are commutative. If A is a cofibrant R-algebra, then [**6**, III.3.8 and VII.6.2] imply that the unit $A \cong Q \wedge_Q A \longrightarrow R \wedge_Q A$ of the adjunction is a weak equivalence. If Q and R are cofibrant and A is a cofibrant commutative Q-algebra, then [**6**, III.3.8 and VII.6.7] give this implication. $\square$

REMARK 3.7. Consider the diagram

$$\Sigma\mathscr{I} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{I}\mathscr{S} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{M},$$

where $\Sigma\mathscr{S}$ is the category of symmetric spectra and $\mathbb{U}$ and $\mathbb{P}$ are the forgetful and prolongation functors of [**20**] (see Example 2.10). By (6.8) below, we have

$$\mathbb{N}^*(\mathbb{R}^n) = (S_S^{-1})^{(n)},$$

where S_S^{-1} is the canonical cofibrant (-1)-sphere in the category of S-modules. It follows that

$$(\mathbb{U} \circ \mathbb{N}^{\#})(M)(\mathbf{n}) \cong \mathscr{M}((S_S^{-1})^{(n)}, M)$$

as Σ_n-spaces. This is the right adjoint $\mathscr{M} \longrightarrow \Sigma\mathscr{S}$ used by Schwede [**31**], and $\mathbb{N} \circ \mathbb{P}$ is its left adjoint. Thus the adjunction studied in [**31**] is the composite of the adjunctions $(\mathbb{P}, \mathbb{U})$ and $(\mathbb{N}, \mathbb{N}^{\#})$.

4. Further Quillen equivalences and homotopical preliminaries

Before turning to the promised proofs, we place our results in context by stating a number of related Quillen equivalences between other model categories of prespectra and spectra and indicating their proofs. We also record some basic results about weak equivalences of spectra and S-modules that are used in the proofs.

We have two categories of prespectra, coordinatized and coordinate-free. The former is classical. It was described in [**20**] as the category of $\mathscr{N}$-spectra, where $\mathscr{N}$ is the discrete category with objects n, $n \geq 0$. We denote this category by $\mathscr{N}\mathscr{S}$. It has a stable model structure and a positive stable model structure [**20**, §§9, 14].

PROPOSITION 4.1. *The forgetful functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{N}\mathscr{S}$ has a left adjoint prolongation functor $\mathbb{P} : \mathscr{N}\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$, and the pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence with respect to either the stable or the positive stable model structures.*

We shall focus on prespectra in the coordinate-free sense of [**19, 6**]. Thus a *prespectrum* X consists of based spaces $X(V)$ and a transitive system of based maps $\sigma : \Sigma^{W-V} X(V) \longrightarrow X(W)$, where V ranges over the finite dimensional sub inner product spaces of a countably infinite dimensional real inner product space U, which we may take to be $U = \mathbb{R}^\infty$. Let $\mathscr{P}$ denote the resulting category of prespectra. A prespectrum X is said to be an *Ω-spectrum* if its adjoint structure maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V} X(W)$ are weak equivalences; it is a *positive Ω-spectrum* if these maps are weak equivalences for $V \neq 0$. Exactly as in [**20**, §§9 14], $\mathscr{P}$ has stable and positive stable model structures in which the respective fibrant objects are the Ω-spectra and the positive Ω-spectra.

REMARK 4.2. We obtain a forgetful functor $\mathbb{U} : \mathscr{P} \longrightarrow \mathscr{N}\mathscr{S}$ by restricting to the subspaces $\mathbb{R}^n$ of U. We also have an underlying coordinate-free prespectrum functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}$. The composite of these two functors is the functor $\mathbb{U}$ of Proposition 4.1. All three functors $\mathbb{U}$ have left adjoints $\mathbb{P}$ given by Example 2.10, and Proposition 4.1 remains true with $\mathbb{U}$ replaced by either of our new functors $\mathbb{U}$.

A prespectrum X is an *inclusion prespectrum* if its adjoint structure maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(X)$ are inclusions. It is a *spectrum* if the $\tilde{\sigma}$ are homeomorphisms. Let $\mathscr{S} \subset \mathscr{P}$ denote the full subcategory of spectra. The category $\mathscr{S}$ has a stable model structure [**6**, VII§§4, 5].

PROPOSITION 4.3. *The forgetful functor $\ell : \mathscr{S} \longrightarrow \mathscr{P}$ has a left adjoint spectrification functor $L : \mathscr{P} \longrightarrow \mathscr{S}$, and the pair (L, ℓ) is a Quillen equivalence with respect to the stable model structures.*

REMARK 4.4. This result applies to both the coordinatized and coordinate-free settings. The restriction of $\mathbb{U} : \mathscr{P} \longrightarrow \mathscr{N}\mathscr{S}$ to the respective subcategories of spectra is an equivalence of categories [**19**, I.2.4]; both $\mathbb{U}$ and its restriction to spectra are the right adjoints of Quillen equivalences of model categories.

Finally, we have a Quillen equivalence relating S-modules to spectra.

PROPOSITION 4.5. *There is a "free functor" $\mathbb{F} : \mathscr{S} \longrightarrow \mathscr{M}$ that has a right adjoint $\mathbb{V} : \mathscr{M} \longrightarrow \mathscr{S}$. The pair $(\mathbb{F}, \mathbb{V})$ is a Quillen equivalence with respect to the stable model structures.*

Propositions 4.3 and 4.5 depend on results about weak equivalences that we explain in the rest of the section. There is an underlying spectrum functor $\mathscr{M} \longrightarrow \mathscr{S}$; that is, an S-module is a spectrum with additional structure. Thus we have forgetful functors from all of our categories to the category of coordinatized prespectra.

DEFINITION 4.6. The *homotopy groups* of a prespectrum, spectrum, orthogonal spectrum, or S-module are the homotopy groups of its underlying coordinatized prespectrum. In any of these categories, a map is a *weak equivalence* if it induces an isomorphism of homotopy groups.

The forgetful functor $\mathscr{M} \longrightarrow \mathscr{S}$ is not itself the right adjoint of a Quillen equivalence, but it is related to $\mathbb{V}$ by a natural weak equivalence. The notion of a *tame* spectrum required in the following result is defined in [**6**, I.2.4]; all cofibrant spectra are tame.

LEMMA 4.7. *For S-modules M, there is a natural weak equivalence of spectra $\tilde{\lambda} : M \longrightarrow \mathbb{V}M$. For tame spectra E, the unit $\eta : E \longrightarrow \mathbb{V}\mathbb{F}E$ is a weak equivalence.*

PROOF. With the notations of [**6**, I.4.1, I.5.1, I.7.1],

$$\mathbb{F}E = S \wedge_{\mathscr{L}} \mathbb{L}E \quad \text{and} \quad \mathbb{V}M = F_{\mathscr{L}}(S, M).$$

The weak equivalence $\tilde{\lambda} : M \longrightarrow \mathbb{V}M$ is given by [**6**, I.8.7]. For tame E, the unit η is the composite of the homotopy equivalence $\eta : E \longrightarrow \mathbb{L}E$ of [**6**, I.4.6], the weak equivalence $\tilde{\lambda} : \mathbb{L}E \longrightarrow F_{\mathscr{L}}(S, \mathbb{L}E)$ of [**6**, I.8.7], and the isomorphism $F_{\mathscr{L}}(S, \mathbb{L}E) \cong F_{\mathscr{L}}(S, S \wedge_{\mathscr{L}} \mathbb{L}E)$ of [**6**, II.2.5]. □

The functor L does not preserve all weak equivalences and, for a general prespectrum X, it is hard to determine if $\eta : X \longrightarrow \ell LX$ is a weak equivalence. Inclusion prespectra are important because of the following result [**19**, I.2.2].

LEMMA 4.8. *Let X be an inclusion prespectrum. Then*

$$LX(V) = \mathrm{colim}_{W \supset V}\, \Omega^{W-V} X(W).$$

The Vth map of the unit $\eta : X \longrightarrow \ell LX$ of the (L, ℓ)-adjunction is the map from the initial term $X(V)$ into the colimit, and η is a weak equivalence of prespectra.

REMARK 4.9. For later use, we note a variant. We call X a *positive inclusion prespectrum* if $\tilde{\sigma}$ is an inclusion when $V \neq 0$. The description of $LX(V)$ is still valid and η is still a weak equivalence.

We record an omnibus result about weak equivalences of spectra; its analogue for prespectra is [**20**, 7.4]. Let Cf and Ff denote the homotopy cofiber and fiber of a map f, defined as usual (for example, in [**20**, 6.8]). An h-cofibration of spectra is a cofibration in the classical sense that the homotopy extension property (HEP) is satisfied.

THEOREM 4.10.
(i) *If $f : X \longrightarrow Y$ is a weak equivalence of tame spectra and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a weak equivalence.*
(i′) *A map of tame spectra is a weak equivalence if and only if its suspension is a weak equivalence, and the natural map $\eta : X \longrightarrow \Omega\Sigma X$ is a weak equivalence for all tame spectra X.*
(ii) *The homotopy groups of a wedge of spectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of weak equivalences of spectra is a weak equivalence.*
(iii) *If $i : A \longrightarrow X$ is an h-cofibration and a weak equivalence and $f : A \longrightarrow Y$ is any map, where A, X, and Y are tame spectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a weak equivalence.*
(iv) *If i and i' are h-cofibrations and the vertical arrows are weak equivalences in the diagram of tame spectra*

$$\begin{array}{ccccc} X & \xleftarrow{\;i\;} & A & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xleftarrow[\;i'\;]{} & A' & \longrightarrow & Y', \end{array}$$

then the induced map of pushouts is a weak equivalence.
(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$ of spectra, each of which is a weak equivalence, then the map from the initial term X_0 into X is a weak equivalence.*
(vi) *For any map $f : X \longrightarrow Y$ of tame spectra, there are natural long exact sequences*

$$\cdots \longrightarrow \pi_q(Ff) \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_{q-1}(Ff) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots,$$

and the natural map $\eta : Ff \longrightarrow \Omega Cf$ is a weak equivalence.

PROOF. Colimits of diagrams in $\mathscr{S}$ are obtained by applying ℓ, taking the colimit in $\mathscr{P}$, and applying L; smash products with spaces are constructed similarly. By Lemma 4.8, the functor L preserves homotopy groups and weak equivalences when applied to inclusion prespectra. Since any wedge of inclusion prespectra (such as spectra) is an inclusion prespectrum, (ii) follows directly from its prespectrum analogue in [**20**, 7.4]. Similarly, an h-cofibration of spectra is a spacewise closed inclusion by [**19**, App.3.9] and the colimit of a sequence of closed inclusions of inclusion prespectra is an inclusion prespectrum, so that (v) follows from its prespectrum analogue in [**20**, 7.4]. This kind of argument fails for the remaining parts because of point-set level pathologies, which are circumvented by the tameness hypotheses. The proof of (i) is similar to the proof of [**6**, I.3.6], which gives the analogue for a CW spectrum A. Part (i′) follows from [**6**, I.3.3]. Part (iii) is a special case of (iv), stated separately for emphasis, and (iv) is [**6**, I.3.5]. □

Sketch proofs of Propositions 4.1, 4.3, and 4.5. In all of these results, it is immediate from the definitions of the model structures that the right adjoints create weak equivalences and preserve q-fibrations. It therefore suffices to show that the units of the adjunctions are weak equivalences when evaluated on cofibrant objects. For Proposition 4.1, this is [**20**, 10.3]. By similar but simpler proofs, Lemma 4.8 and Theorem [**20**, 7.4] imply Proposition 4.3, while Lemma 4.7 and Theorem 4.10 imply Proposition 4.5. □

Finally, we record the analogue of Theorem 4.10 for S-modules.

Theorem 4.11. (i) *If $f : X \longrightarrow Y$ is a weak equivalence of S-modules and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a weak equivalence.*

(i′) *A map of S-modules is a weak equivalence if and only if its suspension is a weak equivalence, and the natural map $\eta : X \longrightarrow \Omega\Sigma X$ is a weak equivalence for all S-modules X.*

(ii) *The homotopy groups of a wedge of S-modules are the direct sums of the homotopy groups of the wedge summands, hence a wedge of weak equivalences of spectra is a weak equivalence.*

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and a weak equivalence of S-modules and $f : A \longrightarrow Y$ is any map of spectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a weak equivalence.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are weak equivalences in a comparison of pushouts diagram of S-modules as in Theorem 4.10(iv), then the induced map of pushouts is a weak equivalence.*

(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$ of S-modules, each of which is a weak equivalence, then the map from the initial term X_0 into X is a weak equivalence.*

(vi) *For any map $f : X \longrightarrow Y$ of S-modules, there are natural long exact sequences*

$$\cdots \longrightarrow \pi_q(Ff) \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_{q-1}(Ff) \longrightarrow \cdots,$$

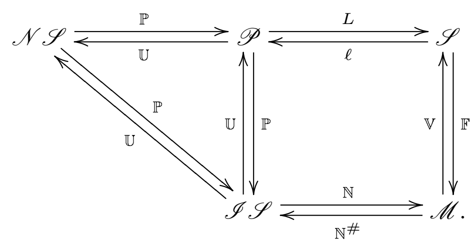
$$\cdots \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots,$$

and the natural map $\eta : Ff \longrightarrow \Omega Cf$ is a weak equivalence.

Proof. Since colimits, homotopy groups, and weak equivalences of S-modules are created by the forgetful functor to spectra, parts (ii) and (v) follow from the corresponding parts of Theorem 4.10. The rest is proven in [**6**, I§6]. It is one of the most remarkable technical features of [**6**] that the tameness hypotheses needed on the spectrum level are no longer necessary on the S-module level. □

5. Model structures and homotopical proofs

Summarizing from the previous section, we see that, even before constructing the functor $\mathbb{N}^*$, we have Quillen equivalences relating the categories $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, $\mathscr{P}$, $\mathscr{S}$, and $\mathscr{M}$, so that all of our homotopy categories are equivalent. Of course, these equivalences are much less highly structured than the one we are after since $\mathscr{N}\mathscr{S}$, $\mathscr{P}$, and $\mathscr{S}$ are not symmetric monoidal under their classical smash products, as defined in the coordinate-free setting in [**19**, III§3]. To help orient the reader, we display our Quillen equivalences in the following (noncommutative) diagram:



We recall the definitions of the relevant model structures on these five categories. We have defined their weak equivalences in Definition 4.6. A map of spectra is a q-fibration if each of its component maps of spaces is a Serre fibration, and the functor $\mathbb{V}$ creates the q-fibrations of S-modules. The (positive) q-fibrations of prespectra or of orthogonal spectra are the (positive) level Serre fibrations such that certain diagrams are homotopy pullbacks [**20**, 9.5]; all that we need to know about the latter condition is that it always holds for maps between (positive) Ω-spectra.

In all of our categories, the q-cofibrations are the maps that satisfy the LLP (left lifting property) with respect to the acyclic q-fibrations. Equivalently, they are the retracts of relative cell complexes in the respective categories. These cell complexes are defined as usual in terms of attaching maps whose domains are appropriate "spheres". We have nth space or Vth space evaluation functors from the categories $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, $\mathscr{P}$, and $\mathscr{S}$ to the category $\mathscr{T}$ of based spaces. These have left adjoint shift desuspension functors, denoted

$$F_n : \mathscr{T} \longrightarrow \mathscr{N}\mathscr{S}, \quad F_V : \mathscr{T} \longrightarrow \mathscr{I}\mathscr{S}, \quad F_V : \mathscr{T} \longrightarrow \mathscr{P}, \quad \text{and} \quad \Sigma^\infty_V : \mathscr{T} \longrightarrow \mathscr{S}.$$

We write $F_n = F_{\mathbb{R}^n}$ in $\mathscr{P}$ and $\mathscr{I}\mathscr{S}$ and $\Sigma^\infty_n = \Sigma^\infty_{\mathbb{R}^n}$ in $\mathscr{S}$. Obvious isomorphisms between right adjoints imply isomorphisms between left adjoints

$$\mathbb{P}F_n \cong F_n \quad \text{and} \quad LF_V \cong \Sigma^\infty_V.$$

The domains of attaching maps are the $F_n S^q$ in $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{P}$ where, for the positive stable model structures, we restrict to $n > 0$. The domains of attaching maps are the $\Sigma^\infty_n S^q$ in $\mathscr{S}$ and the $\mathbb{F}\Sigma^\infty_n S^q$ in $\mathscr{M}$.

We now return to the proofs promised in §3. In §6, we will obtain the following description in terms of shift desuspensions and the functor $\mathbb{F}$ of the values on objects taken by the functor $\mathbb{N}^*$.

LEMMA 5.1. *For an object $V \neq 0$ of $\mathscr{I}$, the S-module $\mathbb{N}^*(V)$ is non-canonically isomorphic to $\mathbb{F}\Sigma^\infty_V S^0$.*

The subtlety in the construction of $\mathbb{N}^*$ lies in its orthogonal functoriality. We cannot just define $\mathbb{N}^*(V)$ to be $\mathbb{F}\Sigma^\infty_V S^0$, since that would not give a functor of V. We begin our proofs with the following observation.

LEMMA 5.2. *For S-modules M, $\mathbb{N}^{\#}M$ is a positive Ω-spectrum.*

PROOF. We have $(\mathbb{N}^{\#}M)(V) = \mathscr{M}(\mathbb{N}^*(V), M)$. For $V \subset W$,

$$\Omega^{W-V}(\mathbb{N}^{\#}M)(W) \cong \mathscr{M}(\Sigma^{W-V}\mathbb{N}^*(W), M)$$

and the adjoint structure map $\tilde{\sigma} : \mathbb{N}^{\#}(V) \longrightarrow \Omega^{W-V}\mathbb{N}^{\#}(W)$ is induced from the evaluation map $\varepsilon : \Sigma^{W-V}\mathbb{N}^*(W) \longrightarrow \mathbb{N}^*(V)$. Let $V \neq 0$. Then ε is a weak equivalence between cofibrant S-modules and $\tilde{\sigma}$ is thus a weak equivalence. □

PROOF OF LEMMA 3.3. By Lemma 5.1 and Theorem 4.5, for an S-module M and an indexing space V in U, we have

$$(\mathbb{N}^{\#}M)(V) \cong \mathscr{M}(\mathbb{F}\Sigma^{\infty}_V S^0, M) \cong \mathscr{S}(\Sigma^{\infty}_V S^0, \mathbb{V}M) \cong \mathscr{T}(S^0, (\mathbb{V}M)(V)) = (\mathbb{V}M)(V),$$

which is weakly equivalent to $M(V)$. These natural weak equivalences imply a natural isomorphism $\pi_*\mathbb{N}^{\#}M \cong \pi_*M$, and it follows that $\mathbb{N}^{\#}$ creates the weak equivalences in $\mathscr{M}$. Alternatively, a map of orthogonal positive Ω-spectra or of S-modules is a weak equivalence if and only if its map on Vth spaces is a weak equivalence for $V \neq 0$ in U, hence our weak equivalences of Vth spaces show directly that a map f of S-modules is a weak equivalence if and only if $\mathbb{N}^{\#}f$ is a weak equivalence of orthogonal spectra. □

PROOF OF LEMMA 3.4. Let $f : M \longrightarrow N$ be a q-fibration of S-modules. We must show that $\mathbb{N}^{\#}f$ is a positive q-fibration of orthogonal spectra. Since $\mathbb{N}^{\#}f$ is a map of positive Ω-spectra, we need only show that the Vth space map of $\mathbb{N}^{\#}f$ is a Serre fibration for $V \neq 0$, and it suffices to show this for $V = \mathbb{R}^n$, $n > 0$. By [**6**, VII.4.6], f is a q-fibration if and only if it satisfies the RLP (right lifting property) with respect to all maps

$$i_0 : \mathbb{F}\Sigma^{\infty}_n CS^q \longrightarrow \mathbb{F}\Sigma^{\infty}_n CS^q \wedge I_+.$$

An easy adjunction argument from the isomorphism $\mathbb{N}^*(\mathbb{R}^n) \cong \mathbb{F}\Sigma^{\infty}_n S^0$ and the fact that $\mathbb{F}$ and the Σ^{∞}_n are right exact shows that

$$f_* : \mathscr{M}(\mathbb{N}^*(\mathbb{R}^n), M) \longrightarrow \mathscr{M}(\mathbb{N}^*(\mathbb{R}^n), N)$$

satisfies the RLP with respect to the maps $i_0 : CS^q \longrightarrow CS^q \wedge I_+$ and is therefore a Serre fibration. □

REMARK 5.3. In principle, the specified RLP states that f_* is a based Serre fibration, whereas what we need to show is that f_* is a classical Serre fibration, that is, a based map that satisfies the RLP in $\mathscr{T}$ with respect to the maps $i_0 : D^q_+ \longrightarrow D^q_+ \wedge I$. However, when $n > 0$, f_* is isomorphic to the loop of a based Serre fibration, and the loop of a based Serre fibration is a classical Serre fibration.

PROOF OF PROPOSITION 3.5. We first prove that $\eta : F_nA \to \mathbb{N}^{\#}\mathbb{N}F_nA$ is a weak equivalence for any based CW complex A; the only case we need is when A is a sphere. Here $F_n = \mathbb{P}F_n$ and it suffices to prove that the adjoint map of prespectra

$$\bar{\eta} : F_nA \to \mathbb{U}\mathbb{N}^{\#}\mathbb{N}F_nA$$

is a weak equivalence. By a check of definitions and use of Lemma 5.1,

$$\mathbb{N}F_nA \cong \mathbb{N}F_nS^0 \wedge A \cong \mathbb{N}^*(\mathbb{R}^n) \wedge A \cong \mathbb{F}\Sigma^{\infty}_n S^0 \wedge A \cong \mathbb{F}\Sigma^{\infty}_n A.$$

Therefore, using Lemma 5.1 and Proposition 4.5, we have weak equivalences

$$\begin{aligned}(\mathbb{N}^{\#}\mathbb{N}F_nA)(\mathbb{R}^q) \cong \mathscr{M}(\mathbb{F}\Sigma^\infty_q S^0, \mathbb{F}\Sigma^\infty_n A) &\cong \mathscr{S}(\Sigma^\infty_q S^0, \mathbb{V}\mathbb{F}\Sigma^\infty_n A)\\ &\simeq \mathscr{S}(\Sigma^\infty_q S^0, \Sigma^\infty_n A) \cong (\Sigma^\infty_n A)(\mathbb{R}^q).\end{aligned}$$

Tracing through definitions, we find that, up to homotopy, the structural maps coincide under these weak equivalences with those of $\Sigma^\infty_n A \cong LF_nA$ and the map $\bar\eta$ induces the same map of homotopy groups as the unit $F_nA \longrightarrow \ell LF_nA$ of the adjunction of Proposition 4.3. Therefore $\bar\eta$ is a weak equivalence. By [**20**, 7.4] and Theorem 4.10, we see that the class of orthogonal spectra for which η is a weak equivalence is closed under wedges, pushouts along h-cofibrations, sequential colimits of h-cofibrations, and retracts. Therefore η is a weak equivalence for all cofibrant orthogonal spectra. □

6. The construction of the functor $\mathbb{N}^*$

We prove Theorem 3.1 here. Implicitly, we shall give two constructions of the functor $\mathbb{N}^*$. The theory of S-modules is based on a functor called the twisted half-smash product, denoted $\ltimes$, the definitive construction of which is due to Cole [**6**, App]. The theory of orthogonal spectra is the theory of diagram spaces with domain category $\mathscr{J}$. Both $\ltimes$ and $\mathscr{J}$ are defined in terms of Thom spaces associated to spaces of linear isometries. We first define $\mathbb{N}^*$ in terms of twisted half-smash products. We then outline the definition of twisted half-smash products in terms of Thom spaces and redescribe $\mathbb{N}^*$ in those terms. That will make the connection with the category $\mathscr{J}$ transparent, since the morphism spaces of $\mathscr{J}$ are Thom spaces closely related to those used to define the relevant twisted half-smash products.

Here we allow the universe U on which we index our coordinate-free prespectra and spectra to vary. We write $\mathscr{P}^U$ and $\mathscr{S}^U$ for the categories of prespectra and spectra indexed on U. We have a forgetful functor $\ell : \mathscr{S}^U \longrightarrow \mathscr{P}^U$ with a left adjoint spectrification functor $L : \mathscr{P}^U \longrightarrow \mathscr{S}^U$. We have a suspension spectrum functor Σ^U that is left adjoint to the zeroth space functor Ω^U. Let $S^U = \Sigma^U(S^0)$. The functors Σ^U and Ω^U are usually denoted Σ^∞ and Ω^∞, but we wish to emphasize the choice of universe rather than its infinite dimensionality. We write Σ^∞ and Ω^∞ when $U = \mathbb{R}^\infty$, and we then write $S^U = S$. More generally, for a finite dimensional sub inner product space V of U, we have a shift desuspension functor $\Sigma^U_V : \mathscr{T} \longrightarrow \mathscr{S}^U$, denoted Σ^∞_V when $U = \mathbb{R}^\infty$. It is left adjoint to evaluation at V.

For inner product spaces U and U', let $\mathscr{I}(U, U')$ be the space of linear isometries $U \longrightarrow U'$, not necessarily isomorphisms. It is contractible when U' is infinite dimensional [**23**, 1.3]. We have a twisted half-smash functor

$$\mathscr{I}(U, U') \ltimes (-) : \mathscr{S}^U \longrightarrow \mathscr{S}^{U'},$$

whose definition we shall recall shortly. It is a "change of universe functor" that converts spectra indexed on U to spectra indexed on U' in a well-structured way.

Now fix $U = \mathbb{R}^\infty$ and consider the universes $V \otimes U$ for $V \in \mathscr{I}$. Identify V with $V \otimes \mathbb{R} \subset V \otimes U$. In the language of [**6**], we define

$$\mathbb{N}^*(V) = S \wedge_{\mathscr{L}} (\mathscr{I}(V \otimes U, U) \ltimes \Sigma^{V\otimes U}_V(S^0)). \tag{6.1}$$

To make sense of this, we must recall some of the definitional framework of [**6**]. We have the linear isometries operad $\mathscr{L}$ with nth space $\mathscr{L}(n) = \mathscr{I}(U^n, U)$. The operad structure maps are given by compositions and direct sums of linear isometries, and they specialize to give a monoid structure on $\mathscr{L}(1)$, a left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$, and a right action of $\mathscr{L}(1)\times\mathscr{L}(1)$ on $\mathscr{L}(2)$. For a spectrum $E\in\mathscr{S}$, $\mathscr{L}(1)\ltimes E$ is denoted $\mathbb{L}E$. The monoid structure on $\mathscr{L}(1)$ induces a monad structure on the functor $\mathbb{L}:\mathscr{S}\longrightarrow\mathscr{S}$.

Definition 6.2. An $\mathbb{L}$-spectrum is an algebra over the monad $\mathbb{L}$. Let $\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra. The functor $\mathbb{L}$ takes values in $\mathbb{L}$-spectra and gives the free $\mathbb{L}$-spectrum functor $\mathbb{L}:\mathscr{S}\longrightarrow\mathscr{S}[\mathbb{L}]$.

By [**6**, I§5], we have an "operadic smash product"

$$E\wedge_{\mathscr{L}}E' = \mathscr{L}(2)\ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)} E\mathbin{\bar\wedge}E' \tag{6.3}$$

between $\mathbb{L}$-spectra E and E', where $E\mathbin{\bar\wedge}E'$ is the external smash product indexed on U^2 [**6**, I§2]. The sphere S is an $\mathbb{L}$-spectrum, and the action of $\mathscr{L}(1)$ by composition on $\mathscr{I}(V\otimes U,U)$ induces a structure of $\mathbb{L}$-spectrum on $\mathscr{I}(V\otimes U,U)\ltimes\Sigma_V^{V\otimes U}(S^0)$.

An $\mathbb{L}$-spectrum E has a unit map $\lambda: S\wedge_{\mathscr{L}}E\longrightarrow E$ that is always a weak equivalence and sometimes an isomorphism [**6**, I§8 and II§1]; we redescribe it in VI§6. In particular, λ is an isomorphism when $E=S$, when $E=S\wedge_{\mathscr{L}}E'$ for any $\mathbb{L}$-spectrum E', and when E is the operadic smash product of two S-modules [**6**, I.8.2, II.1.2].

Definition 6.4. An S-module is an $\mathbb{L}$-spectrum E such that λ is an isomorphism. The smash product $\wedge$ in the category $\mathscr{M}$ of S-modules is the restriction to S-modules of $\wedge_{\mathscr{L}}$. The functor $\mathbb{J}:\mathscr{S}[\mathbb{L}]\longrightarrow\mathscr{M}$ specified by

$$\mathbb{J}E = S\wedge_{\mathscr{L}}E$$

carries $\mathbb{L}$-spectra to weakly equivalent S-modules. The functor $\mathbb{F}:\mathscr{S}\longrightarrow\mathscr{M}$ of Proposition 4.5 is the composite $\mathbb{J}\circ\mathbb{L}$.

We can rewrite (6.1) as

$$\mathbb{N}^*(V)=\mathbb{J}(\mathscr{I}(V\otimes U,U)\ltimes\Sigma_V^{V\otimes U}(S^0)). \tag{6.5}$$

This makes sense of (6.1). It even makes sense when $V=\{0\}$. Here we interpret spectra indexed on the universe $\{0\}$ as based spaces. The space $\mathscr{I}(\{0\},U)$ is a point, namely the inclusion $i^U:\{0\}\longrightarrow U$. The functor $i^U_*=i^U\ltimes(-):\mathscr{T}\longrightarrow\mathscr{S}^U$ is left adjoint to the zeroth space functor, hence $i^U_*\cong\Sigma^U$. Thus (6.5) specializes to give $\mathbb{N}^*(0)=\mathbb{J}S$ and, as we have noted, $\lambda:\mathbb{J}S\longrightarrow S$ is an isomorphism.

The evident homeomorphisms

$$\Sigma^{V'-V}A\wedge\Sigma^{W'-W}B\cong\Sigma^{(V'-V)\oplus(W'-W)}(A\wedge B)$$

for $V\subset V'$ in $V\otimes U$ and $W\subset W'$ in $W\otimes U$, induce an isomorphism

$$\Sigma_V^{V\otimes U}(A)\mathbin{\bar\wedge}\Sigma_W^{W\otimes U}(B)\cong\Sigma_{V\oplus W}^{(V\oplus W)\otimes U}(A\wedge B) \tag{6.6}$$

upon spectrification, where

$$\bar\wedge:\mathscr{S}^{V\otimes U}\times\mathscr{S}^{W\otimes U}\longrightarrow\mathscr{S}^{(V\oplus W)\otimes U}$$

is the external smash product. Using the formal properties [**6**, A.6.2 and A.6.3] of twisted half-smash products, the canonical homeomorphism

$$\mathscr{L}(2)\times_{\mathscr{L}(1)\times\mathscr{L}(1)}(\mathscr{I}(V\otimes U,U)\times\mathscr{I}(W\otimes U,U))\cong\mathscr{I}((V\oplus W)\otimes U,U)$$

given by Hopkins' lemma [**6**, I.5.4], and the associative and unital properties of $\wedge_{\mathscr{L}}$ of [**6**, I§§5,8], we see that the isomorphisms (6.6) induce isomorphisms

$$\phi : \mathbb{N}^*(V) \wedge \mathbb{N}^*(W) \longrightarrow \mathbb{N}^*(V \oplus W). \tag{6.7}$$

We may identify $\mathbb{R}^n \otimes U$ with U^n. With the notations of [**6**, II.1.7], the canonical cofibrant sphere S-modules are $S^n_S = \mathbb{F}S^n$, where S^n is the canonical sphere spectrum. For $n \geq 0$, $S^{-n} = \Sigma^\infty_n S^0$. Thus $\mathbb{N}^*(\mathbb{R}) = S^{-1}_S$ and, for $n \geq 1$,

$$\mathbb{N}^*(\mathbb{R}^n) \cong (S^{-1}_S)^{(n)} \cong S^{-n}_S = \mathbb{F}\Sigma^\infty_n S^0, \tag{6.8}$$

where the middle isomorphism is only canonical up to homotopy. The first isomorphism is Σ_n-equivariant, which is the essential point of Remark 3.7. If $\dim V = n$, $n > 0$, then $\mathbb{N}^*(V)$ is isomorphic to $\mathbb{N}^*(\mathbb{R}^n)$ and is thus cofibrant. Moreover, Σ^∞_V is isomorphic to Σ^∞_n, so that Lemma 5.1 holds.

Intuitively, (6.5) gives a coordinate-free generalization of the canonical cofibrant negative sphere S-modules used in [**6**]. We must still prove the contravariant functoriality in V of $\mathbb{N}^*(V)$, check the naturality of ϕ, and prove that the evaluation maps $\varepsilon : \mathbb{N}^*(V) \wedge S^V \longrightarrow \mathbb{N}^*(0)$ are weak equivalences. While this can be done directly in terms of the definitions on hand, it is more illuminating to review the definition of the half-smash product and relate it directly to the morphism spaces of the category $\mathscr{J}$. We introduce a category Θ of Thom spaces for this purpose. Its objects will be inclusions $V \subset U$, which we secretly think of as symbols $\frac{U}{V}$ since these objects are closely related to the functors Σ^U_V used in our definition of $\mathbb{N}^*$. We think of $T^{U,U'}_{V,V'}$ in the following definition as a slightly abbreviated notation for the morphism space $\Theta(\frac{U}{V}, \frac{U'}{V'})$.

Definition 6.9. Let U and U' be finite or countably infinite dimensional real inner product spaces. Let V and V' be finite dimensional sub inner product spaces of U and U'. Let $\mathscr{I}^{U,U'}_{V,V'}$ be the space of linear isometries $f : U \longrightarrow U'$ such that $f(V) \subset V'$. For $V \subset W$, let $W - V$ denote the orthogonal complement of V in W. Let $E^{U,U'}_{V,V'}$ be the subbundle of the product bundle $\mathscr{I}^{U,U'}_{V,V'} \times V'$ whose points are the pairs (f, x) such that $x \in V' - f(V)$. Let $T^{U,U'}_{V,V'}$ be the Thom space of $E^{U,U'}_{V,V'}$; it is obtained by applying fiberwise one-point compactification and identifying all of the points at ∞. The spaces $T^{U,U'}_{V,V'}$ are the morphism spaces of a based topological *Thom category* Θ whose objects are the inclusions $V \subset U$. Composition

$$\circ : T^{U',U''}_{V',V''} \wedge T^{U,U'}_{V,V'} \longrightarrow T^{U,U''}_{V,V''} \tag{6.10}$$

is defined by $(g, y) \circ (f, x) = (g \circ f, g(x) + y)$. Points $(\mathrm{id}_U, 0)$ give identity morphisms. If $\mathscr{I}^{U,U'}_{V,V'}$ is empty, $T^{U,U'}_{V,V'}$ is a point. For any U and any object $V' \subset U'$,

$$T^{U,U'}_{0,V'} = \mathscr{I}(U, U')_+ \wedge S^{V'}. \tag{6.11}$$

The category Θ is symmetric monoidal with respect to direct sums of inner product spaces. On morphism spaces, the map

$$\oplus : T^{U_1,U_1'}_{V_1,V_1'} \wedge T^{U_2,U_2'}_{V_2,V_2'} \longrightarrow T^{U_1 \oplus U_2, U_1' \oplus U_2'}_{V_1 \oplus V_2, V_1' \oplus V_2'} \tag{6.12}$$

sends $((f_1, x_1), (f_2, x_2))$ to $(f_1 \oplus f_2, x_1 + x_2)$. Note that we have a trivialization isomorphism of bundles

$$E^{U,U'}_{V,V'} \times V \cong \mathscr{I}^{U,U'}_{V,V'} \times V'$$

and thus an "untwisting isomorphism"

$$T^{U,U'}_{V,V'} \wedge S^V \cong \mathscr{I}^{U,U'}_{V,V'+} \wedge S^{V'}. \tag{6.13}$$

The theory of orthogonal spectra is based on the full sub-category of Θ whose objects are the identity inclusions $V \subset V$. If $V \subset V'$, then it is easily verified that

$$T^{V,V'}_{V,V'} \cong O(V')_+ \wedge_{O(V'-V)} S^{V'-V}.$$

Comparing with the definitions in [**20**, 2.1, 4.4], we obtain the following result.

PROPOSITION 6.14. *The full subcategory of Θ whose objects are the identity maps $V \subset V$ is isomorphic as a based symmetric monoidal category to the category $\mathscr{J}$ such that an orthogonal spectrum is a continuous based functor $\mathscr{J} \longrightarrow \mathscr{T}$.*

We regard this isomorphism of categories as an identification.

In contrast, the twisted half-smash product is defined in terms of the full sub category of Θ whose objects are the inclusions $V \subset U$ in which U is infinite dimensional. The following definition and lemma are taken from [**6**, A.4.1–A.4.3].

DEFINITION 6.15. Fix $V \subset U$ and U'. Define a prespectrum $T^{U,U'}_{V,-}$ indexed on U' by letting its V'th space be $T^{U,U'}_{V,V'}$ and letting its structure map for $V' \subset W'$ be induced by passage to Thom spaces from the evident bundle map

$$E^{U,U'}_{V,V'} \oplus (W'-V') \cong E^{U,U'}_{V,W'}|_{\mathscr{I}^{U,U'}_{V,V'}} \longrightarrow E^{U,U'}_{V,W'}.$$

For $V \subset W$, define a map $\tau : \Sigma^{W-V} T^{U,U'}_{W,-} \longrightarrow T^{U,U'}_{V,-}$ of prespectra indexed on U' by letting its V'th map be induced by passage to Thom spaces from the evident bundle map

$$E^{U,U'}_{W,V'} \oplus (W-V) \cong E^{U,U'}_{V,V'}|_{\mathscr{I}^{U,U'}_{W,V'}} \longrightarrow E^{U,U'}_{V,V'}.$$

Observe that $T^{U,U'}_{V,-}$ is an inclusion prespectrum and define $M^{U,U'}_{V,-} = LT^{U,U'}_{V,-}$. (That is, write M consistently for Thom spectra associated to Thom prespectra T.)

LEMMA 6.16. *The spectrified map*

$$L\tau : \Sigma^{W-V} M^{U,U'}_{W,-} \cong L(\Sigma^{W-V} T^{U,U'}_{W,-}) \longrightarrow LT^{U,U'}_{V,-} = M^{U,U'}_{V,-}$$

is an isomorphism of spectra indexed on U'.

The following is a special case of the definition of the twisted half smash product given in [**6**, A.5.1].

DEFINITION 6.17. Let E be a spectrum indexed on U. Define

$$\mathscr{I}(U,U') \ltimes E = \operatorname{colim}_V M^{U,U'}_{V,-} \wedge EV$$

where the colimit (in $\mathscr{S}^{U'}$) is taken over the maps

$$M^{U,U'}_{V,-} \wedge EV \cong \Sigma^{W-V} M^{U,U'}_{W,-} \wedge EV \cong M^{U,U'}_{W,-} \wedge \Sigma^{W-V} EV \longrightarrow M^{U,U'}_{W,-} \wedge EW$$

induced by the structure maps of E.

The following result of Cole [**6**, A.3.9] is pivotal.

PROPOSITION 6.18. *For based spaces A, there is a natural isomorphism*

$$\mathscr{I}(U,U') \ltimes \Sigma^U_V A \cong M^{U,U'}_{V,-} \wedge A$$

of spectra indexed on U'.

The proof is simply the observation that, in this case, the defining colimit stabilizes at the Vth stage. Returning to the fixed choice of $U = \mathbb{R}^\infty$ and taking $A = S^0$, this gives the alternative description

$$\mathbb{N}^*(V) \cong \mathbb{J}M_{V,-}^{V\otimes U,U}. \tag{6.19}$$

We regard this isomorphism as an identification and use it to show the required functoriality of the $\mathbb{N}^*(V)$.

DEFINITION 6.20. Tensoring linear isometries $V \longrightarrow W$ with id_U, we obtain a map $\mu : T_{V,W}^{V,W} \longrightarrow T_{V,W}^{V\otimes U,W\otimes U}$. The evaluation maps $\mathbb{N}^*(W)\wedge \mathscr{J}(W,V) \longrightarrow \mathbb{N}^*(V)$ of the contravariant functor $\mathbb{N}^*$ are defined to be the maps

$$\begin{aligned} \mathbb{J}M_{W,-}^{W\otimes U,U} \wedge T_{V,W}^{V,W} &\xrightarrow{\mathrm{id}\wedge\mu} \mathbb{J}M_{W,-}^{W\otimes U,U} \wedge T_{V,W}^{V\otimes U,W\otimes U} \\ &\cong \mathbb{J}L(T_{W,-}^{W\otimes U,U} \wedge T_{V,W}^{V\otimes U,W\otimes U}) \\ &\xrightarrow{\mathbb{J}L(\circ)} \mathbb{J}L(T_{V,-}^{V\otimes U,U}) = \mathbb{J}M_{V,-}^{V\otimes U,U} \end{aligned}$$

induced by composition in the category Θ.

The naturality of the maps ϕ of (6.7) is now checked by rewriting these maps in terms of Thom complexes, using (6.12). Finally, we have the following lemma.

LEMMA 6.21. *The evaluation map* $\varepsilon : \mathbb{N}^*(V)\wedge S^V \longrightarrow \mathbb{N}^*(0) \cong S$ *of the functor* $\mathbb{N}^*$ *is a weak equivalence. When* $V = \mathbb{R}$, ε *factors as the composite of the canonical isomorphism* $\mathbb{N}^*(\mathbb{R})\wedge S^1 \cong S_S$ *and the canonical cofibrant approximation* $S_S \longrightarrow S$.

PROOF. Using the untwisting isomorphisms

$$T_{V,V'}^{V\otimes U,U} \wedge S^V \cong \mathscr{I}_{V,V'}^{V\otimes U,U} \wedge S^{V'}$$

and applying L, we obtain an isomorphism of $\mathbb{L}$-spectra

$$M_{V,-}^{V\otimes U,U} \wedge S^V \cong \mathscr{I}(V\otimes U,U)_+ \wedge S.$$

Applying $\mathbb{J}$ and using $\mathbb{J}S \cong S$, we find by (6.19) that

$$\mathbb{N}^*(V) \wedge S^V \cong \mathbb{J}((\mathscr{I}(V\otimes U,U)_+ \wedge S) \cong \mathscr{I}(V\otimes U,U)_+ \wedge S. \tag{6.22}$$

Under this isomorphism, the evaluation map corresponds to the homotopy equivalence induced by the evident homotopy equivalence $\mathscr{I}(V\otimes U,U)_+ \longrightarrow S^0$. When $V = \mathbb{R}$, $\mathbb{L}S \cong \mathscr{L}(1)_+ \wedge S$ and the isomorphism just given is the cited canonical isomorphism $\mathbb{N}^*(\mathbb{R}) \wedge S^1 \cong S_S$. □

7. The functor $\mathbb{M}$ and its comparison with $\mathbb{N}$

We begin with the underlying prespectrum and spectrification functors:

$$\mathscr{I}\mathscr{S} \xrightarrow{\mathbb{U}} \mathscr{P} \xrightarrow{L} \mathscr{S}. \tag{7.1}$$

The functor $\mathbb{M}$ is the composite of the following three functors:

$$\mathscr{I}\mathscr{S} \xrightarrow{\mathbb{U}} \mathscr{P}[\mathbb{L}] \xrightarrow{L} \mathscr{S}[\mathbb{L}] \xrightarrow{\mathbb{J}} \mathscr{M}. \tag{7.2}$$

The categories $\mathscr{P}[\mathbb{L}]$ and $\mathscr{S}[\mathbb{L}]$ are the categories of $\mathbb{L}$-prespectra and $\mathbb{L}$-spectra. We have already indicated what $\mathbb{L}$-spectra are, and we shall define $\mathbb{L}$-prespectra shortly. The functors $\mathbb{U}$ and L in (7.2) are restrictions of those of (7.1), and the functor $\mathbb{J}$ is specified in Definition 6.4. Thus, to construct $\mathbb{M}$, we must define $\mathbb{L}$-prespectra and show that the functors $\mathbb{U}$ and L induce functors from orthogonal spectra to $\mathbb{L}$-prespectra and from $\mathbb{L}$-prespectra to $\mathbb{L}$-spectra. The arguments are already implicit in [**23**].

DEFINITION 7.3. For a prespectrum X and a linear isometry $f : U \longrightarrow U$, define a prespectrum f^*X by $(f^*X)(V) = X(fV)$, with structure maps

$$X(fV) \wedge S^{W-V} \xrightarrow{\mathrm{id} \wedge S^f} X(fV) \wedge S^{f(W-V)} \xrightarrow{\sigma} X(fW).$$

Observe that f^*X is a spectrum if X is a spectrum.

DEFINITION 7.4. An $\mathbb{L}$-prespectrum is a prespectrum X together with maps $\xi(f) : X \longrightarrow f^*X$ of prespectra for all linear isometries $f : U \longrightarrow U$ such that $\xi(\mathrm{id}) = \mathrm{id}$, $\xi(f') \circ \xi(f) = \xi(f' \circ f)$, and the function

$$\xi : T^{U,U}_{V,W} \wedge X(V) \longrightarrow X(W)$$

specified by

$$\xi((f, w), x)) = \sigma(\xi(f)(x), w)$$

is a continuous.

In Definition 6.2, we defined a $\mathbb{L}$-spectrum to be an algebra over the monad $\mathbb{L}$. Inspection of the construction of twisted half smash products in §5 (compare [**27**, XXII.5.3]) gives the following consistency statement. While this equivalence of definitions is not difficult, we emphasize that it is central to the mathematics: it converts structures that are defined one isometry at a time into structures that are defined globally in terms of spaces of isometries.

LEMMA 7.5. *An $\mathbb{L}$-spectrum is an $\mathbb{L}$-prespectrum that is a spectrum.*

LEMMA 7.6. *The functor $L : \mathscr{P} \longrightarrow \mathscr{S}$ induces a functor $\mathscr{P}[\mathbb{L}] \longrightarrow \mathscr{S}[\mathbb{L}]$.*

PROOF. For a linear isometry $f : U \longrightarrow U$, the functor $f^* : \mathscr{P} \longrightarrow \mathscr{P}$ and its restriction $f^* : \mathscr{S} \longrightarrow \mathscr{S}$ have left adjoints f_*. The functor f_* on spectra is defined in terms of the functor f_* on prespectra by $f_* = Lf_*\ell$ [**19**, II§1]. Let X be an $\mathbb{L}$–prespectrum. The map $\xi(f)$ has an adjoint map $f_*X \longrightarrow X$; applying L, we obtain a map $f_*LX \longrightarrow LX$, and its adjoint gives an induced map $\xi(f) : LX \longrightarrow f^*LX$. The properties $\xi(\mathrm{id}) = \mathrm{id}$ and $\xi(f' \circ f) = \xi(f') \circ \xi(f)$ are inherited from their prespectrum level analogues. Since the functor L is continuous and commutes with smash products with spaces, the continuity and equivariance condition on ξ in Definition 7.4 are also inherited by LX. □

LEMMA 7.7. *The functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}$ takes values in $\mathscr{P}[\mathbb{L}]$.*

PROOF. We obtain $\xi(f) : X \longrightarrow f^*X$ by applying the functoriality of X and the naturality of σ to the restrictions of linear isometries $f : U \longrightarrow U$ to linear isometric isomorphisms $f : V \longrightarrow f(V)$ for indexing spaces V. It is clear by functoriality that $\xi(\mathrm{id}) = \mathrm{id}$ and $\xi(f' \circ f) = \xi(f') \circ \xi(f)$. The continuity and equivariance condition on ξ in Definition 7.4 follow from the continuity, naturality and equivariance of σ. □

REMARK 7.8. For general $\mathbb{L}$-prespectra, the map $\xi(f) : X(V) \longrightarrow X(fV)$ depends on the linear isometry $f : U \longrightarrow U$ and not just on its restriction $V \longrightarrow f(V)$. For those $\mathbb{L}$-prespectra that come from orthogonal spectra, this map does depend solely on the restriction of f. For this reason, there is no obvious functor $\mathscr{P}[\mathbb{L}] \longrightarrow \mathscr{I}\mathscr{S}$.

The following lemmas give the basic formal properties of the functor $\mathbb{M}$.

LEMMA 7.9. *The functor* $\mathbb{M}$ *is right exact.*

PROOF. The functors $\mathbb{U}$, L, and $\mathbb{J}$ are each right exact. This is obvious for $\mathbb{U}$ from the spacewise specification of colimits and smash products with based spaces, and it holds for L and $\mathbb{J}$ since these functors are continuous left adjoints. □

LEMMA 7.10. *There is a canonical isomorphism* $\lambda : \mathbb{M}(S_{\mathscr{I}}) \longrightarrow S$.

PROOF. Clearly $\mathbb{U}(S_{\mathscr{I}})$ is the usual sphere prespectrum and thus $S = L\mathbb{U}(S_{\mathscr{I}})$. As we have already used, $\mathbb{J}S \cong S$ by [**6**, I.8.2]. □

LEMMA 7.11. *The functor* $\mathbb{M}$ *is lax symmetric monoidal.*

PROOF. We have $\mathbb{M}S_{\mathscr{I}} \cong S$, and we must construct a natural map

$$\phi : \mathbb{M}(X) \wedge \mathbb{M}(X') \longrightarrow \mathbb{M}(X \wedge_{\mathscr{I}} X')$$

for orthogonal spectra X and X'. The functor $\mathbb{J}$ is strong symmetric monoidal, so

$$(\mathbb{J}E) \wedge (\mathbb{J}E') \cong \mathbb{J}(E \wedge_{\mathscr{L}} E')$$

for $\mathbb{L}$-spectra E and E'. Thus it suffices to construct a map of $\mathbb{L}$-spectra

$$\phi : L\mathbb{U}(X) \wedge_{\mathscr{L}} L\mathbb{U}(X') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X'),$$

and ϕ is obtained by passage to coequalizers from a map

$$\xi : \mathscr{L}(2) \ltimes L\mathbb{U}(X) \barwedge L\mathbb{U}(X') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X').$$

To construct ξ, it suffices to construct maps

$$\xi(f) : L\mathbb{U}(X)(V) \wedge L\mathbb{U}(X')(V') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X')(f(V \oplus V'))$$

for linear isometries $f \in \mathscr{L}(2)$ such that the $\xi(f)$ satisfy analogues of the conditions in Definition 7.4 [**27**, XXII.5.3]. The functoriality of X and X' gives maps

$$X(V) \wedge X'(V') \longrightarrow X(f(V)) \wedge X'(f(V')).$$

The universal property (2.13) that relates the external and internal smash product of orthogonal spectra gives a map of $(\mathscr{J} \times \mathscr{J})$-spaces

$$X \barwedge X' \longrightarrow (X \wedge_{\mathscr{I}} X') \circ \oplus,$$

and this gives maps

$$X(f(V)) \wedge X'(f(V')) \longrightarrow (X \wedge_{\mathscr{I}} X')(f(V \oplus V')).$$

We obtain the required maps $\xi(f)$ from the composites

$$X(V) \wedge X'(V') \longrightarrow (X \wedge_{\mathscr{I}} X')(f(V \oplus V'))$$

by passing to prespectra and then to spectra, as in the proof of Lemma 7.6. The coherence properties of the maps ϕ obtained from these maps ξ are shown by formal verifications from the properties of the various smash products. □

Turning to homotopical properties, we have the following observation. Recall Remark 4.9.

LEMMA 7.12. *If* X *is a positive inclusion orthogonal spectrum, then there are natural isomorphisms*

$$\pi_*(X) \cong \pi_*(\mathbb{M}(X)).$$

PROOF. We have a natural weak equivalence $\lambda : \mathbb{M}(X) = \mathbb{J}L\mathbb{U}(X) \longrightarrow L\mathbb{U}(X)$ for any X, and the unit map $\eta : \mathbb{U}X \longrightarrow \ell L\mathbb{U}(X)$ is also a weak equivalence. □

Now the following theorem compares $\mathbb{M}$ and $\mathbb{N}$.

THEOREM 7.13. *There is a symmetric monoidal natural transformation*

$$\alpha : \mathbb{N} \longrightarrow \mathbb{M}$$

such that $\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$ *is a weak equivalence if* X *is cofibrant.*

PROOF. Recall the definition $\mathbb{M}^* = \mathbb{M} \circ \mathbb{D}_{\mathscr{J}} : \mathscr{J} \longrightarrow \mathscr{M}$ (see Definition 2.3 and Notation 2.6). By Corollary 2.7, to construct α, it suffices to construct a natural transformation $\alpha^* : \mathbb{N}^* \longrightarrow \mathbb{M}^*$. Thus consider the orthogonal spectra V^* specified by $V^*(W) = \mathscr{J}(V, W)$. By definition, $\mathbb{M}^*V = \mathbb{M}V^* = \mathbb{J}L\mathbb{U}V^*$. By Proposition 6.14, for $W \subset U$,

$$\mathbb{U}V^*(W) \cong T^{V,W}_{V,W}.$$

For $V \subset W \subset Z$, the structural map agrees under this isomorphism with

$$\oplus : T^{V,W}_{V,W} \wedge S^{Z-W} \cong T^{V,W}_{V,W} \wedge T^{0,Z-W}_{0,Z-W} \longrightarrow T^{V,Z}_{V,Z}.$$

We obtain a map of Thom spaces $T^{V \otimes U,U}_{V,W} \longrightarrow T^{V,W}_{V,W}$ by restricting to V the linear isometries $f : V \otimes U \longrightarrow U$ such that $f(V) \subset W$. These maps define a map of prespectra $T^{V \otimes U,U}_{V,-} \longrightarrow \mathbb{U}V^*$. Applying $\mathbb{J}L$ and using (6.19), there results a map of S-modules

$$\alpha^* : \mathbb{N}^*(V) = \mathbb{J}LT^{V \otimes U,U}_{V,-} \longrightarrow \mathbb{J}L\mathbb{U}V^* = \mathbb{M}^*(V).$$

It is an exercise to verify from Proposition 6.14 and the definitions that these maps specify a natural transformation that is compatible with smash products. Using Theorem 2.5, define

$$\alpha = \alpha^* \otimes_{\mathscr{J}} \mathrm{id} : \mathbb{N}X = \mathbb{N}^* \otimes_{\mathscr{J}} X \longrightarrow \mathbb{M}^* \otimes_{\mathscr{J}} X \cong \mathbb{M}X.$$

Then α is a symmetric monoidal natural transformation, and it remains to prove that $\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$ is a weak equivalence if X is cofibrant. It suffices to show this when X is an $\mathbb{F}I$-cell complex (see [**20**, §6]). Since $\mathbb{M}$ and $\mathbb{N}$ are right exact, it follows by the usual induction up the cellular filtration of X, using commutations with suspension, wedges, pushouts, and colimits, that it suffices to prove that α is a weak equivalence when $X = V^*$. In this case, α reduces to α^*. Again by suspension, it suffices to prove that

$$\Sigma^V \alpha^* : \Sigma^V \mathbb{N}^*(V) \longrightarrow \Sigma^V \mathbb{M}^*(V)$$

is a weak equivalence. We have an untwisting isomorphism (6.22) for the source of $\Sigma^V \alpha^*$ and an analogous isomorphism

$$\mathbb{M}(V^*) \wedge S^V \cong \mathscr{I}(V, U)_+ \wedge S$$

for its target. Under these isomorphisms, $\Sigma^V \alpha^*$ is the smash product with S of the map $\mathscr{I}(V \otimes U, U) \longrightarrow \mathscr{I}(V, U)$ induced by restriction of linear isometries, and this map is a homotopy equivalence since its source and target are contractible. □

REMARK 7.14. By Proposition 2.8, the functor $\mathbb{M}$ has right adjoint $\mathbb{M}^{\#}$. However, $\mathbb{M}$ does not appear to preserve cofibrant objects and does not appear to be part of a Quillen equivalence.

8. A revisionist view of infinite loop space theory

In 1971 [**21**], the second author gave an infinite loop space machine for the passage from space level data to spectra. That machine gave coordinatized spectra as its output. He improved the machine and showed how to feed category level data into it a little later [**22**]. In 1980 [**24**], he retooled the machine to give coordinate-free spectra as its output. The main motivation for the retooling was to show that space level and category level pairing data give rise to pairings $X \wedge Y \longrightarrow Z$ of spectra. Of course, this long preceded the formal introduction of diagram spectra. Nevertheless, their use was implicit in [**21**] and explicit in [**24**], as we now explain.

We first show that the original machine of [**21**] takes values in symmetric spectra. We retain most of the notations of [**21**] and refer to it for details. The machine of [**21**] was based on the little n-cubes operads $\mathscr{C}_n$, which, in the earlier language of PROP's, were introduced by Boardman and Vogt [**3**]. The jth space $\mathscr{C}_n(j)$ consists of j-tuples of little n-cubes with disjoint interiors. A little n-cube is a map $f : I^n \longrightarrow I^n$ that is the product of n linear maps $f_i : I \longrightarrow I$, $f_i(t) = (y_i - x_i)t + x_i$ with $0 \leq x_i < y_i \leq 1$. Obviously Σ_n acts on I^n by permuting coordinates and acts on little n-cubes by conjugation, $(\sigma \cdot f)(t) = \sigma f(\sigma^{-1} t)$ for $t \in I^n$. This means that the $\mathscr{C}_n$ give a symmetric sequence of operads. Therefore they give an associated symmetric sequence of monads C_n on the category $\mathscr{T}$ of based spaces. Given another operad $\mathscr{C}$, in practice an E_∞ or at least spacewise contractible operad, one can form the product operads $\mathscr{D}_n = \mathscr{C} \times \mathscr{C}_n$. Via the action of the symmetric groups on the $\mathscr{C}_n$, this is another symmetric sequence of operads, and it gives rise to another symmetric sequence of monads D_n. Let X be a $\mathscr{C}$-space. By pullback along the projections to $\mathscr{C}$, X is a $\mathscr{D}_n$-space for all n. There is a map of monads $\alpha_n : C_n \longrightarrow \Omega^n \Sigma^n$ for each n, and an adjoint right action of the monad C_n on the n-fold suspension functor Σ^n. With the evident actions of the symmetric groups, the α_n give a map of symmetric sequences of monads. Via the projections $\mathscr{D}_n \longrightarrow \mathscr{C}_n$, these statements remain true with the $\mathscr{C}_n$ replaced by the $\mathscr{D}_n$. Define

$$T_n(X) = B(\Sigma^n, D_n, X).$$

This is a Σ_n-space. Taking the product of a little n-cube with the identity map on I^m gives a little $(n+m)$-cube. This gives a map of operads $\mathscr{C}_n \longrightarrow \mathscr{C}_{n+m}$, and thus a map of operads $\mathscr{D}_n \longrightarrow \mathscr{D}_{n+m}$. The functor Σ^m is given by smashing with S^m, and we obtain a canonical map

$$\sigma : \Sigma^m T_n(X) \cong B(\Sigma^{n+m}, D_n, X) \longrightarrow B(\Sigma^{n+m}, D_{n+m}, X).$$

This map is $(\Sigma_n \times \Sigma_m)$-equivariant. This proves the following result.

THEOREM 8.1. *For a $\mathscr{C}$-space X, the spaces $T_n(X)$ and structure maps σ specify a symmetric spectrum $T(X)$.*

The main theorem in this approach to infinite loop space theory can be stated as follows [**21, 22**].

THEOREM 8.2. *If $\mathscr{C}$ is spacewise contractible, then the adjoint structure maps $\tilde{\sigma} : T_n(X) \longrightarrow \Omega T_{n+1}(X)$ are weak equivalences for $n > 0$, and there is a canonical map $\eta : X \longrightarrow \Omega^n T_n(X)$ that is a group completion for $n > 1$.*

From the point of view of symmetric spectra, this means that $T(X)$ is a positive Ω-spectrum (a fibrant object in the positive stable model structure), and the zeroth space of its associated Ω-spectrum (a fibrant approximation in the stable model structure) is a group completion of X. Taking the original point of view of [**21, 22**], we note that the $\tilde{\sigma}$ are inclusions, so that we can pass to colimits to obtain a spectrum $E(X)$ with nth space $E_n(X) = \operatorname{colim}_m \Omega^m T_{m+n}(X)$ together with a group completion $\eta : X \longrightarrow E_0(X)$. Implicitly, $E(X)$ is obtained from the symmetric spectrum $T(X)$ by applying the forgetful functor to prespectra and then the spectrification functor. We can instead prolong $T(X)$ to an orthogonal spectrum and apply the functor $\mathbb{N}$ (or $\mathbb{M}$) from orthogonal spectra to S-modules, a process which retains more precise information.

In [**28**], this machine is generalized to take $\hat{\mathscr{C}}$-spaces as input, where $\hat{\mathscr{C}}$ is the "category of operators" associated to $\mathscr{C}$. The discussion above applies just as well to the generalized machine, which again gives symmetric spectra as output. The generalized machine accepts Segal's Γ-spaces as special cases of its input. Roughly speaking, the uniqueness theorem of [**28**] says that, up to equivalence, the functor E and natural group completion $\eta : X \longrightarrow E(X)$ from the category of $\hat{\mathscr{C}}$-spaces to the category of Ω-spectra is unique.

There is a $(\Sigma_m \times \Sigma_n)$-equivariant pairing of operads $(\mathscr{C}_m, \mathscr{C}_n) \longrightarrow \mathscr{C}_{m+n}$ [**21**, 8.3]. These pairings fit naturally and easily into the theory of external smash products of symmetric spectra. Using the internalization of the smash product obtained by Kan extension [**16, 20**], this gives the starting point for an elaboration of infinite loop space theory that shows how to pass from pairings of spaces with operad actions (or category of operator actions) to pairings $X \wedge Y \longrightarrow Z$ of symmetric spectra.

While Theorem 8.1 is a new observation, its coordinate-free analogue was explained in detail in 1980 [**24**, §§5, 6], where orthogonal spectra were introduced under the name of $\mathscr{I}_*$-prespectra. Moreover, the analogue was used there to give the elaboration of infinite loop space theory that shows how to pass from pairings of spaces with operad actions (or category of operator actions) to pairings of orthogonal spectra defined in terms of external smash products. Now that we understand the internalization of the smash product, the arguments given there have stronger conclusions. Implicitly the passage from orthogonal spectra to spectra in [**24**] was obtained by applying the forgetful functor to prespectra and then the spectrification functor. This does not preserve point-set level smash products, and we can instead use the functor $\mathbb{N}$ (or just $\mathbb{M}$). We conclude that all statements in [**24**] about the construction of pairings $X \wedge Y \longrightarrow Z$ of spectra in the homotopy category actually give pairings of S-modules that are well-defined and enjoy good algebraic properties on the point-set level.

We briefly recall how the theory of [**24**] goes. For a finite dimensional real inner product space V, there is a Steiner operad $\mathscr{K}_V$ [**33**]. The group $\mathscr{I}(V, V)$ acts on it in a similar fashion to the action of Σ_n on $\mathscr{C}_n$. In fact, the $\mathscr{K}_V$ give the object function of a functor $\mathscr{K}$ from $\mathscr{I}$ to the category of operads [**24**, 6.7]. We can mimic the discussion above, but replacing the little cubes operads $\mathscr{C}_n$ with the Steiner operads $\mathscr{K}_V$, setting $\mathscr{D}_V = \mathscr{C} \times \mathscr{K}_V$. For a $\hat{\mathscr{C}}$-space X, we construct spaces

$$T_V(X) = B(\Sigma^V, \hat{D}, X)$$

and maps

$$\sigma : \Sigma^{W-V} T_V(X) \longrightarrow T_W(X).$$

(Technically, we have suppressed use of a forgetful functor in writing down the bar construction [**24**, p. 325]). We obtain the following conclusion, which is [**24**, 6.1].

THEOREM 8.3. *For a $\hat{\mathscr{C}}$-space X, the spaces $T_V(X)$ and structure maps σ specify an orthogonal spectrum $T(X)$.*

Pairings of operads, categories of operators, $\mathscr{C}$-spaces, $\hat{\mathscr{C}}$-spaces, and permutative categories are studied and interrelated in [**24**, §§1-4]. There are pairings $(\mathscr{K}_V, \mathscr{K}_W) \longrightarrow \mathscr{K}_{V\oplus W}$ analogous to the pairings $(\mathscr{C}_m, \mathscr{C}_n) \longrightarrow \mathscr{C}_{m+n}$. This material provides input for the infinite loop space theory of pairings and is unchanged by the present revisionist attitude towards the output of that theory. By internalizing the output external pairings, we obtain the following reinterpretation of [**24**, 6.2].

THEOREM 8.4. *Let $\wedge : (\mathscr{C}, \mathscr{D}) \longrightarrow \mathscr{E}$ be a pairing of operads. Then pairings $f : (X, Y) \longrightarrow Z$ of a $\hat{\mathscr{C}}$-space X and a $\hat{\mathscr{D}}$-space Y to an $\hat{\mathscr{E}}$-space Z functorially determine maps $Tf : TX \wedge TY \longrightarrow TZ$ of orthogonal spectra.*

In particular, by [**24**, 2.2], this applies to pairings of permutative categories. There is an analogous result [**24**, 6.3] for ring spectra. In [**24**], ring spectra were thought of in the classical, up to homotopy, sense. While [**24**, 6.3] can now be reinterpreted on the point set level, the result then seems to be without application since the resulting input data are too stringent to arise in nature; see [**24**, p. 310]. Thus the present reinterpretation of output data does not obviate the need for the much more elaborate multiplicative infinite loop space theory of [**25**]. That theory shows how to pass from bipermutative categories to E_∞ ring spectra, alias commutative S-algebras. By the comparisons here and in [**20**], it follows that bipermutative categories give rise to commutative symmetric ring spectra and commutative orthogonal ring spectra. It is plausible that a different line of argument might give a direct construction of this sort.

The equivariant generalization of infinite loop space theory shows how to construct G-spectra from equivariant space and category level input. The natural output of the equivariant version of the machine that we have been discussing is given by orthogonal G-spectra, which are the objects of study of the rest of this monograph. We intend to return to equivariant infinite loop space theory elsewhere.

CHAPTER II

Equivariant orthogonal spectra

This chapter parallels [**20**, Part I], and we focus on points of equivariance. It turns out that we need to distinguish carefully between topological G-categories $\mathscr{C}_G$, which are enriched over G-spaces, and their G-fixed topological categories $G\mathscr{C}$, which are enriched over spaces. After explaining this in §1, we define orthogonal G-spectra in §2, discuss their smash product in §3, and reinterpret the definition in terms of diagram spaces in §4. Recall that G is assumed throughout to be a compact Lie group.

1. Preliminaries on equivariant categories

Recall that $\mathscr{T}$ denotes the category of based spaces, where spaces are understood to be compactly generated (= weak Hausdorff k-spaces). Let $G\mathscr{T}$ denote the category of based G-spaces and based G-maps. Then $G\mathscr{T}$ is complete and cocomplete, and it is a closed symmetric monoidal category under its smash product and function G-space functors. For based G-spaces A and B, we write $F(A,B)$ for the function G-space of all continuous maps $A \longrightarrow B$, with G acting by conjugation. Thus

$$G\mathscr{T}(A,B) = F(A,B)^G.$$

That is, a G-map $A \longrightarrow B$ is a fixed point of $F(A,B)$.

It is useful to think of $G\mathscr{T}$ in a different fashion. Let $\mathscr{T}_G$ be the category of based G-spaces (with specified action of G) and non-equivariant maps, which we henceforward call "arrows" to avoid confusion between maps and G-maps. Thus

$$\mathscr{T}_G(A,B) = F(A,B).$$

Then $\mathscr{T}_G$ is enriched over $G\mathscr{T}$: its morphism spaces are G-spaces, and composition is given by G-maps. The objects of $G\mathscr{T}$ and $\mathscr{T}_G$ are the same. If we think of G as acting trivially on the collection of objects (after all, $gA = A$ for all $g \in G$), then we may think of $G\mathscr{T}$ as the G-fixed point category $(\mathscr{T}_G)^G$.

Observe that $\mathscr{T}_G$ is also closed symmetric monoidal under the smash product and function G-space functors, with S^0 as unit. If we ignore the fact that $\mathscr{T}_G$ is enriched over $G\mathscr{T}$, we obtain inverse equivalences of categories $\mathscr{T}_G \longrightarrow \mathscr{T}$ and $\mathscr{T} \longrightarrow \mathscr{T}_G$ by forgetting the action of G on G-spaces and by giving spaces the trivial action by G. Of course, limits and colimits of diagrams of G-spaces (taken in $\mathscr{T}$) only inherit sensible G-actions when the maps in the diagrams are G-maps, so that we are working in $G\mathscr{T}$.

Many of our equivariant categories will come in pairs like this: we will have a category $\mathscr{C}_G$ consisting of G-objects and nonequivariant "arrows", and a category $G\mathscr{C}$ with the same objects and the G-maps between them. We can think of $G\mathscr{C}$ as $(\mathscr{C}_G)^G$, although the notation would be inconvenient. Formally, $\mathscr{C}_G$ will be enriched over $G\mathscr{T}$, so that its hom sets $\mathscr{C}_G(C,D)$ are based G-spaces and composition is given by continuous G-maps. We call such a category a *topological G-category.* As in [**20**], when the morphism spaces of $\mathscr{C}_G$ are given without basepoints, we implicitly give them disjoint G-fixed basepoints.

In all cases that we will encounter, we will have a faithful underlying based G-space functor $\mathbb{S} : \mathscr{C}_G \longrightarrow \mathscr{T}_G$, so that $\mathbb{S}$ embeds $\mathscr{C}_G(C, D)$ as a sub G-space of $F(\mathbb{S}C, \mathbb{S}D)$. We can define a G-map $C \longrightarrow D$ to be a fixed point of $\mathscr{C}_G(C, D)$, and we will have

$$G\mathscr{C}(C, D) = \mathscr{C}_G(C, D)^G \cong \mathbb{S}\mathscr{C}_G(C, D) \cap F(\mathbb{S}C, \mathbb{S}D)^G.$$

We emphasize that it is essential to think in terms of such topological G-categories $\mathscr{C}_G$ even when the categories of ultimate interest are the associated categories $G\mathscr{C}$ of G-objects and G-maps between them. Note that, when constructing model structures, we must work in $G\mathscr{C}$ in order to have limits and colimits.

A *continuous G-functor* $X : \mathscr{C}_G \longrightarrow \mathscr{D}_G$ between topological G-categories is a functor X such that

$$X : \mathscr{C}_G(C, D) \longrightarrow \mathscr{D}_G(X(C), X(D))$$

is a map of G-spaces for all pairs of objects of $\mathscr{C}_G$. In terms of elementwise actions, this means that $gX(f)g^{-1} = X(gfg^{-1})$. It follows that X takes G-maps to G-maps. From now on, all functors defined on topological categories are assumed to be continuous.

A *natural G-map* $\alpha : X \longrightarrow Y$ between G-functors $\mathscr{C}_G \longrightarrow \mathscr{D}_G$ consists of G-maps $\alpha : X(C) \longrightarrow Y(D)$ such that the evident naturality diagrams

$$\begin{array}{ccc} X(C) & \longrightarrow & X(D) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y(C) & \longrightarrow & Y(D) \end{array}$$

commute in $\mathscr{D}_G$ for all arrows (and *not* just all G-maps) $C \longrightarrow D$.

Since the present point of view has not appeared explicitly in previous studies of equivariant stable homotopy theory, we give the definitions of the categories $\mathscr{P}_G$ and $G\mathscr{P}$ of G-prespectra and their full subcategories $\mathscr{S}_G$ and $G\mathscr{S}$ of G-spectra. See [**19**] or [**27**] for more details. In fact, we have such categories for any G-universe U, and we write $\mathscr{P}_G^U$, etc, when necessary for clarity.

Definition 1.1. A *G-universe* U is a sum of countably many copies of each real G-inner product space in some set of irreducible representations of G that includes the trivial representation; U is *complete* if it contains all irreducible representations; U is *trivial* if it contains only trivial representations. An *indexing G-space* in U is a finite dimensional sub G-inner product space of U. When $V \subset W$, we write $W - V$ for the orthogonal complement of V in W. Define $\mathscr{V}(U)$ to be the collection of all real G-inner product spaces that are isomorphic to indexing G-spaces in U.

Write S^V for the one-point compactification of V, and write $\Sigma^V A = A \wedge S^V$ and $\Omega^V A = F(S^V, A)$ for the resulting generalized loop and suspension functors.

Definition 1.2. A G-prespectrum X consists of based G-spaces $X(V)$ for indexing G-spaces $V \subset U$ and based G-maps $\sigma : \Sigma^{W-V} X(V) \longrightarrow X(W)$ for $V \subset W$; here σ is the identity if $V = W$, and the evident transitivity diagram must commute when $V \subset W \subset Z$. An arrow $f : X \longrightarrow Y$ of prespectra consists of based maps $f(V) : X(V) \longrightarrow Y(V)$ that commute with the structure maps σ; f is a G-map if the $f(V)$ are G-maps. A G-spectrum is a G-prespectrum whose adjoint structure G-maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V} X(W)$ are homeomorphisms of G-spaces.

To fit this into the general framework above, define $\mathbb{S}X$ to be the based G-space $\prod_{V\subset U} X(V)$. Then $\mathbb{S} : \mathscr{P}_G \longrightarrow \mathscr{T}_G$ is a faithful functor, and an arrow $f : X \longrightarrow Y$ of prespectra is a G-map if and only if $\mathbb{S}f$ is a fixed point of $F(\mathbb{S}X, \mathbb{S}Y)$. In previous work in this area, the focus is solely on the G-fixed categories $G\mathscr{P}$ and $G\mathscr{S}$.

When U is the trivial universe, $G\mathscr{S}$ is the category of *naive* G-spectra, or spectra with G-actions. When U is a complete universe, $G\mathscr{S}$ is the category of *genuine* G-spectra (and the adjective is omitted): these G-spectra are the objects of the equivariant stable homotopy category of [**19**].

Remark 1.3. The reader experienced in category theory may prefer a different way of thinking about the material of this section, recasting it all in the language of enriched category theory. From the point of view of enriched category theory, we have the category $G\mathscr{C}$ of G-objects and G-maps, which we view as enriched over $\mathscr{T}$: all of our categories are topological, meaning that the categorical hom sets are based topological spaces and composition is given by continuous based maps. We can also view the category $G\mathscr{C}$ as enriched over the category of G-spaces, with the enrichment given by the "enriched hom" G-spaces $\mathscr{C}_G(C, D)$. From that point of view the "category" $\mathscr{C}_G$ is a red herring, an artifact of our special situation rather than something intrinsically relevant to the mathematics. Its "arrows", the points of the $\mathscr{C}_G(C, D)$, are special to the concrete nature of our equivariant situation and should not be thought of as morphisms in a category of their own. Our G-functors and natural G-maps are just examples of the category theorists' $G\mathscr{T}$-enriched functors and $G\mathscr{T}$-enriched natural transformations. The naturality may be expressed conceptually by the commutative diagram of G-spaces

$$\begin{array}{ccc} \mathscr{C}_G(C,D) & \xrightarrow{X} & \mathscr{D}_G(X(C),X(D)) \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathscr{D}_G(Y(C),Y(D)) & \xrightarrow[\alpha^*]{} & \mathscr{D}_G(X(C),Y(D)), \end{array}$$

with no mention of arrows at all. For accessibility and to parallel more closely the nonequivariant theory, we have chosen to avoid introducing the extra language of enriched category theory and to treat $\mathscr{C}_G$ concretely. Our orthogonal G-spectra are G-functors, thought of as objects in a category of diagrams. Their domain categories are of the form $\mathscr{C}_G$ and not $G\mathscr{C}$, with arrows as morphisms. We find it generally more convenient to talk about orthogonal G-spectra concretely as ordinary functors with additional structure rather than as enriched functors in the category theorists' preferred language. The reader familiar with this language may view the use of $\mathscr{C}_G$ as just a notational device to record the use of the $G\mathscr{T}$ enrichment of $G\mathscr{C}$.

2. The definition of orthogonal G-spectra

As with G-spectra, we have several kinds of orthogonal G-spectra, depending on an initial choice of a set of irreducible representations of G. The reader is warned that, as explained in [**20**, 7.1], non-trivial orthogonal G-spectra are never G-spectra in the sense of the Definition 1.2.

Definition 2.1. Let $\mathscr{V} = \mathscr{V}(U)$ for some universe U. Define $\mathscr{I}_G^{\mathscr{V}}$ to be the (unbased) topological G-category whose objects are those of $\mathscr{V}$ and whose arrows are the linear isometric isomorphisms, with G acting by conjugation on the space $\mathscr{I}_G^{\mathscr{V}}(V,W)$ of arrows $V \longrightarrow W$. Let $G\mathscr{I}^{\mathscr{V}}$ be the category with the same objects and the G-linear isometric isomorphisms between them, so that

$$G\mathscr{I}^{\mathscr{V}}(V,W) = \mathscr{I}_G^{\mathscr{V}}(V,W)^G.$$

Define a canonical G-functor $S_G^{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}} \longrightarrow \mathscr{T}_G$ by sending V to S^V. Clearly $\mathscr{I}_G^{\mathscr{V}}$ is a symmetric monoidal category under direct sums of G-inner product spaces, and the functor $S_G^{\mathscr{V}}$ is strong symmetric monoidal.

Variant 2.2. We could relax the conditions on $\mathscr{V}$ by allowing any cofinal subcollection $\mathscr{W}$ of $\mathscr{V}$ that is closed under finite direct sums. Here "cofinal" means that, up to G-isomorphism, every V in $\mathscr{V}$ is contained in some W in $\mathscr{W}$. We shall need the extra generality when we consider change of groups.

We usually abbreviate $\mathscr{I}_G = \mathscr{I}_G^{\mathscr{V}}$ and $S_G = S_G^{\mathscr{V}}$. The case of central interest is $\mathscr{V} = \mathscr{A}\ell\ell$, the collection of all finite dimensional real G-inner product spaces, but we shall work with the general case until we specify otherwise. From here, the basic categorical definitions and constructions of [**20**] go through without essential change. The only new point to keep track of is which arrows are G-maps and which are not. We give a quick summary. We shall not spell out diagrams, referring to [**20**] instead. We choose and fix a skeleton $sk\mathscr{I}_G$ of $\mathscr{I}_G$.

Definition 2.3. An *$\mathscr{I}_G$-space* is a (continuous) G-functor $X : \mathscr{I}_G \longrightarrow \mathscr{T}_G$. Let $\mathscr{I}_G\mathscr{T}$ be the category whose objects are the $\mathscr{I}_G$-spaces X and whose arrows are the natural transformations $X \longrightarrow Y$. Topologize the set $\mathscr{I}_G\mathscr{T}(X,Y)$ of arrows $X \longrightarrow Y$ as a subspace of the product over $V \in sk\mathscr{I}_G$ of the function spaces $F(X(V),Y(V))$ and let G act on it by conjugation; this implicitly specifies an underlying based G-space functor $\mathbb{S} : \mathscr{I}_G\mathscr{T} \longrightarrow \mathscr{T}_G$. Let $G\mathscr{I}\mathscr{T}$ be the category of $\mathscr{I}_G$-spaces and natural G-maps, so that

$$G\mathscr{I}\mathscr{T}(X,Y) = \mathscr{I}_G\mathscr{T}(X,Y)^G.$$

It is essential to keep in mind the distinction between arrows and G-maps of $\mathscr{I}_G$-spaces. We are interested primarily in the G-maps.

Definition 2.4. For $\mathscr{I}_G$-spaces X and Y, define the *"external" smash product* $X \bar{\wedge} Y$ by

$$X \bar{\wedge} Y = \wedge \circ (X \times Y) : \mathscr{I}_G \times \mathscr{I}_G \longrightarrow \mathscr{T}_G;$$

thus $(X \bar{\wedge} Y)(V,W) = X(V) \wedge Y(W)$. For an $\mathscr{I}_G$-space Y and an $(\mathscr{I}_G \times \mathscr{I}_G)$-space Z, define the *external function $\mathscr{I}_G$-space* $\bar{F}(Y,Z)$ by

$$\bar{F}(Y,Z)(V) = \mathscr{I}_G\mathscr{T}(Y, Z\langle V \rangle),$$

where $Z\langle V\rangle(W) = Z(V,W)$.

Remark 2.5. The definition generalizes to give the external smash product functor

$$\mathscr{I}_G^{\mathscr{V}}\mathscr{T} \times \mathscr{I}_G^{\mathscr{V}'}\mathscr{T} \longrightarrow (\mathscr{I}_G^{\mathscr{V}} \times \mathscr{I}_G^{\mathscr{V}'})\mathscr{T}.$$

Definition 2.6. An *$\mathscr{I}_G$-spectrum*, or *orthogonal G-spectrum*, is an $\mathscr{I}_G$-space $X : \mathscr{I}_G \longrightarrow \mathscr{T}_G$ together with a natural structure G-map $\sigma : X \bar{\wedge} S_G \longrightarrow X \circ \oplus$ such that the evident unit and associativity diagrams commute [**20**, §§1,8]. Let

$\mathscr{I}_G\mathscr{S}$ denote the topological G-category of $\mathscr{I}_G$-spectra and arrows $f : X \longrightarrow Y$ that commute with the structure G-maps. Explicitly, the following diagrams must commute, where the σ are G-maps but the f are non-equivariant in general:

$$\begin{array}{ccc} X(V) \wedge S^W & \xrightarrow{\ \sigma\ } & X(V \oplus W) \\ {\scriptstyle f\wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ Y(V) \wedge S^W & \xrightarrow[\ \sigma\ ]{} & Y(V \oplus W) \end{array}$$

If these diagrams commute, then so do the diagrams obtained by replacing f by gf for $g \in G$, so that $\mathscr{I}_G\mathscr{S}(X,Y)$ is indeed a sub G-space of $\mathscr{I}_G\mathscr{T}(X,Y)$. Let $G\mathscr{I}\mathscr{S}$ denote the category of $\mathscr{I}_G$-spectra and the G-maps between them, so that

$$G\mathscr{I}\mathscr{S}(X,Y) = \mathscr{I}_G\mathscr{S}(X,Y)^G.$$

$\mathscr{I}_G$-spectra are G-prespectra by neglect of structure.

DEFINITION 2.7. Let $\mathscr{V} = \mathscr{V}(U)$. Define a discrete subcategory (identity morphisms only) of $\mathscr{I}_G$ whose objects are the indexing G-spaces in U. By restricting functors $\mathscr{I}_G \longrightarrow \mathscr{T}_G$ to this subcategory and using structure maps for $V \oplus (W - V)$, where $V \subset W$, we obtain forgetful functors

$$\mathbb{U} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{P}_G \quad \text{and} \quad \mathbb{U} : G\mathscr{I}\mathscr{S} \longrightarrow G\mathscr{P}.$$

Working with orthogonal G-spectra, we have an equivariant notion of a functor with smash product (FSP). It was used in [**13**] and, implicitly, [**14**].

DEFINITION 2.8. An $\mathscr{I}_G$-FSP is an $\mathscr{I}_G$-space X with a unit G-map $\eta : S \longrightarrow X$ and a natural product G-map $\mu : X \,\overline{\wedge}\, X \longrightarrow X \circ \oplus$ of functors $\mathscr{I}_G \times \mathscr{I}_G \longrightarrow \mathscr{T}_G$ such that the evident unit, associativity, and centrality of unit diagrams commute [**20**, 22.3]. An $\mathscr{I}_G$-FSP is commutative if the evident commutativity diagram also commutes.

We have the topological G-category of $\mathscr{I}_G$-FSP's and its G-fixed point category of G-maps of $\mathscr{I}_G$-FSP's. An $\mathscr{I}_G$-FSP is an $\mathscr{I}_G$-spectrum with additional structure.

LEMMA 2.9. *An $\mathscr{I}_G$-FSP has an underlying $\mathscr{I}_G$-spectrum with structure G-map*

$$\sigma = \mu \circ (\mathrm{id} \,\overline{\wedge}\, \eta) : X \,\overline{\wedge}\, S \longrightarrow X \circ \oplus.$$

We emphasize that all structure maps (σ, η, μ) in the definitions above must be G-maps, while their naturality requires their commutation with arrows.

3. The smash product of orthogonal G-spectra

Just as nonequivariantly, we can reinterpret FSP's in terms of a point-set level internal smash product on the category of orthogonal G-spectra that is associative, commutative, and unital up to coherent natural isomorphism.

THEOREM 3.1. *The category $\mathscr{I}_G\mathscr{S}$ of orthogonal G-spectra has a smash product $\wedge_{S_G}$ and function spectrum functor F_{S_G} under which it is a closed symmetric monoidal category with unit S_G.*

Passing to G-fixed points on morphism spaces, we obtain the following corollary.

COROLLARY 3.2. *The category $G\mathscr{I}\mathscr{S}$ is also closed symmetric monoidal under $\wedge_{S_G}$ and F_{S_G}.*

After this section, we will abbreviate $\wedge_{S_G}$ to $\wedge$ and F_{S_G} to F, but the more cumbersome notations clarify the definitions.

DEFINITION 3.3. A G-monoid X in $\mathscr{I}_G\mathscr{S}$ is a monoid in $G\mathscr{I}\mathscr{S}$; that is, the unit $S_G \longrightarrow X$ and product $X \wedge_{S_G} X \longrightarrow X$ must be G-maps. Allowing arrows of such monoids that are not necessarily G-maps, we obtain the G-category of G-monoids in $\mathscr{I}_G\mathscr{S}$; its associated fixed point category is the category of monoids in $G\mathscr{I}\mathscr{S}$. Similarly, we obtain the G-category of commutative G-monoids in $\mathscr{I}_G\mathscr{S}$; its fixed point category is the category of commutative monoids in $G\mathscr{I}\mathscr{S}$.

The external notion of an $\mathscr{I}_G$-FSP translates to the internal notion of a G-monoid in $\mathscr{I}_G\mathscr{S}$.

THEOREM 3.4. *The categories of $\mathscr{I}_G$-FSP's and of commutative $\mathscr{I}_G$-FSP's are isomorphic to the categories of G-monoids in $\mathscr{I}_G\mathscr{S}$ and of commutative G-monoids in $\mathscr{I}_G\mathscr{S}$.*

We adopt a more familiar language for these objects.

DEFINITION 3.5. A (commutative) orthogonal ring G-spectrum is a (commutative) monoid in $G\mathscr{I}\mathscr{S}$.

Theorem 3.4 asserts that (commutative) orthogonal ring G-spectra are the same as (commutative) $\mathscr{I}_G$-FSP's. That is, they are the same structures, but specified in terms of the internal rather than the external smash product.

We outline the proof of Theorem 3.1, which is the same as in [**20**]. We first construct a smash product $\wedge$ on the category of $\mathscr{I}_G$-spaces [**20**, 21.4]. This internalization of the external smash product is given by left Kan extension and is characterized by the adjunction homeomorphism of based G-spaces

$$\mathscr{I}_G\mathscr{T}(X \wedge Y, Z) \cong (\mathscr{I}_G \times \mathscr{I}_G)\mathscr{T}(X \bar{\wedge} Y, Z \circ \oplus). \tag{3.6}$$

An explicit description of $\wedge$ is given in [**20**, 21.4]. There is one key subtle point. The Kan extension is a kind of colimit, and our G-categories of diagrams do not admit colimits in general. However, the assumption that the maps

$$X : \mathscr{I}_G(V, W) \longrightarrow \mathscr{T}_G(X(V), X(W))$$

given by an $\mathscr{I}_G$-space X must be G-maps ensures that the equivalence relation that defines the Kan extension is G-invariant, producing a well-defined $\mathscr{I}_G$-space $X \wedge Y : \mathscr{I}_G \longrightarrow \mathscr{T}_G$ from $\mathscr{I}_G$-spaces X and Y.

There is a concomitant internal function $\mathscr{I}_G$-space functor F constructed from $\bar{F}$ [**20**, 21.6].

PROPOSITION 3.7. *The category of $\mathscr{I}_G$-spaces is closed symmetric monoidal under $\wedge$ and F. Its unit object is the functor $\mathscr{I}_G \longrightarrow \mathscr{T}_G$ that sends 0 to S^0 and sends $V \neq 0$ to a point.*

We can reinterpret orthogonal G-spectra in terms of the internal smash product.

PROPOSITION 3.8. *The $\mathscr{I}_G$-space S_G is a commutative G-monoid in $\mathscr{I}_G\mathscr{T}$, and the category of orthogonal G-spectra is isomorphic to the category of S_G-modules.*

From here, we imitate algebra, thinking of $\wedge$ and F as analogues of $\otimes$ and Hom.

DEFINITION 3.9. For orthogonal G-spectra X and Y, thought of as right and left S_G-modules, define $X \wedge_{S_G} Y$ to be the coequalizer in the category of $\mathscr{I}_G$-spaces (constructed spacewise) displayed in the diagram

$$X \wedge S_G \wedge Y \underset{\mathrm{id}\wedge\mu'}{\overset{\mu\wedge\mathrm{id}}{\rightrightarrows}} X \wedge Y \longrightarrow X \wedge_{S_G} Y,$$

where μ and μ' are the given actions of S_G on X and Y. Then $X \wedge_{S_G} Y$ inherits an $\mathscr{I}_G$-spectrum structure from the $\mathscr{I}_G$-spectrum structure on X or, equivalently, Y. The function orthogonal G-spectrum $F_{S_G}(X, Y)$ is defined dually in terms of a suitable equalizer [**20**, §22]

$$F_{S_G}(Y, Z) \longrightarrow F(Y, Z) \rightrightarrows F(Y \wedge S_G, Z).$$

Theorem 3.1 follows easily from the definitions and the universal property (3.6).

4. A description of orthogonal G-spectra as diagram G-spaces

As in [**20**, 2.1], there is a category $\mathscr{J}_G = \mathscr{J}_G^{\mathscr{V}}$ constructed from $\mathscr{I}_G$ and S_G such that if we define $\mathscr{J}_G$-spaces exactly as in Definition 2.3, then a $\mathscr{J}_G$*-space* is the same structure as an $\mathscr{I}_G$*-spectrum*. This reduces the study of orthogonal G-spectra to a special case of the conceptually simpler study of diagram G-spaces. Rather than repeat the cited formal definition, we give a more concrete alternative description of $\mathscr{J}_G$ in terms of Thom complexes, implicitly generalizing I.6.14 to the equivariant context.

DEFINITION 4.1. We define the topological G-category $\mathscr{J}_G^{\mathscr{V}}$. The objects of $\mathscr{J}_G^{\mathscr{V}}$ are the same as the objects of $\mathscr{I}_G^{\mathscr{V}}$. For objects V and V', let $\mathscr{I}(V, V')$ be the (possibly empty) G-space of linear isometries from V to V'; G acts by conjugation. Of course, a linear isometry is necessarily a monomorphism, but, in contrast to our definition of the category $\mathscr{I}_G$, we no longer restrict attention to linear isometric isomorphisms. Let $E(V, V')$ be the subbundle of the product G-bundle $\mathscr{I}(V, V') \times V'$ consisting of the points (f, x) such that $x \in V' - f(V)$. The G-space $\mathscr{J}_G^{\mathscr{V}}(V, V')$ of arrows $V \longrightarrow V'$ in $\mathscr{J}_G^{\mathscr{V}}$ is the Thom G-space of $E(V, V')$; it is obtained from the fiberwise one-point compactification of $E(V, V')$ by identifying the points at infinity, and it is interpreted to be a point if $\mathscr{I}(V, V')$ is empty. Define composition

$$\circ : \mathscr{J}_G^{\mathscr{V}}(V', V'') \wedge \mathscr{J}_G^{\mathscr{V}}(V, V') \longrightarrow \mathscr{J}_G^{\mathscr{V}}(V, V'') \tag{4.2}$$

by $(g, y) \circ (f, x) = (g \circ f, g(x) + y)$. The points $(\mathrm{id}_V, 0)$ give identity arrows. Observe that $\mathscr{J}_G^{\mathscr{V}}$ is symmetric monoidal under the operation $\oplus$ specified by $V \oplus V'$ on objects and

$$(f, x) \oplus (f', x') = (f \oplus f', x + x')$$

on arrows. Let $G\mathscr{J}^{\mathscr{V}}$ be the G-fixed category with the same objects, so that

$$G\mathscr{J}^{\mathscr{V}}(V, W) = \mathscr{J}_G^{\mathscr{V}}(V, W)^G.$$

We usually abbreviate $\mathscr{J}_G = \mathscr{J}_G^{\mathscr{V}}$. If $\dim V = \dim V'$, then a linear isometry $V \longrightarrow V'$ is an isomorphism and $\mathscr{J}_G(V, V') = \mathscr{I}_G(V, V')_+$. This embeds $\mathscr{I}_G$ as a sub symmetric monoidal category of $\mathscr{J}_G$. If $V \subset V'$, then

$$\mathscr{J}_G(V, V') \cong O(V')_+ \wedge_{O(V'-V)} S^{V'-V}.$$

In particular, the functor $\mathscr{J}_G(0,-) : \mathscr{I}_G \longrightarrow \mathscr{T}_G$ coincides with S_G. The category of $\mathscr{J}_G$-spaces is symmetric monoidal, as in Proposition 3.7 but with unit S_G, and we have the following result.

THEOREM 4.3. *The symmetric monoidal category of $\mathscr{I}_G$-spectra is isomorphic to the symmetric monoidal category of $\mathscr{J}_G$-spaces.*

Using this reinterpretation, we see immediately that the category $G\mathscr{I}\mathscr{S}$ is complete and cocomplete, with limits and colimits constructed levelwise. The category $\mathscr{I}_G\mathscr{S}$ is tensored and cotensored over the category $\mathscr{T}_G$ of based G-spaces. For an orthogonal G-spectrum X and a based G-space A, the tensor $X \wedge A$ is given by the levelwise smash product, $(X \wedge A)(V) = X(V) \wedge A$, and the cotensor $F(A, X)$ is given similarly by the levelwise function space. We have both

$$\mathscr{I}_G\mathscr{S}(X \wedge A, Y) \cong \mathscr{T}_G(A, \mathscr{I}_G\mathscr{S}(X, Y)) \cong \mathscr{I}_G\mathscr{S}(X, F(A, Y)) \tag{4.4}$$

and, by passage to fixed points,

$$G\mathscr{I}\mathscr{S}(X \wedge A, Y) \cong G\mathscr{T}(A, \mathscr{I}_G\mathscr{S}(X, Y)) \cong G\mathscr{I}\mathscr{S}(X, F(A, Y)). \tag{4.5}$$

From the enriched category point of view of Remark 1.3, these adjunctions give that $G\mathscr{I}\mathscr{S}$ is tensored and cotensored over $G\mathscr{T}$. Here the enriched category point of view is clearly the right one to take. When we specialize these adjunctions to spaces A with trivial G-action $\mathscr{I}_G\mathscr{S}(X, Y)$ coincides with the categorical hom space $G\mathscr{I}\mathscr{S}(X, Y)$. Thus the enrichment over $\mathscr{T}$ takes a more elementary form. We define homotopies between maps of orthogonal G-spectra by use of the cylinders $X \wedge I_+$, and similarly for G-homotopies between G-maps.

We also use $\mathscr{J}_G$ to define represented orthogonal G-spectra that give rise to left adjoints to evaluation functors, as in [**20**, §3].

DEFINITION 4.6. For an object V of $\mathscr{I}_G$, define the orthogonal G-spectrum V^* represented by V by $V^*(W) = \mathscr{J}_G(V, W)$. In particular, $0^* = S_G$. Define the *shift desuspension spectrum functors* $F_V : \mathscr{T}_G \longrightarrow \mathscr{I}_G\mathscr{S}$ and the *evaluation functors* $Ev_V : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{T}_G$ by $F_V A = V^* \wedge A$ and $Ev_V X = X(V)$. Then F_V and Ev_V are left and right adjoint:

$$\mathscr{I}_G\mathscr{S}(F_V A, X) \cong \mathscr{T}_G(A, Ev_V X).$$

In order to mesh with notations in [**19, 27**], we introduce alternative names for these functors.

NOTATIONS 4.7. Let $\Sigma^\infty = F_0$ and $\Omega^\infty = Ev_0$. These are the suspension orthogonal G-spectrum and zeroth space functors. Note that $\Sigma^\infty A = S_G \wedge A$. Similarly, let $\Sigma^\infty_V = F_V$ and $\Omega^\infty_V = Ev_V$; we let $S^{-V} = \Sigma^\infty_V S^0$ and call it the *canonical* $(-V)$*-sphere*.

As in [**20**, 1.8], we have the following commutation with smash products.

LEMMA 4.8. *There is a natural isomorphism*

$$F_V A \wedge F_W B \cong F_{V \oplus W}(A \wedge B).$$

As in [**20**, 1.6], but with the tensor product of functors notation of I§2, we have the following description of general orthogonal G-spectra in terms of represented ones. Observe that V^* varies contravariantly in V, so that we have a contravariant functor $\mathbb{D} : \mathscr{J}_G \longrightarrow \mathscr{I}_G\mathscr{S}$ specified by $\mathbb{D}V = V^*$.

LEMMA 4.9. *The evaluation maps $V^* \wedge X(V) \longrightarrow X$ of $\mathscr{I}_G$-spectra X, thought of as $\mathscr{J}_G$-spaces, induce a natural isomorphism*

$$\mathbb{D} \otimes_{\mathscr{J}_G} X = \int^{V \in sk\mathscr{I}_G} V^* \wedge X(V) \longrightarrow X.$$

The definitions and results of this section have analogues for prespectra. Recall Definition 2.7.

DEFINITION 4.10. Let $\mathscr{V} = \mathscr{V}(U)$. We have a subcategory $\mathscr{K}_G = \mathscr{K}_G^{\mathscr{V}}$ of $\mathscr{J}_G$ such that a $\mathscr{K}_G$-space is the same thing as a G-prespectrum. The objects of $\mathscr{K}_G$ are the indexing G-spaces in U; the G-space $\mathscr{K}_G(V, V')$ of arrows is $S^{V'-V}$ if $V \subset V'$ and a point otherwise. The forgetful G-functor $\mathbb{U} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{P}_G$ has a left adjoint prolongation functor $\mathbb{P}$. With the notation of I.2.10, $\mathbb{P}X = \mathbb{D} \circ \iota \otimes_{\mathscr{K}_G} X$, where $\iota : \mathscr{K}_G \longrightarrow \mathscr{J}_G$ is the inclusion. See also [**20**, §3].

CHAPTER III

Model categories of orthogonal G-spectra

We explain the model structures on the category of orthogonal G-spectra and on its various categories of rings and modules. The material here is parallel to the material of [**20**, §§5-12]. However, since we are focusing on orthogonal spectra, some features that were made more complicated by the inclusion of symmetric spectra in the theory there become simpler here. We focus on points of equivariance. One new equivariant feature is the notion of a G-topological model category, which is an equivariant analogue of the classical notion of a topological (or simplicial) model category. To make sense of this, we must take into account the dichotomy between $\mathscr{C}_G$ and $G\mathscr{C}$: only $G\mathscr{C}$ can have a model structure, but use of $\mathscr{C}_G$ is essential to encode the G-topological structure, which is used to prove the model axioms.

1. The model structure on G-spaces

We take for granted the generalities on nonequivariant topological model categories explained in [**20**, §5]. In particular we have the notion of a compactly generated model category, for which the small object argument for verifying the factorization axioms requires only sequential colimits. All of our examples of model categories will be of this form. However there are a few places where equivariance plays a role. We discuss these and then describe the appropriate model structure on the category $G\mathscr{T}$ of based G-spaces. This must be known and is largely implicit in [**29**], but we include a complete treatment, one that we find amusing, for the reader's convenience.

We begin with a topological G-category $\mathscr{C}_G$ and its G-fixed category $G\mathscr{C}$ of G-maps. We assume that $G\mathscr{C}$ is complete and cocomplete and that $\mathscr{C}_G$ is tensored and cotensored over $G\mathscr{T}$, so that (II.4.4) and (II.4.5) hold with $\mathscr{I}_G\mathscr{S}$ and $G\mathscr{I}$ replaced by $\mathscr{C}_G$ and $G\mathscr{C}$. The discussion in [**20**, §5] applies to $G\mathscr{C}$. One place where equivariance is relevant is in the Cofibration Hypothesis, [**20**, 5.3]. That uses the concept of an h-cofibration in $G\mathscr{C}$, namely a map that satisfies the homotopy extension property (HEP) in $G\mathscr{C}$. Since the maps in $G\mathscr{C}$ are G-maps, the HEP is automatically equivariant. That is, h-cofibrations in $G\mathscr{C}$ satisfy the G-HEP. As in [**20**], we write q-cofibration and q-fibration for model cofibrations and cofibrations, but we write cofibrant and fibrant rather than q-cofibrant and q-fibrant.

A more substantial point of equivariance concerns the notion of a topological model category. As defined in [**20**, 5.12], that notion remembers only that $G\mathscr{C}$ is tensored and cotensored over $\mathscr{T}$, which is insufficient for our applications. We shall return to this point and define the notion of a "G-topological model category" after giving the model structure on $G\mathscr{T}$.

DEFINITION 1.1. Let I be the set of cell h-cofibrations

$$i : (G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+$$

in $G\mathscr{T}$, where $n \geq 0$ (S^{-1} being empty) and H runs through the (closed) subgroups of G. Let J be the set of h-cofibrations

$$i_0 : (G/H \times D^n)_+ \longrightarrow (G/H \times D^n \times I)_+$$

and observe that each such map is the inclusion of a G-deformation retract.

Recall that, for unbased spaces A and B, $(A \times B)_+ \cong A_+ \wedge B_+$. Recall too that, for a based H-space A and a based G-space B,

$$G\mathscr{T}(G_+ \wedge_H A, B) \cong H\mathscr{T}(A, B). \tag{1.2}$$

If A is a G-space, then we have a natural homeomorphism of G-spaces

$$G_+ \wedge_H A \cong (G/H)_+ \wedge A, \tag{1.3}$$

where G acts diagonally on the right; it sends the class of $g \wedge a$ to $gH \wedge ga$. Also, for a based space A regarded as a G-trivial G-space,

$$G\mathscr{T}(A, B) \cong \mathscr{T}(A, B^G) \tag{1.4}$$

and therefore

$$G\mathscr{T}((G/H)_+ \wedge A, B) \cong \mathscr{T}(A, B^H). \tag{1.5}$$

As a right adjoint, the G-fixed point functor preserves limits. It also preserves some, but not all, colimits.

Lemma 1.6. *The functor $(-)^G$ on based G-spaces preserves pushouts of diagrams one leg of which is a closed inclusion and colimits of sequences of inclusions. For a based space A and a based G-space B, $F(A, B)^G \cong F(A, B^G)$. For based G-spaces A and B, $(A \wedge B)^G \cong A^G \wedge B^G$.*

Definition 1.7. A map $f : A \longrightarrow B$ of G-spaces is a weak equivalence or Serre fibration if each $f^H : A^H \longrightarrow B^H$ is a weak equivalence or Serre fibration; by (1.5), f is a Serre fibration if and only if it satisfies the RLP (right lifting property) with respect to the maps in J. Note that a relative G-cell complex is a relative I-cell complex as defined in [**20**, 5.4].

Theorem 1.8. *$G\mathscr{T}$ is a compactly generated proper G-topological model category with respect to the weak equivalences, Serre fibrations, and retracts of relative G-cell complexes. The sets I and J are the generating q-cofibrations and the generating acyclic q-cofibrations.*

We have not yet defined what it means for $G\mathscr{T}$ to be "G-topological", and we shall turn to that concept after proving the rest of the theorem. For the proof, we compare $G\mathscr{T}$ to an appropriate model category of diagram spaces. Thus let $G\mathscr{O}$ be the (unbased) topological category of orbit G-spaces G/H and G-maps. We have the category $G\mathscr{O}^{op}\mathscr{T}$ of $G\mathscr{O}^{op}$-spaces, namely contravariant functors $G\mathscr{O} \longrightarrow \mathscr{T}$. This is an example of a category of diagram spaces, so the theory of [**20**, §6] applies to it; see also Piacenza [**29**]. We have functors

$$\Phi : G\mathscr{T} \longrightarrow G\mathscr{O}^{op}\mathscr{T} \quad \text{and} \quad \Lambda : G\mathscr{O}^{op}\mathscr{T} \longrightarrow G\mathscr{T}$$

specified by $\Phi(A)(G/H) = A^H$ and $\Lambda(D) = D(G/e)$. It is the contravariance of $\Phi(A)$ as a functor on $G\mathscr{O}$ that motivates the use of $G\mathscr{O}^{op}$. Clearly $\Lambda \circ \Phi = \mathrm{Id}$. In fact, we have the following elementary observation.

Lemma 1.9. *The functor Φ is full and faithful.*

Lemma 1.6 implies the following further properties of Φ.

LEMMA 1.10. *The functor Φ preserves limits. It also preserves pushouts of diagrams one leg of which is a closed inclusion and colimits of sequences of inclusions. For a based space A and a based G-space B, $\Phi F(A,B) \cong F(A, \Phi B)$. For based G-spaces A and B, $\Phi(A \wedge B) \cong \Phi(A) \wedge \Phi(B)$, where $(\Phi(A) \wedge \Phi(B))(G/H) = A^H \wedge B^H$.*

Note that $\Phi(G/H_+)(G/K) = (G/H)_+^K \cong G\mathscr{O}(G/K, G/H)_+$, so that $\Phi(G/H_+)$ is a represented diagram.

DEFINITION 1.11. A map $f : D \longrightarrow E$ of $G\mathscr{O}^{op}$-spaces is a *level equivalence* or *level fibration* if each $f(G/H)$ is a weak equivalence or Serre fibration. Let $G\mathscr{O}I$ and $G\mathscr{O}J$ be the sets of maps of the form $\Phi(G/H_+) \wedge i$ and $\Phi(G/H_+) \wedge j$, where $i \in I$ and $j \in J$. A *relative $G\mathscr{O}$-cell complex* is a relative $G\mathscr{O}I$-cell complex (see [**20**, 5.4]).

The following result is a special case of [**20**, 6.5]; most of it is in Piacenza [**29**].

THEOREM 1.12. *The category $G\mathscr{O}^{op}\mathscr{T}$ is a compactly generated proper topological model category with respect to the level equivalences, level fibrations, and retracts of relative $G\mathscr{O}$-cell complexes. The sets $G\mathscr{O}I$ and $G\mathscr{O}J$ are the generating q-cofibrations and generating acyclic q-cofibrations.*

PROOF OF THEOREM 1.8. By definition, a map f of G-spaces is a weak equivalence or Serre fibration if and only if Φf is a level equivalence or level Serre fibration. The amusing thing is that we have an analogue for cofibrations. By the preservation properties of Φ already specified, it is clear that Φ carries G-cell complexes to $G\mathscr{O}$-cell complexes. Moreover, because Φ is full and faithful, it is elementary to check inductively that if $g : \Phi(X) \longrightarrow D$ is a relative $G\mathscr{O}$-cell complex, then $g = \Phi(f)$ for a unique relative G-cell complex f; compare [**27**, VI.6.2]. We now see that both lifting axioms, both factorization axioms, and the left and right properness for $G\mathscr{T}$ follow directly from the corresponding results for $G\mathscr{O}^{op}\mathscr{T}$. $\square$

We must still explain what it means for $G\mathscr{T}$ to be a "G-topological" model category. We revert to our general categories $\mathscr{C}_G$ and $G\mathscr{C}$, and we suppose that $G\mathscr{C}$ has a given model structure. For maps $i : A \longrightarrow X$ and $p : E \longrightarrow B$ in $G\mathscr{C}$, let

$$\mathscr{C}_G(i^*, p_*) : \mathscr{C}_G(X, E) \longrightarrow \mathscr{C}_G(A, E) \times_{\mathscr{C}_G(A,B)} \mathscr{C}_G(X, B) \tag{1.13}$$

be the map of G-spaces induced by $\mathscr{C}_G(i, \mathrm{id})$ and $\mathscr{C}_G(\mathrm{id}, p)$ by passage to pullbacks.

DEFINITION 1.14. A model category $G\mathscr{C}$ is *G-topological* if the map $\mathscr{C}_G(i^*, p_*)$ is a Serre fibration (of G-spaces) when i is a q-cofibration and p is a q-fibration and is a weak equivalence (as a map of G-spaces) when, in addition, either i or p is a weak equivalence.

The point is that we must go beyond the category $G\mathscr{C}$ to the category $\mathscr{C}_G$ to formulate this equivariant notion. From the point of view of enriched category theory of Remark 1.3, this is the obviously right enriched version of the standard definitions of a simplicial or topological model category. It follows on passage to G-fixed point spaces that $G\mathscr{C}$ is also nonequivariantly topological, in the sense of [**20**, 5.12], but we need the equivariant version. The nonequivariant version has the following significance.

LEMMA 1.15. *The pair (i, p) has the lifting property if and only if $G\mathscr{C}(i^*, p_*)$ is surjective.*

As in [**20**, §5], we will need two pairs of analogues of the maps $\mathscr{C}_G(i^*, p_*)$. First, for a map $i : A \longrightarrow B$ of based G-spaces and a map $j : X \longrightarrow Y$ in $G\mathscr{C}$, passage to pushouts gives a map

$$i\Box j : (A \wedge Y) \cup_{A\wedge X} (B \wedge X) \longrightarrow B \wedge Y \tag{1.16}$$

and passage to pullbacks gives a map

$$F_\Box(i,j) : F(B,X) \longrightarrow F(A,X) \times_{F(A,Y)} F(B,Y), \tag{1.17}$$

where $\wedge$ and F denote the tensor and cotensor in $\mathscr{C}_G$.

Second, assume that $\mathscr{C}_G$ is a closed symmetric monoidal category with product $\wedge_\mathscr{C}$ and internal function object functor $F_\mathscr{C}$. Then, for maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ in $G\mathscr{C}$, passage to pushouts gives a map

$$i\Box j : (Y \wedge_\mathscr{C} W) \cup_{X\wedge_\mathscr{C} W} (X \wedge_\mathscr{C} Z) \longrightarrow Y \wedge_\mathscr{C} Z, \tag{1.18}$$

and passage to pullbacks gives a map

$$F_\Box(i,j) : F_\mathscr{C}(Y,W) \longrightarrow F_\mathscr{C}(X,W) \times_{F_\mathscr{C}(X,Z)} F_\mathscr{C}(Y,Z). \tag{1.19}$$

Inspection of definitions gives adjunctions relating these maps.

LEMMA 1.20. *Let $i : A \longrightarrow B$ be a map of based G-spaces and let $j : X \longrightarrow Y$ and $p : E \longrightarrow F$ be maps in $G\mathscr{C}$. Then there are natural isomorphisms of G-maps*

$$\mathscr{C}_G((i\Box j)^*, p_*) \cong \mathscr{T}_G(i^*, \mathscr{C}_G(j^*, p_*)_*) \cong \mathscr{C}_G(j^*, F_\Box(i,p)_*).$$

Therefore, passing to G-fixed points, $(i\Box j, p)$ has the lifting property in $G\mathscr{C}$ if and only if $(i, \mathscr{C}_G(j^*, p_*))$ has the lifting property in $G\mathscr{T}$.

LEMMA 1.21. *Let i, j, and p be maps in $G\mathscr{C}$, where $\mathscr{C}_G$ is closed symmetric monoidal. Then there is a natural isomorphism of G-maps*

$$\mathscr{C}_G((i\Box j)^*, p_*) \cong \mathscr{C}_G(i^*, F_\Box(j,p)_*).$$

Returning to $\mathscr{T}_G$ and using Lemma 1.20, we see by a formal argument that the following lemma is equivalent to the assertion that $G\mathscr{T}$ is G-topological.

LEMMA 1.22. *Let $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ be q-cofibrations of G-spaces. Then $i\Box j$ is a q-cofibration and is acyclic if i or j is acyclic.*

PROOF. By passage to wedges, pushouts, colimits, and retracts, it suffices to prove the first part for a pair of generating q-cofibrations. Here the conclusion holds because products $G/H \times G/K$ with the diagonal action are triangulable as (finite) G-CW complexes. This is trivial when G is finite and holds for general compact Lie groups by [**17**]. It suffices to prove the second part when i is a generating acyclic q-cofibration, but then i is the inclusion of a G-deformation retract and the conclusion is clear. □

2. The level model structure on orthogonal G-spectra

We here give the category $G\mathscr{I}\mathscr{S}$ of orthogonal G-spectra and G-maps a level model structure, following [**20**, §3]; maps will mean G-maps throughout. We need three definitions, the first of which concerns nondegenerate basepoints. A G-space is said to be nondegenerately based if the inclusion of its basepoint is an unbased h-cofibration (satisfies the G-HEP in the category of unbased G-spaces). As in [**34**, Prop. 9], a based h-cofibration between nondegenerately based G-spaces is an unbased h-cofibration. Each morphism space $\mathscr{J}_G(V,W)$ is nondegenerately based.

DEFINITION 2.1. An orthogonal G-spectrum X is nondegenerately based if each $X(V)$ is a nondegenerately based G-space.

DEFINITION 2.2. Define FI to be the set of all maps $F_V i$ with $V \in sk\mathscr{I}_G$ and $i \in I$. Define FJ to be the set of all maps $F_V j$ with $V \in sk\mathscr{I}_G$ and $j \in J$, and observe that each map in FJ is the inclusion of a G-deformation retract.

DEFINITION 2.3. We define five properties of maps $f : X \longrightarrow Y$ of orthogonal G-spectra.

(i) f is a *level equivalence* if each map $f(V) : X(V) \longrightarrow Y(V)$ of G-spaces is a weak equivalence.
(ii) f is a *level fibration* if each map $f(V) : X(V) \longrightarrow Y(V)$ of G-spaces is a Serre fibration.
(iii) f is a *level acyclic fibration* if it is both a level equivalence and a level fibration.
(iv) f is a *q-cofibration* if it satisfies the LLP with respect to the level acyclic fibrations.
(v) f is a *level acyclic q-cofibration* if it is both a level equivalence and a q-cofibration.

THEOREM 2.4. *The category $G\mathscr{I}\mathscr{S}$ of orthogonal G-spectra is a compactly generated proper G-topological model category with respect to the level equivalences, level fibrations, and q-cofibrations. The sets FI and FJ are the generating q-cofibrations and the generating acyclic q-cofibrations, and the following identifications hold.*

(i) *The level fibrations are the maps that satisfy the RLP with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes, and all orthogonal G-spectra are level fibrant.*
(ii) *The level acyclic fibrations are the maps that satisfy the RLP with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.*
(iii) *The q-cofibrations are the retracts of relative FI-cell complexes.*
(iv) *The level acyclic q-cofibrations are the retracts of relative FJ-cell complexes.*

Moreover, every cofibrant orthogonal G-spectrum X is nondegenerately based.

The proof is the same as that of [**20**, 6.5]. As there, the following analogue of [**20**, 5.5] plays a role.

LEMMA 2.5. *Every q-cofibration is an h-cofibration.*

The following analogue of [**20**, 3.7] also holds. The proof depends on II.4.8 and Lemma 1.22 and thus on the fact that products of orbit spaces are triangulable as G-CW complexes.

LEMMA 2.6. *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are q-cofibrations, then*

$$i \Box j : (Y \wedge W) \cup_{X \wedge W} (X \wedge Z) \longrightarrow Y \wedge Z$$

is a q-cofibration which is level acyclic if either i or j is level acyclic. In particular, if Z is cofibrant, then $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is a q-cofibration, and the smash product of cofibrant orthogonal G-spectra is cofibrant.

Let $[X,Y]^\ell_G$ denote the set of maps $X \longrightarrow Y$ in the level homotopy category $\mathrm{Ho}_\ell G\mathscr{I}\mathscr{S}$ and let $\pi(X,Y)_G$ denote the set of homotopy classes of maps $X \longrightarrow Y$. Then $[X,Y]^\ell_G \cong \pi(\Gamma X, Y)_G$, where $\Gamma X \longrightarrow X$ is a cofibrant approximation of X.

Fiber and cofiber sequences of orthogonal G-spectra behave the same way as for based G-spaces, starting from the usual definitions of homotopy cofibers Cf and homotopy fibers Ff [**20**, 6.8]. We record the analogue of [**20**, 6.9]. Most of the proof is the same as there. Some statements, such as the last clause of (i), are most easily proven by using (1.5) and Lemma 1.6 to reduce them to their nonequivariant counterparts by levelwise passage to fixed points.

THEOREM 2.7. (i) *If A is a based G-CW complex and X is a nondegenerately based orthogonal G-spectrum, then $X \wedge A$ is nondegenerately based and*

$$[X \wedge A, Y]^\ell_G \cong [X, F(A,Y)]^\ell_G$$

for any Y. If $f : X \longrightarrow Y$ is a level equivalence of nondegenerately based orthogonal G-spectra, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a level equivalence.

(ii) *For nondegenerately based X_i, $\bigvee_i X_i$ is nondegenerately based and*

$$[\bigvee_i X_i, Y]^\ell_G \cong \prod_i [X_i, Y]^\ell_G$$

for any Y. A wedge of level equivalences of nondegenerately based orthogonal G-spectra is a level equivalence.

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and $f : A \longrightarrow Y$ is any map of orthogonal G-spectra, where A, X, and Y are nondegenerately based, then $X \cup_A Y$ is nondegenerately based and the cobase change $j : Y \longrightarrow X \cup_A Y$ is an h-cofibration. If i is a level equivalence, then j is a level equivalence.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are level equivalences in the following commutative diagram of nondegenerately based orthogonal G-spectra, then the induced map of pushouts is a level equivalence.*

$$\begin{array}{ccccc} X & \xleftarrow{\;i\;} & A & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xleftarrow[\;i'\;]{} & A' & \longrightarrow & Y' \end{array}$$

(v) *If X is the colimit of a sequence of h-cofibrations $i_n : X_n \longrightarrow X_{n+1}$ of nondegenerately based orthogonal G-spectra, then X is nondegenerately based and there is a $\lim^1$ exact sequence of pointed sets*

$$* \longrightarrow \lim{}^1 [\Sigma X_n, Y]^\ell_G \longrightarrow [X,Y]^\ell_G \longrightarrow \lim [X_n, Y]^\ell_G \longrightarrow *$$

for any Y. If each i_n is a level equivalence, then the map from the initial term X_0 into X is a level equivalence.

(vi) *If $f : X \longrightarrow Y$ is a map of nondegenerately based orthogonal G-spectra, then Cf is nondegenerately based and there is a natural long exact sequence*

$$\cdots \to [\Sigma^{n+1} X, Z]^\ell_G \to [\Sigma^n Cf, Z]^\ell_G \to [\Sigma^n Y, Z]^\ell_G \to [\Sigma^n X, Z]^\ell_G \to \cdots \to [X,Z]^\ell_G.$$

We shall also need a variant of the level model structure, called the positive level model structure, as in [**20**, §14]. It is obtained by ignoring representations V that do not contain a positive dimensional trivial representation. We can obtain a similar model structure by ignoring only $V = 0$, but that would not give the right model structure for some of our applications.

DEFINITION 2.8. Define the positive analogues of the classes of maps specified in Definition 2.3 by restricting attention to those levels V with $V^G \neq 0$.

DEFINITION 2.9. Let F^+I and F^+J be the sets of maps in FI and FJ that are specified in terms of the functors F_V with $V^G \neq 0$.

THEOREM 2.10. *The category $G\mathscr{I}\mathscr{S}$ is a compactly generated proper G-topological model category with respect to the positive level equivalences, positive level fibrations, and positive level q-cofibrations. The sets F^+I and F^+J are the generating sets of positive q-cofibrations and positive level acyclic q-cofibrations. The positive q-cofibrations are the q-cofibrations that are homeomorphisms at all levels V such that $V^G = 0$.*

PROOF. As in [**20**, §14], this is a special case of a general relative version of Theorem 2.4. For the last statement, the positive q-cofibrations are the retracts of the relative F^+I-cell complexes, and a relative FI-cell complex is a homeomorphism at levels V with $V^G \neq 0$ if and only if no standard cells $F_V i$ with $V^G \neq 0$ occur in its construction. □

VARIANTS 2.11. There are other variants. Rather than using varying categories $\mathscr{I}_G^{\mathscr{V}}$, we could work with orthogonal G-spectra defined with respect to $\mathscr{V} = \mathscr{A}\ell\ell$ and define a "$\mathscr{V}$-level model structure" by restricting to those levels V that are isomorphic to representations in $\mathscr{V}$ when defining level equivalences, level fibrations, and the generating sets of q-cofibrations and acyclic q-cofibrations. This allows us to change $\mathscr{V}$ by changing the model structure on a single fixed category of orthogonal G-spectra; see Remark 1.9.

REMARK 2.12. Everything in this section applies verbatim to the category $G\mathscr{P}$ of G-prespectra. Recall II.2.7 and II.4.10. Because $\mathscr{K}_G$ contains all objects of $\mathscr{J}_G$, the forgetful functor $\mathbb{U} : G\mathscr{I} \longrightarrow G\mathscr{P}$ creates the level equivalences and level fibrations of orthogonal G-spectra. That is, a map f of orthogonal G-spectra is a level equivalence or level fibration if and only if $\mathbb{U}f$ is a level equivalence or level fibration of prespectra. In particular $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair [**20**, A.1].

3. The homotopy groups of G-prespectra

By II.2.7, an orthogonal G-spectrum has an underlying G-prespectrum indexed on a universe U such that $\mathscr{V}(U) = \mathscr{V}$. The homotopy groups of orthogonal G-spectra are defined to be the homotopy groups of their underlying G-prespectra, and we discuss the homotopy groups of G-prespectra here. We first define Ω-G-spectra (more logically, prespectra).

DEFINITION 3.1. A G-prespectrum X is an *Ω-G-spectrum* if each of its adjoint structure maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(W)$ is a weak equivalence of G-spaces. An orthogonal G-spectrum is an *orthogonal Ω-G-spectrum* if each of its adjoint structure maps is a weak equivalence or, equivalently, if its underlying G-prespectrum is an Ω-G-spectrum.

It is convenient to write

$$\pi_q^H(A) = \pi_q(A^H)$$

for based G-spaces A.

DEFINITION 3.2. For subgroups H of G and integers q, define the homotopy groups $\pi_q^H(X)$ of a G-prespectrum X by

$$\pi_q^H(X) = \mathrm{colim}_V \, \pi_q^H(\Omega^V X(V)) \ \text{ if } \ q \geq 0,$$

where V runs over the indexing G-spaces in U, and

$$\pi_{-q}^H(X) = \mathrm{colim}_{V \supset \mathbb{R}^q} \, \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \ \text{ if } \ q > 0.$$

A map $f : X \longrightarrow Y$ of G-prespectra is a *π_*-isomorphism* if it induces isomorphisms on all homotopy groups. A map of orthogonal G-spectra is a π_*-isomorphism if its underlying map of G-prespectra is a π_*-isomorphism.

As H varies, the $\pi_q^H(X)$ define a contravariant functor from the homotopy category $hG\mathscr{O}$ of orbits to the category of Abelian groups, but the functoriality need not be considered in the development of the model structures. We shall later use the terms "π_*-isomorphism" and "weak equivalence" interchangeably, but we prefer to use the term π_*-isomorphism here to avoid confusion among the different model structures on G-prespectra and orthogonal G-spectra. We state the results of this section for G-prespectra but, since the forgetful functor $\mathbb{U}$ preserves all relevant constructions, they apply equally well to orthogonal G-spectra. The previous section gives $G\mathscr{P}$ a level model structure.

LEMMA 3.3. *A level equivalence of G-prespectra is a π_*-isomorphism.*

PROOF. Since each S^V is triangulable as a finite G-CW complex [**17**], this follows from the fact that if A is a G-CW complex and $f : B \longrightarrow C$ is a weak equivalence of G-spaces, then $f_* : F(A,B) \longrightarrow F(A,C)$ is a weak equivalence of G-spaces. □

The nonequivariant version [**20**, 7.3] of the following partial converse is trivial. The equivariant version is the key result, [**19**, I.7.12], in the classical development of the equivariant stable homotopy category, and it is also the key result here. While the result there is stated for G-spectra, the argument is entirely homotopical and applies verbatim to Ω-G-spectra. To make this paper more nearly self-contained, we will rework the proof in §9.

THEOREM 3.4. *A π_*-isomorphism between Ω-G-spectra is a level equivalence.*

Using that space-level constructions commute with passage to fixed points, as in Lemma 1.6, all parts of the following equivariant analogue of [**20**, 7.4] either follow from or are proven in the same way as the corresponding part of that result. As there, the nondegenerate basepoint hypotheses in Theorem 2.7 are not needed here.

THEOREM 3.5. (i) *A map of G-prespectra is a π_*-isomorphism if and only if its suspension is a π_*-isomorphism.*

(ii) *The homotopy groups of a wedge of G-prespectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms of G-prespectra is a π_*-isomorphism.*

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and a π_*-isomorphism of G-prespectra and $f : A \longrightarrow Y$ is any map of G-prespectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a π_*-isomorphism.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are π_*-isomorphisms in the comparison of pushouts diagram of Theorem 2.7(iv), then the induced map of pushouts is a π_*-isomorphism.*

(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.*

(vi) *For any map $f : X \longrightarrow Y$ of G-prespectra and any $H \subset G$, there are natural long exact sequences*

$$\cdots \longrightarrow \pi_q^H(Ff) \longrightarrow \pi_q^H(X) \longrightarrow \pi_q^H(Y) \longrightarrow \pi_{q-1}^H(Ff) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \pi_q^H(X) \longrightarrow \pi_q^H(Y) \longrightarrow \pi_q^H(Cf) \longrightarrow \pi_{q-1}^H(X) \longrightarrow \cdots,$$

and the natural map $\eta : Ff \longrightarrow \Omega Cf$ is a π_*-isomorphism.

Equivariant stability requires consideration of general representations $V \in \mathscr{V}$, rather than just the trivial representation as in (i).

THEOREM 3.6. *Let $V \in \mathscr{V}$. A map $f : X \longrightarrow Y$ of G-prespectra is a π_*-isomorphism if and only if $\Sigma^V f : \Sigma^V X \longrightarrow \Sigma^V Y$ is a π_*-isomorphism*

We prove half of the theorem in the following lemma, which will be used in our development of the stable model structure.

LEMMA 3.7. *Let $V \in \mathscr{V}$. If $f : X \longrightarrow Y$ is a map of G-prespectra such that $\Sigma^V f : \Sigma^V X \longrightarrow \Sigma^V Y$ is a π_*-isomorphism, then f is a π_*-isomorphism.*

PROOF. By Proposition 3.9 below, $\Omega^V \Sigma^V f$ is a π_*-isomorphism. The conclusion follows by naturality from the following lemma. □

LEMMA 3.8. *For G-prespectra X and $V \in \mathscr{V}$, the unit $\eta : X \longrightarrow \Omega^V \Sigma^V X$ of the (Σ^V, Ω^V) adjunction is a π_*-isomorphism.*

PROOF. Up to isomorphism, we may write the universe U as $U' \oplus V^\infty$. We choose an expanding sequence of indexing G-spaces U'_i in U' whose union is U'. For $q \geq 0$,

$$\pi_q^H(X) = \mathrm{colim}_{i,j}\, \pi_q^H(\Omega^{U'_i \oplus V^j} X(U'_i \oplus V^j))$$

and

$$\pi_q^H(\Omega^V \Sigma^V X) = \mathrm{colim}_{i,j}\, \pi_q^H(\Omega^{U'_i \oplus V^j} \Omega^V \Sigma^V X(U'_i \oplus V^j)).$$

The unit η induces a map from the first colimit to the second, and the structure maps $\Sigma^V X(U'_i \oplus V^j) \longrightarrow X(U'_i \oplus V^{j+1})$ induce a map from the second colimit to the first. These are inverse isomorphisms. A similar argument applies when $q < 0$. □

PROPOSITION 3.9. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of G-prespectra and A is a finite based G-CW complex, then $F(\mathrm{id}, f) : F(A, X) \longrightarrow F(A, Y)$ is a π_*-isomorphism.*

PROOF. By inspection of colimits, using the standard adjunctions, we see that

$$\pi_*^H(F(A, X)) \cong \pi_*^G(F(G/H_+ \wedge A, X)). \tag{3.10}$$

Thus, since $G/H_+ \wedge A$ is a finite G-CW complex [**17**], we may focus on π_*^G. Since the functor $F(-, X)$ converts cofiber sequences of G-spaces to fiber sequences of G-prespectra, we see by the first long exact sequence in Theorem 3.5(vi) and commutation relations with suspension that the result holds in general if it holds when $A = G/K_+$ for any K. Here (3.10) gives that $\pi_*^G(F(G/K_+, X)) \cong \pi_*^K(X)$, and the conclusion follows. □

The analogue for smash products is a little more difficult and gives the converse of Lemma 3.7 that is needed to complete the proof of Theorem 3.6.

THEOREM 3.11. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of G-prespectra and A is a based G-CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

We can reduce this to the case $A = G/H_+$ by use of Theorem 3.5, but that case seems hard to handle directly. We prove a partial result here. The rest will drop out model theoretically in the next section.

LEMMA 3.12. *If $f : X \longrightarrow Y$ is a level equivalence of G-prespectra and A is a based G-CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

PROOF. We consider π_q^H for $q \geq 0$. The case $q < 0$ is similar. Let $U = \mathbb{R}^\infty \oplus U'$, where $(U')^G = 0$. We may write

$$\pi_q^H(X \wedge A) = \mathrm{colim}_{V \subset U'} \,\mathrm{colim}_{W \subset \mathbb{R}^\infty} \,\pi_q^H(\Omega^V \Omega^W (X(V \oplus W) \wedge A)).$$

For fixed V and $K \subset G$, the $X(V \oplus W)^K$ for varying W specify a nonequivariant prespectrum $X[V]^K$ indexed on $\mathbb{R}^\infty$, and we have

$$(\Omega^W(X(V \oplus W) \wedge A))^K \cong \Omega^W(X(V \oplus W)^K \wedge A^K).$$

Since f is a level equivalence, it induces a nonequivariant π_*-isomorphism $f[V]^K : X[V]^K \longrightarrow Y[V]^K$. By the nonequivariant version [**20**, 7.4(i)] of Theorem 3.11,

$$f[V]^K \wedge \mathrm{id} : X[V]^K \wedge A^K \longrightarrow Y[V]^K \wedge A^K$$

is a π_*-isomorphism. Therefore, for each V, the induced map of G-spaces

$$\mathrm{hocolim}_W \,\Omega^W(X(V \oplus W) \wedge A) \longrightarrow \mathrm{hocolim}_W \,\Omega^W(Y(V \oplus W) \wedge A)$$

is a weak G-equivalence. Applying Ω^V to this map still gives a weak G-equivalence. Passage to homotopy groups π_q^H and then to colimits over V gives the result. □

4. The stable model structure on orthogonal G-spectra

We give the categories of orthogonal G-spectra and G-prespectra stable model structures and prove that they are Quillen equivalent. The arguments are like those in the nonequivariant context of [**20**], except that we work with π_*-isomorphisms rather than the equivariant analogue of the stable equivalences used there. As we explain in §6, that analogue gives a formally equivalent condition for a map to be a π_*-isomorphism. All of the statements and most of the proofs are identical in $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$. Definition 2.3 specifies the level equivalences, level fibrations, level acyclic fibrations, q-cofibrations, and level acyclic q-cofibrations in these categories.

DEFINITION 4.1. Let $f : X \to Y$ be a map of orthogonal G-spectra or G-prespectra.

(i) f is an *acyclic q-cofibration* if it is a π_*-isomorphism and a q-cofibration.
(ii) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iii) f is an *acyclic q-fibration* if it is a π_*-isomorphism and a q-fibration.

THEOREM 4.2. *The categories $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$ are compactly generated proper G-topological model categories with respect to the π_*-isomorphisms, q-fibrations, and q-cofibrations. The fibrant objects are the Ω-G-spectra.*

The set of generating q-cofibrations is the set FI specified in Definition 1.1. The set K of generating acyclic q-cofibrations properly contains the set FJ specified there. As nonequivariantly [**20**, §§8, 9], it is defined in terms of the following maps $\lambda_{V,W}$, which turn out to be π_*-isomorphisms.

DEFINITION 4.3. For $V, W \in \mathscr{V}$, define $\lambda_{V,W} : F_{V\oplus W}S^W \longrightarrow F_V S^0$ to be the adjoint of the map

$$S^W \longrightarrow (F_V S^0)(V \oplus W) \cong O(V \oplus W)_+ \wedge_{O(W)} S^W$$

that sends w to the class of $e \wedge w$, where $e \in O(V \oplus W)$ is the identity element.

LEMMA 4.4. *For any orthogonal G-spectrum or G-prespectrum X,*

$$\lambda^*_{V,W} : \mathscr{I}_G\mathscr{S}(F_V S^0, X) \longrightarrow \mathscr{I}_G\mathscr{S}(F_{V\oplus W}S^W, X)$$

coincides with $\tilde{\sigma} : X(V) \longrightarrow \Omega^W X(V \oplus W)$ under the canonical homeomorphisms

$$X(V) = \mathscr{T}_G(S^0, X(V)) \cong \mathscr{I}_G\mathscr{S}(F_V S^0, X)$$

and

$$\Omega^W X(V \oplus W) = \mathscr{T}_G(S^W, X(V \oplus W)) \cong \mathscr{I}_G\mathscr{S}(F_{V\oplus W}S^W, X).$$

PROOF. With $X = F_V S^0$, $\tilde{\sigma}$ may be identified with a map

$$\tilde{\sigma} : \mathscr{I}_G\mathscr{S}(F_V S^0, F_V S^0) \longrightarrow \mathscr{I}_G\mathscr{S}(F_{V\oplus W}S^W, F_V S^0),$$

and $\lambda_{V,W}$ is the image of the identity map under $\tilde{\sigma}$. □

The following result is the equivariant version of [**20**, 8.6].

LEMMA 4.5. *For all based G-CW complexes A, the maps*

$$\lambda_{V,W} \wedge \mathrm{id} : F_{V\oplus W}\Sigma^W A \cong F_{V\oplus W}S^W \wedge A \longrightarrow F_V S^0 \wedge A \cong F_V A$$

are π_*-isomorphisms.

PROOF. We prove this separately in the two cases.

G-PRESPECTRA. As in [**20**, 4.1], we have $(F_V A)(Z) = S^{Z-V} \wedge A$, where $S^{Z-V} = *$ if V is not contained in Z. Thus $F_V A$ is essentially a reindexing of the suspension G-prespectrum of A. The map $\lambda_{V,W}(Z)$ is the identity unless Z contains V but does not contain W, when it is the inclusion $* \longrightarrow S^{Z-V}$. Passing to colimits, we see that $\lambda_{V,W} \wedge \mathrm{id}$ is a π_*-isomorphism.

ORTHOGONAL G-SPECTRA. As in [**20**, 4.4], for $Z \supset V$ we have

$$(F_V A)(Z) = O(Z)_+ \wedge_{O(Z-V)} S^{Z-V} \wedge A.$$

By Lemma 3.7, it suffices to prove that $\Sigma^V(\lambda_{V,W} \wedge \mathrm{id})$ is a π_*-isomorphism. When Z contains $V \oplus W$, $\Sigma^V \lambda_{V,W}(Z)$ can be identified with the quotient map

$$O(Z)_+ \wedge_{O(Z-(V\oplus W))} S^Z \longrightarrow O(Z)_+ \wedge_{O(Z-V)} S^Z.$$

Under G-homeomorphisms (1.3), $\Sigma^V \lambda_{V,W}(Z)$ corresponds to the map

$$\pi \wedge \mathrm{id} : O(Z)/O(Z - (V \oplus W))_+ \wedge S^Z \longrightarrow O(Z)/O(Z-V)_+ \wedge S^Z,$$

where π is the evident quotient map. Smashing with A, restricting to indexing G-spaces in a universe U, taking homotopy groups, and passing to colimits, we obtain an isomorphism between copies of equivariant stable homotopy groups of A. □

Recall the operation □ from (1.16).

DEFINITION 4.6. Let $M\lambda_{V,W}$ be the mapping cylinder of $\lambda_{V,W}$. Then $\lambda_{V,W}$ factors as the composite of a q-cofibration $k_{V,W} : F_{V\oplus W}S^W \longrightarrow M\lambda_{V,W}$ and a deformation retraction $r_{V,W} : M\lambda_{V,W} \longrightarrow F_V S^0$. Let $j_{V,W} : F_V S^0 \longrightarrow M\lambda_{V,W}$ be the evident homotopy inverse of $r_{V,W}$. Restricting to V and W in $sk\mathscr{I}_G$, let K be the union of FJ and the set of all maps of the form $i\square k_{V,W}$, $i \in I$.

We need a characterization of the maps that satisfy the RLP with respect to K. It is the equivariant analogue of [**20**, 9.5], but we give the proof since this is the place where we need the new notion of a G-topological model category.

DEFINITION 4.7. A commutative diagram of based G-spaces

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array}$$

in which p and q are Serre fibrations is a *homotopy pullback* if the induced map $D \longrightarrow A \times_B E$ is a weak equivalence of G-spaces.

PROPOSITION 4.8. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is a level fibration and the diagram*

$$\begin{array}{ccc} EV & \xrightarrow{\tilde{\sigma}} & \Omega^W E(V\oplus W) \\ {\scriptstyle pV}\downarrow & & \downarrow{\scriptstyle \Omega^W p(V\oplus W)} \\ BV & \xrightarrow[\tilde{\sigma}]{} & \Omega^W B(V\oplus W) \end{array} \tag{4.9}$$

is a homotopy pullback for all V and W.

PROOF. The map p has the RLP with respect to FJ if and only if it is a level fibration. By Lemma 1.20, p has the RLP with respect to $i\square k_{V,W}$ if and only if $\mathscr{I}_G\mathscr{S}(k^*_{V,W}, p_*)$ has the RLP with respect to I, which means that $\mathscr{I}_G\mathscr{S}(k^*_{V,W}, p_*)$ is an acyclic Serre fibration of G-spaces. Since $k_{V,W}$ is a q-cofibration and p is a level fibration, $\mathscr{I}_G\mathscr{S}(k^*_{V,W}, p_*)$ is a Serre fibration because the level model structure is G-topological. We conclude that p satisfies the RLP with respect to K if and only if p is a level fibration and each $\mathscr{I}_G\mathscr{S}(k^*_{V,W}, p_*)$ is a weak equivalence. Since $k_{V,W} \simeq j_{V,W}\lambda_{V,W}$ and $j_{V,W}$ is a homotopy equivalence, this holds if and only if $\mathscr{I}_G\mathscr{S}(\lambda^*_{V,W}, p_*)$ is a weak equivalence. Using the fact that $\lambda_{V,W}$ corresponds to $\tilde{\sigma}$ under adjunction, we see that the map $\mathscr{I}_G\mathscr{S}(\lambda^*_{V,W}, p_*)$ is isomorphic to the map

$$EV \longrightarrow BV \times_{\Omega^W B(V\oplus W)} \Omega E(V\oplus W)$$

and is thus a weak equivalence if and only if (4.9) is a homotopy pullback. $\square$

From here, the proof of Theorem 4.2 is virtually identical to that of its nonequivariant version in [**20**, §9]. We record the main steps of the argument since they give the order of proof and encode useful information about the q-fibrations and q-cofibrations. Rather than repeat the proofs, we point out the main input. The following corollary is immediate.

COROLLARY 4.10. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is an Ω-G-spectrum.*

It is at this point that the key result, Theorem 3.4, comes into play.

COROLLARY 4.11. *If $p : E \longrightarrow B$ is a π_*-isomorphism that satisfies the RLP with respect to K, then p is a level acyclic fibration.*

PROOF. If $F = p^{-1}(*)$, then F is an Ω-G-spectrum and $F \longrightarrow *$ is a π_*-isomorphism. Thus, by Theorem 3.4, F is level acyclic. The rest is proven as in [**20**, 9.8]. □

PROPOSITION 4.12. *Let $f : X \longrightarrow Y$ be a map of orthogonal G-spectra.*

(i) *f is an acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(ii) *f is a q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is an orthogonal Ω-G-spectrum.*
(iii) *f is an acyclic q-fibration if and only if it is a level acyclic fibration.*

PROOF. Lemma 4.5 implies that the maps $i\Box k_{V,W}$ in K are π_*-isomorphisms. Thus all maps in K are π_*-isomorphisms. Now Theorem 3.5 implies that all retracts of relative K-cell complexes are π_*-isomorphisms. The rest is as in [**20**, 9.9]. □

The proof of the model axioms is completed as in [**20**, §9]. The properness of the model structure is implied by the following more general statements.

LEMMA 4.13. *Consider the following commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

(i) *If the diagram is a pushout in which i is an h-cofibration and f is a π_*-isomorphism, then g is a π_*-isomorphism.*
(ii) *If the diagram is a pullback in which j is a level fibration and g is a π_*-isomorphism, then f is a π_*-isomorphism.*

The following model theoretical observation leads to the proof of Theorem 3.11.

LEMMA 4.14. *If A is a based G-CW complex A, then $(- \wedge A, F(A,-))$ is a Quillen adjoint pair on $G\mathscr{I}\mathscr{S}$ or $G\mathscr{P}$ with its stable model structure.*

PROOF. Since the functor $F(A,-)$ preserves fibrations, level equivalences, and homotopy pullbacks, it preserves q-fibrations and acyclic q-fibrations by their characterizations in Propositions 4.8 and 4.12. □

THE PROOF OF THEOREM 3.11. Let $f : X \longrightarrow Y$ be a π_*-isomorphism and A be a based G-CW complex. We must show that $f \wedge \mathrm{id}_A$ is a π_*-isomorphism. By cofibrant approximation in the level model structure and use of Lemma 3.12, we may assume that X and Y are cofibrant. However, as a Quillen left adjoint, the functor $(-) \wedge A$ preserves weak equivalences between cofibrant objects. □

The following result, which is immediate from Lemmas 4.14 and 3.8, implies that the homotopy category with respect to the stable model structure really is an "equivariant stable homotopy category", in the sense that the functors Σ^V and Ω^V on it are inverse equivalences of categories for $V \in \mathscr{V}$.

THEOREM 4.15. *For $V \in \mathscr{V}$, the pair (Σ^V, Ω^V) is a Quillen equivalence.*

Finally, we have the following promised comparison theorem.

THEOREM 4.16. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence between the categories $G\mathscr{P}$ and $G\mathscr{I}\mathscr{S}$ with their stable model structures.*

PROOF. In view of III.2.12 and [**20**, A.2], we need only show that the unit $\eta : X \longrightarrow \mathbb{UP}X$ of the adjunction is a π_*-isomorphism when X is a cofibrant G-prespectrum. As in [**20**, 10.3], Theorems 3.5 and 3.6 imply that it suffices to prove this when $X = \Sigma^V F_V A$ for an indexing G-space V and a based G-CW complex A. The functors F_V for $\mathscr{P}_G$ and $\mathscr{I}_G\mathscr{S}$ are related by $F_V \cong \mathbb{P}F_V$, by inspection of their right adjoints, and these functors commute with smash products with based G-spaces. We have the commutative diagram

$$\begin{array}{ccc} F_V\Sigma^V A & \xrightarrow{\lambda_{0,V}} & F_0 A \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \mathbb{U}F_V\Sigma^V A & \xrightarrow[\mathbb{U}\lambda_{0,V}]{} & \mathbb{U}F_0 A. \end{array}$$

The right vertical arrow is an isomorphism and the maps $\lambda_{0,V}$ and $\mathbb{U}\lambda_{0,V}$ are π_*-isomorphisms by Lemma 4.5. □

5. The positive stable model structure

In §2, we explained the positive level model structure, and we need the concomitant positive stable model structure, as in [**20**, §14].

DEFINITION 5.1. A G-prespectrum or orthogonal G-spectrum X is a positive Ω-G-spectrum if $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(W)$ is a weak equivalence for $V^G \neq 0$.

DEFINITION 5.2. Define *acyclic positive q-cofibrations*, *positive q-fibrations*, and *acyclic positive q-fibrations* as in Definition 4.1, but starting with the positive level classes of maps specified in Definition 2.8.

THEOREM 5.3. *The categories $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$ are compactly generated proper G-topological model categories with respect to the π_*-isomorphisms, positive q-fibrations, and positive q-cofibrations. The positive fibrant objects are the positive Ω-G-spectra.*

The set of generating positive q-fibrations is the set F^+I specified in Definition 2.9. The set of generating acyclic positive q-cofibrations is the union, K^+, of the set F^+J specified there and the set of maps of the form $i\Box k_{V,W}$ with $i \in I$ and $V^G \neq 0$ from Definition 4.6.

The proof of Theorem 5.3 depends on the positive analogue of Theorem 3.4.

THEOREM 5.4. *A π_*-isomorphism between positive Ω-G-spectra is a positive level equivalence.*

This can be shown by restricting the proof of Theorem 3.4 in §9 to positive Ω-G-spectra and positive levels. For orthogonal G-spectra, there is an illuminating alternative argument. Indeed, for $V \in \mathscr{V}$ and an orthogonal G-spectrum X, the map $\lambda = \lambda_{0,V} : F_V S^V \longrightarrow F_0 S^0 = S$ induces a map

$$\lambda^* : X \cong F(S, X) \longrightarrow F(F_V S^V, X). \tag{5.5}$$

Standard adjunctions imply that

$$F(F_V S^V, X)(W) \cong \Omega^V X(V \oplus W),$$

and this leads to the following relationship between orthogonal Ω-G-spectra and orthogonal positive Ω-G-spectra.

LEMMA 5.6. *If E is a positive orthogonal Ω-G-spectrum, then $F(F_1 S^1, E)$ is an orthogonal Ω-G-spectrum and λ^* is a positive level equivalence.*

Therefore, for orthogonal G-spectra, Theorem 5.4 can be proven by applying Theorem 3.4 to $F(F_1 S^1, -)$. From here, Theorem 5.3 is proven by the same arguments as for the stable model structure, but with everything restricted to positive levels. Similarly, the proof of the following comparison result is the same as the proof of Theorem 4.16.

THEOREM 5.7. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence between the categories $G\mathscr{P}$ and $G\mathscr{I}\mathscr{S}$ with their positive stable model structures.*

The relationship between the stable model structure and the positive stable model structure is given by the following equivariant analogue of [**20**, 14.6].

PROPOSITION 5.8. *The identity functor from $G\mathscr{I}\mathscr{S}$ with its positive stable model structure to $G\mathscr{I}\mathscr{S}$ with its stable model structure is the left adjoint of a Quillen equivalence, and similarly for $G\mathscr{P}$.*

6. Stable equivalences of orthogonal G-spectra

Let $[X,Y]^{\ell}_G$ and $[X,Y]^{+\ell}_G$ denote the set of maps $X \longrightarrow Y$ in the homotopy category of $G\mathscr{I}\mathscr{S}$ or $G\mathscr{P}$ with respect to the level model structure or the positive level model structure and let $[X,Y]_G$ denote the set of maps $X \longrightarrow Y$ in the homotopy category with respect to the stable (or, equivalently, positive stable) model structure. The following observation applies to both $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$.

THEOREM 6.1. *The following conditions on a map $f : X \longrightarrow Y$ are equivalent.*

(i) *f is a π_*-isomorphism.*
(ii) *$f^* : [Y,E]^{\ell}_G \longrightarrow [X,E]^{\ell}_G$ is an isomorphism for all Ω-G-spectra E.*
(iii) *$f^* : [Y,E]^{+\ell}_G \longrightarrow [X,E]^{+\ell}_G$ is an isomorphism for all positive Ω-G-spectra E.*

PROOF. We prove that (i) and (ii) are equivalent. The same proof shows that (i) and (iii) are equivalent. Since the q-cofibrations are the same in the level and stable model structures and level equivalences are π_*-isomorphisms, the identity functor is the left adjoint of a Quillen adjoint pair from the level model structure to the stable model structure. Since Ω-G-spectra are stably fibrant, this implies that

$$[X,E]_G \cong [X,E]^{\ell}_G \tag{6.2}$$

when E is an Ω-G-spectrum. Since every object of the stable homotopy category is isomorphic to an Ω-G-spectrum, the Yoneda lemma gives that f is an isomorphism in that category if and only if $f^* : [Y,E]_G \longrightarrow [X,E]_G$ is an isomorphism for all Ω-G-spectra E. In view of (6.2), this says that (i) and (ii) are equivalent. □

The maps that satisfy (ii) or (iii) are called *stable equivalences* or *positive stable equivalences*. The nonequivariant analogues of the stable equivalences play a central role in the theory of symmetric spectra [**16, 20**], where the previous theorem fails, but they need not be introduced explicitly in the study of orthogonal spectra. An easy formal argument shows that a stable equivalence between Ω-G-spectra is a level equivalence. Nonequivariantly, there is a direct proof [**20**, 8.8] that a π_*-isomorphism of symmetric spectra or orthogonal spectra is a stable equivalence. That argument precedes the development of the stable model structure there. However, the argument depends on the trivial nonequivariant version of the key result Theorem 3.4 and is of no help in the present approach to the equivariant stable homotopy category.

7. Model categories of ring and module G-spectra

In this section and the next, we study model structures induced by the stable or positive stable model structure on $G\mathscr{I}\mathscr{S}$. We prove here that the categories of orthogonal ring spectra and of modules over an orthogonal ring spectrum are Quillen model categories. The proofs are essentially the same as those in the nonequivariant case given in [**20**, §§12, 14], but the inclusion of the case of symmetric spectra there dictated a more complicated line of argument than is necessary here. We give an outline.

In the language of [**32**], we show that the monoid and pushout-product axioms hold for orthogonal G-spectra. As in [**19**, 11.2], the following elementary complement to Lemmas 2.5 and 2.6 is used repeatedly.

Lemma 7.1. *If $i : X \longrightarrow Y$ is an h-cofibration of orthogonal G-spectra and Z is any orthogonal G-spectrum, then $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is an h-cofibration.*

The following lemma is the key step in the proof of the cited axioms. Its nonequivariant analogue is part of the proof of [**20**, 12.3].

Lemma 7.2. *Let Y be an orthogonal G-spectrum such that $\pi_*(Y) = 0$. Then $\pi_*(F_V S^V \wedge Y) = 0$ for any $V \in \mathscr{V}$.*

Proof. Let $\gamma_V = \lambda_{0,V} : F_V S^V \longrightarrow F_0 S^0 = S$ be the canonical π_*-isomorphism of Lemma 4.5. Let $\alpha \in \pi_q^H(F_V S^V \wedge Y)$. Taking $q \geq 0$ for definiteness, the proof for $q < 0$ being similar, choose a map $S^q \longrightarrow (\Omega^W(F_V S^V \wedge Y)(W))^H$ that represents α. By standard adjunctions, this map is determined by a map of orthogonal G-spectra

$$f : F_W(G/H_+ \wedge S^q \wedge S^W) \longrightarrow F_V S^V \wedge Y.$$

Since $\pi_*^H(Y) = 0$, we can choose W large enough that the composite

$$(\gamma_V \wedge \mathrm{id}) \circ f : F_W(G/H_+ \wedge S^q \wedge S^W) \longrightarrow F_V S^V \wedge Y \longrightarrow S \wedge Y \cong Y$$

is null homotopic. Let $g = (\gamma_V \wedge \mathrm{id}) \circ f$ and let g' be the map

$$\mathrm{id} \wedge g \colon F_{V \oplus W}(S^V \wedge G/H_+ \wedge S^q \wedge S^W) \cong F_V S^V \wedge F_W(G/H_+ \wedge S^q \wedge S^W) \longrightarrow F_V S^V \wedge Y$$

obtained from g by smashing with $F_V S^V$. Then g' is also null homotopic. Now let

$$f' : F_{V \oplus W}(S^V \wedge G/H_+ \wedge S^q \wedge S^W) \cong F_V S^V \wedge F_W(G/H_+ \wedge S^q \wedge S^W) \longrightarrow F_V S^V \wedge Y$$

be the composite $f \circ (\gamma_V \wedge \mathrm{id})$. Then f' also represents α. We show that $\alpha = 0$ by showing that the maps f' and g' are homotopic. We can rewrite f' and g' as the composites of the map

$$\mathrm{id} \wedge f : F_V S^V \wedge F_W(G/H_+ \wedge S^q \wedge S^W) \longrightarrow F_V S^V \wedge F_V S^V \wedge Y$$

and the maps $F_VS^V \wedge F_VS^V \wedge Y \longrightarrow F_VS^V \wedge Y$ obtained by applying γ_V to the first or second factor F_VS^V. Thus, it suffices to show that the maps $\gamma_V \wedge \mathrm{id}$ and $\mathrm{id} \wedge \gamma_V$ from $F_{V\oplus V}S^{V\oplus V} \cong F_VS^V \wedge F_VS^V$ to F_VS^V are homotopic. The adjoints

$$S^{V\oplus V} \longrightarrow (F_VS^V)(V \oplus V) = O(V \oplus V)_+ \wedge_{e\times O(V)} S^{V\oplus V}$$

of these two maps are the G-maps that send $(v, v') \in S^{V\oplus V}$ to $\tau \wedge (v', v)$ and to $e \wedge (v, v')$, where $\tau \in O(V \oplus V)$ is the evident transposition on $V \oplus V$. Writing out the homeomorphism of $O(V \oplus V)$-spaces

$$O(V \oplus V)_+ \wedge_{e\times O(V)} S^{V\oplus V} \cong O(V \oplus V)/O(V)_+ \wedge S^{V\oplus V}$$

given by (1.3), we see that it is a G-map. Under this homeomorphism, our two G-maps send $s \in S^{V\oplus V}$ to $\tau\wedge s$ and to $e\wedge s$. The elements e and τ are in $O(V\oplus V)^G$. By writing V as a sum of irreducible representations $V_i^{n_i}$ and considering G-invariant isometries, we find that $O(V \oplus V)/O(V)^G$ is the product of groups $O(2n_i)/O(n_i)$, $U(2n_i)/U(n_i)$ or $Sp(2n_i)/Sp(n_i)$, depending on the type of V_i (compare for example [**8**, 3.6]) and is thus connected. A path connecting e and τ in $O(V \oplus V)/O(V)^G$ determines a homotopy between our two G-maps. □

Proposition 7.3. *If X is a cofibrant orthogonal G-spectrum, then the functor $X \wedge (-)$ preserves π_*-isomorphisms.*

Proof. When $X = F_VS^V$, this is implied by Lemma 7.2, as we see by using the usual mapping cylinder construction to factor a given π_*-isomorphism as a composite of an h-cofibration and a G-homotopy equivalence and comparing long exact sequences given by Lemma 7.1 and Theorem 3.5(vi). As in the proof of [**20**, 12.3], the general case follows by use of Theorems 3.5, 3.6, and 3.11. □

As in [**20**, 12.5 and 12.6], this together with other results already proven implies the monoid and pushout-product axioms. These apply to $G\mathscr{I}\mathscr{S}$ with both its stable and its positive stable model structures.

Proposition 7.4 (Monoid axiom). *For any acyclic (positive) q-cofibration $i : X \longrightarrow Y$ of orthogonal G-spectra and any orthogonal G-spectrum Z, the map $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is a π_*-isomorphism and an h-cofibration. Moreover, cobase changes and sequential colimits of such maps are also π_*-isomorphisms and h-cofibrations.*

Proposition 7.5 (Pushout-product axiom). *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are (positive) q-cofibrations of orthogonal G-spectra and i is a π_*-isomorphism, then the (positive) q-cofibration $i\Box j : (Y \wedge W) \cup_{X\wedge W} (X \wedge Z) \longrightarrow Y \wedge Z$ is a π_*-isomorphism.*

As in [**20**, §§12, 14], the methods and results of [**32**], together with Proposition 5.8, entitle us to the following conclusions. More explicitly, [**20**, 5.13] specifies conditions for the category of algebras over a monad in a compactly generated topological model category $\mathscr{C}$ to inherit a structure of topological model category, and that result generalizes to G-topological model categories. The pushout-product and monoid axioms allow the verification of the conditions in the cases on hand.

Theorem 7.6. *Let R be an orthogonal ring G-spectrum, and consider the stable model structure on $G\mathscr{I}\mathscr{S}$.*

(i) *The category of left R-modules is a compactly generated proper G-topological model category with weak equivalences and q-fibrations created in $G\mathscr{I}\mathscr{S}$.*

(ii) *If R is cofibrant as an orthogonal G-spectrum, then the forgetful functor from R-modules to orthogonal G-spectra preserves q-cofibrations, hence every cofibrant R-module is cofibrant as an orthogonal G-spectrum.*
(iii) *If R is commutative, the symmetric monoidal category $G\mathscr{I}\mathscr{S}_R$ of R-modules also satisfies the pushout-product and monoid axioms.*
(iv) *If R is commutative, the category of R-algebras is a compactly generated right proper G-topological model category with weak equivalences and q-fibrations created in $G\mathscr{I}\mathscr{S}$.*
(v) *If R is commutative, every q-cofibration of R-algebras whose source is cofibrant as an R-module is a q-cofibration of R-modules, hence every cofibrant R-algebra is cofibrant as an R-module.*
(vi) *If $f : Q \longrightarrow R$ is a weak equivalence of orthogonal ring G-spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-modules and of R-modules.*
(vii) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative orthogonal ring G-spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-algebras and of R-algebras.*

Parts (i), (iii), (iv), (vi), and (vii) also hold for the positive stable model structure.

Parts (ii) and (v) do not hold for the positive stable model structure, in which S_G is not cofibrant. As in [**20**, 12.7], we have the following generalization of Proposition 7.3, which is needed in the proofs of parts (vi) and (vii) of the theorem.

PROPOSITION 7.7. *For a cofibrant right R-module M, the functor $M \wedge_R N$ of N preserves π_*-isomorphisms.*

8. The model category of commutative ring G-spectra

Let $\mathbb{C}$ be the monad on orthogonal G-spectra that defines commutative orthogonal ring G-spectra. Thus $\mathbb{C}X = \bigvee_{i\geq 0} X^{(i)}/\Sigma_i$, where $X^{(i)}$ denotes the ith smash power, with $X^{(0)} = S_G$.

THEOREM 8.1. *The category of commutative orthogonal ring G-spectra is a compactly generated proper G-topological model category with q-fibrations and weak equivalences created in the positive stable model category of orthogonal G-spectra. The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ are the generating sets of q-cofibrations and acyclic q-cofibrations.*

This is a consequence of the following two results, which (together with two general results on colimits [**6**, I.7.2, VII.2.10]) verify the criteria for inheritance of a model structure specified in [**20**, 5.13].

LEMMA 8.2. *The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ satisfy the Cofibration Hypothesis.*

PROOF. The Cofibration Hypothesis is specified in [**20**, 5.3]. Its verification here amounts to the following results:

(i) The functor $\mathbb{C}$ preserves h-cofibrations.
(ii) The cobase change $R \longrightarrow R \wedge_{\mathbb{C}X} \mathbb{C}Y$ associated to a wedge $X \longrightarrow Y$ of maps in FI^+ and a map $\mathbb{C}X \longrightarrow R$ of commutative orthogonal ring G-spectra is an h-cofibration.
(iii) Sequential colimits of maps of commutative orthogonal ring G-spectra that are h-cofibrations are computed as the colimits of their underlying orthogonal G-spectra.

Here (i) and (iii) are easy, but (ii) requires the methods of [**6**, VII§3], as explained in the proof of [**20**, 15.9]. □

LEMMA 8.3. *Every relative $\mathbb{C}K^+$-cell complex is a π_*-isomorphism.*

PROOF. First, one needs that each map in $\mathbb{C}K^+$ is a π_*-isomorphism or, more generally, that the functor $\mathbb{C}$ preserves π_*-isomorphisms between positive cofibrant orthogonal G-spectra. That is a consequence of the second statement of the following lemma. From there, as explained in [**20**, §15], the proof reduces to showing that if $R \longrightarrow R'$ is a relative $\mathbb{C}F^+I$-cell complex, then the functor $(-) \wedge_R R'$ preserves π_*-isomorphisms. In turn, using the methods of [**6**, VII§4], that reduces to showing that the functor $\mathbb{C}F_V A \wedge (-)$ on orthogonal G-spectra preserves π_*-isomorphisms when A is a based G-CW complex and $V^G \neq 0$. That is a consequence of the first statement of the following lemma. Compare [**20**, 15.6, 15.7]. □

LEMMA 8.4. *Let A be a based G-CW complex, X be an orthogonal G-spectrum, and $V^G \neq 0$. Then the quotient map*

$$q : (E\Sigma_{i+} \wedge_{\Sigma_i} (F_V A)^{(i)}) \wedge X \longrightarrow ((F_V A)^{(i)}/\Sigma_i) \wedge X$$

is a π_*-isomorphism. If X is a positive cofibrant orthogonal G-spectrum, then

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$$

is a π_*-isomorphism.

PROOF. For the first statement, we show that q is an eventual level G-homotopy equivalence. Precisely, we prove that the Wth map of q is a G-homotopy equivalence for all W that, up to G-isomorphism, contain V^i. We may as well assume that $W \supset V^i$. Then, by [**20**, 4.4] and inspection of coequalizers,

$$((F_V A)^{(i)} \wedge X)(W) \cong O(W)_+ \wedge_{O(W-V^i)} (A^{(i)} \wedge X(W - V^i)).$$

The action of $\sigma \in \Sigma_i$ is to permute the factors in $A^{(i)}$ and to act through $\sigma \oplus \mathrm{id}_{W-V^i}$ on $O(W)$, where $\sigma \in O(V^i)$ permutes the i summands of V in V^i. Since Σ_i acts on $O(W)$ as a subgroup of $O(V^i)$, the action commutes with the action of $O(W - V^i)$. Therefore, the G-map at level W in the first statement is obtained by passing to orbits over $\Sigma_i \times O(W - V^i)$ from the projection

$$(E\Sigma_i \times O(W))_+ \wedge (A^{(i)} \wedge X(W - V^i)) \longrightarrow O(W)_+ \wedge (A^{(i)} \wedge X(W - V^i)).$$

This map is equivariant with respect to the evident actions of the semi-direct product $\Gamma = G \ltimes (\Sigma_i \times O(W - V^i))$, where G acts on $\Sigma_i \times O(W - V^i)$ through its (right) action on $O(W - V^i)$. Therefore, to show that the level G-map is a G-homotopy equivalence, it suffices to show that the projection

$$E\Sigma_{i+} \wedge O(W)_+ \longrightarrow O(W)_+$$

is a Γ-homotopy equivalence. Since both sides are Γ-CW complexes, it suffices to observe that this map becomes a homotopy equivalence on passage to Λ-fixed points for each $\Lambda \subset \Gamma$. The Λ-fixed points of the source and target are empty unless $\Lambda \subset G \ltimes O(W - V^i)$, in which case Λ acts trivially on the contractible space $E\Sigma_i$ and the projection is a Λ-homotopy equivalence. This proves the first statement. For the second statement, we may assume that X is an F^+I-cell spectrum, and the proof then is an induction on i and on the cellular filtration of X that is essentially the same as the proof of [**6**, III.5.1]. □

9. Level equivalences and π_*-isomorphisms of Ω-G-spectra

We must prove Theorem 3.4. It asserts that a π_*-isomorphism $f : X \longrightarrow Y$ between Ω-G-spectra indexed on any given universe U is a level equivalence. The proof is a somewhat streamlined version of the proof of [**19**, 7.12]. Consider the fiber Ff. By the long exact sequence of homotopy groups, $\pi_*^H(Ff) = 0$ for all H. Since Ff is constructed by taking levelwise fibers, it suffices by the level exact sequences of homotopy groups to prove that Ff is level acyclic. For the case of π_0^H, this verification uses that, up to homotopy, the map $f(V) : X(V) \longrightarrow Y(V)$ is the loop of the map $f(V+\mathbb{R}) : X(V+\mathbb{R}) \longrightarrow Y(V+\mathbb{R})$, where $\mathbb{R} \subset U$ and $\mathbb{R} \cap V = 0$. Since Ff is again an Ω-G-spectrum, it suffices to prove the following result.

Lemma 9.1. *If X is an Ω-G-spectrum such that $\pi_n^H(X) = 0$ for all integers n and all $H \subset G$, then $\pi_n^H X(V) = 0$ for all $n \geq 0$, $H \subset G$, and $V \subset U$.*

We need an observation about Ω-G-spectra. It is adapted from [**23**, p. 30].

Lemma 9.2. *Let X be an Ω-G-spectrum and let V and V' be indexing G-spaces in U that are isomorphic as H-inner product spaces. Then the G-spaces $X(V)$ and $X(V')$ are weakly equivalent as H-spaces.*

Proof. Choose an indexing G-space Z that contains V and V' and let W and W' be the orthogonal complements of V and V' in Z. Then W and W' are indexing G-spaces that are isomorphic as H-inner product spaces. Together with structural G-equivalences of X, any choice of H-isomorphism gives a diagram

$$X(V) \longrightarrow \Omega^W X(Z) \cong \Omega^{W'} X(Z) \longleftarrow X(V')$$

that displays the claimed weak H-equivalence. □

The main tool in the proof of Lemma 9.1 is a familiar fiber sequence. For any V, let $S(V)$ be the unit sphere in the unit disk $D(V)$. We may identify S^V with the G-space $D(V)/S(V) = D(V)_+/S(V)_+$, and the quotient map $D(V)_+ \longrightarrow S^0$ is a G-homotopy equivalence. For any G-space A, application of the functor $F(-,A)$ to the cofiber sequence $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$ gives a fiber sequence of G-spaces, and passage to G-fixed points gives a fiber sequence of spaces.

Proof of Lemma 9.1. By cofibrant approximation in the level model structure, we may as well assume that X is a cell G-prespectrum, so that each $X(V)$ has the homotopy type of a G-CW complex. We may identify $\mathbb{R}^\infty$ with U^G and so fix $\mathbb{R}^d \subset U$ for all $d \geq 0$. We observe first that $\pi_*^H X(\mathbb{R}^d) = 0$ for all d and H. Indeed, by the definition of the homotopy groups of an Ω-G-spectrum, we have

$$\pi_n^H(X) \cong \pi_n^H \Omega^d X(\mathbb{R}^d) = \pi_{n+d}^H X(\mathbb{R}^d) = 0$$

if $n \geq 0$ and, if $0 < n \leq d$,

$$\pi_{-n}^H(X) = \pi_0^H \Omega^{d-n} X(\mathbb{R}^d) = \pi_{d-n}^H X(\mathbb{R}^d) = 0.$$

If $V = V^G$, then V is G-isomorphic to some $\mathbb{R}^d$. Thus, by Lemma 9.2, $\pi_*^H X(V) = 0$ for all H in this case as well. Similarly, any V is e-isomorphic to some $\mathbb{R}^d$, and thus $\pi_* X(V) = \pi_*^e X(V) = 0$ for all V.

Since G is a compact Lie group, it contains no infinite descending chain of (closed) subgroups and we can argue by induction over subgroups. Assume inductively that, for all V and all proper subgroups K of H, $\pi_*^K X(V) = 0$. The inductive step is to prove that $\pi_*^H X(V) = 0$, and we have already proven this when $H = e$.

Fix V, let W be the orthogonal complement of V^G in V, and let Z be the orthogonal complement of W^H in W. Then W is a G-space and Z is an H-space. Let $d = \dim(W^H)$. We begin by proving that $\pi_n^H \Omega^{W^H} X(V) = \pi_{n+d}^H X(V) = 0$ for $n > 0$. We have the fiber sequence

$$F(S^Z, \Omega^{W^H} X(V))^H \longrightarrow F(D(Z)_+, \Omega^{W^H} X(V))^H \longrightarrow F(S(Z)_+, \Omega^{W^H} X(V))^H.$$

The middle term is equivalent to $\Omega^{W^H} X(V)^H$. In the left term, the H-space $F(S^Z, \Omega^{W^H} X(V))$ is H-homeomorphic to the G-space $\Omega^W X(V)$. We have a structural G-equivalence $X(V^G) \longrightarrow \Omega^W X(V)$ since $V = V^G + W$ is a direct sum splitting of G-spaces. Thus, up to homotopy, our fiber sequence may be written

$$X(V^G)^H \longrightarrow \Omega^{W^H} X(V)^H \longrightarrow F(S(Z)_+, \Omega^{W^H} X(V))^H.$$

We have shown that $\pi_*^H X(V^G) = 0$. Thus to show that $\pi_n^H \Omega^{W^H} X(V) = 0$ for $n > 0$, it suffices to show that $\pi_n^H F(S(Z)_+, \Omega^{W^H} X(V)) = 0$ for $n > 0$. We may triangulate $S(Z)$ as a finite (unbased) H-CW complex [**17**]. By construction, $S(Z)^H$ is the empty set, so the triangulation only has cells of the form $D^m \times H/K$ with K a proper subgroup of H. By the induction hypothesis, $\pi_n^K(\Omega^{W^H} X(V)) = \pi_{n+d}^K X(V) = 0$ for $n \geq 0$. It follows by induction on the number of cells in the chosen triangulation that $\pi_n^H F(S(Z)_+, \Omega^{W^H} X(V)) = 0$ for $n > 0$.

We have proven that $\pi_n^H X(V) = 0$ for $n > d$. Choose a copy of $\mathbb{R}^{d+1}$ in $\mathbb{R}^\infty$ such that $V \cap \mathbb{R}^{d+1} = 0$. Then $V + \mathbb{R}^{d+1} = (V^G + \mathbb{R}^{d+1}) + W$. Applied to $V + \mathbb{R}^{d+1}$, the argument just given shows that $\pi_n^H X(V + \mathbb{R}^{d+1}) = 0$ for $n > d$. Since $X(V)$ is G-equivalent to $\Omega^{d+1} X(V + \mathbb{R}^{d+1})$, we conclude that $\pi_n^H X(V) = 0$ for all n. This completes the proof. □

Let $\mathscr{F}$ be a family of subgroups of G, that is a set of subgroups closed under passage to subgroups and conjugates. The inductive nature of the argument makes it clear that the following generalization holds. We shall need it later.

THEOREM 9.3. *Let $f : X \longrightarrow Y$ be a map of Ω-G-prespectra. Assume that $f_* : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism for all $H \in \mathscr{F}$. Then, for $V \subset U$, $f(V)_* : \pi_*^H(X(V)) \longrightarrow \pi_*^H(Y(V))$ is an isomorphism for all $H \in \mathscr{F}$.*

CHAPTER IV

Orthogonal G-spectra and S_G-modules

1. Introduction and statements of results

Taking G-spectra and orthogonal G-spectra to be indexed on a complete universe, we shall prove the following precise analogues of the results stated in I§1.

THEOREM 1.1. *There is a strong symmetric monoidal functor* $\mathbb{N} : G\mathscr{I}\mathscr{S} \longrightarrow G\mathscr{M}$ *and a lax symmetric monoidal functor* $\mathbb{N}^{\#} : G\mathscr{M} \longrightarrow G\mathscr{I}\mathscr{S}$ *such that* $(\mathbb{N}, \mathbb{N}^{\#})$ *is a Quillen equivalence between* $G\mathscr{I}\mathscr{S}$ *and* $G\mathscr{M}$*. The induced equivalence of homotopy categories preserves smash products.*

THEOREM 1.2. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of orthogonal ring G-spectra and S_G-algebras.*

THEOREM 1.3. *For a cofibrant orthogonal ring G-spectrum R, the pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of R-modules and of* $\mathbb{N}R$*-modules.*

COROLLARY 1.4. *For an S_G-algebra R, the categories of R-modules and of* $\mathbb{N}^{\#}R$*-modules are Quillen equivalent.*

THEOREM 1.5. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of commutative orthogonal ring G-spectra and of commutative S_G-algebras.*

THEOREM 1.6. *Let R be a cofibrant commutative orthogonal ring spectrum. The categories of R-modules, R-algebras, and commutative R-algebras are Quillen equivalent to the categories of* $\mathbb{N}R$*-modules,* $\mathbb{N}R$*-algebras, and commutative* $\mathbb{N}R$*-algebras.*

COROLLARY 1.7. *Let R be a commutative S_G-algebra. The categories of R-modules, R-algebras, and, if R is cofibrant, commutative R-algebras are Quillen equivalent to the categories of* $\mathbb{N}^{\#}R$*-modules,* $\mathbb{N}^{\#}R$*-algebras, and commutative* $\mathbb{N}^{\#}R$*-algebras.*

These comparisons shed new light on the original Lewis–May theory of G-spectra [**19**]. A subtle and somewhat mysterious aspect of the theory concerns when to use sequential indexing and when to use coordinate-free indexing. The objects, G-spectra or, in the modern version, S_G-modules, are intrinsically coordinate-free. However, their homotopy groups can be $\mathbb{Z}$-graded: one can define $RO(G)$-graded homotopy groups, but they play no role in the model theoretic foundations. The theory of CW objects is at the heart of the matter. These are special kinds of cell objects. There is no explicit model structure on G-spectra in the earlier literature but, in the model structure that is implicit in [**6, 19**], the cofibrations are the retracts of the cell G-complexes, which are "coordinatized" kinds of objects, in the sense that they are defined in terms of sphere G-spectra $G/H_+ \wedge S^n$ for integers n, just as in the theory of G-spaces or in the nonequivariant theory. We call this the *cellular model structure*. It is *not* the model structure relevant to the results above.

It has often been wondered why integers, which implicitly encode trivial virtual representations, appear in the definitions of cells, rather than general virtual representations. As we shall see in §2, the answer is that there is a choice. There are two model structures with the same (stable) weak equivalences. The cofibrations in the (implicit) classical model structure are the retracts of the relative cell G-complexes. We shall present a second model structure, called the *generalized cellular model structure*. In this structure, the cofibrations are the retracts of generalized relative cell G-complexes, which are "coordinate-free" kinds of objects in the sense that they are are defined in terms of G-spectra $G/H_+ \wedge S^\alpha$ for general virtual representations α of G. These two model structures are Quillen equivalent. More precisely, the identity functor is the left adjoint of a Quillen equivalence from $G\mathscr{S}$ or $G\mathscr{M}$ with its cellular model structure to $G\mathscr{S}$ or $G\mathscr{M}$ with its generalized cellular model structure. Of course, these model structures determine the same homotopy category since that depends only on the weak equivalences. It is the generalized cellular model structure on $G\mathscr{M}$ that is relevant to the results above.

The relevant model structure on the orthogonal G-spectrum side is the positive stable model structure of III§5. In fact, there is no model structure on the category of orthogonal G-spectra that corresponds to the cellular, as opposed to the generalized cellular, model structure on the category of S_G-modules. We cannot expect to define cell orthogonal spectra with just the $G/H_+ \wedge F_m S^n$ as domains of attaching maps. These only detect the homotopy groups of the G-spaces $X(\mathbb{R}^m)$ of an orthogonal G-spectrum X. Taking the weak equivalences to be the π_*-isomorphisms, as we must, these space level homotopy groups do not have enough information built into them to prove the model axioms. Technically, with such cells, one cannot use the small object argument to factor a map as the composite of an acyclic relative cell complex and a fibration. Of course, the $G/H_+ \wedge F_V S^n$ do detect the homotopy groups of all of the G-spaces $X(V)$. This is intrinsic to the mathematics and is closely related to the change of universe issues discussed in V§1. The problem does not arise with G-spectra or S_G-modules X because the homotopy groups of their G-spaces $X(V)$ are equivariant stable homotopy groups: all information has been built in by use of the spectrification functor $L : \mathscr{P} \longrightarrow \mathscr{S}$; compare I.4.8.

We have left adjoints of Quillen equivalences from G-spectra to S_G-modules, both with either the cellular or the generalized cellular model structure, from S_G-modules with the cellular model structure to S_G-modules with the generalized cellular model structure, and from orthogonal G-spectra with the positive stable model structure to S_G-modules with the generalized cellular model structure. Since we have left adjoints with the same target, these Quillen equivalences cannot be composed. However, all homotopy category level information can be transported back and forth along the induced equivalences of homotopy categories. In particular, information proven using the classical cellular model structure on G-spectra can be transported along the equivalences to give information about orthogonal G-spectra.

Some such proofs can also be carried out using the generalized kind of cell structure, but others cannot. In view of the prevalence of arguments based on inductive verifications on cell complexes, it is in principle preferable to have a model structure with as few cofibrations as possible. More fundamentally, very many arguments, both equivariant and nonequivariant, depend on the use of CW complexes rather than just cell complexes. It is worth emphasizing that this fundamentally important refinement of cell theory is invisible to the model category formalities. This refinement is available with the classical cell complexes but not with the generalized cell complexes. The refinement makes sense only when one has the cellular approximation theorem, and one does not have such a result for generalized cell S_G-modules or for cell orthogonal G-spectra, even when one requires cells to be attached only to cells of lower dimension. The essential point is the space level fact that if $m < n$, then every map $S^m \longrightarrow S^n$ is null homotopic, hence null G-homotopic, whereas if V and W are representations of dimensions $m < n$, then G-maps $S^V \longrightarrow S^W$ need not be null G-homotopic. In fact, in $\mathscr{S}_G$ and $\mathscr{M}_G$, using G-CW objects, the statement and proof of the cellular approximation theorem [**19**, I.5.8] are virtually identical to their classical space level analogues; similarly, every object is weakly equivalent to a G-CW object [**19**, I.5.12].

For this reason, we have made no attempt to rederive the deeper results of equivariant stable homotopy theory in terms of orthogonal G-spectra; just as with the nonequivariant theory of diagram spectra, the new category should be viewed as complementary to the old one, rather than as a replacement for it. For an explicit example of a result proven with G-CW spectra that we do not know how to prove directly in the category of orthogonal G-spectra, we cite the theorem of [**18**] that an ordinary $\mathbb{Z}$-graded cohomology theory on G-spaces extends to an $RO(G)$-graded cohomology theory if and only if its system of coefficients extends to a Mackey functor. For finite groups G, one can use equivariant infinite loop space theory to construct the relevant Eilenberg-Mac Lane orthogonal G-spectra, as we intend to explain elsewhere, but for general compact Lie groups G the only known proof is the original one. That uses the cellular cochains of G-CW spectra with coefficients in a Mackey functor to construct a $\mathbb{Z}$-graded cohomology theory on G-spectra. Application of Brown's representability theorem to its zeroth term then gives the Eilenberg-Mac Lane G-spectrum that represents the extension to an $RO(G)$-graded cohomology theory.

We construct the functors $\mathbb{N}$ and $\mathbb{N}^{\#}$ of Theorem 1.1 in §3 and prove the comparison results 1.2 – 1.7 in §4. We show that the functor $\mathbb{N}$ is equivalent to a more intuitive comparison functor $\mathbb{M}$ in the brief §5. In §6 we consider families and cofamilies of subgroups of G. We define and compare new model structures on the categories of G-spectra, S_G-modules, and orthogonal G-spectra whose associated homotopy categories localize information at or away from a chosen collection of subgroups. Some of these model structures are given by Bousfield localizations, which admit a simple model theoretical construction in all of our categories.

2. Model structures on the category of S_G-modules

We shall not repeat the basic definitions given in [**6, 27**] and recalled nonequivariantly in I§5. The papers [**5, 10, 11**] give an introductory overview, and the basic equivariant reference is [**27**, XXIV]. However, we must modify its perspective on equivariance, in line with II§1.

As observed in II.1.2, we have categories $\mathscr{P}_G$ and $\mathscr{S}_G$ of G-prespectra and G-spectra. Their arrows are nonequivariant, but their spaces of arrows are G-spaces; that is, they are enriched over $G\mathscr{T}$. They have associated G-fixed categories $G\mathscr{P}$ and $G\mathscr{S}$, which are enriched over $\mathscr{T}$. There is a spectrification G-functor $L : \mathscr{P}_G \longrightarrow \mathscr{S}_G$ that is left adjoint to the evident forgetful functor ℓ.

There is a sphere G-spectrum S_G, and a closed symmetric monoidal category of S_G-modules. Its objects are G-spectra with additional structure, as specified in [**27**, XXIV.1.2, 1.5]. We give a summary. There is a monad $\mathbb{L}$ on $\mathscr{S}_G$ (and not just on $G\mathscr{S}$). The unit and product maps $\eta : E \longrightarrow \mathbb{L}E$ and $\mu : \mathbb{L}\mathbb{L}E \longrightarrow \mathbb{L}E$ are G-maps, and the action map $\xi : \mathbb{L}E \longrightarrow E$ of an $\mathbb{L}$-spectrum is required to be a G-map. There is a smash product of $\mathbb{L}$-spectra, denoted $\wedge_{\mathscr{L}}$, that is associative and commutative; it has a natural unit G-map $\lambda : S_G \wedge_{\mathscr{L}} E \longrightarrow E$ that is always a weak equivalence and sometimes an isomorphism. (We redescribe it in VI§6). An S_G-module is an $\mathbb{L}$-spectrum E for which λ is an isomorphism, and $\wedge_{\mathscr{L}}$ restricts to a smash product $\wedge$ between S_G-modules. We have a G-space of nonequivariant arrows $f : E \longrightarrow E'$ of $\mathbb{L}$-spectra, where f must satisfy $f \circ \xi = \xi' \circ \mathbb{L}f$. Its G-fixed point space is the space of G-maps $E \longrightarrow E'$. Arrows and G-maps of S_G-modules are just arrows and G-maps of the underlying $\mathbb{L}$-spectra.

We let $\mathscr{M}_G$ denote the category of S_G-modules and their arrows. This category is enriched over $G\mathscr{T}$. We let $G\mathscr{M}$ denote its G-fixed category; its objects are the S_G-modules and its morphisms are the G-maps.

All of these categories depend on a choice of a universe U: G-spectra are indexed on the indexing G-spaces in U, as in II.1.2. When suppressed from the notation, as above, U is generally assumed to be complete. However, everything in this section applies verbatim to an arbitrary universe U.

We have defined the stable and positive stable model structures on $G\mathscr{P}$ in Chapter III. We now consider model structures on the categories $G\mathscr{S}$ and $G\mathscr{M}$ of G-spectra and S_G-modules. We write $\Sigma^\infty_V : \mathscr{T}_G \longrightarrow \mathscr{S}_G$ for the shift desuspension G-spectrum functor LF_V, which again is left adjoint to evaluation at V. As in I.4.5 and I.4.6, we have an adjunction $(\mathbb{F}, \mathbb{V})$ relating the categories $\mathscr{S}_G$ and $\mathscr{M}_G$. It induces an adjunction between the G-fixed categories $G\mathscr{S}$ and $G\mathscr{M}$.

PROPOSITION 2.1. *Define $\mathbb{F} : \mathscr{S}_G \longrightarrow \mathscr{M}_G$ by $\mathbb{F}E = S_G \wedge_{\mathscr{L}} \mathbb{L}E$ and $\mathbb{V} : \mathscr{M}_G \longrightarrow \mathscr{S}_G$ by $\mathbb{V}M = F_{\mathscr{L}}(S_G, M)$. Then $\mathbb{F}$ and $\mathbb{V}$ are left and right adjoint, and there is a natural weak equivalence of G-spectra $\tilde{\lambda} : M \longrightarrow \mathbb{V}M$.*

DEFINITION 2.2. We define spheres and cells in $G\mathscr{S}$ and $G\mathscr{M}$.

(i) A sphere G-spectrum is a G-spectrum of the form $\Sigma^\infty_q(G/H_+ \wedge S^n)$, where $q \geq 0$, $n \geq 0$, and $H \subset G$. A generalized sphere G-spectrum is a G-spectrum of the form $\Sigma^\infty_V(G/H_+ \wedge S^n)$, where V is an indexing G-space V in U, $n \geq 0$, and $H \subset G$. Write $S^n_G = \Sigma^\infty S^n$ and $S^{-n}_G = \Sigma^\infty_n S^0$ if $n \geq 0$ and write $S^{-V}_G = \Sigma^\infty_V S^0$ for an indexing G-space V.

(ii) A sphere S_G-module or a generalized sphere S_G-module is an S_G-module of the form $\mathbb{F}E$, where E is a sphere G-spectrum or a generalized sphere G-spectrum.

(iii) A generating q-cofibration or generalized generating q-cofibration in $G\mathscr{S}$ or $G\mathscr{M}$ is a map of the form $E \longrightarrow CE$, where E is a sphere object or generalized sphere object and CE is the cone on E.

(iv) A generating acyclic q-cofibration or generalized generating acyclic q-cofibration is a map of the form $i_0 : CE \longrightarrow CE \wedge I_+$, where E is a sphere object or generalized sphere object.

REMARK 2.3. It would serve no purpose to consider sphere G-spectra of the more general form $F_V(G/H_+ \wedge S^W)$ for G-representations W since S^W is triangulable as a finite G-CW complex [**17**]. The use of functors F_V rather than just functors F_n is the fundamental distinction.

The definition of homotopy groups for G-spectra takes a simpler form than for G-prespectra, as in [**6**, I.4.4]. Write $\pi(E, E')_G$ for the set of homotopy classes of maps $E \longrightarrow E'$ in $G\mathscr{S}$. Then, for $H \subset G$, $n \in \mathbb{Z}$, and $E \in G\mathscr{S}$,

$$\pi_n^H(E) = \pi(G/H_+ \wedge S^n, E)_G. \tag{2.4}$$

Equivalently, for $n \geq 0$,

$$\pi_n^H(E) = \pi_n^H(E(0)) \quad \text{and} \quad \pi_{-n}^H(E) = \pi_0^H(E(\mathbb{R}^n)). \tag{2.5}$$

The homotopy groups of an S_G-module are the homotopy groups of its underlying G-spectrum.

REMARK 2.6. The homotopy groups of a G-spectrum E are the same as those of the G-prespectrum ℓE. A G-prespectrum T is a (positive) inclusion G-prespectrum if each adjoint structure map $T(V) \longrightarrow \Omega^{W-V} T(W)$ (with $V^G \neq 0$) is an inclusion. As in I.4.8, the unit $T \longrightarrow \ell L T$ of the adjunction is then a weak equivalence; see [**19**, I.2.2]. This applies to cofibrant G-prespectra, for example.

DEFINITION 2.7. Consider the categories $G\mathscr{S}$ and $G\mathscr{M}$ of G-spectra and S_G-modules.

(i) A map in either category is a weak equivalence if it induces an isomorphism on all homotopy groups π_n^H.
(iia) A map is a q-cofibration if it is a retract of a relative cell G-complex defined in terms of generating q-cofibrations.
(iib) A map is a generalized q-cofibration if it is a retract of a relative cell G-complex defined in terms of generalized generating q-cofibrations.
(iiia) A map is a q-fibration if it satisfies the RLP with respect to the generating acyclic q-cofibrations.
(iiib) A map is a restricted q-fibration if it satisfies the RLP with respect to the generalized generating acyclic q-cofibrations.

REMARK 2.8. Exactly as in I§4, the results of III.3.5 imply the corresponding statements for weak equivalences of G-spectra and of S_G-modules. That is, the evident equivariant analogues of I.4.10 and I.4.11 hold.

THEOREM 2.9. *Consider the categories $G\mathscr{S}$ and $G\mathscr{M}$ of G-spectra and S_G-modules.*

(i) *These categories are G-topological model categories with respect to the weak equivalences, q-cofibrations, and q-fibrations; we call this the cellular model structure.*
(ii) *These categories are also G-topological model categories with respect to the weak equivalences, generalized q-cofibrations, and restricted q-fibrations; we call this the generalized cellular model structure.*
(iii) *The identity functors of $G\mathscr{S}$ and $G\mathscr{M}$ are the left adjoints of Quillen equivalences from the cellular model structure to the generalized cellular model structure.*
(iv) *With either the cellular or the generalized cellular model structures on both categories, the pair $(\mathbb{F}, \mathbb{V})$ is a Quillen equivalence between $G\mathscr{S}$ and $G\mathscr{M}$.*

(v) *With the generalized cellular model structure on $G\mathscr{S}$ and the stable model structure on $G\mathscr{P}$, (L, ℓ) is a Quillen equivalence between $G\mathscr{P}$ and $G\mathscr{S}$.*

With either model structure, $G\mathscr{S}$ is right proper and $G\mathscr{M}$ is proper.

Proof. One can mimic the proofs of the model axioms in Chapter III or in [**6**, V§5]. With the second strategy, one starts with different definitions. One redefines the q-fibrations of G-spectra to be the levelwise Serre G-fibrations for levels n or levels V, and one redefines the q-fibrations of S_G-modules to be the maps f such that $\mathbb{V}f$ is a q-fibration of G-spectra; for both G-spectra and S_G-modules, one redefines the q-cofibrations to be the maps that satisfy the LLP with respect to the acyclic q-fibrations. With the first proof, this characterization of the q-fibrations and q-cofibrations follows. The second proof capitalizes on the facts that the generating acyclic q-cofibrations are inclusions of deformation retracts and that homotopic maps induce isomorphisms of homotopy groups. These facts make the proofs of the model axioms almost completely formal. The only point that requires comment in the equivariant setting is the proof of [**6**, VII.5.8], where one needs to know that a map is an acyclic (restricted) q-fibration if and only if it satisfies the RLP with respect to the (generalized) generating q-cofibrations. The interesting point is that the same weak equivalences work for both statements, and this is a direct consequence of III.3.4.

To prove that our adjoint pairs are Quillen equivalences, it suffices to prove that their right adjoints create weak equivalences and preserve q-fibrations and that the unit of the adjunction is a weak equivalence on cofibrant objects [**20**, A.2]. The statements about right adjoints are immediate; for (iv), the definitions imply that $\mathbb{V}$ creates the q-fibrations in $G\mathscr{M}$. The statement about the unit of the adjunction is trivial in part (iii), follows as in I.4.6 from [**6**, I.4.6, I.8.7, II.2.5] in part (iv), and follows from Remark 2.6 in part (v).

The last statement is proven using long exact sequences of homotopy groups of fiber and cofiber sequences, as in [**20**, 9.10] or [**6**, I.6.6]. □

Remark 2.10. It seems unlikely that $G\mathscr{S}$ is left proper. The problem is that a cofiber sequence of G-spectra is only known to give rise to a long exact sequence of homotopy groups under the mild hypothesis of tameness [**6**, I.3.4], whereas any cofiber sequence of S_G-modules gives rise to a long exact sequence of homotopy groups [**6**, I.6.4], as in I.4.10 and I.4.11.

Note that S_G itself is cofibrant as an object of $G\mathscr{S}$ but is not cofibrant as an object of $G\mathscr{M}$, where $\mathbb{F}(S_G)$ is a cofibrant approximation of S_G.

Of course, the reason for the introduction of $G\mathscr{M}$ is that, unlike $G\mathscr{S}$, it is a closed symmetric monoidal category under the smash product, so that we can define rings, called S_G-algebras, and modules over them; see also [**7, 26**]. Exactly as in [**6**, VII§§4,5], we have model categories of such highly structured equivariant ring and module spectra.

Theorem 2.11. *The following categories admit cellular and generalized cellular G-topological model structures whose weak equivalences and q-fibrations or restricted q-fibrations are created in $G\mathscr{M}$.*

(i) *The category of S_G-algebras.*
(ii) *The category of commutative S_G-algebras.*
(iii) *The category of modules over an S_G-algebra R.*

(iv) *The category of algebras over a commutative S_G-algebra R.*
(v) *The category of commutative algebras over a commutative S_G-algebra R.*

In all cases, the q-cofibrations are the retracts of relative cellular or generalized cellular objects in the specified category. All of these categories are right proper, and the categories in (iii) are also left proper.

A general notion of cellular object that applies in all cases is given in [**6**, VII.4.11]. The fact that all of our categories are right proper is inherited from $G\mathscr{M}$, as is the fact that the categories in (iii) are left proper.

REMARK 2.12. The analogues for orthogonal G-spectra of the categories in (ii), (iii), and (v) are left proper [**20**, 12.1, 15.2], but not the analogues of the categories in (i) and (iv). The reason that commutative orthogonal ring G-spectra behave better than commutative S_G-algebras can be seen by comparing [**20**, 15.5, 15.6] with [**6**, III.5.1].

3. The construction of the functors $\mathbb{N}$ and $\mathbb{N}^{\#}$

The following equivariant analogue of I.3.1 is the essential step in the construction of the adjoint pair $(\mathbb{N}, \mathbb{N}^{\#})$. We assume that our given fixed universe U is complete, but the result holds more generally; see Remark 3.8. Recall that the G-category $\mathscr{I}_G\mathscr{S}$ of orthogonal G-spectra is isomorphic to the G-category $\mathscr{J}_G\mathscr{T}$ of $\mathscr{J}_G$-spaces, where $\mathscr{J}_G$ is the category constructed from the category $\mathscr{I}_G$ and the $\mathscr{I}_G$-space S_G in II.4.1.

THEOREM 3.1. *There is a strong symmetric monoidal contravariant G-functor $\mathbb{N}^* : \mathscr{J}_G \longrightarrow \mathscr{M}_G$. If $V^G \neq 0$, then $\mathbb{N}^*(V)$ is (non-canonically) isomorphic to $\mathbb{F}S_G^{-V}$, and the evaluation G-map*

$$\varepsilon : \mathbb{N}^*(V) \wedge S^V = \mathbb{N}^*(V) \wedge \mathscr{J}_G(0, V) \longrightarrow \mathbb{N}^*(0) \cong S_G$$

of the functor is a weak equivalence.

PROOF. The proof is similar to the nonequivariant argument in I§6, and we shall not give full details. Two equivalent constructions of $\mathbb{N}^*$ were given in I§6, and both apply equivariantly. The first is in terms of twisted half-smash products. The required equivariant theory of twisted half-smash products is given by Cole in [**27**, XXII], except that he writes entirely in terms of G-maps. Reinterpretation in terms of the equivariant context of II§1 is straightforward.

As in I§6, we use superscripts to denote relevant universes. Consider the universes $V \otimes U$ for $V \in \mathscr{I}_G$, together with their subspaces $V \cong V \otimes \mathbb{R}$. We set

$$\mathbb{N}^*(V) = S_G \wedge_{\mathscr{L}} (\mathscr{I}(V \otimes U, U) \ltimes \Sigma_V^{V \otimes U}(S^0)). \tag{3.2}$$

Here, for inner product G-spaces U and U', $\mathscr{I}(U, U')$ is the G-space of linear isometries $U \longrightarrow U'$. It is G-contractible when there is a G-linear isometry $U \longrightarrow U'$ [**23**, 1.3]. Since U is complete, there is such a G-linear isometry $V \otimes U \longrightarrow U$ for any V. The functor $\mathbb{J}$ given by

$$\mathbb{J}E = S_G \wedge_{\mathscr{L}} E \tag{3.3}$$

converts $\mathbb{L}$-spectra to weakly equivalent S_G-modules; we rewrite

$$\mathbb{N}^*(V) = \mathbb{J}(\mathscr{I}(V \otimes U, U) \ltimes \Sigma_V^{V \otimes U}(S^0)). \tag{3.4}$$

The functoriality in V can be proven directly or by comparison with the alternative second construction, which is given in terms of Thom G-spectra. Replacing inner product spaces in I.6.8-6.18 by G-inner product spaces, still using general linear isometries but taking the action of G by conjugation into account, we obtain the alternative description

$$\mathbb{N}^*(V) \cong \mathbb{J}M^{V\otimes U,U}_{V,-}. \tag{3.5}$$

With this form of the definition, the functoriality in V is given by I.6.20.

The definition (3.4) makes sense even when $V = 0$, with G-spectra indexed on 0 interpreted as G-spaces. Inspection of definitions shows that $\mathbb{N}^*(0) \cong S_G$, as required for $\mathbb{N}^*$ to be strong symmetric monoidal. The natural isomorphism

$$\phi : \mathbb{N}^*(V) \wedge \mathbb{N}^*(W) \longrightarrow \mathbb{N}^*(V \oplus W) \tag{3.6}$$

required of a strong symmetric monoidal functor is constructed as in I.6.7.

The argument for the identification of the $\mathbb{N}(\mathbb{R}^n)$ for $n > 0$ given in I§6 does not generalize to an equivariant identification of the $\mathbb{N}(V)$ for $V^G \neq 0$. However, since U is complete, $V \otimes U$ is complete when $V^G \neq 0$, and we can choose an isomorphism of G-universes $f_V : V \otimes U \longrightarrow U$ that restricts to the identification $V \otimes \mathbb{R} \cong V$. Then f_V induces an isomorphism

$$\mathscr{I}(V \otimes U, U) \ltimes \Sigma^{V\otimes U}_V(S^0) \cong \mathscr{I}(U,U) \ltimes \Sigma^U_V S^0 = \mathbb{L}S^{-V}_G.$$

Applying $\mathbb{J}$ and using the definition in (3.4), we obtain the required isomorphism

$$\mathbb{N}^*(V) \longrightarrow \mathbb{J}\mathbb{L}S^{-V}_G = \mathbb{F}S^{-V}_G. \tag{3.7}$$

Comparing with I.6.21, we see that $\varepsilon : \mathbb{N}^*(V) \wedge S^V \longrightarrow S_G$ corresponds under this isomorphism to the composite of the identification $\mathbb{F}S^{-V}_G \wedge S^V \cong \mathbb{F}S_G$ (see [**19**, I.4.2]) and the cofibrant approximation $\mathbb{F}S_G \longrightarrow S_G$. Observe that, although f_V and the isomorphism (3.7) depend non-canonically on V, when we suspend (3.7) by S^V and map to S_G, the resulting weak equivalence, namely ε, is natural in V. Concretely, this holds since we are projecting the contractible G-spaces $\mathscr{I}(V\otimes U, U)$ and $\mathscr{I}(U,U)$ to a point, which makes the choice of isomorphism between them invisible. □

REMARK 3.8. Let U be any universe and let $\mathscr{V} = \mathscr{V}(U)$ be the collection of all G-inner product spaces that are isomorphic to indexing G-spaces in U. Taking $\mathscr{J}_G$ and $\mathscr{M}_G$ to be defined with respect to $\mathscr{V}$ and U, the construction of the G-functor $\mathbb{N}^*$ still applies. We say that U is *closed under tensor products* if the tensor product of representations in U is isomorphic to a representation in U; clearly this holds if and only if $\mathscr{V}$ is closed under tensor products. In this case, $V \otimes U \cong U$ if $V \subset U$ and $V^G \neq 0$. This implies that we still have the crucial isomorphism (3.7), and we conclude that Theorem 3.1 remains valid as stated. All of the results in the introduction hold in this generality. For example, these results hold when U is the trivial universe.

REMARK 3.9. Again, let U be any universe. The following observations, which are due to the referee, show that the condition $V^G \neq 0$ can be relaxed to the less restrictive assumption that $V \neq 0$ in Theorem 3.1 and its generalization in the previous remark. Assume that $V \neq 0$.

(i) Since U contains a copy of V and V is isomorphic to its dual V^*, $V \otimes U$ contains a copy of $V \otimes V^* \cong \mathrm{Hom}(V, V)$ and hence contains a copy of the trivial representation. Therefore $V \otimes U$ is a universe in the sense of II.1.1.

(ii) If U is closed under tensor products, then $V \otimes U \cong U$. Indeed, for any $W \subset U$, there is a $Z \subset U$ such that $Z \cong V \otimes W$. In view of (i), $V \otimes Z \subset V \otimes U$ is isomorphic to $\mathrm{Hom}(V, V) \otimes W$ and thus contains a copy of W. By the definition of a universe in II.1.1, this implies that $V \otimes U \cong U$.

The categorical discussion of I§2 applies verbatim in the equivariant context, provided that all functors are required to be (continuous) G-functors between "bicomplete" (topological) G-categories. Here a G-category $\mathscr{C}_G$ is said to be bicomplete if it is tensored and cotensored over the G-category $\mathscr{T}_G$ of based G-spaces and if its associated G-fixed category $G\mathscr{C}$ is complete and cocomplete. A G-functor $\mathbb{F} : \mathscr{C}_G \longrightarrow \mathscr{C}'_G$ is defined to be "right exact" if it preserves tensors and if its restriction $\mathbb{F} : G\mathscr{C} \longrightarrow G\mathscr{C}'$ to G-maps is right exact. The cited discussion then gives the following formal consequence of Theorem 3.1.

Theorem 3.10. *Define G-functors $\mathbb{N} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{M}_G$ and $\mathbb{N}^\# : \mathscr{M}_G \longrightarrow \mathscr{I}_G\mathscr{S}$ by letting $\mathbb{N}(X) = \mathbb{N}^* \otimes_{\mathscr{J}_G} X$ and $(\mathbb{N}^\# M)(V) = \mathscr{M}_G(\mathbb{N}^*(V), M)$. Then $(\mathbb{N}, \mathbb{N}^\#)$ is an adjoint pair of G-functors such that $\mathbb{N}$ is strong symmetric monoidal and $\mathbb{N}^\#$ is lax symmetric monoidal.*

4. The proofs of the comparison theorems

As in I§3, to prove Theorem 1.1 it suffices to prove the following three results.

Lemma 4.1. *The functor $\mathbb{N}^\#$ preserves homotopy groups and creates weak equivalences.*

Lemma 4.2. *The functor $\mathbb{N}^\#$ preserves q-fibrations.*

Proposition 4.3. *The unit $\eta : X \longrightarrow \mathbb{N}^\#\mathbb{N}X$ of the adjunction is a weak equivalence for all cofibrant orthogonal G-spectra X.*

As in the proof of its nonequivariant analogue in I§5, Lemma 4.1 is implied by the following result.

Lemma 4.4. *For S_G-modules M, $\mathbb{N}^\# M$ is a positive Ω-G-spectrum.*

Proof. The statement means that if $V \subset W$, $V^G \neq 0$, then the structure map

$$\tilde{\sigma} : (\mathbb{N}^\# M)(V) = \mathscr{M}(\mathbb{N}^*(V), M) \longrightarrow \mathscr{M}(\Sigma^{W-V}\mathbb{N}^*(W), M) \cong \Omega^{W-V}(\mathbb{N}^\# M)(W)$$

is a weak equivalence. This holds because $\tilde{\sigma}$ is induced by the evaluation map $\varepsilon : \mathbb{N}^*(W) \wedge S^{W-V} \longrightarrow \mathbb{N}^*(V)$, which is a weak equivalence between cofibrant S_G-modules by Theorem 3.1. □

In Lemma 4.2, we are saying that $\mathbb{N}^\#$ carries restricted q-fibrations of S_G-modules to q-fibrations of orthogonal G-spectra in the positive stable model structure, and the proof is the same formal argument as in I§5.

To prove Theorem 1.1, we only need Proposition 4.3 for orthogonal G-spectra that are cofibrant in the positive stable model structure, but it holds more generally for orthogonal G-spectra that are cofibrant in the stable model structure. The proof is the same as in the nonequivariant case I§5. As there, we first prove the result when $X = F_V A$ for a G-CW complex A, and we then deduce it for cell orthogonal

G-spectra. For the first step, it is convenient to work on the prespectrum level, using the Quillen equivalence $(\mathbb{P}, \mathbb{U})$ between $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$ of III.4.16 and III.5.7.

We display our Quillen equivalences in the following (noncommutative) diagram:

$$\begin{array}{ccc} G\mathscr{P} & \underset{\ell}{\overset{L}{\rightleftarrows}} & G\mathscr{S} \\ \mathbb{U}\uparrow\downarrow\mathbb{P} & & \mathbb{V}\uparrow\downarrow\mathbb{F} \\ G\mathscr{I}\mathscr{S} & \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} & G\mathscr{M}. \end{array}$$

As in I§3, to prove Theorems 1.2, 1.3, 1.5, and 1.6, we observe that Lemmas 4.1 and 4.2 and Proposition 4.3 imply their analogues for the adjoint pairs $(\mathbb{N}, \mathbb{N}^{\#})$ induced on the categories of multiplicatively enriched objects considered in the cited theorems. Since the weak equivalences and q-fibrations in the multiplicatively enriched categories are created in the underlying categories (of orthogonal G-spectra or of S_G-modules), this is obvious for the lemmas. For the proposition, we must relate cofibrancy of multiplicatively enriched objects with cofibrancy of their underlying orthogonal G-spectra. For Theorems 1.2 and 1.3, III.7.6 gives what is needed. For Theorem 1.5, we argue as in I§3. The essential point is comparison of the second statement of III.8.4 with the following analogue for S_G-modules, whose proof is precisely parallel to the proof of its nonequivariant analogue in [**6**, III.5.1].

LEMMA 4.5. *For a cofibrant S_G-module M (in the generalized cellular model structure), the quotient map*

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} M^{(i)} \longrightarrow M^{(i)}/\Sigma_i$$

is a weak equivalence.

Corollaries 1.4 and 1.7 are also proven as in I§3, noting that the result I.3.6 remains valid in the equivariant setting, with the same proof.

5. The functor $\mathbb{M}$ and its comparison with $\mathbb{N}$

Exactly as in I§7, there is another G-functor $\mathbb{M} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{M}_G$ which, although less convenient for the comparison theorems, is considerably more natural intuitively. We have the forgetful G-functor $\mathbb{U}$ from orthogonal G-spectra to G-prespectra and the spectrification G-functor L from G-prespectra to G-spectra. As in I.7.4 and I.7.5, we can define equivariant notions of $\mathbb{L}$-prespectra and $\mathbb{L}$-spectra. The latter notion is the one that forms the basis for the definition of S_G-modules. Writing $\mathscr{P}_G[\mathbb{L}]$ and $\mathscr{S}_G[\mathbb{L}]$ for the resulting G-categories, we verify as in I.7.6 and I.7.7 that the forgetful G-functor $\mathbb{U}$ takes values in $\mathscr{P}_G[\mathbb{L}]$ and the spectrification G-functor L induces a G-functor $L : \mathscr{P}_G[\mathbb{L}] \longrightarrow \mathscr{S}_G[\mathbb{L}]$. Moreover, we have the G-functor $\mathbb{J} : \mathscr{S}_G[\mathbb{L}] \longrightarrow \mathscr{M}_G$ specified by $\mathbb{J}E = S_G \wedge_{\mathscr{L}} E$.

DEFINITION 5.1. Define the G-functor $\mathbb{M} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{M}_G$ to be the composite

$$\mathscr{I}_G\mathscr{S} \xrightarrow{P} \mathscr{P}_G[\mathbb{L}] \xrightarrow{L} \mathscr{S}_G[\mathbb{L}] \xrightarrow{J} \mathscr{M}_G,$$

where P denotes the underlying prespectrum functor regarded as taking values in the category of $\mathbb{L}$-prespectra.

As in I.7.9–7.11, the functor $\mathbb{M}$ has formal properties much like those of $\mathbb{N}$.

LEMMA 5.2. *The G-functor $\mathbb{M}$ is right exact and lax symmetric monoidal, with $\mathbb{M}S_G \cong S_G$ (where S_G on the left is the sphere orthogonal G-spectrum).*

As in I.7.12, although $\mathbb{M}$ does not appear to preserve cofibrant objects, it has the following basic homotopical property. Recall Remark 2.6.

LEMMA 5.3. *For positive inclusion orthogonal G-spectra X, there is a natural isomorphism*

$$\pi_*^H(X) \cong \pi_*^H(\mathbb{M}(X)).$$

Arguing as in I.7.13, we obtain the following comparison theorem.

THEOREM 5.4. *There is a symmetric monoidal natural G-map*

$$\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$$

such that $\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$ is a weak equivalence if X is cofibrant.

One advantage of $\mathbb{M}$ over $\mathbb{N}$ is that it is quite convenient for the study of change of groups, as was exploited implicitly by Greenlees and May in [**13**]. We turn to considerations of change of group and universe in our new model theoretic framework after generalizing the theory to families.

6. Families, cofamilies, and Bousfield localization

We discuss in model theoretical terms the familiar idea of concentrating G-spaces or G-spectra at or away from a family of subgroups. We also relate this idea to Bousfield localization. The theory works the same way for S_G-modules and for orthogonal G-spectra; we often use the neutral term "object", and we let $G\mathscr{C}$ stand for either $G\mathscr{M}$ or $G\mathscr{I}\mathscr{S}$. Similar arguments lead to weaker conclusions in the category of G-spectra, due to Remark 2.10. We write $[X, Y]_G$ for maps $X \longrightarrow Y$ in the homotopy category Ho$G\mathscr{C}$ with respect to the stable model structure.

We call weak equivalences G-equivalences in this section. For $H \subset G$, we say that a G-map is an H-equivalence if it is a weak equivalence when regarded as an H-map; we will treat restriction to subgroups systematically later. Let $\mathscr{F}$ be a family of subgroups of G, namely a set of subgroups closed under passage to conjugates and subgroups. There is a universal $\mathscr{F}$-space $E\mathscr{F}$. It is a G-CW complex such that $(E\mathscr{F})^H$ is contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$. Of course, its cells must be of orbit type G/H with $H \in \mathscr{F}$. The following definitions make sense for based G-spaces as well as for objects in $G\mathscr{C}$.

DEFINITION 6.1. (i) A map $f : X \longrightarrow Y$ is a $\mathscr{F}$-equivalence if it is an H-equivalence for all $H \in \mathscr{F}$.
(ii) An object X is an $\mathscr{F}$-object if the map $\pi : E\mathscr{F}_+ \wedge X \longrightarrow X$ induced by the projection $E\mathscr{F}_+ \longrightarrow S^0$ is a G-equivalence.

DEFINITION 6.2. Let E be a cofibrant object of $G\mathscr{C}$ or a cofibrant based G-space.
(i) A map $f : X \longrightarrow Y$ is an E-equivalence if $\mathrm{id} \wedge f : E \wedge X \longrightarrow E \wedge Y$ is a G-equivalence.
(iii) Z is E-local if $f^* : [Y, Z]_G \longrightarrow [X, Z]_G$ is an isomorphism for all E-equivalences $f : X \longrightarrow Y$.

(iv) An E-localization of X is an E-equivalence $\lambda : X \longrightarrow Y$ from X to an E-local object Y.

Consider $G\mathscr{M}$ with either the cellular or the generalized cellular model structure and consider $G\mathscr{I}\mathscr{S}$ with either the stable or the positive stable model structure. Indexing can be on any universe. Writing $G\mathscr{C}$ for any of these model categories, we have the following slightly digressive, but important, omnibus theorem. Nonequivariantly, it is proven for S-modules in [**6**, VIII§1]. With only minor variations, the argument there applies equivariantly to all of the model structures we are considering. It does not apply to G-spectra because it uses that $G\mathscr{C}$ is left proper.

THEOREM 6.3. *Let E be a cofibrant object of $G\mathscr{C}$ or a cofibrant based G-space. Then $G\mathscr{C}$ has an E-model structure whose equivalences are the E-equivalences and whose E-cofibrations are the q-cofibrations of the given model structure. The E-fibrant objects are the E-local objects, and E-fibrant approximation constructs a Bousfield localization $\lambda : X \longrightarrow L_E X$ of X at E.*

Taking $E = E\mathscr{F}_+$, we call the resulting model structures *Bousfield $\mathscr{F}$-model structures*. Here Bousfield localization takes the following elementary form.

PROPOSITION 6.4. *The map $\xi : X \longrightarrow F(E\mathscr{F}_+, X)$ induced by the projection $E\mathscr{F}_+ \longrightarrow S^0$ is an $E\mathscr{F}_+$-localization of X.*

PROOF. The map ξ is an $E\mathscr{F}_+$-equivalence by [**12**, 17.2], and it is immediate by adjunction that $F(E\mathscr{F}_+, X)$ is $E\mathscr{F}_+$-local. □

Completion theorems in equivariant stable homotopy theory are concerned with the comparison of this Bousfield localization with another, more algebraically computable, Bousfield localization. See, for example, [**9**, 4.1], [**10**]. However, the Bousfield $\mathscr{F}$-model structures are not the most natural ones to consider in the context of families. In the model structures on $G\mathscr{C}$, the generating q-cofibrations and generating acyclic q-cofibrations are obtained by applying functors F_V or Σ^∞_V to certain maps $G/H_+ \wedge A \longrightarrow G/H_+ \wedge B$ of G-spaces. We can restrict attention to those $H \in \mathscr{F}$ in all of these definitions. We refer to $\mathscr{F}$-cofibrations rather than q-cofibrations for the retracts of the resulting relative $\mathscr{F}$-cell complexes. In contrast with Theorem 6.3, the following theorem, with left properness deleted, applies just as well to G-spectra as to S_G-modules and orthogonal G-spectra.

THEOREM 6.5. *The category $G\mathscr{C}$ is a compactly generated proper G-topological model category with weak equivalences the $\mathscr{F}$-equivalences and with generating $\mathscr{F}$-cofibrations and generating acyclic $\mathscr{F}$-cofibrations obtained from the original generating q-cofibrations and generating acyclic q-cofibrations by restricting to orbits G/H with $H \in \mathscr{F}$.*

We refer to these as *$\mathscr{F}$-model structures*. The proofs of the model axioms are the same as in the case $\mathscr{F} = \mathscr{A}\ell\ell$. Following the approach of Chapter III, we first give $G\mathscr{T}$ an $\mathscr{F}$-model structure using its $\mathscr{F}$-equivalences and $\mathscr{F}$-cofibrations. The $\mathscr{F}$-fibrations are the maps that give Serre fibrations on passage to H-fixed points for $H \in \mathscr{F}$. We then give G-prespectra and orthogonal G-spectra level $\mathscr{F}$-model structures using level $\mathscr{F}$-equivalences and level $\mathscr{F}$-fibrations; the resulting $\mathscr{F}$-cofibrations are as described above. Finally, we give G-prespectra and orthogonal G-spectra stable $\mathscr{F}$-model structures using the $\mathscr{F}$-equivalences and $\mathscr{F}$-cofibrations. Similarly, the approach of [**6**] applies verbatim to give G-spectra and

S_G-modules cellular and generalized cellular $\mathscr{F}$-model structures. All of our comparison theorems have $\mathscr{F}$ versions that admit the same proofs.

To relate the $\mathscr{F}$-model structure to the Bousfield $\mathscr{F}$-model structure, we must show that the two structures have the same weak equivalences. This is not at all obvious. We need the following lemma in the proof.

LEMMA 6.6. *If A is a based $\mathscr{F}$-CW complex and X is a cell object of $G\mathscr{C}$, then $A \wedge X$ is an $\mathscr{F}$-cell object.*

PROOF. This is shown by inspection of the combinatorial structure of cell objects. The essential point is that, for $H \in \mathscr{F}$ and any K, we can triangulate the product $G/H \times G/K$ as a finite G-CW complex by [**17**], and in any such triangulation the only orbit types that can occur are G/L with $L \in \mathscr{F}$. While the details from here are straightforward, the reader should be aware that the underlying spaces of $\mathscr{F}$-cell objects in any of our categories are generally not $\mathscr{F}$-spaces, so that the conclusion is much less obvious than its space level analogue. □

PROPOSITION 6.7. *The following conditions on a map $f : X \longrightarrow Y$ are equivalent.*

(i) *f is an $\mathscr{F}$-equivalence.*
(ii) *$f_* : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism for $H \in \mathscr{F}$.*
(iii) *f is an $E\mathscr{F}_+$-equivalence.*

PROOF. Parts (i) and (ii) are equivalent by definition. For $H \in \mathscr{F}$, the G-map $E\mathscr{F}_+ \longrightarrow S^0$ is an H-homotopy equivalence and therefore (iii) implies (i). We must prove that (i) implies (iii). Thus let $f : X \longrightarrow Y$ be an $\mathscr{F}$-equivalence. We must show that $\mathrm{id} \wedge f : E\mathscr{F}_+ \wedge X \longrightarrow E\mathscr{F}_+ \wedge Y$ is a G-equivalence, and it is certainly an $\mathscr{F}$-equivalence. Since $E\mathscr{F}_+$ is a G-CW complex, smashing with it preserves G-equivalences. Using functorial cofibrant approximation in the original model structure, we may assume without loss of generality that X and Y are cell objects. By Lemma 6.6, $E\mathscr{F}_+ \wedge X$ and $E\mathscr{F}_+ \wedge Y$ are then $\mathscr{F}$-cell objects. In the case of S_G-modules, where all objects are fibrant, we conclude that $\mathrm{id} \wedge f$ is a G-homotopy equivalence because it is an $\mathscr{F}$-equivalence between $\mathscr{F}$-cofibrant objects. (In earlier terminology, we are invoking the $\mathscr{F}$-Whitehead theorem [**19**, II.2.2]). In the case of $G\mathscr{I}\mathscr{S}$, using functorial fibrant approximation we may assume further that X and Y are orthogonal Ω-G-spectra. Using III.9.3, we conclude that f is a level $\mathscr{F}$-equivalence. On the space level, it is clear that (i) implies (iii), and we conclude that $\mathrm{id} \wedge f$ is a level G-equivalence and hence a π_*-isomorphism. □

REMARK 6.8. For a fibrant object X of $G\mathscr{C}$, we have $\pi_*^H(X) = \pi_*(X^H)$, just as for G-spaces. For orthogonal G-spectra, this is V.3.2 below. For G-spectra (all of which are fibrant), it is [**19**, I.4.5], and the analogue for S_G-modules follows (see VI.3.4 below). Thus, when X and Y are fibrant, a map $f : X \longrightarrow Y$ is an $\mathscr{F}$-equivalence if and only $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all $H \in \mathscr{F}$.

The Bousfield $\mathscr{F}$-model structures have more cofibrations and the same weak equivalences as the $\mathscr{F}$-model structures. This implies the following result.

THEOREM 6.9. *The identity functor $G\mathscr{C} \longrightarrow G\mathscr{C}$ is the left adjoint of a Quillen equivalence from the $\mathscr{F}$-model structure to the Bousfield $\mathscr{F}$-model structure.*

Now return to the notion of an $\mathscr{F}$-object in Definition 6.1. Observe that this is an intrinsic notion, independent of any model structure.

THEOREM 6.10. *An object X is an $\mathscr{F}$-object if and only if its $\mathscr{F}$-cofibrant approximation $\gamma : \Gamma X \longrightarrow X$ is a G-equivalence.*

PROOF. Since $E\mathscr{F}_+ \wedge G/H_+ \longrightarrow G/H_+$ is a G-homotopy equivalence if $H \in \mathscr{F}$, an $\mathscr{F}$-cell complex is an $\mathscr{F}$-object. The standard functorial construction gives ΓX as an $\mathscr{F}$-cell complex. The conclusion follows from the evident commutative diagram

$$\begin{array}{ccc} E\mathscr{F}_+ \wedge \Gamma X & \xrightarrow{\pi} & \Gamma X \\ \Big\downarrow{\scriptstyle \mathrm{id} \wedge \gamma} & & \Big\downarrow{\scriptstyle \gamma} \\ E\mathscr{F}_+ \wedge X & \xrightarrow[\pi]{} & X, \end{array}$$

in which $\mathrm{id} \wedge \gamma$ and the top map π are G-equivalences for any X by Lemma 6.6 and Theorem 6.7. □

Observe that, in $G\mathscr{M}$, this holds for both the cellular and the generalized cellular $\mathscr{F}$-model structures. Let $\mathrm{Ho}\mathscr{F}\mathscr{C}$ denote the homotopy category associated to the $\mathscr{F}$-model structure on $G\mathscr{C}$, or, equivalently, the Bousfield $\mathscr{F}$-model structure, and write $[X, Y]_{\mathscr{F}}$ for the set of maps $X \longrightarrow Y$ in this category. The results above imply the following description of $\mathrm{Ho}\mathscr{F}\mathscr{C}$.

THEOREM 6.11. *Smashing with $E\mathscr{F}_+$ defines an isomorphism*

$$[X, Y]_{\mathscr{F}} \cong [E\mathscr{F}_+ \wedge X, E\mathscr{F}_+ \wedge Y]_G$$

and thus gives an equivalence of categories from $Ho\mathscr{F}\mathscr{C}$ to the full subcategory of objects $E\mathscr{F}_+ \wedge X$ in $HoG\mathscr{C}$.

PROOF. If ΓX and ΓY are $\mathscr{F}$-cofibrant approximations of X and Y, then

$$[X, Y]_{\mathscr{F}} \cong [\Gamma X, \Gamma Y]_G \cong [E\mathscr{F}_+ \wedge \Gamma X, E\mathscr{F}_+ \wedge \Gamma Y]_G \cong [E\mathscr{F}_+ \wedge X, E\mathscr{F}_+ \wedge Y]_G.$$

Indeed, the definition of an $\mathscr{F}$-cell object, Lemma 6.6, and Theorem 6.10 imply that ΓX is an $\mathscr{F}$-object, that ΓX and $E\mathscr{F}_+ \wedge \Gamma X$ are cofibrant in our original model structure, and that $E\mathscr{F}_+ \wedge \Gamma X$ is a cofibrant approximation of $E\mathscr{F}_+ \wedge X$. Formally, we are using that the identity functor is the right adjoint of a Quillen adjoint pair relating the original model structure to the $\mathscr{F}$-model structure. □

There is an analogous theory for cofamilies, namely complements $\mathscr{F}'$ of families. Thus $\mathscr{F}'$ is the set of subgroups of G not in $\mathscr{F}$. We define $\tilde{E}\mathscr{F}$ to be the cofiber of $E\mathscr{F}_+ \longrightarrow S^0$. Then $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and is S^0 if $H \notin \mathscr{F}$. In contrast to the situation for G-spaces, the evident analogue of Proposition 6.7 is false for G-spectra. This motivates the following variant of Definition 6.1.

DEFINITION 6.12.

(i) A map $f : X \longrightarrow Y$ is an $\mathscr{F}'$-equivalence if it is an $\tilde{E}\mathscr{F}$-equivalence.

(ii) An object X is an $\mathscr{F}'$-object if the map $\lambda : X \longrightarrow \tilde{E}\mathscr{F} \wedge X$ induced by the inclusion $S^0 \longrightarrow \tilde{E}\mathscr{F}$ is a G-equivalence.

Again, take $G\mathscr{C}$ to be $G\mathscr{M}$ or $G\mathscr{I}\mathscr{S}$ with one of our usual model structures. We do not obtain model structures by restricting attention to orbits G/H with $H \in \mathscr{F}'$, but we do still have the Bousfield $\mathscr{F}'$-model structure obtained by taking $E = \tilde{E}\mathscr{F}$ in Theorem 6.3. We have the following analogue of Theorem 6.10.

THEOREM 6.13. *The following conditions on an object X are equivalent.*

(i) *X is an $\mathscr{F}'$-object.*
(ii) *X is an $\tilde{E}\mathscr{F}$-local object.*
(iii) *The $\tilde{E}\mathscr{F}$-fibrant approximation $\lambda : X \longrightarrow L_{\tilde{E}\mathscr{F}}X$ is a G-equivalence.*
(iv) *$\pi_*^H(X) = 0$ for $H \in \mathscr{F}$.*

For such an X and any Y, $\lambda^* : [\tilde{E}\mathscr{F} \wedge Y, X]_G \longrightarrow [Y, X]_G$ is an isomorphism.

PROOF. This is a strengthened version of [**19**, II.9.2]. Assume (i). Then the composite

$$[Y, X]_G \longrightarrow [\tilde{E}\mathscr{F} \wedge Y, \tilde{E}\mathscr{F} \wedge X]_G \longrightarrow [Y, \tilde{E}\mathscr{F} \wedge X]_G \cong [Y, X]_G$$

is the identity, where the first map is given by smashing with $\tilde{E}\mathscr{F}$, the second is λ^*, and the isomorphism is given by λ_*. Therefore, if $f : Y \longrightarrow Y'$ is an $\tilde{E}\mathscr{F}$-equivalence, then $f^* : [Y', X]_G \longrightarrow [Y, X]_G$ is a retract of the isomorphism $f^* : [\tilde{E}\mathscr{F} \wedge Y', \tilde{E}\mathscr{F} \wedge X]_G \longrightarrow [\tilde{E}\mathscr{F} \wedge Y, \tilde{E}\mathscr{F} \wedge X]_G$ and is thus an isomorphism. This shows that (i) implies (ii), and (ii) and (iii) are equivalent by Theorem 6.3 and the uniqueness of localizations. If $H \in \mathscr{F}$, then $G/H_+ \wedge \tilde{E}\mathscr{F}$ is G-contractible by a check of fixed point spaces, hence $G/H_+ \wedge Y \longrightarrow *$ is an $\tilde{E}\mathscr{F}$-equivalence for any Y. Letting Y run through the spheres S^n, this shows that (ii) implies (iv). Finally, assume (iv). We must prove (i). Smashing with the G-CW complex $\tilde{E}\mathscr{F}$ preserves G-equivalences, so by cofibrant approximation we may assume that X is a cell complex. Clearly (i) holds if and only if $E\mathscr{F}_+ \wedge X$ is trivial and, by Lemma 6.6, $E\mathscr{F}_+ \wedge X$ is an $\mathscr{F}$-cell complex. If $H \in \mathscr{F}$, then $E\mathscr{F}_+ \longrightarrow S^0$ is an H-homotopy equivalence, hence $\pi_*^H(E\mathscr{F}_+ \wedge X) = 0$ by hypothesis. Thus $E\mathscr{F}_+ \wedge X$ is $\mathscr{F}$-equivalent to the trivial object and is therefore trivial. Since $\lambda : \tilde{E}\mathscr{F} \longrightarrow \tilde{E}\mathscr{F} \wedge \tilde{E}\mathscr{F}$ is a G-homotopy equivalence by a check on fixed points, $\lambda : Y \longrightarrow \tilde{E}\mathscr{F} \wedge Y$ is an $\mathscr{F}'$-equivalence for any Y. Therefore the last statement follows from (ii). □

Again, in $G\mathscr{M}$ this holds for both the cellular and the generalized cellular Bousfield $\mathscr{F}'$-model structures. Let Ho$\mathscr{F}'\mathscr{C}$ denote the homotopy category associated to the Bousfield $\mathscr{F}'$-model structure on $G\mathscr{C}$ and write $[X, Y]_{\mathscr{F}'}$ for the set of maps $X \longrightarrow Y$ in this category. The previous theorem implies the following one.

THEOREM 6.14. *Smashing with $\tilde{E}\mathscr{F}$ defines an isomorphism*

$$[X, Y]_{\mathscr{F}'} \cong [\tilde{E}\mathscr{F} \wedge X, \tilde{E}\mathscr{F} \wedge Y]_G$$

and thus gives an equivalence of categories from Ho$\mathscr{F}'\mathscr{C}$ to the full subcategory of objects $\tilde{E}\mathscr{F} \wedge X$ in Ho$G\mathscr{C}$.

CHAPTER V

“Change” functors for orthogonal G-spectra

We develop the analogues for orthogonal G-spectra of the central structural features of equivariant stable homotopy theory: change of universe, change of group, fixed point and orbit spectra, and geometric fixed point spectra. The last notion has turned out to be very important in many applications, and its treatment in [**19**, II§9] is decidedly ad hoc and conceptually unsatisfactory. The geometric fixed point functor on orthogonal spectra turns out to be far more satisfactory.

1. Change of universe

Change of universe plays a fundamental role in the homotopical theory of [**19**], and, as explained in [**7, 26**], it takes a precise point-set level form in the theory of S_G-modules. The theory for orthogonal G-spectra takes a similarly precise point-set level form. The key fact is the following implication of the definition of an orthogonal G-spectrum.

Lemma 1.1. *Let V and W be G-inner product spaces in $\mathscr{V}$ of the same dimension. Then, for orthogonal G-spectra X, the evaluation G-map*

$$\mathscr{J}_G(V,W) \wedge X(V) \longrightarrow X(W)$$

of the G-functor X induces a G-homeomorphism

$$\alpha : \mathscr{J}_G(V,W) \wedge_{O(V)} X(V) \longrightarrow X(W).$$

Its domain is homeomorphic, but not necessarily G-homeomorphic, to $X(V)$.

Proof. Since V and W have the same dimension, $\mathscr{J}_G(V,W) = \mathscr{I}_G(V,W)_+$, and $\mathscr{I}_G(V,W)$ is a free right $O(V)$-space generated by any chosen linear isometric isomorphism $f : V \longrightarrow W$. We see that α is a homeomorphism, hence a G-homeomorphism, by noting that the map that sends y in $X(W)$ to the equivalence class of $(f, X(f^{-1})(y))$ gives the inverse homeomorphism. Mapping x to the equivalence class of (f,x) gives the homeomorphism $X(V) \cong \mathscr{J}_G(V,W) \wedge_{O(V)} X(V)$. □

Change of universe appears in several equivalent guises. We could apply the general theory of prolongation functors left adjoint to forgetful functors, using the equivariant version of I.2.10 and [**20**, §3], but we prefer to be more explicit. We mimic the analogous theory of [**7**].

Definition 1.2. Let $\mathscr{V}$ and $\mathscr{V}'$ be collections of representations as in II.1.1 and II.2.1. Thus both collections contain all trivial representations. Define a G-functor $I_{\mathscr{V}'}^{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}'}\mathscr{S} \longrightarrow \mathscr{I}_G^{\mathscr{V}}\mathscr{S}$ by letting

$$(I_{\mathscr{V}'}^{\mathscr{V}}X)(V) = \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)$$

for $X \in \mathscr{I}_G^{\mathscr{V}'}\mathscr{S}$ and $V \in \mathscr{V}$ with $\dim V = n$. The evaluation G-maps of the G-functor $I_{\mathscr{V}'}^{\mathscr{V}}X : \mathscr{J}_G^{\mathscr{V}} \longrightarrow \mathscr{T}_G$ are given as follows. By II.4.1, we see that

$$\mathscr{J}_G^{\mathscr{V}'}(\mathbb{R}^n, \mathbb{R}^p) = \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, \mathbb{R}^p)$$

and that, for $W \in \mathscr{V}$ with $\dim W = p$, composition gives a G-homeomorphism

$$\mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^p, W) \wedge_{O(p)} \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, \mathbb{R}^p) \longrightarrow \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, W).$$

The evaluation G-map

$$\mathscr{J}_G^{\mathscr{V}}(V, W) \wedge (I_{\mathscr{V}'}^{\mathscr{V}}X)(V) \longrightarrow (I_{\mathscr{V}'}^{\mathscr{V}}X)(W)$$

is the following composite, in which the first and last map are given by composition G-maps of $\mathscr{J}_G^{\mathscr{V}}$ and evaluation G-maps of X, while the isomorphism is given by the inverse of the G-homeomorphism just noted:

$$\begin{aligned} &\mathscr{J}_G^{\mathscr{V}}(V, W) \wedge \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n) \\ &\qquad \longrightarrow \quad \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, W) \wedge_{O(n)} X(\mathbb{R}^n) \\ &\qquad \cong \quad \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^p, W) \wedge_{O(p)} \mathscr{J}_G^{\mathscr{V}'}(\mathbb{R}^n, \mathbb{R}^p) \wedge_{O(n)} X(\mathbb{R}^n) \\ &\qquad \longrightarrow \quad \mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^p, W) \wedge_{O(p)} X(\mathbb{R}^p). \end{aligned}$$

If $\mathscr{V} = \mathscr{V}(U)$ and $\mathscr{V}' = \mathscr{V}(U')$ for universes U and U', then $\mathscr{V} \subset \mathscr{V}'$ if and only if there is a G-linear isometry $U \longrightarrow U'$. This is the starting point for the change of universe functors in [**19**]. By inspection or by [**20**, 2.1], the inclusion then induces a full and faithful strong symmetric monoidal functor $\mathscr{J}_G^{\mathscr{V}} \longrightarrow \mathscr{J}_G^{\mathscr{V}'}$, and it is in this case that the theory of prolongation functors of I.2.10 or [**20**, §3] applies. Here we have a more natural looking but equivalent form of the definition of $I_{\mathscr{V}'}^{\mathscr{V}}$, namely

$$(I_{\mathscr{V}'}^{\mathscr{V}}X)(V) = X(V) \tag{1.3}$$

for $X \in \mathscr{I}_G^{\mathscr{V}'}\mathscr{S}$ and $V \in \mathscr{V}$. Evaluation G-homeomorphisms

$$\mathscr{J}_G^{\mathscr{V}}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n) = \mathscr{J}_G^{\mathscr{V}'}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n) \longrightarrow X(V)$$

give a natural isomorphism comparing the two descriptions of $I_{\mathscr{V}'}^{\mathscr{V}}X$. Here Definition 1.2 also gives a functor $I_{\mathscr{V}}^{\mathscr{V}'}$. By the following theorem, it is an inverse isomorphism to $I_{\mathscr{V}'}^{\mathscr{V}}$, hence is left (and right) adjoint to $I_{\mathscr{V}}^{\mathscr{V}'}$ and therefore coincides with the prolongation functor given by I.2.10. Writing $F_V^{\mathscr{V}}A$ to indicate the universe of shift desuspension functors, it follows by inspection of right adjoints and use of the inverse isomorphism property that

$$I_{\mathscr{V}'}^{\mathscr{V}}F_V^{\mathscr{V}'}A \cong F_V^{\mathscr{V}}A \quad \text{and} \quad I_{\mathscr{V}}^{\mathscr{V}'}F_V^{\mathscr{V}}A \cong F_V^{\mathscr{V}'}A \tag{1.4}$$

for $V \in \mathscr{V}$ and any based G-space A.

Returning to general collections, write $\Sigma^{\mathscr{V}} : \mathscr{T}_G \longrightarrow \mathscr{I}_G^{\mathscr{V}}\mathscr{S}$ for the suspension G-spectrum functor. The following result is analogous to [**7**, 2.3, 2.4].

Theorem 1.5. *Consider collections $\mathscr{V}$, $\mathscr{V}'$ and $\mathscr{V}''$.*

(i) $I_{\mathscr{V}'}^{\mathscr{V}} \circ \Sigma^{\mathscr{V}'}$ *is naturally isomorphic to* $\Sigma^{\mathscr{V}}$.
(ii) $I_{\mathscr{V}'}^{\mathscr{V}} \circ I_{\mathscr{V}''}^{\mathscr{V}'}$ *is naturally isomorphic to* $I_{\mathscr{V}''}^{\mathscr{V}}$.
(iii) $I_{\mathscr{V}}^{\mathscr{V}}$ *is naturally isomorphic to the identity functor.*
(iv) *The functor* $I_{\mathscr{V}'}^{\mathscr{V}}$ *commutes with smash products with based* G*-spaces.*
(v) *The functor* $I_{\mathscr{V}'}^{\mathscr{V}}$ *is strong symmetric monoidal.*

Therefore $I^{\mathscr{V}}_{\mathscr{V}'}$ is an equivalence of categories with inverse $I^{\mathscr{V}'}_{\mathscr{V}}$. Moreover, $I^{\mathscr{V}}_{\mathscr{V}'}$ is homotopy preserving, hence $I^{\mathscr{V}}_{\mathscr{V}'}$ and $I^{\mathscr{V}'}_{\mathscr{V}}$ induce inverse equivalences of the homotopy categories obtained by passing to homotopy classes of maps.

Proof. Evaluation G-homeomorphisms $\mathscr{J}^{\mathscr{V}}_G(\mathbb{R}^n, V) \wedge S^n \longrightarrow S^V$ give (i). For (ii), if $\dim V = n$, we have G-homeomorphisms

$$\begin{aligned}(I^{\mathscr{V}}_{\mathscr{V}'} \circ I^{\mathscr{V}'}_{\mathscr{V}''})(X)(V) &= \mathscr{J}^{\mathscr{V}}_G(\mathbb{R}^n, V) \wedge_{O(n)} \mathscr{J}^{\mathscr{V}'}_G(\mathbb{R}^n, \mathbb{R}^n) \wedge_{O(n)} X(\mathbb{R}^n)\\ &\cong \mathscr{J}^{\mathscr{V}}_G(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)\end{aligned}$$

since $\mathscr{J}^{\mathscr{V}'}_G(\mathbb{R}^n, \mathbb{R}^n) = O(n)_+$. Part (iii) is clear from (1.3). Part (iv) is obvious and implies the last statement. For (v), consideration of $\mathscr{V} \cup \mathscr{V}'$ shows that we may assume without loss of generality that $\mathscr{V}' \subset \mathscr{V}$. Then, as the inverse of $\mathscr{I}^{\mathscr{V}'}_{\mathscr{V}}$, $\mathscr{I}^{\mathscr{V}}_{\mathscr{V}'}$ is a prolongation functor, and (v) holds by I.2.14 or [**20**, 3.3]. □

We turn to the relationship with model structures. It is important to realize what Lemma 1.1 does not imply: a map $f : X \longrightarrow Y$ can be a weak equivalence at level $\mathbb{R}^n$ for all n but still not be a level equivalence. The point is that the H-fixed point functors do not commute with passage to orbits over $O(n)$.

Similarly, it is important to realize what the last statement of Theorem 1.5 does not imply: the functors $I^{\mathscr{V}}_{\mathscr{V}'}$ do not preserve either level equivalences or π_*-isomorphisms in general. Therefore, there is no reason to expect the homotopy categories associated to the model structures to be equivalent. However, (1.3) and the characterization of q-fibrations and acyclic q-fibrations given in III.4.12 imply the following result.

Theorem 1.6. *If $\mathscr{V} \subset \mathscr{V}'$, then the functor $I^{\mathscr{V}}_{\mathscr{V}'} : G\mathscr{I}^{\mathscr{V}'}_G \longrightarrow G\mathscr{I}^{\mathscr{V}}_G$ preserves level equivalences, level fibrations, q-fibrations, and acyclic q-fibrations, and similarly for the positive analogues of these classes of maps. Therefore $(I^{\mathscr{V}'}_{\mathscr{V}}, I^{\mathscr{V}}_{\mathscr{V}'})$ is a Quillen adjoint pair of functors relating the respective level, positive level, stable, and positive stable model structures.*

There is another way to think about change of universe. For $\mathscr{V} \subset \mathscr{V}'$, we can define new $\mathscr{V}$-model structures on the category of $\mathscr{I}^{\mathscr{V}'}_G$-spectra. For the $\mathscr{V}$-level model structure (or positive $\mathscr{V}$-level model structure), we define weak equivalences and fibrations by restricting attention to levels in $\mathscr{V}$; equivalently, the $\mathscr{V}$-level equivalences and fibrations are created by the forgetful functor $I^{\mathscr{V}}_{\mathscr{V}'}$. We define the $\mathscr{V}$-cofibrations of $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ to be the G-maps that satisfy the LLP with respect to the $\mathscr{V}$-level acyclic fibrations. Compare [**20**, 6.10]. We then let the $\mathscr{V}$-stable equivalences and the $\mathscr{V}$-fibrations be created by $I^{\mathscr{V}}_{\mathscr{V}'}$. Thus the $\mathscr{V}$-stable equivalences are the $\mathscr{V}$ π_*-isomorphisms, that is, the maps that induce isomorphisms of the homotopy groups defined using only those $V \in \mathscr{V}$ in the relevant colimits. Arguing as for the stable model structure, we obtain the following result.

Theorem 1.7. *For $\mathscr{V} \subset \mathscr{V}'$, the category $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ of $\mathscr{I}^{\mathscr{V}'}_G$-spectra and natural G-maps has a $\mathscr{V}$-stable model structure in which the functor $I^{\mathscr{V}}_{\mathscr{V}'}$ creates the $\mathscr{V}$-stable equivalences and the $\mathscr{V}$-fibrations. The acyclic $\mathscr{V}$-fibrations coincide with the $\mathscr{V}$-level acyclic fibrations, and the $\mathscr{V}$-cofibrations are the maps that satisfy the LLP with respect to the acyclic $\mathscr{V}$-fibrations. The pair $(I^{\mathscr{V}'}_{\mathscr{V}}, I^{\mathscr{V}}_{\mathscr{V}'})$ is a Quillen equivalence between $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$ with its stable model structure and $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ with its $\mathscr{V}$-stable model structure. The analogous statements for positive $\mathscr{V}$-stable model structures hold.*

Here the Quillen equivalence is easily proven using the usual characterization [**20**, A.2]. It is a rare example of an interesting "Quillen equivalence" of model categories that is an actual equivalence of underlying categories. There is another observation to make along the same lines.

COROLLARY 1.8. *For $\mathscr{V} \subset \mathscr{V}'$, the identity functor* $\mathrm{Id} : G\mathscr{I}^{\mathscr{V}'}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ *is the right adjoint of a Quillen adjoint pair relating the (positive) stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ to the (positive) $\mathscr{V}$-stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$.*

Thus the forgetful functor $I^{\mathscr{V}}_{\mathscr{V}'} : G\mathscr{I}^{\mathscr{V}'}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{V}}\mathscr{S}$ relating the (original) stable model structures factors through the $\mathscr{V}$-stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$. That is, the Quillen adjoint pair of Theorem 1.6 is the composite of the Quillen adjoint pair of Corollary 1.8 and the Quillen adjoint equivalence of Theorem 1.7.

REMARK 1.9. There is yet another way to think about change of universe. Fix $\mathscr{I}_G = \mathscr{I}_G^{\mathscr{A}\ell\ell}$. Then, for any $\mathscr{V}$, the (positive) $\mathscr{V}$-stable model structure on the category $G\mathscr{I}\mathscr{S}$ is Quillen equivalent to the (positive) stable model structure on $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$, and similarly for the various categories of rings and modules. However, to make sense of some of the constructions in the following sections, we must work with the more general categories of $\mathscr{I}_G^{\mathscr{V}}$-spectra, with their intrinsic model structures.

REMARK 1.10. In addition to changes of $\mathscr{V}$, we must deal with changes of the choice of "indexing G-spaces" within a given $\mathscr{V}$, as in II.2.2. Thus let $\mathscr{W} \subset \mathscr{V}$ be a cofinal set of G-inner product spaces that is closed under finite direct sums and contains the $\mathbb{R}^n$. We have a forgetful functor $I^{\mathscr{W}}_{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}}\mathscr{S} \longrightarrow \mathscr{I}_G^{\mathscr{W}}\mathscr{S}$ specified as in (1.3). It can also be specified as in Definition 1.2 and, arguing as in that definition and Theorem 1.5, $I^{\mathscr{W}}_{\mathscr{V}}$ is an equivalence of categories with inverse equivalence $I^{\mathscr{V}}_{\mathscr{W}}$. We can carry out all of our model category theory in the more general context. The functor $I^{\mathscr{W}}_{\mathscr{V}} : G\mathscr{I}^{\mathscr{V}}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{W}}\mathscr{S}$ preserves q-fibrations, and cofinality ensures that $I^{\mathscr{W}}_{\mathscr{V}}$ creates the stable equivalences in $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$. We conclude that $(I^{\mathscr{V}}_{\mathscr{W}}, I^{\mathscr{W}}_{\mathscr{V}})$ is a Quillen equivalence.

2. Change of groups

Let H be a subgroup of G and write $\iota : H \longrightarrow G$ for the inclusion. For a G-space A, let ι^*A denote A regarded as an H-space via ι. We want analogues for orthogonal G-spectra of such space level observations as (III.1.2) – (III.1.5).

This involves change of universe as well as change of groups. If $\mathscr{V} = \{V\}$ is a collection of representations of G, then $\iota^*\mathscr{V} = \{\iota^*V\}$ is a collection of representations of H. According to our conventions in II.2.1, G-summands of representations in $\mathscr{V}$ are in $\mathscr{V}$, but this need not be true of H-summands of representations in $\iota^*\mathscr{V}$. For example, not all H-representations are in $\iota^*\mathscr{A}\ell\ell(G)$. However, we can let $\mathscr{W}$ be the collection of H-representations that are isomorphic to summands of representations in $\iota^*\mathscr{V}$. Since $\iota^*\mathscr{V}$ is cofinal in $\mathscr{W}$ and closed under finite direct sums, Remark 1.10 applies. For example, if $\mathscr{V} = \mathscr{A}\ell\ell(G)$, then $\mathscr{W} = \mathscr{A}\ell\ell(H)$ since any H-representation is a summand of a G-representation.

To fix ideas and simplify notation, we work with $\mathscr{A}\ell\ell(H)$ when defining orthogonal H-spectra, and we do not introduce notation for the change of universe functor that passes from orthogonal H-spectra indexed on $\iota^*\mathscr{A}\ell\ell(G)$ to orthogonal H-spectra indexed on $\mathscr{A}\ell\ell(H)$.

Definition 2.1. For an orthogonal G-spectrum X, let $\iota^* X$ be the orthogonal H-spectrum that is specified by $(\iota^* X)(\iota^* V) = \iota^* X(V)$ for representations V of G and is then extended to all representations of H by Remark 1.10.

Lemma 2.2. *The functor ι^* preserves level fibrations, level equivalences, q-cofibrations, π_*-isomorphisms, and q-fibrations.*

Proof. Most of this is clear from the characterizations of the various classes of maps. An exception is the assertion that ι^* preserves q-cofibrations, which is less obvious. Clearly ι^* preserves colimits and satisfies $\iota^*(F_V(A)) = F_{\iota^* V}(\iota^* A)$. When $A = G/K_+ \wedge S^n$, $\iota^* A = \iota^*(G/K)_+ \wedge S^n$. If G is finite, then $\iota^*(G/K)$ is isomorphic to a disjoint union of H-orbits, the choice of isomorphism depending on a double coset decomposition of $H\backslash G/K$. Fixing such choices gives a decomposition of the underlying H-space of a G-cell complex as an H-cell complex. For a general compact Lie group G, $\iota^*(G/K)$ can be decomposed, non-canonically, as a finite H-CW complex [**17**]. Again, fixing choices of such decompositions allows us to decompose the underlying H-space of a G-cell complex as an H-cell complex. □

We claim that the functor ι^* has both a left and a right adjoint. On the space level, for H-spaces B, the left adjoint of ι^* is given by $G_+ \wedge_H B$ and the right adjoint is given by the G-space of H-maps $F_H(G_+, B)$. For G-spaces A and A', we have obvious identifications of H-spaces

$$\iota^* F(A, A') = F(\iota^* A, \iota^* A') \quad \text{and} \quad \iota^*(A \wedge A') = \iota^* A \wedge \iota^* A'.$$

On passage to left and right adjoints, respectively, these formally imply natural isomorphisms of G-spaces

$$(G_+ \wedge_H B) \wedge A \cong G_+ \wedge_H (B \wedge \iota^* A)$$

and

$$F(A, F_H(G_+, B)) \cong F_H(G_+, F(\iota^* A, B)),$$

and it is easy to write down explicit isomorphisms.

Proposition 2.3. *Let X be an orthogonal G-spectrum and Y be an orthogonal H-spectrum. Let $G_+ \wedge_H Y$ be the orthogonal G-spectrum specified by*

$$(G_+ \wedge_H Y)(V) = G_+ \wedge_H Y(\iota^* V)$$

for representations V of G. Then there is an adjunction

$$G\mathscr{I}\mathscr{S}(G_+ \wedge_H Y, X) \cong H\mathscr{I}\mathscr{S}(Y, \iota^* X),$$

which is a Quillen adjoint pair relating the respective (positive) level and stable model structures. Moreover, there is a natural isomorphism

$$(G_+ \wedge_H Y) \wedge X \cong G_+ \wedge_H (Y \wedge \iota^* X).$$

In particular,

$$G/H_+ \wedge X \cong G_+ \wedge_H \iota^* X.$$

Proof. The evaluation G-maps of the G-functor $G_+ \wedge_H Y : \mathscr{J}_G \longrightarrow \mathscr{T}_G$ are induced from the evaluation H-maps of the H-functor Y via

$$\begin{aligned} \mathscr{J}_G(V, W) \wedge (G_+ \wedge_H Y(\iota^* V)) &\cong G_+ \wedge_H (\iota^* \mathscr{J}_G(V, W) \wedge Y(\iota^* V)) \\ &\cong G_+ \wedge_H (\mathscr{J}_H(\iota^* V, \iota^* W) \wedge Y(\iota^* V)) \\ &\longrightarrow G_+ \wedge_H Y(\iota^* W). \end{aligned}$$

Lemma 2.2 implies the statement about model structures, and the rest is clear. □

PROPOSITION 2.4. *Let X be an orthogonal G-spectrum and Y be an orthogonal H-spectrum. Let $F_H(G_+, Y)$ be the orthogonal G-spectrum specified by*

$$F_H(G_+, Y)(V) = F_H(G_+, Y(\iota^*V))$$

for representations V of G. Then there is an adjunction

$$G\mathscr{I}\mathscr{S}(X, F_H(G_+, Y)) \cong H\mathscr{I}\mathscr{S}(\iota^*X, Y),$$

which is a Quillen adjoint pair relating the respective (positive) level and stable model structures. Moreover, there is a natural isomorphism

$$F(X, F_H(G_+, Y)) \cong F_H(G_+, F(\iota^*X, Y)).$$

In particular,

$$F(G/H_+, X) \cong F_H(G_+, \iota^*X).$$

PROOF. The adjoints of the evaluation G-maps of the G-functor $F_H(G_+, Y)$ are induced from the adjoints of the evaluation H-maps of Y via

$$\begin{array}{rcl} F_H(G_+, Y(\iota^*V)) & \longrightarrow & F_H(G_+, F(\mathscr{J}_H(\iota^*V, \iota^*W), Y(\iota^*W))) \\ & \cong & F_H(G_+, F(\iota^*\mathscr{J}_G(V, W), Y(\iota^*W))) \\ & \cong & F(\mathscr{J}_G(V, W), F_H(G_+, Y(\iota^*W))). \quad \square \end{array}$$

3. Fixed point and orbit spectra

We relate orthogonal G-spectra to orthogonal spectra via fixed point and orbit functors, just as for G-spaces.

DEFINITION 3.1. For an orthogonal G-spectrum X, define $X^G(V) = X(V)^G$ for an inner product space V regarded as a trivial representation of G. Regarding a linear isometry $f : V \longrightarrow W$ as a G-linear isometry between trivial representations, we see that $X(f)$ is a G-map since X is a G-functor. Therefore X^G defines a functor $\mathscr{J} \longrightarrow \mathscr{T}$. More formally, let $G\mathscr{I}^{\text{triv}}\mathscr{S}$ denote the category of orthogonal G-spectra indexed only on trivial G-representations. We call the objects of $G\mathscr{I}^{\text{triv}}\mathscr{S}$ "naive" orthogonal G-spectra, in contrast to the genuine orthogonal G-spectra of $G\mathscr{I}\mathscr{S}$. The G-fixed point functor is the composite of the change of universe functor

$$G\mathscr{I}\mathscr{S} = G\mathscr{I}^{\mathscr{A}\ell\ell}\mathscr{S} \longrightarrow G\mathscr{I}^{\text{triv}}\mathscr{S}$$

and the G-fixed point functor

$$G\mathscr{I}^{\text{triv}}\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}.$$

For $H \subset G$, define $X^H = (\iota^*X)^H$.

The following fundamental result relating equivariant and nonequivariant homotopy groups is immediate from the definitions.

PROPOSITION 3.2. *Let E be an orthogonal Ω-G-spectrum. Then*

$$\pi_*^H(E) \cong \pi_*(E^H).$$

For any orthogonal G-spectrum X, $\pi_*^H(X) \cong \pi_*^H(RX)$, where RX is a fibrant approximation of X in the stable or positive stable model structure.

Giving spaces trivial G-action, we obtain a functor

$$\varepsilon^* : \mathscr{I}\mathscr{S} \longrightarrow G\mathscr{I}^{triv}\mathscr{S}. \tag{3.3}$$

We then have the following fixed-point adjunction and its composite with the evident change of universe adjunction.

PROPOSITION 3.4. *Let X be a naive orthogonal G-spectrum and Y be a nonequivariant orthogonal spectrum. There is a natural isomorphism*

$$G\mathscr{I}^{triv}\mathscr{S}(\varepsilon^*Y, X) \cong \mathscr{I}\mathscr{S}(Y, X^G).$$

For (genuine) orthogonal G-spectra X, there is a natural isomorphism

$$G\mathscr{I}\mathscr{S}(i_*\varepsilon^*Y, X) \cong \mathscr{I}\mathscr{S}(Y, (i^*X)^G),$$

where $i_* = I_{triv}^{\mathscr{A}\ell\ell}$ and $i^* = I_{\mathscr{A}\ell\ell}^{triv}$. Both of these adjunctions are Quillen adjoint pairs relating the respective (positive) level and stable model structures.

The last statement means that passage to fixed points preserves q-fibrations and acyclic q-fibrations. We have the following observation about q-cofibrations. In the following two results, we agree to be less pedantic and to write $(-)^G$ for the composite of i^* and passage to G-fixed points. With this notation, the counit of the second adjunction is a natural G-map $i_*\varepsilon^*X^G \longrightarrow X$.

PROPOSITION 3.5. *For a representation V and a G-space A, $(F_VA)^G = *$ unless G acts trivially on V, when $(F_VA)^G \cong F_V(A^G)$ as a nonequivariant orthogonal spectrum. The functor $(-)^G$ preserves q-cofibrations, but not acyclic q-cofibrations.*

PROOF. For a trivial representation W, $(F_VA)^G(W) = G\mathscr{J}(V,W) \wedge A^G$. If V is non-trivial, there are no non-trivial G-linear isometries $V \longrightarrow W$ and $G\mathscr{J}(V,W) = *$, whereas $G\mathscr{J}(V,W) = \mathscr{J}(V,W)$ if V is trivial. This gives the first statement. Since the functor $(-)^G$ preserves the colimits used to construct relative cell orthogonal G-spectra, by III.1.6, it follows that it preserves preserves q-cofibrations. For non-trivial representations V of G, the maps $k_{0,V}$ of III.4.6 are acyclic q-cofibrations, whereas $k_{0,V}^G$ is equivalent to $* \longrightarrow S$. □

WARNING 3.6. The last statement of Proposition 3.5 implies that the functor $(-)^G$ is not a Quillen left adjoint. This functor does not behave homotopically as one might expect from the results of [**19**]. The reason is that it does not commute with fibrant replacement (whereas all objects are fibrant in the context of [**19**]), and we must replace orthogonal G-spectra by weakly equivalent orthogonal Ω-G-spectra before passing to fixed points in order to obtain the correct homotopy groups.

The following two results are in marked contrast to the situation in [**19, 27**], where the (categorical) fixed point functor does not satisfy analogous commutation relations. The point is that these results do not imply corresponding commutation results on passage to homotopy categories, in view of Warning 3.6.

Taking $V = 0$, Proposition 3.5 has the following implication.

COROLLARY 3.7. *For based G-spaces A,*

$$(\Sigma^\infty A)^G \cong \Sigma^\infty(A^G).$$

(This isomorphism of orthogonal spectra does not imply an isomorphism in $HoG\mathscr{I}\mathscr{S}$).

Note that the functors i_* and ε^* are strong symmetric monoidal.

PROPOSITION 3.8. *For orthogonal G-spectra X and Y, there is a natural map of (nonequivariant) orthogonal spectra*

$$\alpha : X^G \wedge Y^G \longrightarrow (X \wedge Y)^G,$$

and α is an isomorphism if X and Y are cofibrant. (This isomorphism of orthogonal spectra does not imply an isomorphism in $HoG\mathscr{I}\mathscr{S}$).

PROOF. The map α is adjoint to the evident natural G-map

$$i_*\varepsilon^*(X^G \wedge Y^G) \cong (i_*\varepsilon^* X^G) \wedge (i_*\varepsilon^* Y^G) \longrightarrow X \wedge Y.$$

Using the properties of $(-)^G$ given in III.1.6, the second statement follows from Proposition 3.5 and the natural isomorphism

$$F_V A \wedge F_W B \cong F_{V\oplus W}(A \wedge B)$$

of Lemma 4.8. □

We can obtain a sharper version of Proposition 3.4. Let NH denote the normalizer of H in G and let $WH = NH/H$. We can obtain an H-fixed point functor from orthogonal G-spectra to WH-spectra. It factors as a composite

$$G\mathscr{I}\mathscr{S} \longrightarrow NH\mathscr{I}\mathscr{S} \longrightarrow NH\mathscr{I}^{H\text{-triv}}\mathscr{S} \longrightarrow WH\mathscr{I}\mathscr{S}$$

of a change of group functor as in Definition 2.1, a change of universe functor, and a fixed point functor, all three of which are right adjoints.

It is useful to be more general about the last two functors. Thus let N be any normal subgroup of G, let $J = G/N$, and let $\varepsilon : G \longrightarrow J$ be the quotient homomorphism. In the situation above, we are thinking of the normal subgroup H of NH with quotient group WH.

DEFINITION 3.9. Let $G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ be the category of orthogonal G-spectra indexed on N-trivial representations of G. Define $\varepsilon^* : J\mathscr{I}\mathscr{S} \longrightarrow G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ by regarding J-spaces as N-trivial G-spaces. Define $(-)^N : G\mathscr{I}^{N\text{-triv}}\mathscr{S} \longrightarrow J\mathscr{I}\mathscr{S}$ by passage to N-fixed points spacewise, $(X^N)(V) = X(V)^N$ for a J-representation V regarded as an N-trivial G-representation.

PROPOSITION 3.10. *Let $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ and $Y \in J\mathscr{I}\mathscr{S}$. There is a natural isomorphism*

$$G\mathscr{I}^{N\text{-triv}}\mathscr{S}(\varepsilon^*Y, X) \cong J\mathscr{I}\mathscr{S}(Y, X^N).$$

For (genuine) orthogonal G-spectra X, there is a natural isomorphism

$$G\mathscr{I}\mathscr{S}(i_*\varepsilon^*Y, X) \cong J\mathscr{I}\mathscr{S}(Y, (i^*X)^N),$$

where $i_* = I^{\mathscr{A}\ell\ell}_{N\text{-triv}}$ and $i^* = I^{N\text{-triv}}_{\mathscr{A}\ell\ell}$. Both of these adjunctions are Quillen adjoint pairs relating the respective (positive) level and stable model structures.

Similarly, we can define orbit spectra. Here again, we must first restrict to trivial representations. However, since this change of universe functor is a right adjoint and passage to orbits is a left adjoint, the composite functor appears to be of no practical value (just as in [**19**]).

DEFINITION 3.11. For $X \in G\mathscr{I}^{\text{triv}}\mathscr{S}$, define X/G by $(X/G)(V) = X(V)/G$ for an inner product space V. More generally, for $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$, define $X/N \in J\mathscr{I}\mathscr{S}$ by $(X/N)(V) = X(V)/N$ for a J-representation V regarded as an N-trivial G-representation.

PROPOSITION 3.12. *Let $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ and $Y \in J\mathscr{I}\mathscr{S}$. There is a natural isomorphism*

$$G\mathscr{I}^{N\text{-triv}}\mathscr{S}(X, \varepsilon^*Y) \cong J\mathscr{I}\mathscr{S}(X/N, Y).$$

This adjunction is a Quillen adjoint pair relating the respective (positive) level and stable model structures.

REMARK 3.13. The left and right adjoints of ε^* in this section and of ι^* in the previous section can be regarded as special cases of a composite construction that applies to an arbitrary homomorphism $\alpha : H \longrightarrow G$ of compact Lie groups. Let $N = \mathrm{Ker}(\alpha)$ and $K = H/N$. We have a quotient homomorphism $\varepsilon : H \longrightarrow K$ and an inclusion $\iota : K \longrightarrow G$ induced by α. Since $\alpha = \iota \circ \varepsilon$, $\alpha^* = \varepsilon^* \circ \iota^*$. Therefore, if $X \in G\mathscr{I}\mathscr{S}$ and $Y \in H\mathscr{I}^{N\text{-triv}}\mathscr{S}$, we have the composite adjunctions

$$G\mathscr{I}\mathscr{S}(G_+ \wedge_K Y/N, X) \cong H\mathscr{I}^{N\text{-triv}}\mathscr{S}(Y, \alpha^* X)$$

and

$$G\mathscr{I}\mathscr{S}(X, F_K(G_+, Y^N)) \cong H\mathscr{I}^{N\text{-triv}}\mathscr{S}(\alpha^* X, Y).$$

4. Geometric fixed point spectra

There are actually two G-fixed point functors on orthogonal G-spectra, just as there are on G-spectra [**19**, II§9] and [**27**, XVI§3], namely the "categorical" one already defined and another "geometric" one. Because the categorical fixed point functor here seems to enjoy some of the basic properties that motivated the introduction of the geometric fixed point functor in the classical setting, the discussion requires some care. We want a version of the G-fixed point functor for which the commutation relations of Corollary 3.7 and Proposition 3.8 are true, but which also preserves acyclic q-cofibrations, so that these properties remain true after passage to fibrant-cofibrant approximation of cofibrant objects.

In this section, we work from the beginning in the general context of a normal subgroup N of G with quotient group J. The reader may wish to focus on the special case $N = G$, in which case J is the trivial group. However, G plays two quite different roles in that case, and the general case clarifies issues of equivariance. We need some categorical preliminaries.

DEFINITION 4.1. Let E denote the extension

$$e \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \varepsilon\ } J \longrightarrow e.$$

We define a category $\mathscr{J}_E$ enriched over the category $J\mathscr{T}$ of based J-spaces. The objects of $\mathscr{J}_E$ are the G-representations V. The J-space $\mathscr{J}_E(V,W)$ of arrows $V \longrightarrow W$ is the N-fixed point space $\mathscr{J}_G(V,W)^N$. Thus, if we ignore the J-action, then

$$\mathscr{J}_E(V,W) = N\mathscr{J}(\iota^* V, \iota^* W).$$

A non-basepoint arrow $(f,x) : V \longrightarrow W$ is an N-linear isometry $f : V \longrightarrow W$ together with a point $x \in W^N - f(V^N)$. Observe that $\mathscr{J}_E = G\mathscr{J}$ when $N = G$ and $\mathscr{J}_E = \mathscr{J}_G$ when $N = e$. Let

$$\phi : \mathscr{J}_E \longrightarrow \mathscr{J}_J$$

be the N-fixed point J-functor. It sends the G-representation V to the J-representation V^N and sends an arrow $(f,x) : V \longrightarrow W$ to the N-fixed point arrow $(f^N, x) \in \mathscr{J}_J(V^N, W^N)$. Let

$$\nu : \mathscr{J}_J \longrightarrow \mathscr{J}_E$$

be the J-functor that sends a J-representation V to V regarded as a G-representation by pullback along ε and is given on morphism spaces $\mathscr{J}_J(V,W)$ by identity maps; this makes sense since every linear isometry $V \longrightarrow W$ is an N-map. Observe that

$$\phi \circ \nu = \mathrm{Id} : \mathscr{J}_J \longrightarrow \mathscr{J}_J.$$

DEFINITION 4.2. Let $\mathscr{J}_E\mathscr{T}$ denote the category of $\mathscr{J}_E$-spaces, namely (continuous) J-functors $\mathscr{J}_E \longrightarrow \mathscr{T}_J$. Note that a $\mathscr{J}_E$-space Y has structural J-maps

$$Y(V) \wedge S^{W^N - V^N} \longrightarrow Y(W)$$

for $V \subset W$. Let

$$\mathbb{U}_\phi : \mathscr{J}_J\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T} \quad \text{and} \quad \mathbb{U}_\nu : \mathscr{J}_E\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T}$$

be the forgetful functors induced by ϕ and ν. By I.2.10 or [**20**, 3.2], left Kan extension along ϕ and ν gives prolongation functors

$$\mathbb{P}_\phi : \mathscr{J}_E\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T} \quad \text{and} \quad \mathbb{P}_\nu : \mathscr{J}_J\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T}$$

left adjoint to $\mathbb{U}_\phi$ and $\mathbb{U}_\nu$. Since $\phi \circ \nu = \mathrm{Id}$, $\mathbb{U}_\nu \circ \mathbb{U}_\phi = \mathrm{Id}$ and therefore $\mathbb{P}_\phi \circ \mathbb{P}_\nu \cong \mathrm{Id}$.

With these definitions in place, we can define the geometric fixed point functors.

DEFINITION 4.3. Define a fixed point functor $\mathrm{Fix}^N : \mathscr{J}_G\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T}$ by sending an orthogonal G-spectrum X to the $\mathscr{J}_E$-space $\mathrm{Fix}^N X$ with

$$(\mathrm{Fix}^N X)(V) = X(V)^N$$

and with evaluation J-maps

$$X(V)^N \wedge \mathscr{J}_G(V, W)^N \longrightarrow X(W)^N$$

obtained by passage to N-fixed points from the evaluation G-maps of X. Define the geometric fixed point functor

$$\Phi^N : \mathscr{J}_G\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T}$$

to be the composite $\mathbb{P}_\phi \circ \mathrm{Fix}^N$. Define a natural J-map $\gamma : X^N \longrightarrow \Phi^N X$ of orthogonal J-spectra by observing that the categorical fixed point functor can be reinterpreted as $X^N = \mathbb{U}_\nu \mathrm{Fix}^N X$ and letting γ be the map

$$\mathbb{U}_\nu \eta : X^N = \mathbb{U}_\nu \mathrm{Fix}^N X \longrightarrow \mathbb{U}_\nu \mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X = \mathbb{P}_\phi \mathrm{Fix}^N X = \Phi^N X, \tag{4.4}$$

where $\eta : \mathrm{Id} \longrightarrow \mathbb{U}_\phi \mathbb{P}_\phi$ is the unit of the prolongation adjunction.

We have the following analogue of Proposition 3.5.

PROPOSITION 4.5. *For a representation V of G and a G-space A,*

$$\Phi^N(F_V A) \cong F_{V^N} A^N.$$

The functor Φ^N preserves q-cofibrations and acyclic q-cofibrations.

PROOF. By the definitions, we have

$$(\mathrm{Fix}^N F_V A)(W) = \mathscr{J}_G(V, W)^N \wedge A^N.$$

Thus $\mathrm{Fix}^N F_V A = F_V A^N$ where F_V on the right is left adjoint to the Vth J-space evaluation functor on the category $\mathscr{J}_E\mathscr{T}$. We have $\mathbb{P}_\phi \circ F_V = F_{V^N}$ by the equality of their right adjoints, and the first statement follows. By inspection from III.4.6, the functor Φ^N preserves generating q-cofibrations and generating acyclic q-cofibrations. It also preserves the colimits used to construct relative cell orthogonal G-spectra, by III.1.6, and it therefore preserves q-cofibrations and acyclic q-cofibrations. □

Analogues of Corollary 3.7 and Proposition 3.8 follow readily.

COROLLARY 4.6. *For based G-spaces A,*

$$\Phi^N \Sigma^\infty A \cong \Sigma^\infty(A^N),$$

where Σ^∞ on the left and right are the suspension spectrum functors from G-spaces to G-spectra and from J-spaces to J-spectra.

The functors $\mathbb{P}_\phi$ and $\mathbb{P}_\nu$ are strong symmetric monoidal, by I.2.14 or [**20**, 3.3].

PROPOSITION 4.7. *For orthogonal G-spectra X and Y, there is a natural J-map*

$$\alpha : \Phi^N X \wedge \Phi^N Y \longrightarrow \Phi^N(X \wedge Y)$$

of orthogonal J-spectra, and α is an isomorphism if X and Y are cofibrant.

PROOF. By the definition of internal smash products [**20**, 21.4], there are canonical maps of G-spaces

$$X(V) \wedge Y(W) \longrightarrow (X \wedge Y)(V \oplus W).$$

Passing to N-fixed point spaces, we obtain a natural J-map

$$\mathrm{Fix}^N X \,\bar{\wedge}\, \mathrm{Fix}^N Y \longrightarrow \mathrm{Fix}^N(X \wedge Y) \circ \oplus$$

of $(\mathscr{J}_E \times \mathscr{J}_E)$-spaces. We obtain α by applying $\mathbb{P}_\phi$ to the adjoint J-map

$$\mathrm{Fix}^N X \wedge \mathrm{Fix}^N Y \longrightarrow \mathrm{Fix}^N(X \wedge Y)$$

of $\mathscr{J}_E$-spaces. It follows easily from Proposition 4.5 that α is an isomorphism when $X = F_V A$ and $Y = F_W B$, and it follows inductively that α is an isomorphism when X and Y are cofibrant. □

In the previous section, we interpreted the homotopy groups of the categorical fixed points of a fibrant approximation of X as the homotopy groups of X. We now interpret the homotopy groups of the geometric fixed points of a cofibrant approximation of X as a different kind of homotopy groups of X. For this, we introduce homotopy groups of $\mathscr{J}_E$-spaces.

DEFINITION 4.8. Let Y be a $\mathscr{J}_E$-space and X be an orthogonal G-spectrum. Let $K \subset J$ and write $K = H/N$, where $N \subset H \subset G$.

(i) Define

$$\pi_q^K(Y) = \mathrm{colim}_V\, \pi_q^K \Omega^{V^N} Y(V) \ \text{ if } \ q \geq 0,$$

where V runs over the indexing G-spaces in the universe U, and

$$\pi_{-q}^K(Y) = \mathrm{colim}_{V \supset \mathbb{R}^q}\, \pi_0^K \Omega^{V^N - \mathbb{R}^q} Y(V) \ \text{ if } \ q > 0.$$

(ii) Define a natural homomorphism

$$\zeta : \pi_*^K(\mathbb{U}_\nu Y) \longrightarrow \pi_*^K(Y)$$

by restricting colimit systems to N-fixed indexing G-spaces.

(iii) Define

$$\rho_q^K(X) = \pi_q^K(\mathrm{Fix}^N X),$$

so that $\rho_q^K(X) = \mathrm{colim}_V\, \pi_q^K \Omega^{V^N} X(V)^N$ for $q \geq 0$, and similarly for $q < 0$.

(iv) Define a natural homomorphism

$$\psi : \pi_*^K(X^N) \longrightarrow \pi_*^H(X)$$

by restricting colimit systems to N-fixed indexing G-spaces W, using

$$(\Omega^W X(W)^N)^K \cong (\Omega^W X(W))^H.$$

(v) Define a natural homomorphism

$$\omega : \pi_*^H(X) \longrightarrow \rho_*^K(X)$$

by sending an element of $\pi_q^H(X)$, $q \geq 0$, that is represented by an H-map $f : S^q \wedge S^V \longrightarrow X(V)$ to the element of $\rho_q^K(X)$ that is represented by the K-map $f^N : S^q \wedge S^{V^N} \longrightarrow X(V)^N$, and similarly for $q < 0$.

Define π_*-isomorphisms of $\mathscr{J}_E$-spaces and ρ_*-isomorphisms of orthogonal G-spectra in the evident way.

If X is an orthogonal Ω-G-spectrum, then ψ is a natural isomorphism. In this case, we may identify ζ and ω in view of the following immediate observation.

Lemma 4.9. *The homomorphism*

$$\zeta : \pi_*^K(X^N) = \pi_*^K(\mathbb{U}_\nu \mathrm{Fix}^N X) \longrightarrow \pi_*^K(\mathrm{Fix}^N X) = \rho_*^K(X)$$

is the composite of $\psi : \pi_*^K(X^N) \longrightarrow \pi_*^H(X)$ *and* $\omega : \pi_*^H(X) \longrightarrow \rho_*^K(X)$.

We also have the following observation.

Lemma 4.10. *For orthogonal J-spectra Z, the homomorphism*

$$\zeta : \pi_*^K(Z) = \pi_*^K(\mathbb{U}_\nu \mathbb{U}_\phi Z) \longrightarrow \pi_*^K(\mathbb{U}_\phi Z)$$

is an isomorphism.

Proof. We may rewrite the colimits in Definition 4.8 as iterated colimits by first considering indexing J-spaces W in U^N and then considering indexing G-spaces V in U such that $V^N = W$. Thus, if $q \geq 0$, then

$$\pi_q^K(Y) = \operatorname*{colim}_{W \subset U^N} \ \operatorname*{colim}_{V \subset U,\, V^N = W} \pi_q^K \Omega^W Y(V)$$

for a $\mathscr{J}_E$-space Y. When $Y = \mathbb{U}_\phi Z$, $Y(V) = Z(V^N)$ and this colimit reduces to

$$\pi_q^K(Y) \cong \mathrm{colim}_{W \subset U^N} \pi_q^K \Omega^W Z(W) = \pi_q^K(Z).$$

The proof for $q < 0$ is similar. $\square$

Via the naturality of ζ, this leads to the following identification of γ_*, where $\gamma = \mathbb{U}_\nu \eta$ as in (4.4). Observe that the unit η of the prolongation adjunction for ϕ induces a natural map

$$\eta_* : \rho_*^K(X) = \pi_*^K(\mathrm{Fix}^N X) \longrightarrow \pi_*^K(\mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X) \xrightarrow{\zeta^{-1}} \pi_*^K(\mathbb{P}_\phi \mathrm{Fix}^N X) = \pi_*^K(\Phi^N X).$$

Lemma 4.11. *Let $K = H/N$, where $N \subset H$. For orthogonal Ω-G-spectra X, the map $\gamma_* : \pi_*^K(X^N) \longrightarrow \pi_*^K(\Phi^N X)$ is the composite*

$$\pi_*^K(X^N) \cong \pi_*^H(X) \xrightarrow{\omega} \rho_*^K(X) \xrightarrow{\eta_*} \pi_*^K(\Phi^N X).$$

We have the following basic identification of homotopy groups.

Proposition 4.12. *The map $\eta_* : \rho_*^K(X) \longrightarrow \pi_*^K(\Phi^N X)$ is an isomorphism for cofibrant orthogonal G-spectra X.*

Proof. The functor Φ^N preserves cofiber sequences, wedges, and colimits of sequences of h-cofibrations. Therefore both functors ρ_*^K and $\pi_*^K \circ \Phi^N$ convert cofiber sequences to long exact sequences, convert wedges to direct sums, and convert colimits of sequences of h-cofibrations to colimits of groups. Thus to show that η_* is an isomorphism on all cofibrant objects, it suffices to show that it is an isomorphism on objects $X = F_Z A$, where Z is a G-representation and A is a G-CW complex. We treat the case $q \geq 0$, the case $q < 0$ being similar. Here η_* is the map

$$\operatorname*{colim}_{W} \operatorname*{colim}_{V^N = W} \pi_q^K \Omega^W(\mathscr{J}_E(Z,V) \wedge A^N) \longrightarrow \operatorname{colim}_{W \subset U^N} \pi_q^K \Omega^W(\mathscr{J}_J(Z^N,W) \wedge A^N)$$

induced by the functor ϕ, where $W \subset U^N$ and $V \subset U$. It suffices to prove that, for fixed W, the map

$$\operatorname*{hocolim}_{V^N = W} \mathscr{J}_E(Z,V) \longrightarrow \mathscr{J}_J(Z^N,W)$$

is a J-homotopy equivalence. Via II.4.1, Definition 4.1 leads to explicit descriptions of the relevant J-spaces. Write $Z = Z^N \oplus Z'$ and $V = W \oplus V'$, where $V^N = W$. The space $\mathscr{J}_E(Z,V)$ is the N-fixed point space of the Thom complex of a certain G-bundle. It can be identified as the Thom complex of an N-fixed point J-bundle over the base J-space

$$\mathscr{I}(Z,V)^N \cong \mathscr{I}(Z^N,W) \times \mathscr{I}(Z',V')^N.$$

This bundle is just the product of the J-space $\mathscr{I}(Z',V')^N$ with the J-bundle over $\mathscr{I}(Z^N,W)$ whose Thom complex is the J-space $\mathscr{J}_J(Z^N,W)$. Using this, we see that the map

$$\mathscr{J}_E(Z,V) \longrightarrow \mathscr{J}_J(Z^N,W)$$

can be identified with the projection

$$\mathscr{J}_J(Z^N,W) \wedge \mathscr{I}(Z',V')^N_+ \longrightarrow \mathscr{J}_J(Z^N,W).$$

Thus it suffices to prove that the space $\operatorname{hocolim} \mathscr{I}(Z',V')^N$ is J-contractible. This is standard. The maps of the colimit system are h-cofibrations of J-spaces, and

$$\operatorname{colim} \mathscr{I}(Z',V')^N \cong \mathscr{I}(Z', \operatorname{colim} V')^N$$

is the J-space of N-linear isometries $Z' \longrightarrow \operatorname{colim} V'$. It is J-contractible by the proof of [**19**, II.1.5]. □

Corollary 4.13. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of orthogonal G-spectra, then f is a ρ_*-isomorphism.*

Proof. If X and Y are q-cofibrant, this is immediate from Propositions 4.5 and 4.12. Since a level weak equivalence is a ρ_*-isomorphism, the general case follows by use of cofibrant approximation in the level model structure. □

To see that the geometric fixed point functor bears the same homotopical relationship to the categorical fixed point functor as in the classical case [**19**, II§3], we need the following notations and lemmas.

Notations 4.14. Let $\mathscr{F} = \mathscr{F}[N]$ be the family of subgroups of G that do not contain N; when $N = G$, this is the family of proper subgroups of G. let $E\mathscr{F}$ be the universal $\mathscr{F}$-space, and let $\tilde{E}\mathscr{F}$ be the cofiber of the quotient map $E\mathscr{F}_+ \longrightarrow S^0$ that collapses $E\mathscr{F}$ to the non-basepoint. Then $(\tilde{E}\mathscr{F})^H = S^0$ if $H \supset N$ and $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$. The map $S^0 \longrightarrow \tilde{E}\mathscr{F}$ induces a natural map $\lambda : X \longrightarrow X \wedge \tilde{E}\mathscr{F}$ of orthogonal G-spectra.

Although trivial to prove, the following lemma is surprisingly precise.

Lemma 4.15. *For orthogonal G-spectra X, the map*

$$\Phi^N \lambda : \Phi^N X \longrightarrow \Phi^N(X \wedge \tilde{E}\mathscr{F})$$

is a natural isomorphism of orthogonal J-spectra.

Proof. For G-spaces A, $\mathrm{Fix}^N(X \wedge A) \cong (\mathrm{Fix}^N X) \wedge A^N$. Since $(\tilde{E}\mathscr{F})^N = S^0$, the conclusion follows. □

Lemma 4.16. *Let $K = H/N$, where $N \subset H$. For cofibrant orthogonal G-spectra X, the map $\omega : \pi_*^H(X \wedge \tilde{E}\mathscr{F}) \longrightarrow \rho_*^K(X \wedge \tilde{E}\mathscr{F})$ is an isomorphism.*

Proof. For based G-CW complexes A and B, the inclusion $A^N \longrightarrow A$ and the map $\lambda : B \longrightarrow B \wedge \tilde{E}\mathscr{F}$ induce bijections

$$[A, B \wedge \tilde{E}\mathscr{F}]_G \longrightarrow [A^N, B \wedge \tilde{E}\mathscr{F}]_G \longleftarrow [A^N, B]_G$$

(e.g. [**19**, II.9.3]). Here $[A^N, B]_G \cong [A^N, B^N]_J$ and $B^N \cong (B \wedge \tilde{E}\mathscr{F})^N$. The composite isomorphism

$$[A, B \wedge \tilde{E}\mathscr{F}]_G \longrightarrow [A^N, B^N]_J$$

sends a G-map f to the J-map f^N. Using that $G/H \cong J/K$ is N-fixed, this specializes to show that, for $q \geq 0$,

$$\omega : \mathrm{colim}_V\, \pi_q^H \Omega^V(X(V) \wedge \tilde{E}\mathscr{F}) \longrightarrow \mathrm{colim}_V\, \pi_q^K \Omega^{V^N}(X(V) \wedge \tilde{E}\mathscr{F})^N$$

is a colimit of isomorphisms. The argument for $q < 0$ is similar. □

The following analogue of [**19**, II.9.8] gives an isomorphism in the homotopy category $\mathrm{Ho}J\mathscr{S}$ between the geometric N-fixed point functor and the composite of the categorical N-fixed point functor with the smash product with $\tilde{E}\mathscr{F}$. Let $\xi : X \longrightarrow RX$ be a fibrant replacement functor on orthogonal G-spectra, so that ξ is an acyclic cofibration and RX is an orthogonal Ω-G-spectrum.

Proposition 4.17. *For cofibrant orthogonal G-spectra X, the diagram*

$$R(X \wedge \tilde{E}\mathscr{F})^N \xrightarrow{\gamma} \Phi^N R(X \wedge \tilde{E}\mathscr{F}) \xleftarrow{\Phi^N(\xi\lambda)} \Phi^N(X)$$

displays a pair of natural π_*-isomorphisms of orthogonal J-spectra.

Proof. Since $\Phi^N\lambda$ is an isomorphism by Lemma 4.15 and $\Phi^N\xi$ is an acyclic cofibration by Proposition 4.5, we need only consider γ. Let $K = H/N$ and consider the diagram

$$\begin{array}{ccccc} & \pi_*^H(X \wedge \tilde{E}\mathscr{F}) & \xrightarrow{\omega} & \rho_*^K(X \wedge \tilde{E}\mathscr{F}) & \\ & \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \xi_*} & \\ \pi_*^K R(X \wedge \tilde{E}\mathscr{F})^N \cong & \pi_*^H R(X \wedge \tilde{E}\mathscr{F}) & \xrightarrow[\omega]{} & \rho_*^K R(X \wedge \tilde{E}\mathscr{F}) & \xrightarrow[\eta_*]{} \pi_*^K \Phi^N R(X \wedge \tilde{E}\mathscr{F}). \end{array}$$

The maps ξ_* are isomorphisms since ξ is an acyclic cofibration. The top map ω is an isomorphism by Lemma 4.16, hence the bottom map ω is an isomorphism. Since $R(X \wedge \tilde{E}\mathscr{F})$ is cofibrant, η_* is an isomorphism by Proposition 4.12. Since the bottom composite is γ_*, by Lemma 4.11, this proves the result. □

CHAPTER VI

"Change" functors for S_G-modules and comparisons

We explain the analogues for G-spectra and S_G-modules of the functors on orthogonal G-spectra that we discussed in Chapter V. It turns out that passage from the definitions for G-spectra in [**19**, II§§1-4] to definitions for S_G-modules is not at all automatic. We show further that the comparisons among G-spectra, orthogonal G-spectra, and S_G-modules respect the change of universe, change of group, fixed point spectra, and orbit spectra functors. Technically, these comparisons are the heart of our work. They imply that such fundamental homotopical results as the Wirthmüller isomorphism and the Adams isomorphism, which are proven for G-spectra in [**19**], apply verbatim to orthogonal G-spectra and S_G-modules.

1. Comparisons of change of group functors

Let $\iota : H \longrightarrow G$ be an inclusion of a closed subgroup in G and write ι^* for functors that assign H-action to an object with G-action. We fix a G-universe U throughout this section, and we have the H-universe $\iota^* U$. We may regard G-spectra indexed on U as H-spectra indexed on $\iota^* U$, thus obtaining a forgetful functor

$$\iota^* : G\mathscr{S} \longrightarrow H\mathscr{S}$$

(where we omit the implicit fixed choice of universes from the notation). This functor has a left and a right adjoint [**19**, II.4.1]. Because of the action of the groups on the universes, these functors are given by suitably twisted half-smash product and function spectra functors, which were denoted by $G \ltimes_H (-)$ and $F[H, -)$ in [**19**]. We change notation and call these functors $G_+ \wedge_H (-)$ and $F_H(G_+, -)$ here. This is consistent with the usual notation for these functors on the space level and with the notation we have used for these functors on the orthogonal spectrum level. Write S_H for the sphere H-spectrum indexed on $\iota^* U$; it may be identified with $\iota^* S_G$. The corresponding functors relating S_G-modules and S_H-modules have not yet been defined in the literature. We first show that the functors relating G-spectra and H-spectra induce corresponding functors relating S_G-modules and S_H-modules, and we then compare these change of group functors on the categories of G-spectra, S_G-modules, and orthogonal G-spectra.

The monad $\mathbb{L}$ used in the definition of S_G-modules is given by the twisted half-smash product $\mathscr{I}(U,U) \ltimes (-)$ on G-spectra, and it has an adjoint comonad $\mathbb{L}^{\#}$ given by the twisted function spectrum functor $F[\mathscr{I}(U,U), -)$ [**6**, I§4]. The category $S_G[\mathbb{L}]$ of $\mathbb{L}$-spectra is defined to be the category of $\mathbb{L}$-algebras, and it can be identified with the category of $\mathbb{L}^{\#}$-coalgebras [**6**, I.4.3]. Recall that an S_G-module is an $\mathbb{L}$-spectrum M whose unit map $\lambda : \mathbb{J}M \longrightarrow M$ is an isomorphism.

By an abuse of notation, we let $\iota^*\mathbb{L}$ denote the monad $\mathscr{I}(\iota^*U, \iota^*U) \ltimes (-)$ on H-spectra and let $\iota^*\mathbb{L}^\#$ denote its adjoint comonad $F[\mathscr{I}(\iota^*U, \iota^*U), -)$. We continue to write $\mathbb{J}$ for its $\iota^*\mathbb{L}$-spectrum version $S_H \wedge_{\mathscr{L}} (-)$, so that an S_H-module N is an $\iota^*\mathbb{L}$-spectrum whose unit map $\lambda : \mathbb{J}N \longrightarrow N$ is an isomorphism.

PROPOSITION 1.1. *The functor $\iota^* : G\mathscr{S} \longrightarrow H\mathscr{S}$ and its left and right adjoints $G_+ \wedge_H (-)$ and $F_H(G_+, -)$ induce a functor $\iota^* : G\mathscr{M} \longrightarrow H\mathscr{M}$ and its left and right adjoints $G_+ \wedge_H (-)$ and $F_H(G_+, -)$.*

PROOF. By [**19**, VI.1.8], we have commutation isomorphisms relating the monads $\mathbb{L}$ and $\iota^*\mathbb{L}$ to the functors ι^*, $G_+ \wedge_H (-)$, and $F_H(G_+, -)$. That is, for G-spectra D and H-spectra E, we have

$$\iota^*(\mathbb{L}D) \cong (\iota^*\mathbb{L})(\iota^*D)$$
$$G_+ \wedge_H (\iota^*\mathbb{L}E) \cong \mathbb{L}(G_+ \wedge_H E)$$
$$F_H(G_+, \iota^*\mathbb{L}^\# E) \cong \mathbb{L}^\# F_H(G_+, E).$$

Diagram chases show that these isomorphisms are compatible with the monad and comonad structures and thus ι^* carries $\mathbb{L}$-algebras to $\iota^*\mathbb{L}$-algebras, $G_+ \wedge_H (-)$ carries $\iota^*\mathbb{L}$-algebras to $\mathbb{L}$-algebras, and $F_H(G_+, -)$ carries $\iota^*\mathbb{L}^\#$-coalgebras to $\mathbb{L}^\#$-coalgebras. For $\mathbb{L}$-algebras M and $\iota^*\mathbb{L}$-algebras N, we have an isomorphism

$$\iota^*\mathbb{J}M \cong \mathbb{J}\iota^*M$$

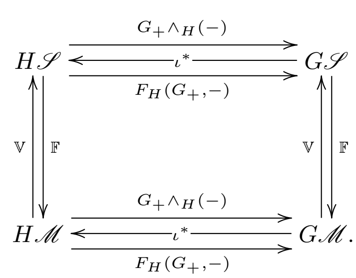
under which $\iota^*\lambda$ agrees with λ and an isomorphism

$$G_+ \wedge_H (\mathbb{J}N) \cong \mathbb{J}(G_+ \wedge_H N)$$

under which $G_+ \wedge_H \lambda$ agrees with λ. Therefore ι^* carries S_G-modules to S_H-modules and $G_+ \wedge_H (-)$ carries S_H-modules to S_G-modules. As is typical in the theory of S_G-modules [**6**, II§2], the right adjoint to $\iota^* : G\mathscr{M} \longrightarrow H\mathscr{M}$ is obtained as the composite of the functor $F_H(G_+, -)$ from S_H-modules to $\mathbb{L}^\#$-coalgebras and the functor $\mathbb{J}$ from $\mathbb{L}^\#$-coalgebras to S_G-modules, the latter being right adjoint to the evident forgetful functor by [**6**, II.2.5]. □

These functors are compatible with the Quillen equivalence $(\mathbb{F}, \mathbb{V})$ relating G-spectra and S_G-modules described in II.2.1. The following result holds for either the cellular or the generalized cellular model structures.

THEOREM 1.2. *Consider the following diagram:*



Each square of left adjoints and each square of right adjoints commutes up to natural isomorphism, and the $(G_+ \wedge_H (-), \iota^*)$ and $(\iota^*, F_H(G_+, -))$ are pairs of Quillen adjoints. Therefore the induced adjoint pairs on homotopy categories agree under the induced adjoint equivalences $HoH\mathscr{S} \simeq HoH\mathscr{M}$ and $HoG\mathscr{S} \simeq HoG\mathscr{M}$.

PROOF. It is clear from the previous proof that $\mathbb{F}\iota^* \cong \iota^*\mathbb{F}$ and $\mathbb{F}(G_+ \wedge_H (-)) \cong (G_+ \wedge_H (-))\mathbb{F}$, and it follows by adjunction that $F_H(G_+, -)\mathbb{V} \cong \mathbb{V}F_H(G_+, -)$ and $\iota^*\mathbb{V} \cong \mathbb{V}\iota^*$. It is clear from the definitions that the functors ι^* preserve weak equivalences and q-fibrations. As in V.2.2, because orbit spaces ι^*G/K are triangulable as finite H-CW complexes [**17**], the functors ι^* also preserves q-cofibrations. $\square$

Turning to the comparison with orthogonal G-spectra, we have the following precise analogue of the previous theorem. Recall IV.3.8. We assume that our given universe U is closed under tensor products. Then ι^*U is also closed under tensor products. We use the collections $\mathscr{V}(U)$ and $\iota^*\mathscr{V}(U)$ to define the categories $G\mathscr{I}\mathscr{S}$ and $H\mathscr{I}\mathscr{S}$ of orthogonal G-spectra and orthogonal H-spectra; for H, this entails a change of indexing spaces isomorphism that is discussed at the start of V§2.

THEOREM 1.3. *Consider the following diagram:*

$$\begin{array}{ccc}
H\mathscr{S}\mathscr{S} & \begin{array}{c} \xrightarrow{G_+ \wedge_H (-)} \\ \xleftarrow{\iota^*} \\ \xrightarrow[F_H(G_+,-)]{} \end{array} & G\mathscr{I}\mathscr{S} \\
\mathbb{N}^{\#}\Big\uparrow\Big\downarrow\mathbb{N} & & \mathbb{N}^{\#}\Big\uparrow\Big\downarrow\mathbb{N} \\
H\mathscr{M} & \begin{array}{c} \xrightarrow{G_+ \wedge_H (-)} \\ \xleftarrow{\iota^*} \\ \xrightarrow[F_H(G_+,-)]{} \end{array} & G\mathscr{M}.
\end{array}$$

Each square of left adjoints and each square of right adjoints commutes up to natural isomorphism, and the $(G_+ \wedge_H (-), \iota^*)$ and $(\iota^*, F_H(G_+, -))$ are pairs of Quillen adjoints. Therefore the induced adjoint pairs on homotopy categories agree under the induced adjoint equivalences $HoH\mathscr{I}\mathscr{S} \simeq HoH\mathscr{M}$ and $HoG\mathscr{I}\mathscr{S} \simeq HoG\mathscr{M}$.

PROOF. By inspection, $\iota^*\mathbb{N}^*(V) \cong \mathbb{N}^*(\iota^*V)$, naturally in V. It follows that $\iota^*\mathbb{N} \cong \mathbb{N}\iota^*$ and $\mathbb{N}^{\#}\iota^* \cong \iota^*\mathbb{N}^{\#}$. Therefore $F_H(G_+, -)\mathbb{N}^{\#} \cong \mathbb{N}^{\#}F_H(G_+, -)$ and $\mathbb{N}(G_+ \wedge_H (-)) \cong (G_+ \wedge_H (-))\mathbb{N}$. Again, it is clear that the functors ι^* preserve weak equivalences and (restricted) q-fibrations, and it follows from the triangulability of orbits that ι^* preserves (generalized) q-cofibrations. $\square$

The Wirthmuller isomorphism explains the homotopical behavior of the functors $F_H(G_+, Y)$. On homotopy categories, there is a natural isomorphism

$$F_H(G_+, Y) \simeq G_+ \wedge_H \Sigma^{-L(H)}Y, \tag{1.4}$$

where $L(H)$ is the tangent H-representation at the identity coset of G/H. This is proven for H-spectra Y in [**19**, II§6], use of H-CW spectra being convenient in the proof. By the results above, it follows for S_H-modules and for orthogonal H-spectra. Writing $[-, -]_G$ for morphisms in homotopy categories, we have

$$[G_+ \wedge_H Y, X]_G \cong [Y, \iota^*X]_H \tag{1.5}$$

and

$$[\iota^*X, Y]_H \cong [X, G_+ \wedge_H \Sigma^{-L(H)}Y]_G \tag{1.6}$$

for G-objects X and H-objects Y.

2. Comparisons of change of universe functors

Change of universe functors are studied on S_G-modules in [**7**] and [**27**, XXIV§3]. Let U and U' be any two G-universes and let $\mathscr{V} = \mathscr{V}(U)$ and $\mathscr{V}' = \mathscr{V}(U')$. Using superscripts to identify categories, we have strong symmetric monoidal equivalences of categories

$$I_{\mathscr{V}'}^{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}'}\mathscr{S} \longrightarrow \mathscr{I}_G^{\mathscr{V}}\mathscr{S}$$

and

$$I_{U'}^{U} : \mathscr{M}_G^{U'} \longrightarrow \mathscr{M}_G^{U}.$$

Since these functors enjoy virtually identical formal properties, it is to be expected that the adjoint pairs $(\mathbb{N}, \mathbb{N}^{\#})$ connecting their sources and targets commute with them. Unfortunately, this expectation is over-optimistic, and the precise comparison is one of the most subtle aspects of the entire theory. In fact, despite the formal similarity, we have no direct comparison between these change of universe functors in general.

We focus on the special case that is relevant to the applications. We consider an inclusion of universes $i : U \longrightarrow U'$, so that $\mathscr{V} \subset \mathscr{V}'$. We write i^* for the forgetful functor $I_{\mathscr{V}'}^{\mathscr{V}}$. It is specified by

$$(i^*X)(V) = X(V) \ \text{ for } \ X \in \mathscr{I}_G^{U'}\mathscr{S} \ \textit{and} \ V \in \mathscr{V}.$$

We have similar forgetful functors

$$i^* : \mathscr{P}_G^{U'} \longrightarrow \mathscr{P}_G^{U} \quad \text{and} \quad i^* : \mathscr{S}_G^{U'} \longrightarrow \mathscr{S}_G^{U}$$

specified by

$$(i^*T)(V) = T(iV) \ \text{ for } \ T \in \mathscr{P}_G^{U'} \ \text{ and } \ V \subset U.$$

It is clear that the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccccc}
\mathscr{S}_G^{U'} & \xrightarrow{\ \ell\ } & \mathscr{P}_G^{U'} & \xleftarrow{\ \mathbb{U}\ } & \mathscr{I}_G^{\mathscr{V}'} \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\mathscr{S}_G^{U} & \xrightarrow[\ \ell\ ]{} & \mathscr{P}_G^{U} & \xleftarrow[\ \mathbb{U}\ ]{} & \mathscr{I}_G^{\mathscr{V}}.
\end{array}$$

Since the forgetful functors $\mathbb{U}$ and ℓ are the right adjoints of Quillen equivalences, this implies the compatibility of the three functors i^* under the induced equivalences of homotopy categories.

The analogue $I_{U'}^{U}$ of i^* on $\mathscr{M}_G^{U'}$ is not induced by restriction of the functor i^* on $\mathscr{S}_G^{U'}$. Its inverse functor $I_U^{U'}$ is compared homotopically with the left adjoint i_* of i^* in [**7**] and the comparison suggests adjointly that the functor $I_{U'}^{U}$ is compatible with the functors i^* in the diagram above after passage to homotopy categories. However, it is awkward to construct this derived functor in a way that allows a direct comparison between $I_{U'}^{U}$ and the functors i^*, and we need such a comparison in our study of fixed point functors in the next section.

We solve this problem by showing that $i^* : \mathscr{S}_G^{U'} \longrightarrow \mathscr{S}_G^{U}$ induces a new forgetful functor $i^* : \mathscr{M}_G^{U'} \longrightarrow \mathscr{M}_G^{U}$ that has a left adjoint $i_* : \mathscr{M}_G^{U} \longrightarrow \mathscr{M}_G^{U'}$ and giving a direct comparison between the new functor i_* and $I_U^{U'}$. To do this, we require that the universe U' be the direct sum of U and the orthogonal complement $U^{\perp}$ of U in U'. This holds in all cases of interest. The following result holds for either the cellular or the generalized cellular model structures.

THEOREM 2.1. *Let $U' = U \oplus U^{\perp}$. Then there is an adjoint pair of G-functors with left adjoint $i_* : \mathscr{M}_G^U \longrightarrow \mathscr{M}_G^{U'}$ and right adjoint $i^* : \mathscr{M}_G^{U'} \longrightarrow \mathscr{M}_G^U$. On passage to G-fixed categories, the functors i_* and i^* give a Quillen adjoint pair such that the following diagrams of left and of right adjoints commute up to natural isomorphism:*

$$\begin{array}{ccc} G\mathscr{S}^U & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & G\mathscr{S}^{U'} \\ \mathbb{V}\uparrow\downarrow\mathbb{F} & & \mathbb{V}\uparrow\downarrow\mathbb{F} \\ G\mathscr{M}^U & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & G\mathscr{M}^{U'}. \end{array}$$

The adjoint pair (i_*, i^*) relating $\mathscr{M}_G^U$ and $\mathscr{M}_G^{U'}$ is defined in terms of an adjoint pair (i_*, i^*) relating the respective categories $S_G^U[\mathbb{L}]$ and $S_G^{U'}[\mathbb{L}]$ of $\mathbb{L}$-spectra. Here the right adjoint i^* is defined as the restriction of the functor $i^* : \mathscr{S}_G^{U'} \longrightarrow \mathscr{S}_G^U$ of the spectrum level adjoint pair (i_*, i^*) discussed in [**19**, II.1.3]. We first show that this restriction makes sense.

LEMMA 2.2. *The functor $i^* : \mathscr{S}_G^{U'} \longrightarrow \mathscr{S}_G^U$ carries $\mathbb{L}^{U'}$-algebras to $\mathbb{L}^U$-algebras and thus restricts to a functor $i^* : \mathscr{S}_G^{U'}[\mathbb{L}] \longrightarrow \mathscr{S}_G^U[\mathbb{L}]$.*

PROOF. Define a map of monoids

$$\alpha : \mathscr{I}(U, U) \longrightarrow \mathscr{I}(U', U')$$

by $\alpha(f) = f \oplus \mathrm{id}_{U^{\perp}}$. By the theory of twisted half-smash products [**6**, App], there results a monad $\alpha \ltimes (-)$ on $\mathscr{S}_G^{U'}$ and an induced map $\bar{\alpha}$ from this monad to the monad $\mathbb{L}^{U'} = \mathscr{I}(U', U') \ltimes (-)$. The following diagram of spaces commutes trivially:

$$\begin{array}{ccc} \mathscr{I}(U,U) & \xrightarrow{\alpha} & \mathscr{I}(U',U') \\ & {\scriptstyle \mathscr{I}(\mathrm{id},i)}\searrow \quad \swarrow{\scriptstyle \mathscr{I}(i,\mathrm{id})} & \\ & \mathscr{I}(U,U'). & \end{array}$$

By [**6**, A.6.2], this implies an isomorphism of functors $i_*\mathbb{L}^U \cong \alpha \ltimes i_*(-)$. Composing with $\bar{\alpha}$ there results a natural transformation

$$\beta : i_*\mathbb{L}^U \longrightarrow \mathbb{L}^{U'} i_*$$

of functors $\mathscr{S}_G^U \longrightarrow \mathscr{S}_G^{U'}$. Taking adjoints, we obtain a natural transformation

$$\tilde{\beta} : i^*\mathbb{L}_{U'}^{\#} \longrightarrow \mathbb{L}_U^{\#} i^*$$

of functors $\mathscr{S}_G^{U'} \longrightarrow \mathscr{S}_G^U$. If M' is an $\mathbb{L}^{U'}$-algebra and thus an $\mathbb{L}_{U'}^{\#}$-coalgebra, say with structure map $\nu' : M' \longrightarrow \mathbb{L}_{U'}^{\#} M'$, then i^*M' is an $\mathbb{L}_U^{\#}$-coalgebra with structure map $\nu = \tilde{\beta} \circ i^*\nu'$. This proves the result. $\square$

DEFINITION 2.3. We define the left adjoint $i_* : \mathscr{S}_G^U[\mathbb{L}] \longrightarrow \mathscr{S}_G^{U'}[\mathbb{L}]$ of i^*. For an $\mathbb{L}^U$-spectrum M with action $\xi : \mathbb{L}^U M \longrightarrow M$, let i_*M be the coequalizer of

$$\mathbb{L}^{U'} i_* \mathbb{L}^U M \xrightarrow{\mathbb{L}^{U'} i_* \xi} \mathbb{L}^{U'} i_* M$$

and the composite

$$\mathbb{L}^{U'} i_* \mathbb{L}^U M \xrightarrow{\mathbb{L}^{U'}\beta} \mathbb{L}^{U'}\mathbb{L}^{U'} i_* M \xrightarrow{\mu'} \mathbb{L}^{U'} i_* M,$$

where μ' is the product of the monad $\mathbb{L}^{U'}$. Now easy diagram chases show that $\mu' : \mathbb{L}^{U'}\mathbb{L}^{U'} i_* M \longrightarrow \mathbb{L}^{U'} i_* M$ passes to coequalizers to define an action of $\mathbb{L}'$ on $i_* M$ and that the resulting functor i_* is left adjoint to i^*.

To define the functors i_* and i^* relating $\mathscr{M}_G^U$ and $\mathscr{M}_G^{U'}$, we need the following lemma. Since the proof is technical and distracting, we defer it to §6.

LEMMA 2.4. *For any $\mathbb{L}^U$-spectrum E, the unit maps*

$$i_* \mathbb{J} E \xleftarrow{\lambda} \mathbb{J} i_* \mathbb{J} E \xrightarrow{\mathbb{J} i_* \lambda} \mathbb{J} i_* E$$

are isomorphisms.

This has the following immediate consequence.

LEMMA 2.5. *The functor $i_* : \mathscr{S}_G^U[\mathbb{L}] \longrightarrow \mathscr{S}_G^{U'}[\mathbb{L}]$ carries S_G^U-modules to $S_G^{U'}$-modules and thus restricts to a functor $i_* : \mathscr{M}_G^U \longrightarrow \mathscr{M}_G^{U'}$.*

Formally, $i_* \mathbb{U} = \mathbb{U} i_*$, where $\mathbb{U} : \mathscr{M}_G^U \longrightarrow \mathscr{S}_G^U[\mathbb{L}]$ is the forgetful functor.

DEFINITION 2.6. Define $i^* : \mathscr{M}_G^{U'} \to \mathscr{M}_G^U$ to be $\mathbb{J} i^* \mathbb{U}$. Since $\mathbb{U}$ is an embedding of a full subcategory and $\mathbb{J}$ is the right adjoint of $\mathbb{U}$ [**6**, II.1.3], we see that i^* is the right adjoint of i_*.

PROOF OF THEOREM 2.1. We now pass to fixed point categories. Lemma 2.4 implies the commutativity of the diagrams of left and right adjoints displayed in the theorem. The adjoint pair (i_*, i^*) on G-spectra is a Quillen pair since the functor i^* preserve weak equivalences and q-fibrations. It follows that the adjoint pair (i_*, i^*) on S_G-modules is also a Quillen adjoint pair. □

The strong symmetric monoidal analogue $I_U^{U'}$ of i_* defined in [**7**] is the functor appropriate to the study of highly structured ring and module spectra, and we have the following comparison. There is a class $\bar{\mathscr{E}}_{S_G^U}$ of S_G^U-modules that contains all of the cofibrant objects in all of our categories of highly structured ring and module spectra over cofibrant commutative S_G-algebras and enjoys especially good homotopy theoretic properties. It was defined equivariantly [**7**, 3.1], following [**6**, VII.6.4], but it is best to reinterpret $\bar{\mathscr{E}}_{S_G^U}$ to mean the equivariant analogue of the larger class defined by Basterra [**2**, 9.3], which enjoys the same good properties.

PROPOSITION 2.7. *There is a natural map $\alpha : i_* M \longrightarrow I_U^{U'} M$ of $S_G^{U'}$-modules that is a weak equivalence for all $M \in \bar{\mathscr{E}}_{S_G^U}$. Moreover, both maps in the diagram*

$$i_* M \wedge i_* N \xrightarrow{\alpha\wedge\alpha} I_U^{U'} M \wedge I_U^{U'} N \cong I_U^{U'}(M \wedge N) \xleftarrow{\alpha} i_*(M \wedge N)$$

are weak equivalences for all $M, N \in \bar{\mathscr{E}}_{S_G^U}$. Therefore the induced functor $i_* : HoG\mathscr{M}^U \longrightarrow HoG\mathscr{M}^{U'}$ is strong symmetric monoidal.

PROOF. The definition of i_* on S-modules M can be written concisely as

$$i_* M = \mathscr{I}(U', U') \ltimes_{\mathscr{I}(U,U)} M, \tag{2.8}$$

where the twisted half-smash product is defined with respect to the map

$$\mathscr{I}(i, \mathrm{id}) : \mathscr{I}(U', U') \longrightarrow \mathscr{I}(U, U').$$

According to [**7**, 2.3], the functor $I_U^{U'} : \mathscr{M}_G^U \longrightarrow \mathscr{M}_G^{U'}$ is defined by

$$I_U^{U'} M = \mathscr{I}(U, U') \ltimes_{\mathscr{I}(U,U)} M. \tag{2.9}$$

This makes sense because the corresponding functor $\mathscr{S}^U[\mathbb{L}] \longrightarrow \mathscr{S}^{U'}[\mathbb{L}]$ carries S_G^U-modules to $S_G^{U'}$-modules [**7**, 2.4]. Therefore $\mathscr{I}(i, \mathrm{id})$ induces a natural map of $S_G^{U'}$-modules

$$\alpha : i_* M \longrightarrow I_U^{U'} M. \tag{2.10}$$

We have an evident commutative diagram of underlying G-spectra

$$\begin{array}{ccc} \mathscr{I}(U', U') \ltimes M & \longrightarrow & \mathscr{I}(U', U') \ltimes_{\mathscr{I}(U,U)} M \\ \downarrow & & \downarrow \alpha \\ \mathscr{I}(U, U') \ltimes M & \longrightarrow & \mathscr{I}(U, U') \ltimes_{\mathscr{I}(U,U)} M \end{array}$$

in which the horizontal arrows are quotient maps. The left vertical arrow is a homotopy equivalence of G-spectra for all "tame" G-spectra M, in particular for all S_G-modules $M \in \bar{\mathscr{E}}_{S_G^U}$, by [**6**, I.2.5]. The bottom horizontal arrow is proven to be a weak equivalence for all $M \in \bar{\mathscr{E}}_{S_G^U}$ in [**7**, p.148], and essentially the same argument shows that the top horizontal arrow and thus also α is a weak equivalence. The proof that $\alpha \wedge \alpha$ is a weak equivalence when applied to S_G^U-modules in $\bar{\mathscr{E}}_{S_G^U}$ is similar; compare [**6**, VII§6] or [**2**, §9]. □

We now assume that our universes U and $U' = U \oplus U^{\perp}$ are closed under tensor products and consider the resulting Quillen equivalences $(\mathbb{N}, \mathbb{N}^{\#})$. We have a class $\bar{\mathscr{E}}_{\mathscr{I}_G}$ of orthogonal G-spectra parallel to the class $\bar{\mathscr{E}}_{S_G}$ of S_G-modules. Precisely, let $\mathscr{E}_{\mathscr{I}_G}$ be the subclass of objects of $\mathscr{I}_G\mathscr{S}$ consisting of S_G together with the orthogonal G-spectra of the form X^j/Σ, where X is a positive cell orthogonal G-spectrum and Σ is a subgroup of Σ_j. Let $\bar{\mathscr{E}}_{\mathscr{I}_G}$ be the smallest class of orthogonal G-spectra that contains $\mathscr{E}_{\mathscr{I}_G}$ and is closed under wedges, pushouts along h-cofibrations, sequential colimits of h-cofibrations, finite smash products, and homotopy equivalences. It is clear from the arguments in III§§5, 7, 8 that the class $\bar{\mathscr{E}}_{\mathscr{I}_G}$ contains all of the positive cofibrant objects in all of our categories of highly structured ring and module spectra over cofibrant commutative orthogonal ring spectra. The following analogue of Theorem 2.1 is technically parallel to Proposition 2.7.

THEOREM 2.11. *Write $i_* = I_{\mathscr{V}}^{\mathscr{V}'} : \mathscr{I}_G^{\mathscr{V}} \longrightarrow \mathscr{I}_G^{\mathscr{V}'}$ and $i^* = I_{\mathscr{V}'}^{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}'} \longrightarrow \mathscr{I}_G^{\mathscr{V}}$. There are natural maps of $S_G^{U'}$-modules*

$$i_* \mathbb{N} X \xrightarrow{\alpha} I_U^{U'} \mathbb{N} X \xleftarrow{\beta} \mathbb{N} i_* X$$

that are weak equivalences for all X in $\bar{\mathscr{E}}_{\mathscr{I}_G}$. Passing to homotopy categories, the left and right adjoints in the following diagram commute up to natural isomorphism.

$$\begin{array}{ccc} HoG\mathscr{I}^{\mathscr{V}} & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & HoG\mathscr{I}^{\mathscr{V}'} \\ {\scriptstyle \mathbb{N}^\#}\uparrow\downarrow{\scriptstyle \mathbb{N}} & & {\scriptstyle \mathbb{N}^\#}\uparrow\downarrow{\scriptstyle \mathbb{N}} \\ HoG\mathscr{M}^{U} & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & HoG\mathscr{M}^{U'}. \end{array}$$

PROOF. Since $\mathbb{N}$ takes positive cell orthogonal G-spectra to cell S_G-modules and is strong symmetric monoidal and a topological left adjoint, $\mathbb{N}$ carries the class $\bar{\mathscr{E}}_{\mathscr{I}_G}$ into the class $\bar{\mathscr{E}}_{S_G}$. The map α is given by Proposition 2.7, and that result gives that α is a weak equivalence on objects in $\bar{\mathscr{E}}_{\mathscr{I}_G}$.

We construct β by use of observations on "right exact" functors from I§2. Recall that $i_*F_V^{\mathscr{V}}S^0 \cong F_V^{\mathscr{V}'}S^0$, by V.1.4, and that the functors $\mathbb{N}$ are defined in terms of the functors $\mathbb{N}^*$ of IV.3.4. We have the contravariant functors $\mathbb{N}^*_{U'}i^*$ and $I_U^{U'}\mathbb{N}^*_U$ from $\mathscr{J}_G^{\mathscr{V}}$ to $\mathscr{M}_G^{U'}$. The first satisfies

$$\mathbb{N}_{U'}i_*(F_VS^0) = \mathbb{N}^*_{U'}i^*(V) = \mathbb{J}\mathscr{I}(V\otimes U', U') \ltimes \Sigma_V^{V\otimes U'}S^0. \tag{2.12}$$

By [**7**, 2.2], composition induces a homeomorphism of G-spaces

$$\gamma : \mathscr{I}(U,U') \times_{\mathscr{I}(U,U)} \mathscr{I}(V\otimes U, U) \longrightarrow \mathscr{I}(V\otimes U, U').$$

Via (2.9), this implies that the second satisfies

$$I_U^{U'}\mathbb{N}_U(F_VS^0) = I_U^{U'}\mathbb{N}^*_U(V) \cong \mathbb{J}\mathscr{I}(V\otimes U, U') \ltimes \Sigma_V^{V\otimes U}S^0. \tag{2.13}$$

The functors $\mathbb{N}_{U'}i_*$ and $I_U^{U'}\mathbb{N}_U$ are continuous left adjoints and therefore right exact. Since their composites with the functor that sends $V \in \mathscr{J}_G^{\mathscr{V}}$ to F_VS^0 are $\mathbb{N}_{U'}i_*$ and $I_U^{U'}\mathbb{N}_U$, we obtain natural isomorphisms

$$\mathbb{N}_{U'}i_*X = \mathbb{N}_{U'}i^* \otimes_{\mathscr{J}_G^{\mathscr{V}}} X \quad \text{and} \quad I_U^{U'}\mathbb{N}_UX = I_U^{U'}\mathbb{N}^*_U \otimes_{\mathscr{J}_G^{\mathscr{V}}} X.$$

For $V \in \mathscr{V}$, let $i(V) = \mathrm{id}\otimes i : V\otimes U \longrightarrow V\otimes U'$. The functors $i(V)_*\Sigma_V^{V\otimes U}$ and $\Sigma_V^{V\otimes U'}$ are isomorphic since both are left adjoint to the Vth space functor. By [**6**, A.6.2], it follows that the map

$$\mathscr{I}(i(V),\mathrm{id}) : \mathscr{I}(V\otimes U', U') \longrightarrow \mathscr{I}(V\otimes U, U')$$

induces a natural map

$$\beta^* : \mathbb{N}^*_{U'}i^*(V) \longrightarrow I_U^{U'}\mathbb{N}^*_U(V).$$

By adjunction, β^* induces the required natural map $\beta : \mathbb{N}i_*X \longrightarrow I_U^{U'}\mathbb{N}X$. When $X = F_VA$, where $V^G \neq 0$, we see by use of (2.12), (2.13), and the "untwisting theorem" of [**6**, A.5.5] that β is obtained by applying $\mathbb{J}$ to a map of $\mathbb{L}$-spectra whose underlying map of G-spectra is isomorphic to the map

$$\mathscr{I}(i(V),\mathrm{id})_+ \wedge \mathrm{id} : \mathscr{I}(V\otimes U', U')_+ \wedge \Sigma_V^{U'}A \longrightarrow \mathscr{I}(V\otimes U, U')_+ \wedge \Sigma_V^{U'}A.$$

Since $\mathscr{I}(V\otimes U', U')$ and $\mathscr{I}(V\otimes U, U')$ are G-contractible G-spaces [**6**, XI.1.5], this map is a homotopy equivalence of G-spectra, hence β is a weak equivalence. By passage to wedges, pushouts, sequential colimits, and retracts, it follows that β is a weak equivalence when X is positive cofibrant. By III.2.6, finite smash products of positive cofibrant orthogonal G-spectra are positive cofibrant. We pass to orbit orthogonal G-spectra X^j/Σ by use of III.8.4, and we pass to general orthogonal G-spectra $X \in \bar{\mathscr{E}}_{\mathscr{I}_G}$ by an analysis of their structure that is similar to, but simpler than, the analogous analysis in the category of S-modules given in [**6**, VII§6]. □

REMARK 2.14. On homotopy categories Ho$G\mathscr{S}$, Ho$G\mathscr{M}$, and Ho$G\mathscr{I}\mathscr{S}$, the functors i_* are all strong symmetric monoidal. For Ho$G\mathscr{S}$ this is implicit in [**19**, II.3.14], for Ho$G\mathscr{M}$ this is part of Proposition 2.7, and for Ho$G\mathscr{I}\mathscr{S}$ the functor i_* on $G\mathscr{I}\mathscr{S}$ is already strong symmetric monoidal. Diagram chases show that the isomorphisms $i_*X \wedge i_*Y \longrightarrow i_*(X \wedge Y)$ in these three settings agree under the equivalences induced by functors $\mathbb{F}$ and $\mathbb{N}$.

3. Comparisons of fixed point and orbit G-spectra functors

First assume given a trivial G-universe. We denote it U^G and let $\mathscr{V}^G = \mathscr{V}(U^G)$, so that $\mathscr{V}^G$ is just the collection of all finite dimensional inner product spaces, with trivial action by G. For G-spectra and S_G-modules indexed on U^G and for orthogonal G-spectra indexed on $\mathscr{V}^G$, we pass to fixed points and orbits levelwise. That is, in all three settings, for an object X,

$$X^G(V) = X(V)^G \quad \text{and} \quad (X/G)(V) = X(V)/G. \tag{3.1}$$

In the case of S_G-modules, since G acts trivially on $\mathscr{I}(U^G, U^G)$, it is clear that X^G and X/G inherit S-module structures from the S_G-module structure on X. Formally, for G-spectra X and $\mathbb{L}$-spectra Y, we have isomorphisms of functors

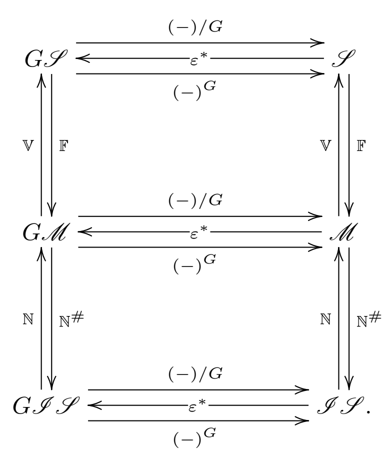
$$(\mathbb{L}X)^G \cong \mathbb{L}(X^G), \quad (\mathbb{L}X)/G \cong \mathbb{L}(X/G), \quad (\mathbb{J}Y)^G \cong \mathbb{J}(Y^G), \quad (\mathbb{J}Y)/G \cong \mathbb{J}(Y/G).$$

Write ε^* for functors that assign trivial G-action to nonequivariant objects. For S_G-modules X and S-modules Y, we then have adjunctions

$$G\mathscr{M}(\varepsilon^*Y, X) \cong \mathscr{M}(Y, X^G) \quad \text{and} \quad G\mathscr{M}(X, \varepsilon^*Y) \cong \mathscr{M}(X/G, Y). \tag{3.2}$$

Since G acts trivially on U^G, the cellular and generalized cellular model structures coincide on $G\mathscr{M}$, and these are Quillen adjoint pairs since in both cases the right adjoints are easily seen to preserve weak equivalences and q-fibrations. The same holds for G-spectra, and orthogonal G-spectra work similarly by V§3. Inspections of definitions give the following elementary comparisons.

THEOREM 3.3. *Consider the following diagram, in which all spectra are indexed on a trivial universe:*



Each square of left adjoints and each square of right adjoints commutes up to natural isomorphism, and the $(\varepsilon^*, (-)^G)$ *and* $((-)/G, \varepsilon^*)$ *are pairs of Quillen adjoints. Therefore the induced adjoint pairs on homotopy categories agree under the induced adjoint equivalences* $HoG\mathscr{S} \simeq HoG\mathscr{M} \simeq HoG\mathscr{I}\mathscr{S}$ *and* $Ho\mathscr{S} \simeq Ho\mathscr{M} \simeq Ho\mathscr{I}\mathscr{S}$.

Now let U be a complete G-universe, or any G-universe closed under tensor products, and let $\mathscr{V} = \mathscr{V}(U)$. We have the G-fixed universe U^G, and the theory of the previous section applies to the inclusion $i : U^G \longrightarrow U$.

DEFINITION 3.4. For G-spectra, S_G-modules, or orthogonal G-spectra X indexed on the universe U (or on $\mathscr{V}(U)$), define $X^G = (i^*X)^G$.

We do not define orbit G-spectra this way, preferring to regard them as defined only on G-fixed universes. We write $\varepsilon^\# = i_*\varepsilon^*$ in all three contexts; that is, we interpret $\varepsilon^\#$ as a functor $\mathscr{S}^{U^G} \longrightarrow G\mathscr{S}^U$, $\mathscr{M}^{U^G} \longrightarrow G\mathscr{M}^U$, or $\mathscr{I}^{\mathscr{V}^G} \longrightarrow G\mathscr{I}^{\mathscr{V}}$. In all three contexts, $\varepsilon^\#$ is left adjoint to $(-)^G$. For example, for S_G^U-modules X and S^{U^G}-modules Y,

$$G\mathscr{M}^U(\varepsilon^\# Y, X) \cong \mathscr{M}^{U^G}(Y, X^G). \tag{3.5}$$

In view of Theorem 2.1, this is a Quillen adjoint. Deleting the universes from the notation and combining Theorems 2.1, 2.11, and 3.3, we obtain somewhat more precise information than is given in the statement of the following result.

THEOREM 3.6. *The fixed point functors*

$$HoG\mathscr{S} \to Ho\mathscr{S}, \quad HoG\mathscr{M} \to Ho\mathscr{M}, \quad HoG\mathscr{I}\mathscr{S} \to Ho\mathscr{I}\mathscr{S}$$

and their left adjoints $\varepsilon^\#$ *agree under the equivalences of their domain and target categories induced by Quillen equivalences* $(\mathbb{F}, \mathbb{V})$ *and* $(\mathbb{N}, \mathbb{N}^\#)$.

REMARK 3.7. In all three contexts, the counit of the adjunction $(\varepsilon^\#, (-)^G)$ is a natural G-map $\varepsilon^\# X^G \longrightarrow X$. It is the analogue of the inclusion $A^G \longrightarrow A$ for based G-spaces A. Working in the trivial universe, ε^* clearly commutes with smash products, and i_* commutes with smash products up to natural isomorphism in the respective homotopy categories by Remark 2.14. For G-spectra, S_G-modules, or orthogonal G-spectra X and Y, there results a natural map

$$\varepsilon^\#(X^G \wedge Y^G) \cong (\varepsilon^\# X^G) \wedge (\varepsilon^\# Y^G) \longrightarrow X \wedge Y$$

in the respective homotopy category. Its adjoint is a natural map

$$X^G \wedge Y^G \longrightarrow (X \wedge Y)^G.$$

These maps agree under the equivalences induced by the pairs $(\mathbb{F}, \mathbb{V})$ and $(\mathbb{N}, \mathbb{N}^\#)$. We have similarly compatible natural maps

$$X^G \wedge A^G \longrightarrow (X \wedge A)^G$$

and

$$\Sigma^\infty(A^G) \longrightarrow (\Sigma^\infty A)^G$$

where Σ^∞ on the left and right refer to the universes U^G and U. These maps are not equivalences. They are discussed on G-spectra in [**6**, 3.14], and we have nothing new to say about their analogues for S_G-modules; we could just as well lift along the equivalence induced by $(\mathbb{F}, \mathbb{V})$. The analogues for orthogonal G-spectra are surprisingly well-behaved on the point-set level, as explained in V§3.

If $WH = NH/H$, we can obtain H-fixed point functors from G-objects to WH-objects by forgetting down to NH-objects, changing to an H-trivial universe, and taking H-fixed points. The first step is discussed in §1, and we discuss the second and third steps more generally. Thus let N be a normal subgroup of G, let $J = G/N$, and let $\varepsilon : G \longrightarrow J$ be the quotient homomorphism. We are thinking of the normal subgroup H of NH with quotient group WH.

Fix a complete G-universe U and consider its N-fixed subuniverse U^N. Write $\mathscr{V}^N = \mathscr{V}(U^N)$. The universe U is the direct sum of U^N and its orthogonal complement, which is the sum of all irreducible sub G-spaces V of U such that $U^N \neq 0$. Moreover both universes U^N and U are closed under tensor products. Therefore, using IV.3.8, the results of the previous section apply to the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$. Write ε^* for any functor that assigns to J-objects the same objects regarded as N-fixed G-objects.

Definition 3.8. Define $\varepsilon^{\#} = i_*\varepsilon^*$; this specifies functors

$$J\mathscr{S}^{U^N} \longrightarrow G\mathscr{S}^U, \quad J\mathscr{M}^{U^N} \longrightarrow G\mathscr{M}^U, \quad \text{and} \quad J\mathscr{I}^{\mathscr{V}^N}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{V}}\mathscr{S}.$$

For G-objects X indexed on the N-fixed universe U^N (or on $\mathscr{V}^N$), define J-objects X^N indexed on U^N by $(X^N)(V) = X(V)^N$. For G-objects X indexed on the complete G-universe U, define $X^N = (i^*X)^N$. This specifies N-fixed point functors

$$G\mathscr{S}^U \longrightarrow J\mathscr{S}^{U^N}, \quad G\mathscr{M}^U \longrightarrow J\mathscr{M}^{U^N}, \quad \text{and} \quad G\mathscr{I}^{\mathscr{V}}\mathscr{S} \longrightarrow J\mathscr{I}^{\mathscr{V}^N}\mathscr{S}.$$

In all three cases, this functor $(-)^N$ is right adjoint to $\varepsilon^{\#}$, and $(\varepsilon^{\#}, (-)^N)$ is a Quillen adjoint pair. For example, for S_G-modules X and S_J-modules Y,

$$G\mathscr{M}^U(\varepsilon^{\#}Y, X) \cong J\mathscr{M}^{U^N}(Y, X^N). \tag{3.9}$$

Theorem 3.6 generalizes to this situation.

Theorem 3.10. *The N-fixed point functors*

$$HoG\mathscr{S} \to HoJ\mathscr{S}, \quad HoG\mathscr{M} \to HoJ\mathscr{M}, \quad HoG\mathscr{I}\mathscr{S} \to HoJ\mathscr{I}\mathscr{S}$$

and their left adjoints $\varepsilon^{\#}$ agree under the equivalences of their domain and target categories induced by Quillen equivalences $(\mathbb{F}, \mathbb{V})$ *and* $(\mathbb{N}, \mathbb{N}^{\#})$.

If we start in the N-fixed point universe, we can be more precise. Here we define orbit J-spectra levelwise, just as we defined fixed point J-spectra.

Definition 3.11. For G-objects X indexed on the N-fixed universe U^N, define the J-object X/N indexed on U^N by $(X/N)(V) = X(V)/N$.

We have the evident adjunctions, and Theorem 3.3 generalizes directly.

Theorem 3.12. *Consider the following diagram, in which all spectra are indexed on the N-fixed universe U^N:*

$$\begin{array}{ccc}
G\mathscr{S} & \begin{array}{c}\xrightarrow{(-)/N} \\ \xleftarrow{\ \varepsilon^*\ } \\ \xrightarrow[(-)^N]{} \end{array} & J\mathscr{S} \\
\mathbb{V}\uparrow\downarrow\mathbb{F} & & \mathbb{V}\uparrow\downarrow\mathbb{F} \\
G\mathscr{M} & \begin{array}{c}\xrightarrow{(-)/N} \\ \xleftarrow{\ \varepsilon^*\ } \\ \xrightarrow[(-)^N]{} \end{array} & J\mathscr{M} \\
\mathbb{N}\uparrow\downarrow\mathbb{N}^{\#} & & \mathbb{N}\uparrow\downarrow\mathbb{N}^{\#} \\
G\mathscr{I}\mathscr{S} & \begin{array}{c}\xrightarrow{(-)/N} \\ \xleftarrow{\ \varepsilon^*\ } \\ \xrightarrow[(-)^N]{} \end{array} & J\mathscr{I}\mathscr{S}.
\end{array}$$

Each square of left adjoints and each square of right adjoints commutes up to natural isomorphism, and the $(\varepsilon^*, (-)^N)$ *and* $((-)/N, \varepsilon^*)$ *are pairs of Quillen adjoints. The induced adjoint pairs on homotopy categories agree under the induced adjoint equivalences* $HoG\mathscr{S} \simeq HoG\mathscr{M} \simeq HoG\mathscr{I}\mathscr{S}$ *and* $HoJ\mathscr{S} \simeq HoJ\mathscr{M} \simeq HoJ\mathscr{I}\mathscr{S}$.

4. N-free G-spectra and the Adams isomorphism

Following [**19**, II§2], which is clarified by our present model theoretic framework, we relate families to change of universe and use this relation to describe and compare N-free G-spectra, S_G-modules, and orthogonal G-spectra. This allows us to transport the Adams isomorphism, which is perhaps the deepest foundational result in equivariant stable homotopy theory, from G-spectra to S_G-modules and orthogonal G-spectra.

THEOREM 4.1. *Let* $i : U' \longrightarrow U$ *be an inclusion of* G*-universes and consider the family* $\mathscr{F} = \mathscr{F}(U, U')$ *of subgroups* H *of* G *such that there exists an* H*-linear isometry* $U \longrightarrow U'$.

(i) $H \in \mathscr{F}$ *if and only if* U *is* H*-isomorphic to* U'.
(ii) $\mathscr{I}(U, U')$ *is a universal* $\mathscr{F}$*-space.*
(iii) $i_* : Ho\mathscr{F}\mathscr{S}^{U'} \longrightarrow Ho\mathscr{F}\mathscr{S}^{U}$ *is an equivalence of categories.*

PROOF. Parts (i) and (ii) are [**19**, II.2.4 and II.2.11]. Part (iii) is [**19**, II.2.6]. We give the idea. First, for an $\mathscr{F}$-cofibrant $X' \in G\mathscr{S}^{U'}$ and any $Y' \in G\mathscr{S}^{U'}$,

$$i_* : [X', Y']_{\mathscr{F}} \cong [X', Y']_G \longrightarrow [i_*X', i_*Y']_G \cong [i_*X', i_*Y']_{\mathscr{F}}$$

is a bijection. To see this, one uses (i) to prove that the unit $Y' \longrightarrow i^*i_*Y'$ of the (i_*, i^*) adjunction is an $\mathscr{F}$-equivalence [**19**, II.1.9]. Second, for an $\mathscr{F}$-cell complex X in $G\mathscr{S}^U$, there is an $\mathscr{F}$-cell complex $X' \in G\mathscr{S}^{U'}$ and an equivalence $i_*X' \longrightarrow X$. In fact, using ordinary rather than generalized cell structures, we can construct X inductively so that its cells are in bijective correspondence with those of X'. Other choices of X such that i_*X is weakly equivalent to X' are $E\mathscr{F}_+ \wedge i^*X'$ and, using (ii), $\mathscr{I}(U, U') \ltimes X'$; see [**19**, II.2.14]. $\square$

Now return to the consideration of a normal subgroup N of G with quotient group J. Let U be a complete G-universe and let $U' = U^N$. Using these universes, the results of the §2 allow us to transport the conclusion of the previous theorem to both S_G-modules and orthogonal G-spectra.

DEFINITION 4.2. Define $\mathscr{F}(N)$ to be the family of subgroups H of G such that $H \cap N = e$. For G-spectra, S_G-modules, or orthogonal G-spectra indexed on either U or U^N, an $\mathscr{F}(N)$-object is called an N-free G-object, and an $\mathscr{F}(N)$-cell complex is called an N-free G-cell complex.

Thus an N-free G-cell complex is built up out of cells of orbit types G/H such that $H \cap N = e$. This correctly captures the intuition. Note that we are free to use either the cellular or the generalized cellular interpretation of a G-cell complex here. The following elementary observation [**19**, II.2.4] ties things together.

LEMMA 4.3. *The families $\mathscr{F}(U, U^N)$ and $\mathscr{F}(N)$ are the same.*

THEOREM 4.4. *For a normal subgroup N of G,*

$$i_* : Ho\mathscr{F}(N)\mathscr{S}^{U^N} \longrightarrow Ho\mathscr{F}(N)\mathscr{S}^U$$

is an equivalence of categories, and similarly for S_G-modules and for orthogonal G-spectra.

In either universe, we can identify $\mathrm{Ho}\mathscr{F}(N)\mathscr{S}$ with the full subcategory of N-free G-spectra in $\mathrm{Ho}G\mathscr{S}$. The previous result is summarized by the slogan that "N-free G-spectra live in the N-trivial universe". Using Theorem 3.12, it gives

$$[X/N, Y]_J \cong [X, \varepsilon^* Y]_G \cong [i_* X, \varepsilon^{\#} Y]_G \tag{4.5}$$

for an N-free G-object X and any G-object Y, both indexed on U^N. We can ask about the behavior with the order of variables reversed, and the Adams isomorphism relating the orbit and fixed point functors gives the answer. On homotopy categories, there is a natural isomorphism

$$X/N \cong (\Sigma^{-A} i_* X)^N \tag{4.6}$$

for an N-free G-object X indexed on U^N, where A is the G-representation given by the tangent space of N at e. Use of i_* to pass to the complete universe before taking fixed points is essential. This result is proven for G-spectra X in [**19**, II§7]. Here the cellular model structure has a considerable advantage over the generalized cellular model structure, but the conclusion carries over to our other categories. Using (3.9), this implies that

$$[Y, X/N]_J \cong [\varepsilon^{\#} Y, \Sigma^{-A} i_* X]_G. \tag{4.7}$$

5. The geometric fixed point functor and quotient groups

The geometric fixed point functor was defined on G-spectra in [**19**, II§9], where it was shown to commute up to equivalence with smash products and the suspension spectrum functor; see also [**27**, XVI§§3,6]. Recall that $\mathscr{F}[N]$ denotes the family of subgroups H such that H does not contain N. Note that $\mathscr{F}(N) \subset \mathscr{F}[N]$, with equality only if $N = e$. For G-spectra X, there is an equivalence

$$\Phi^N X \simeq (\tilde{E}\mathscr{F}[N] \wedge X)^N. \tag{5.1}$$

In the case of S_G-modules, it seems best to define $\Phi^N X = (X \wedge \tilde{E}\mathscr{F}[N])^N$, although this obscures the simple space level intuition behind the notion. On orthogonal G-spectra, we have given a natural geometric definition of Φ^N and have derived

(5.1) from that definition. Since (5.1) holds in all cases, the Φ^N agree under our equivalences between homotopy categories.

An important role of the original geometric fixed point functor is its use to prove an equivalence between the homotopy category of J-spectra indexed on U^N and the homotopy category of G-spectra indexed on U that are concentrated over N, namely G-spectra X such that $\pi_*^H(X) = 0$ unless H contains N. Specialization of IV.6.13 and IV.6.14 gives the following starting point.

THEOREM 5.2. *A G-spectrum X is concentrated over N if and only if the map $\lambda : X \longrightarrow \tilde{E}\mathscr{F}[N] \wedge X$ is a weak equivalence. Smashing with $\tilde{E}\mathscr{F}[N]$ defines an equivalence of categories from $Ho\mathscr{F}'[N]\mathscr{S}$ to the full subcategory of G-spectra concentrated over N in $HoG\mathscr{S}$.*

THEOREM 5.3. *There is an adjoint equivalence from $HoJ\mathscr{S}$ to the full subcategory of G-spectra concentrated over N in $HoG\mathscr{S}$.*

By the last statement of Theorem IV.6.13, for a G-spectrum X concentrated over N and any J-spectrum Y,

$$(5.4) \qquad \lambda^* : [\tilde{E}\mathscr{F}[N] \wedge \varepsilon^{\#}Y, X]_G \longrightarrow [\varepsilon^{\#}Y, X]_G \cong [Y, X^N]_J$$

is an isomorphism. This gives the required adjunction, and its unit and counit are proven to be equivalences in [**19**, II.§9]. By the comparisons we have given, Theorems 5.2 and 5.3 apply verbatim to S_G-modules and orthogonal G-spectra.

6. Technical results on the unit map $\lambda : \mathbb{J}E \longrightarrow E$

Finally, we return to the proof of Lemma 2.4. The argument relies on the following lemmas, which are in essence special cases of the fundamental lemmas of [**6**] that make the smash product of S-modules associative and unital, namely "Hopkins' lemma" [**6**, I.5.4] and "the accidental isomorphism lemma" [**6**, I.8.1].

LEMMA 6.1. *Let U, U_1, U_2 be G-universes, and let W be a G-inner product space (either finite or countably infinite dimensional). The G-map*

$$\delta : \mathscr{I}(U \oplus W, U_1) \times_{\mathscr{I}(U,U)} \mathscr{I}(U_2, U) \longrightarrow \mathscr{I}(U_2 \oplus W, U_1)$$

specified by $\delta(g, f) = g \circ (f \oplus \mathrm{id}_W)$ is a homeomorphism of G-spaces.

PROOF. It suffices to show that δ is a nonequivariant homeomorphism. We can choose (nonequivariant) isometric isomorphisms between U and U_1 and U_2. Under these isomorphisms, we can identify δ with the analogous map

$$\mathscr{I}(U \oplus W, U) \times_{\mathscr{I}(U,U)} \mathscr{I}(U, U) \longrightarrow \mathscr{I}(U \oplus W, U),$$

which is clearly a homeomorphism. □

LEMMA 6.2. *Let U, U', U_1 be G-universes, and let W be a G-inner product space (either finite or countably infinite dimensional). The G-map*

$$\mathscr{I}(U \oplus U' \oplus W, U_1)/(\mathscr{I}(U,U) \times \mathscr{I}(U',U')) \longrightarrow \mathscr{I}(U' \oplus W, U_1)/\mathscr{I}(U',U')$$

induced by restriction of isometries is a G-homeomorphism.

PROOF. The previous lemma gives a G-homeomorphism

$$\delta : \mathscr{I}(U' \oplus W, U_1) \times_{\mathscr{I}(U',U')} \mathscr{I}(U \oplus U', U') \cong \mathscr{I}(U \oplus U' \oplus W, U_1)$$

of right $(\mathscr{I}(U,U)\times\mathscr{I}(U',U'))$-spaces. We claim that

$$\mathscr{I}(U\oplus U',U')/(\mathscr{I}(U,U)\times\mathscr{I}(U',U'))$$

is the one-point space. Indeed, it suffices to show this nonequivariantly, and after choosing an isometric isomorphism $U\cong U'$, this is just [**6**, I.8.1]. Therefore, after passing to orbits over $\mathscr{I}(U,U)\times\mathscr{I}(U',U')$, δ induces a homeomorphism

$$\mathscr{I}(U'\oplus W,U_1)/\mathscr{I}(U',U')\longrightarrow\mathscr{I}(U\oplus U'\oplus W,U_1)/(\mathscr{I}(U,U)\times\mathscr{I}(U',U')).$$

Its inverse is induced by restriction of isometries. □

The proof of Lemma 2.4 requires a clarification of the definitions of $\mathbb{J}E$ and the unit map $\lambda:\mathbb{J}E\longrightarrow E$ given in [**6**, I.8.3]. We work in a given universe U, using the linear isometries operad $\mathscr{L}$ such that $\mathscr{L}(j)=\mathscr{I}(U^j,U)$. By definition,

$$\mathbb{J}E=S\wedge_{\mathscr{L}}E=\mathscr{L}(2)\ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)}S_G\,\bar{\wedge}\,E,$$

where $\bar{\wedge}$ is the external smash product. Here $S_G\cong\mathscr{L}(0)\ltimes S^0$. The structure map $\gamma:\mathscr{L}(2)\times\mathscr{L}(0)\times\mathscr{L}(1)\longrightarrow\mathscr{L}(1)$ of $\mathscr{L}$ induces a map

$$\hat{\gamma}:\hat{\mathscr{L}}(1)=\mathscr{L}(2)\times_{\mathscr{L}(1)\times\mathscr{L}(1)}\mathscr{L}(0)\times\mathscr{L}(1)\longrightarrow\mathscr{L}(1),$$

which is a G-homotopy equivalence [**6**, XI.2.2]. Form the orbit space $\mathscr{L}(2)/\mathscr{L}(1)$ with respect to the right action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$ given by $(g,e)\longrightarrow g\circ(e\oplus\mathrm{id})$, and let $\mathscr{L}(1)$ act on the right of $\mathscr{L}(2)/\mathscr{L}(1)$ by $([g],f)\longrightarrow[g\circ(\mathrm{id}\oplus f)]$. Then $\hat{\gamma}$ factors as the composite of the homeomorphism

$$\bar{\gamma}:\hat{\mathscr{L}}(1)\longrightarrow\mathscr{L}(2)/\mathscr{L}(1)$$

given by $\bar{\gamma}(g,0,f)=[g\circ(\mathrm{id}\oplus f)]$ and the map

$$i_2:\mathscr{L}(2)/\mathscr{L}(1)\longrightarrow\mathscr{L}(1)$$

obtained by restricting isometries $U\oplus U$ to the second summand U. These maps are both G and $\mathscr{L}(1)$-equivariant. Using [**6**, I.2.2(ii)], which describes iterated twisted half-smash products, and the isomorphism $\bar{\gamma}$, we obtain an identification

$$\mathbb{J}E\cong\mathscr{L}(2)/\mathscr{L}(1)\ltimes_{\mathscr{L}(1)}E.$$

Under this identification, $\lambda:\mathbb{J}E\longrightarrow E$ coincides with the map

$$i_2\ltimes_{\mathscr{L}(1)}\mathrm{id}:\mathscr{L}(2)/\mathscr{L}(1)\ltimes_{\mathscr{L}(1)}E\longrightarrow\mathscr{L}(1)\ltimes_{\mathscr{L}(1)}E\cong E.$$

THE PROOF OF LEMMA 2.4. Recall that we are given a universe $U'=U\oplus U^{\perp}$. Using 6.1, the description (2.8) of i_*, and [**6**, A.6.2], we obtain natural isomorphisms of $i_*\mathbb{J}E$, $\mathbb{J}i_*\mathbb{J}E$, and $\mathbb{J}i_*E$ with $A_1\ltimes_{\mathscr{L}^U(1)}E$, $A_2\ltimes_{\mathscr{L}^U(1)}E$, and $A_3\ltimes_{\mathscr{L}^U(1)}E$, where A_1, A_2, and A_3 are the $(G\ltimes\mathscr{L}^U(1))$-spaces over $\mathscr{I}(U,U')$ specified by

$$A_1=\mathscr{I}(U\oplus U\oplus U^{\perp},U')/\mathscr{L}^U(1)$$

$$A_2=\mathscr{I}(U'\oplus U\oplus U\oplus U^{\perp},U')/(\mathscr{L}^{U'}(1)\times\mathscr{L}^U(1)),\quad\text{and}$$

$$A_3=\mathscr{I}(U'\oplus U\oplus U^{\perp},U')/\mathscr{L}^{U'}(1).$$

The maps

$$i_*\mathbb{J}E\xleftarrow{\lambda}\mathbb{J}i_*\mathbb{J}E\xrightarrow{\mathbb{J}i_*\lambda}\mathbb{J}i_*E$$

are induced by appropriate restriction maps $A_1\longleftarrow A_2\longrightarrow A_3$. By Lemma 6.2, these maps are G-homeomorphisms. □

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