

EQUIVARIANT INFINITE LOOP SPACE THEORY, THE SPACE LEVEL STORY

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ABSTRACT. We rework and generalize equivariant infinite loop space theory, which shows how to construct G -spectra from G -spaces with suitable structure. There is a classical version which gives classical Ω - G -spectra for any topological group G , but our focus is on the construction of genuine Ω - G -spectra when G is finite. We also show what is and is not true when G is a compact Lie group.

We give new information about the Segal and operadic equivariant infinite loop space machines, supplying many details that are missing from the literature, and we prove by direct comparison that the two machines give equivalent output when fed equivalent input. The proof of the corresponding nonequivariant uniqueness theorem, due to May and Thomason, works for classical G -spectra for general G but fails for genuine G -spectra. Even in the nonequivariant case, our comparison theorem is considerably more precise, giving an illuminating direct point-set level comparison.

We have taken the opportunity to update this general area, equivariant and nonequivariant, giving many new proofs, filling in some gaps, and giving a number of corrections to results and proofs in the literature.

CONTENTS

Introduction	2
Outline and commentary	7
Omissions	10
Acknowledgements	10
1. Preliminaries	11
1.1. Preliminaries about G -spaces and Hopf G -spaces	11
1.2. Preliminaries about G -cofibrations and simplicial G -spaces	12
1.3. Categorical preliminaries and basepoints	14
1.4. Spectrum level preliminaries	16
2. The simplicial and conceptual versions of the Segal machine	18
2.1. Definitions: the input of the Segal machine	19
2.2. The simplicial version of the Segal machine	23
2.3. The conceptual version of the Segal machine	24
2.4. \mathcal{F}_G and a factorization of the conceptual Segal machine	26
3. The homotopical version of the Segal machine	29
3.1. The categorical bar construction	29
3.2. Specializations to spaces and G -spaces	32
3.3. The classical homotopical Segal machine	33
3.4. The genuine homotopical Segal machine	36

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4.	The generalized Segal machine	38
4.1.	G -categories of operators \mathcal{D} over \mathcal{F} and \mathcal{D}_G over \mathcal{F}_G	39
4.2.	The equivalence between $\mathcal{D}[\mathcal{T}_G]$ and $\mathcal{D}_G[\mathcal{T}_G]$	41
4.3.	Comparisons of \mathcal{D} - G -spaces and \mathcal{E} - G -spaces for $\nu: \mathcal{D} \rightarrow \mathcal{E}$	42
4.4.	Comparisons of inputs and outputs of the generalized Segal machine	45
5.	The generalized operadic machine	48
5.1.	The Steiner operads	49
5.2.	The operadic machine	50
5.3.	G -categories of operators associated to a G -operad \mathcal{C}_G	52
5.4.	The monads \mathbb{D} and \mathbb{D}_G associated to the G -categories \mathcal{D} and \mathcal{D}_G	54
5.5.	Comparisons of inputs and outputs of the operadic machine	58
6.	The equivalence between the Segal and operadic machines	60
6.1.	The statement of the comparison theorem	60
6.2.	The proof of the comparison theorem	60
6.3.	Construction and analysis of the map π	63
6.4.	The contravariant functors $I_+ \wedge S^{V \bullet}$ on $\mathcal{D}_{G,V}$	64
6.5.	Construction and analysis of the map ι	66
6.6.	Construction of the map ω	68
7.	Proofs of technical results about the operadic machine	70
7.1.	Levelwise G -equivalences of categories of operators over \mathcal{F}_G	70
7.2.	The structure of $\mathbb{D}X$	72
7.3.	The proof that \mathbb{D} preserves \mathbb{H} - G -spaces	76
7.4.	The proof that \mathbb{D} preserves \mathbb{F}_\bullet -equivalences	77
8.	Proofs of technical results about the Segal machine	84
8.1.	Combinatorial analysis of $A^\bullet \otimes_{\mathcal{F}} X$	85
8.2.	The proof of the group completion property	88
8.3.	The positive linearity theorem	89
8.4.	The proof that the wedge axiom holds	94
8.5.	The heart of the Segal machine	97
9.	General topological groups and compact Lie groups	103
9.1.	Classical G -spectra for topological groups G	103
9.2.	The Segal machine for compact Lie groups	104
9.3.	The operadic machine for compact Lie groups	106
9.4.	The comparison theorem for compact Lie groups	107
10.	Epilogue: model categorical interpretations	108
10.1.	Model structures on G -spaces and diagram G -spaces	108
10.2.	Quillen equivalences among input model categories	110
10.3.	\mathcal{F}_G - G -spaces, \mathcal{W}_G - G -spaces, and orthogonal G -spectra	111
10.4.	The bar construction model theoretically	113
11.	Appendix: Bearding functors $\mathcal{D} \rightarrow G\mathcal{U}_*$	114
12.	Appendix: Realization of levelwise G -cofibrations and G -equivalences	116
	References	119

INTRODUCTION

Equivariant homotopy theory is much richer than nonequivariant homotopy theory. Equivariant generalizations of nonequivariant theory are often non-trivial and

often admit several variants. Nonequivariantly, symmetric monoidal (or equivalently permutative) categories and E_∞ spaces give rise to spectra. There are several “machines” that take such categorical or space level input and deliver spectra as output [8, 45, 72] and there are comparison theorems showing that all such machines are equivalent [47, 48, 57].

Equivariantly, there are different choices of G -spectra that can be taken as output of infinite loop space machines. One choice is classical G -spectra, which are simply spectra with a G -action. That is, in the simplest formulation, they are sequences of based G -spaces E_n and based G -maps $\Sigma E_n \rightarrow E_{n+1}$; they are Ω - G -spectra if the adjoint maps $E_n \rightarrow \Omega E_{n+1}$ are weak G -equivalences. They have H -fixed point spectra for $H \subset G$, and a map of classical G -spectra is a weak equivalence if it induces a nonequivariant weak equivalence on each fixed point spectrum. For finite groups G , or compact Lie groups G , these are often called “naive” G -spectra in the literature. We use the word “classical” rather than “naive” to avoid prejudice.

Classical G -spectra really are “naive”, in the sense that they do not know about crucial structure. For example, one cannot prove any version of Poincaré duality in the cohomology theories they represent. They are indexed on the natural numbers, and n should be thought of geometrically as a stand-in for \mathbb{R}^n or S^n , with trivial G -action. When G is a compact Lie group, we also have genuine G -spectra, which are indexed on representations or, more precisely, real vector spaces (better, inner product spaces) V with an action of G . In the simplest formulation, they are systems of based G -spaces E_V and based G -maps $\Sigma^W E_V \rightarrow E_{V \oplus W}$; they are Ω - G -spectra if the adjoint maps $E_V \rightarrow \Omega^W \Omega E_{V \oplus W}$ are weak G -equivalences. They have H -fixed point spectra for $H \subset G$, and a map of genuine G -spectra is a weak equivalence if it induces a nonequivariant weak equivalence on each fixed point spectrum.

Since suspending and looping with respect to the spheres S^V does not commute with passage to H -fixed points when the action of G on V is nontrivial, the H -fixed spectra retain homotopical information about the representations of G . That information is encoded in the notion of equivalence just given. One can also restrict attention to subclasses of representations and obtain a plethora of kinds of G -spectra intermediate between the classical ones (indexed only on trivial representations) and the genuine ones (indexed on all finite dimensional representations).

We shall refer to “genuine” structures when dealing with structures that relate to deloopings with respect to all finite-dimensional representations, not just the trivial ones. To simplify the exposition, we shall assume that G is finite until [Section 9](#). There we shall make clear that everything that comes earlier works just as well for compact Lie groups, provided that we restrict attention everywhere to representations whose isotropy groups H are of finite index in G .

Thus, with G finite, our main theme is equivariant infinite loop space machines whose inputs are G -spaces X with extra structure and whose outputs are genuine Ω - G -spectra whose zeroth spaces are equivariant “group completions” of X . We recall the homological definition of a group completion of a (homotopy associative and commutative) Hopf space in [Definition 1.6](#), and we explain how unique such a group completion is in [Remark 1.7](#). We recall the corresponding homological definition of a group completion of a Hopf G -space in [Definition 1.8](#).

Nonequivariantly, infinite loop space machines give a “recognition principle”, the point of which is to recognize what extra structure on a Hopf space X is sufficient

to ensure that its group completion is an infinite loop space and to do so in a way that facilitates concrete calculations. One machine is the operadic one developed by the first author in [45] and another is the approach using Γ -spaces introduced by Segal in [72]. The opposite category of Segal's Γ is the category \mathcal{F} (denoted elsewhere as \mathbf{Fin}_*) of finite based sets, and we shall call Γ -spaces \mathcal{F} -spaces. We require them to take values in the category \mathcal{T} of nondegenerately based spaces when applying infinite loop space machines.¹

We emphasize the group completion property. We say that a Hopf space X is grouplike if $\pi_0(X)$ is a group, and then X is its own group completion. It is emphatically *not* the case in the most interesting examples that X itself is grouplike, and the process of group completion has a dramatic effect on homotopy groups. The book [13] gives reams of concrete calculations based on Dyer-Lashof operations visible on the input and output of the operadic machine and visibly preserved by group completion. For example, the homologies of free E_∞ -algebras $\mathbb{C}X$ and their group completions $QX = \Omega^\infty \Sigma^\infty X$ are computed with all structure in sight (Hopf algebra structure, Dyer Lashof and Steenrod operations) as explicit functors of the homology of X . In particular, $\mathbb{C}S^0$ is the disjoint union of the classifying spaces of the symmetric groups Σ_n , whereas the homotopy groups of QS^0 are the stable homotopy groups of spheres. The group completion property here is the Barratt-Priddy-Quillen theorem, a dramatic example discovered in the 1970's. Many other applications have followed since.

The paper [22] shows how the Dyer-Lashof operations can also be constructed, albeit considerably less conveniently, using the Segal machine. A key point is that the higher homotopies implicit in the Segal machine are given by *properties* (certain maps are homotopy equivalences with unprescribed homotopy inverses) whereas the higher homotopies implicit in the operadic machine are given by *structure*. It is structure that most easily leads to explicit formulas. It pays to have both machines, and thus it becomes necessary to know how they compare.

Categories of operators were introduced in [57]² to give a home for a common generalization of operadic input, namely algebras over an E_∞ operad, and Segallic input, namely special \mathcal{F} -spaces. To give context, we recall from [57] the nonequivariant definition of an infinite loop space machine from and the uniqueness theorem that characterizes them.

Definition 0.1 ([57, Definition 2.1]). Let $\mathcal{D}[\mathcal{T}]$ be the category of special \mathcal{D} -spaces, where \mathcal{D} is an E_∞ category of operators; an object $X \in \mathcal{D}[\mathcal{T}]$ has a first space X_1 . Let \mathbf{Sp}_c be the category of connective Ω -spectra. An infinite loop space machine is a functor

$$\mathbb{E}: \mathcal{D}[\mathcal{T}] \longrightarrow \mathbf{Sp}_c$$

together with a natural group completion $\iota: X_1 \longrightarrow \mathbb{E}_0 X$.

Remark 0.2. Machines appear most naturally in an equivalent form that more readily generalizes equivariantly. The target category can be *positive* connective Ω -spectra, for which the adjoint structure maps $E_n \longrightarrow \Omega E_{n+1}$ are weak equivalences

¹We warn the reader that when revising this paper in 2021, we found several previously unnoticed subtleties in the verification that basepoints are nondegenerate, hence we have been much more careful about basepoints in the current version.

²Modulo multicategorical generalization, Lurie's ∞ -operads [41] are essentially defined as categories of operators. The equivariant generalization of categories of operators is central to our work here and should lead to an equivariant version of ∞ -operads.

for $n \geq 1$. We then require a natural map $\nu: X_1 \rightarrow E_0X$ such that the composite of ν and $E_0X \rightarrow \Omega E_1X$ is a group completion.

A category of operators \mathcal{D} comes with a map $\xi: \mathcal{D} \rightarrow \mathcal{F}$ of categories of operators, and there is a Segal machine \mathbb{S} defined on special \mathcal{F} -spaces. There is also a (derived) functor $\xi_*: \mathcal{D}[\mathcal{T}] \rightarrow \mathcal{F}[\mathcal{T}]$. The following result compares machines, showing that any machine is equivalent to the Segal machine.

Theorem 0.3 ([57, Corollary 2.6]). *For any infinite loop space machine \mathbb{E} , there is a natural equivalence of spectra between $\mathbb{E}X$ and $\mathbb{S}\xi_*X$.*

Remark 0.4. On homotopy categories, any such machine \mathbb{E} restricts to an equivalence from the full subcategory of grouplike special \mathcal{D} -spaces to the category of connective spectra. To see that, it suffices to prove it for any one machine, and it has been proven for both the operadic and Segal machines. We call such a result a “characterization principle” rather than a recognition principle. One can prove a characterization principle without any consideration of group completion (in the homological sense) of non-grouplike objects. Quite a few papers, starting with Bousfield-Friedlander [9], consider characterization theorems without ever considering group completion. Others consider group completion in senses alternative to our homological version.

The proof of [Theorem 0.3](#) depends heavily on the combinatorial properties of the original Segal machine [72] and does not work in the genuine context, where one must implicitly keep track of deloopings with respect to the representation spheres S^V for finite dimensional G -representations V . Our main result is a similar comparison theorem in the equivariant context. It can be stated roughly as follows. A precise statement is given in [Theorem 6.1](#).

Main Theorem 0.5. *Given equivalent input, the equivariant versions of the Segal and operadic infinite loop space machines give equivalent Ω - G -spectra as output.*

The genuine equivariant generalization of the operadic approach was worked out by Costenoble and Waner [16], following an earlier but unpublished treatment of Hauschild, May, and Waner. Their treatment was quite brief since the point of their paper was a new machine defined on appropriate presheaves of spaces on the orbit category of G rather than on G -spaces and since they had the unpublished treatment on hand. The operadic approach has been worked out more fully and quite recently in [25], which can be viewed in part as a prequel to this paper. It includes discussion of the recognition principle for V -fold loop spaces for representations V of G , as we will recall briefly in [Section 5.2](#).

The genuine equivariant version of Segal’s approach was first studied by Segal [73] in an influential, albeit flawed,³ unpublished preprint. The first published version is due to Shimakawa [75]. He followed a modification due to Woolfson [82] of Segal’s original nonequivariant machine [72] and he corrected [73] to give the first proof that the construction gives genuine Ω - G -spectra when fed appropriate input. In order to give a self-contained treatment, we give the details of the proof, following [73, 75], in [Section 8.5](#). As discussed in [Remark 3.25](#), Shimakawa outlined one proof of the group completion property [77, p.357]. We give details of a quite

³The definition in [73] of the specialness condition on \mathcal{F} - G -spaces needed to get Ω - G -spectra as output is incorrect, and the mistake was followed in several later papers published by others.

different proof, and the length of our treatment of the Segal machine is largely due to subtleties in proving that property. The work of Segal and Shimakawa was later revisited by Blumberg [6], whose focus was on axiomatizing what can be said in the case of compact Lie groups G . As said before, we defer discussion of the compact Lie case to [Section 9](#).

We shall introduce G -categories of operators in [Sections 4.1](#). They come in two flavors, one with a map $\xi: \mathcal{D} \rightarrow \mathcal{F}$ and the other with a map $\xi_G: \mathcal{D}_G \rightarrow \mathcal{F}_G$, where \mathcal{F}_G is the category of finite based G -sets. We shall develop equivalent infinite loop G -space machines starting from either flavor. With definitions of as yet undefined terms to be supplied later, here is the equivariant analogue of [Definition 0.1](#). We say that a G -spectrum E is a positive Ω - G -spectrum if its adjoint structure maps $E(V) \rightarrow \Omega^W E(V \oplus W)$ are weak G -equivalences when $V^G \neq 0$ and is connective if its fixed point spectra are all connective.

Definition 0.6. Let G be a finite group. Let $\mathcal{D}[G\mathcal{F}]$ be the category of \mathbb{F}_\bullet -special \mathcal{D} -spaces (as defined in [Definition 2.14](#)), where \mathcal{D} is an E_∞ G -category of operators over \mathcal{F} (as defined in [Definition 4.23](#)); an object $X \in \mathcal{D}[G\mathcal{F}]$ has a first G -space $X(\mathbf{1})$. Let $G\mathbf{Sp}_{\text{cp}}$ be the category of connective positive Ω - G -spectra. An infinite loop G -space machine is a functor

$$\mathbb{E}: \mathcal{D}[G\mathcal{F}] \rightarrow G\mathbf{Sp}_{\text{cp}}$$

together with a natural map of G -spaces

$$\nu: X(\mathbf{1}) \rightarrow \mathbb{E}X(S^0)$$

such that the composite of ν with the adjoint structure map

$$\mathbb{E}(S^0) \rightarrow \Omega^V \mathbb{E}X(S^V)$$

is a group completion for all V such that $V^G \neq 0$.

Replacing \mathcal{D} and \mathcal{F} by \mathcal{D}_G and \mathcal{F}_G and replacing \mathbb{F}_\bullet -special by special (as defined in [Definition 2.35](#)) gives the definition of an infinite loop G -space machine starting from an E_∞ G -category of operators over \mathcal{F}_G . Incorporating a characterization principle, we require that, on homotopy categories, \mathbb{E} restricts to an equivalence from the full subcategory of grouplike special \mathcal{D}_G -spaces to the category of connective positive Ω - G -spectra. Either way, equivariant infinite loop space machines restrict to give underlying classical machines, and the group completion property is seen there.

Remark 0.7. Machines appear most naturally as prescribed in [Definition 0.6](#). However, if we set $\mathbb{E}'X(V) = \Omega \mathbb{E}X(V \oplus \mathbb{R})$, then the adjoint structure maps $\mathbb{E}X(V) \rightarrow \mathbb{E}'X(V)$ specify an equivalence from $\mathbb{E}X$ to an Ω - G -spectrum whose zeroth space is a group completion of $X(\mathbf{1})$.

There are implications common to any equivariant infinite loop space machine, and we refer the reader to [\[25, §2.3\]](#) for a discussion. For example, the group completion property directly implies that any machine commutes with products and with passage to fixed points.

Our goal here is to give a self-contained development of the Segal and operadic machines, both generalized to categories of operators, and to prove that they are equivalent. Due to their very different constructions, these machines have different advantages and disadvantages, as already indicated in the nonequivariant case.

The Segal machine works simplicially and seems to be the machine of choice in proving a motivic recognition principle. In the framework of ∞ -categories, Elmanto, Hoyois, Khan, Sosnilo, and Yakerson [19] have adapted it to construct a motivic infinite loop space machine. The operadic machine generalizes directly to give machines that manufacture intermediate types of G -spectra from intermediate types of input data, but we have not worked out a Segal type analogue. Due to its more topological flavor, the operadic machine was used to produce genuine G -spectra from elementary categorical data in [25], where the machine was used to give categorical proofs of topological results, especially a strong categorical version of the Barratt-Priddy-Quillen theorem.

Outline and commentary. Aside from Section 1, which gives preliminaries relevant to both machines, all sections have their own introductions. We therefore limit the rest of the introduction to a brief overview.

We have several ways to generalize the Segal machine equivariantly, and we have comparisons among them. We develop the one closest to Segal's original version in Section 2, highlighting the role of its inductive simplicial definition in proving the group completion property. This version of the machine has not previously been developed equivariantly, and we shall use it to prove the equivariant group completion property for our other versions. Following Segal [72] nonequivariantly, we compare that machine to a conceptual version that is defined by categorical prolongation of functors defined on \mathcal{F} first to functors defined on \mathcal{F}_G and from there to functors defined on the category \mathcal{W}_G of finite based G -CW complexes. Reproving a result of Shimakawa [76], we show in Theorem 2.38 that the input categories of \mathcal{F} - G -spaces and of \mathcal{F}_G - G -spaces are equivalent. The comparison of the relevant "specialness" conditions and of equivalences sheds considerable light on the underlying homotopy theory and is revisited model categorically in Section 10.

The homotopical conditions needed to make the conceptual machine useful are seldom satisfied by the examples that arise in nature. We recall the homotopically well-behaved Segal machine in Section 3. It is based on the categorical two-sided bar construction. This version was first defined nonequivariantly by Woolfson [82], before the relevant bar constructions had been formalized. However, the obvious equivariant generalization of his definition starting from \mathcal{F} - G -spaces fails to be well-behaved homotopically. Again following Shimakawa [75, 76, 77], we instead focus on an equivariant generalization that starts from the category of \mathcal{F}_G - G -spaces. The definition of our preferred homotopical Segal machines starting from \mathcal{F} or from \mathcal{F}_G is given in Definition 3.24. The statement that they give equivalent infinite loop space machines in the sense of Definition 0.6 is made more precise and proven in Theorem 3.34. We summarize the conclusions in the following rough statement.

Theorem 0.8. *There are equivalent Segal infinite loop G -space machines that take \mathbb{F}_\bullet -special \mathcal{F} - G -spaces or special \mathcal{F}_G - G -spaces to (genuine) Ω - G -spectra.*

To explain ideas without undue technical clutter, we defer the longer proofs about the Segal machine from Sections 2 and 3 to Section 8.

To prove that the Segal and operadic machines are equivalent, we must of course first redevelop and generalize both so that they do in fact accept the same input. The generalizations follow the corresponding nonequivariant theory in May and Thomason [57]. We have E_∞ G -categories of operators \mathcal{D} over \mathcal{F} and E_∞ G -categories of operators \mathcal{D}_G over \mathcal{F}_G , and we have algebras over each. Again, by

Theorem 4.11. these categories of input data are equivalent. Still ignoring operads, we develop and compare Segal machines that produce genuine G -spectra from such algebras in [Section 4](#). For an E_∞ G -category of operators \mathcal{D} over \mathcal{F} with prolonged E_∞ G -category of operators \mathcal{D}_G over \mathcal{F}_G , we show that the input categories of \mathcal{F} , \mathcal{F}_G , \mathcal{D} , and \mathcal{D}_G G -spaces are all equivalent in [Theorems 4.24](#) and [4.25](#). We show that the Segal machines starting from these four input categories are all equivalent in [Theorem 4.31](#). Again, we give a rough statement of the conclusions.

Theorem 0.9. *For a category of operators \mathcal{D} over \mathcal{F} with associated category of operators \mathcal{D}_G over \mathcal{F}_G , there are equivalent Segal infinite loop space machines that take \mathbb{F}_\bullet -special \mathcal{D} - G -spaces or special \mathcal{D}_G - G -spaces to (genuine) Ω - G -spectra.*

We turn to the operadic machine in [Section 5](#). We first recall the equivariant Steiner operads and the classical operadic machine whose input is \mathcal{C}_G -algebras for an E_∞ operad \mathcal{C}_G . We show how to interpret it as an infinite loop space machine in the sense of [Definition 0.6](#) in [Theorem 5.6](#) and [Remark 0.7](#). This machine is defined in terms of a monadic 2-sided bar construction, starting from the monad \mathbb{C}_G associated to \mathcal{C}_G whose algebras are the \mathcal{C}_G -algebras. We show how to construct G -categories of operators \mathcal{D} over \mathcal{F} and \mathcal{D}_G over \mathcal{F}_G from \mathcal{C}_G , and we show how to construct monads \mathbb{D} and \mathbb{D}_G from \mathcal{D} and \mathcal{D}_G whose algebras are the \mathcal{D} -algebras and \mathcal{D}_G -algebras. That allows us to define infinite loop space machines on \mathcal{D} -algebras and \mathcal{D}_G -algebras in [Definition 5.28](#). In marked contrast to the Segalic context, we show in [Proposition 5.19](#) that the categories of \mathbb{D} -algebras and of \mathbb{D}_G -algebras are equivalent. Thus there is no particular need to consider \mathcal{D}_G rather than \mathcal{D} when developing the operadic machine, although use of \mathcal{D}_G is more convenient for purposes of comparison.

We give formal comparisons of \mathcal{C}_G -algebras, \mathcal{D} -algebras, and \mathcal{D}_G -algebras in [Propositions 5.21](#) and [5.22](#) and homotopical comparisons in [Theorem 5.23](#). These allow us to conclude in [Corollary 5.31](#) that the machines defined on \mathcal{C}_G -spaces and on special \mathcal{D}_G - G -spaces are equivalent. This is conceptually the same as in May and Thomason [\[57\]](#), but the key equivariant proof is more intricate and is deferred to [Section 7](#). The following rough statement summarizes the conclusions.

Theorem 0.10. *For an E_∞ G -operad \mathcal{C}_G , there is an infinite loop G -space machine that takes \mathcal{C}_G -algebras to (genuine) Ω - G -spectra. If \mathcal{D} and \mathcal{D}_G are the categories of operators over \mathcal{F} and \mathcal{F}_G constructed from \mathcal{C}_G , this machine extends to equivalent infinite loop G -space machines defined on \mathbb{F}_\bullet -special \mathcal{D} - G -spaces or on special \mathcal{D}_G - G -spaces.*

The comparison of the Segal and operadic machines starting from the same input is given in [Section 6](#), where [Main Theorem 0.5](#) is proven. The comparison seems quite amazing to us. Even nonequivariantly, it is far more precise than the comparison given in [\[57\]](#). The Steiner operads are variants of the little cubes and little discs operads that share the good properties and lack the bad properties of each of those, as explained in [\[54, §3\]](#). Nonequivariantly, they have played an important role in infinite loop space theory ever since their introduction in 1979. We see here that they mediate between the Segal and operadic infinite loop space machines just as if they had been invented for that purpose. That is wholly unexpected and truly uncanny.

A little more background may help explain why we find such a precise point-set level comparison so surprising. Expanding on the property versus structure

dichotomy, in the Segal machine, higher homotopies are encoded in the specialness *property* of the structure map $\delta: X_n \rightarrow X_1^n$ relating the n th space to the n th power of the first space of an \mathcal{F} - G -space. In the operadic machine, they are encoded in the *structure* given by the action maps $\mathcal{C}_G(n) \times X^n \rightarrow X$ of a \mathcal{C}_G -algebra, which is why the operadic machine is more useful for purposes of computation. Actions by the G -category of operators \mathcal{D}_G associated to \mathcal{C}_G encode both sources of higher homotopies in the same structure and yet, up to equivalence, carry no more information than either does alone.

The difference is perhaps illuminated by thinking about the commutativity operad \mathcal{N} with $\mathcal{N}(j) = *$. The G -category $\mathcal{D}(\mathcal{N})$ over \mathcal{F} is just \mathcal{F} itself. An \mathcal{N} - G -space $X = X_1$ is the same thing as an \mathcal{F} - G -space with $X_n = X_1^n$. Both just give X the structure of a commutative monoid in $G\mathcal{T}$, and that is far too restricted to give the domain of an infinite loop space machine: the fixed point spaces of the infinite loop G -spaces resulting from such input are equivalent to products of Eilenberg-MacLane G -spaces. From the point of view of the Segal machine, we are replacing \mathcal{N} - G -spaces with homotopically well-behaved \mathcal{F} - G -spaces as input. From the point of view of the operadic machine, we are replacing \mathcal{N} - G -spaces by \mathcal{C}_G -spaces for any chosen E_∞ operad \mathcal{C}_G .

In [Section 9](#), we explain what is true for general topological groups G and for compact Lie groups G . For general topological groups G , there are classical Segal and operadic infinite loop space machines. They construct classical Ω - G -spectra from special \mathcal{F} - G -spaces or from algebras over classical E_∞ operads. When generalized to equivariant categories of operators, they are equivalent. Restricting to compact Lie groups, we reach the following conclusion. See [Theorem 9.8](#) for a precise statement of the extended version of [Main Theorem 0.5](#).

Theorem 0.11. *Without exception, all results in this paper generalize from finite groups to compact Lie groups, provided that we restrict attention to representations whose isotropy groups have finite index in G .*

That is a severe restriction, but it is a very natural one here: those are the subgroups H such that G/H is a finite G -set. With this restriction, all of our details when G is finite make sense and apply equally well when G is compact Lie.

As noted by the referee, model categorical interpretations were originally conspicuous by their absence. We have sketchily rectified that in [Section 10](#), which is an epilogue by the senior author. As we explain there, it is implicit or explicit in the literature, for example in [\[42, 43, 70, 64\]](#), that chains of Quillen equivalences connect all of the input categories of our machines and that the machines themselves construct connective Quillen equivalences, meaning that they are Quillen adjunctions that induce equivalences of homotopy categories between the full subcategories of grouplike objects on the input side and the full subcategory of connective G -spectra on the output side. The references just cited proceed by analogy with the classical nonequivariant treatment of Bousfield and Friedlander [\[9\]](#). As there, the model category theory of the cited references makes no mention of the group completion property. That property is the main point of the theory and its applications, but it is irrelevant model categorically.

Even aside from the missing group completion property, the model theoretic discussion of the Segal and operadic machines will expose some limitations of thinking model categorically in the present context; see [Remarks 10.15](#) and [10.17](#). As we will explain, it is folklore that, with appropriate mixed model structures, the bar

constructions that we use to approximate the input data of both machines can be interpreted as highly structured model categorical cofibrant approximations. That structure is central to our work, but the model theoretic interpretation is not.

Philosophically, in this paper we are interested in understanding and preserving as much point-set level structure as possible, which we find illuminating and useful. While of course more abstract frameworks have their advantages, such point-set level precision is precisely what they are designed to avoid.

Omissions. We do not consider categorical input in this paper, but that feeds in effortlessly. It is shown in [25] that permutative G -categories give natural input to our operadic machine, and we showed how to strictify symmetric monoidal G -categories to equivalent permutative G -categories in [31].

More deeply, we do not considerable *multiplicative* infinite loop space theory here. That is needed to give ring and module G -spectra, rather than just the additive G -spectra we obtain here. Starting from G -space level input, we solve that problem in [28], again with Guillou. There we develop a variant of the Segal machine that starts from and is equivalent to the version given here, but that gives a symmetric monoidal functor from the category of \mathcal{F} - G -spaces to the category of G -spectra. Therefore any multiplicative structure present on the level of \mathcal{F} - G -spaces gives rise to corresponding structure on the level of G -spectra.

However, that multiplicative machine has a serious defect: it does not accept categorical input crucial to the applications to equivariant algebraic K -theory that we have in mind. For that, it seems essential to work 2-categorically with lax functors, or at least pseudofunctors, rather than just with categories and functors, That requires quite different categorical underpinnings than are discussed in this paper or in [25, 28]. It is treated in [26]. A 2-monadic alternative treatment will be given in [44].

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1. PRELIMINARIES

1.1. Preliminaries about G -spaces and Hopf G -spaces. We repeat that we assume that G is finite until [Section 9](#). We let \mathcal{U} be the category of compactly generated weak Hausdorff spaces, let \mathcal{U}_* be the category of based spaces in \mathcal{U} , and let \mathcal{T} be its subcategory of nondegenerately based spaces. We let $G\mathcal{U}$, $G\mathcal{U}_*$ and $G\mathcal{T}$ be the categories of G -spaces, based G -spaces, and nondegenerately based G -spaces, with left action by G ; basepoints of G -spaces are fixed by G . Maps in these categories are G -maps. While $G\mathcal{U}_*$ is convenient for formal arguments, $G\mathcal{T}$ is required when questions of homotopy type are at issue, and we shall focus on that category. In our treatment of the Segal machine, we will also need to consider the full subcategory $G\mathcal{W} \subset G\mathcal{T}$ of based finite G -CW complexes.

Properties of G -spaces are very often defined by passage to fixed point spaces. For example, a G -space X is said to be G -connected if X^H is (path) connected for all $H \subset G$.

Definition 1.1. Let $f: K \rightarrow L$ be a map of G -spaces. We say that f is a *weak G -equivalence* if $f^H: K^H \rightarrow L^H$ is a weak equivalence for all $H \subset G$. A family of subgroups of G is a set of subgroups closed under subconjugacy. For a family \mathbb{F} of subgroups of G , we say that f is a *weak \mathbb{F} -equivalence* if f is a weak H -equivalence for all $H \in \mathbb{F}$. We often omit the word weak, taking it to be understood throughout.

The following families are central to equivariant bundle theory and to the analysis of equivariant infinite loop space machines. They will be used ubiquitously. Let Σ_n denote the n th symmetric group.

Definition 1.2. For a subgroup H of G and a homomorphism $\alpha: H \rightarrow \Sigma_n$, let Λ_α be the subgroup $\{(h, \alpha(h)) \mid h \in H\}$ of $G \times \Sigma_n$; thus Λ_α is the graph of α . Note that projection to the first coordinate gives an isomorphism $\pi: \Lambda_\alpha \rightarrow H$. All subgroups Λ of $G \times \Sigma_n$ such that $\Lambda \cap \Sigma_n = \{e\}$ are of this form. Let \mathbb{F}_n denote the family of all such subgroups. Taking α to be trivial, we see that $H \in \mathbb{F}_n$ for all n and all $H \subset G$.

Definition 1.3. Let Y be a $(G \times \Sigma_n)$ -space and $\alpha: G \rightarrow \Sigma_n$ be a homomorphism. We define Y^α to be the G -space with underlying space Y and with a new G -action \cdot_α given by $g \cdot_\alpha y = (g, \alpha(g)) \cdot y$. Thus Y^α is Y with G -action twisted by α . Said another way Y^α is just notation for Y with its action by the subgroup Λ_α of $G \times \Sigma_n$, pulled back along the isomorphism $\pi^{-1}: G \rightarrow \Lambda_\alpha$. Thus, on passage to fixed point spaces,

$$(Y^\alpha)^G \cong Y^{\Lambda_\alpha}.$$

Of course, we can replace G by $H \subset G$ in [Definition 1.3](#). We use this definition to fix conventions about finite G -sets.

Convention 1.4. Let \mathbf{n} denote the based set $\{0, 1, \dots, n\}$ with basepoint 0. For a group G and a homomorphism $\alpha: G \rightarrow \Sigma_n$, note that \mathbf{n}^α is the based G -set specified by letting G act on \mathbf{n} by $g \cdot i = \alpha(g)(i)$ for $1 \leq i \leq n$. Conversely, a based G -action on \mathbf{n} determines a G -homomorphism α by the same formula. Every based finite G -set with n non-basepoint elements is isomorphic to one of the form \mathbf{n}^α for some α . We understand based finite G -sets to be of this form throughout.

We highlight the following special case of [Definition 1.3](#).

Example 1.5. Let X be a based G -space and consider the based $(G \times \Sigma_n)$ -space X^n . Then we have the identification $(X^n)^\alpha = X^{\mathbf{n}^\alpha}$, where $X^{\mathbf{n}^\alpha}$ denotes the set of based maps $\mathbf{n}^\alpha \rightarrow X$ with G acting by conjugation. Explicitly, this is the cartesian power X^n with twisted action by G given by

$$g(x_1, \dots, x_n) = (gx_{\alpha(g^{-1})(1)}, \dots, gx_{\alpha(g^{-1})(n)}).$$

We also need some preliminaries about H -spaces, which we call Hopf spaces to avoid confusion with subgroups of G . Recall that a Hopf space is a based space X with a product such that the basepoint is a two-sided unit up to homotopy. For simplicity, we assume once and for all that our Hopf spaces are homotopy associative and homotopy commutative⁴, since that holds in our examples. We say that a Hopf space is grouplike if, in addition, $\pi_0(X)$ is a group, necessarily abelian.

Definition 1.6. A Hopf map $f: X \rightarrow Y$ is a *group completion* if Y is grouplike, $f_*: \pi_0(X) \rightarrow \pi_0(Y)$ is the Grothendieck group of the commutative monoid $\pi_0(X)$, and for every field of coefficients, $f_*: H_*(X) \rightarrow H_*(Y)$ is the algebraic localization obtained by inverting the elements of the submonoid $\pi_0(X)$ of $H_*(X)$.⁵

Remark 1.7. Universal properties of group completions are studied in [11, §1]. The discussion there implies that if Y and Y' are group completions of X , then Y and Y' are weakly homotopy equivalent.

Definition 1.8. A Hopf G -space is a based G -space X with a product G -map such that its basepoint e is a two-sided unit element, in the sense that left or right multiplication by e is a weak G -equivalence $X \rightarrow X$. Then each X^H is a Hopf space, and we assume as before that each X^H is homotopy associative and commutative. A Hopf G -space X is grouplike if each X^H is grouplike. A Hopf G -map $f: X \rightarrow Y$ is a group completion if Y is grouplike and the fixed point maps f^H are all nonequivariant group completions. Clearly a group completion of a G -connected Hopf G -space is a weak G -equivalence.

1.2. Preliminaries about G -cofibrations and simplicial G -spaces. Since G -cofibrations play an important role in our work, we insert some standard remarks about them.⁶

Remark 1.9. A map is a G -cofibration if it satisfies the G -homotopy extension property and a basepoint $* \in X$ is nondegenerate if the inclusion $* \rightarrow X$ is a G -cofibration. Since we are working in $G\mathcal{U}$, a G -cofibration is an inclusion with closed image [51, Problem 5.1]. By [7, Proposition A.2.1] (or [51, p. 43]), if $i: A \rightarrow B$ is a closed inclusion of G -spaces, then i is a G -cofibration if and only if (B, A) is a G -NDR pair. Using this criterion, we see that i is then also an H -cofibration for any subgroup H of G and that passage to orbits or to fixed points over H gives a cofibration. Moreover, just as nonequivariantly, a pushout of a map of G -spaces along a G -cofibration is a G -cofibration.

⁴It would suffice to assume that left and right translation by any element are homotopic.

⁵Segal [72, §4] describes the notion of group completion a bit differently, in a form less amenable to equivariant generalization, and he makes several reasonable restrictive hypotheses in his proof of the group completion property. In particular, he assumes that X is a topological monoid and that $\pi_0(X)$ contains a cofinal free abelian monoid.

⁶They are part of the h -model structure on $G\mathcal{U}$, as in [55] nonequivariantly.

Definition 1.10. Let X_* be a simplicial G -space with G -space of n -simplices X_n . The n th latching space of X is given by

$$L_n X = \bigcup_{i=0}^{n-1} s_i(X_{n-1}).$$

It is a G -space, and the inclusion $L_n X \rightarrow X_n$ is a G -map. We say that X_* is Reedy cofibrant if this map is a G -cofibration for each n .

With different nomenclature, this concept was studied nonequivariantly in the early 1970's (e.g. [45, §11], [46, Appendix], [72, Appendix A]). We will use the following standard results.

Lemma 1.11. *A simplicial G -space X_* is Reedy cofibrant if all degeneracy operators s_i are G -cofibrations.*

Proof. The nonequivariant statement is proven by an inductive application of Lillig's union theorem stating that the union of cofibrations is a cofibration [40] (or [45, Lemma A.6]). The proof can be found in [72, proof of A.5] or [39, proof of 2.4.(b)]. The equivariant proof is the same, using the equivariant version of Lillig's theorem, which is a particular case of [8, Theorem A.2.7]. \square

The converse to Lemma 1.11 is proved in [65, Proposition 4.11], but we shall not use it.

Theorem 1.12. *Let $f_*: X_* \rightarrow Y_*$ be a map of Reedy cofibrant simplicial G -spaces such that each f_n is a weak G -equivalence. Then the realization $|f_*|: |X_*| \rightarrow |Y_*|$ is a weak G -equivalence.*

Proof. Nonequivariantly this is in [46, Theorem A.4]. Since passage to fixed points is a finite limit, it commutes with geometric realization. Therefore the equivariant version follows by application of the nonequivariant version to fixed point spaces. \square

The following companion result is well-known, but since we could not find a proof in the published literature, we provide one in Section 12.⁷

Theorem 1.13. *Let $f_*: X_* \rightarrow Y_*$ be a map of Reedy cofibrant simplicial G -spaces such that each f_n is a G -cofibration. Then the realization $|f_*|: |X_*| \rightarrow |Y_*|$ is a G -cofibration.*

Remark 1.14. Without exception, every simplicial G -space used in this paper is Reedy cofibrant. In each case, we can check from the definitions and the fact that we are working with nondegenerately based G -spaces that all s_i are G -cofibrations. For the examples appearing in the Segal machine, the verifications are straightforward. For the examples appearing in the operadic machine, the verifications follow those in [45, Proposition A.10] and elaborations of the arguments there.

⁷We were inspired by [65], an unpublished masters thesis, which gives a detailed exposition of simplicial spaces. However, its statement of Theorem 1.13 is missing a necessary hypothesis.

1.3. Categorical preliminaries and basepoints. Some familiarity with enriched category theory, especially in equivariant contexts, may be helpful. A more thorough treatment of the double enrichment present here is given in [27]. We first recall some general definitions that start with a closed symmetric monoidal category \mathcal{V} with unit object U and product denoted by \otimes . Closed means that we have internal function objects $\underline{\mathcal{V}}(V, W)$ in \mathcal{V} giving an adjunction

$$\underline{\mathcal{V}}(X \otimes Y, Z) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z))$$

in \mathcal{V} . We write $\underline{\mathcal{V}}$ instead of \mathcal{V} when we wish to emphasize the closed structure.

A \mathcal{V} -category \mathcal{E} is a category enriched in \mathcal{V} . This means that for each pair (m, n) of objects of \mathcal{E} there is an object $\mathcal{E}(m, n)$ of \mathcal{V} and there are unit and composition maps $I: U \rightarrow \mathcal{E}(m, m)$ and $C: \mathcal{E}(n, p) \otimes \mathcal{E}(m, n) \rightarrow \mathcal{E}(m, p)$ satisfying the identity and associativity axioms. A \mathcal{V} -functor $F: \mathcal{E} \rightarrow \mathcal{E}'$ between \mathcal{V} -categories is a functor enriched in \mathcal{V} . This means that for each pair (m, n) , there is a map

$$F: \mathcal{E}(m, n) \rightarrow \mathcal{E}'(F(m), F(n))$$

in \mathcal{V} , and these maps are compatible with the unit and composition of \mathcal{E} and \mathcal{E}' . A \mathcal{V} -transformation $\eta: F \rightarrow F'$ is given by maps $\eta_m: U \rightarrow \mathcal{E}'(F(m), F'(m))$ in \mathcal{V} such that the evident naturality diagram commutes in \mathcal{V} .

$$\begin{array}{ccc} \mathcal{E}(m, n) & \xrightarrow{F} & \mathcal{E}'(F(m), F(n)) \\ \downarrow F' & & \downarrow (\eta_n)_* \\ \mathcal{E}'(F'(m), F'(n)) & \xrightarrow{(\eta_m)_*} & \mathcal{E}'(F(m), F'(n)) \end{array}$$

As is standard, we use the same name for morphism spaces (or G -spaces) as for the underlying morphism sets (or G -sets). In view of the double enrichment, in spaces and in G -spaces, that is present equivariantly (and is discussed in general in [27]), we carefully establish notations for the ambient categories in which we work.

Notations 1.15. Nonequivariantly, $\mathcal{U}(X, Y)$ denotes the space of maps $X \rightarrow Y$. Equivariantly, for G -spaces X and Y , $G\mathcal{U}(X, Y)$ denotes the space of G -maps $X \rightarrow Y$ of the category $G\mathcal{U}$. In conformity with the categorical notation $\underline{\mathcal{V}}$, we let $\underline{G}\mathcal{U}(X, Y)$ denote the G -space of maps $X \rightarrow Y$, with G acting by conjugation: for a map $f: X \rightarrow Y$, $(gf)(x) = g \cdot f(g^{-1} \cdot x)$. For based spaces or based G -spaces, we have the analogous based spaces $\mathcal{U}_*(X, Y)$ and $G\mathcal{U}_*(X, Y)$ and the analogous based G -space $\underline{G}\mathcal{U}_*(X, Y)$; here the basepoints are the trivial based maps.

We are especially interested in the categories \mathcal{T} and $G\mathcal{T}$, but they are not closed since their hom spaces need not be nondegenerately based in general. Therefore, for nondegenerately based spaces X and Y , we agree to write $\mathcal{T}(X, Y) = \mathcal{U}_*(X, Y)$ for the based space of maps $X \rightarrow Y$. That expresses the enrichment of \mathcal{T} in \mathcal{U}_* , but not necessarily in \mathcal{T} . For nondegenerately based G -spaces X and Y , we write $G\mathcal{T}(X, Y) = G\mathcal{U}_*(X, Y)$ for the based space of based G -maps $X \rightarrow Y$, expressing the enrichment of $G\mathcal{T}$ in $G\mathcal{U}_*$. We write $\underline{\mathcal{T}}_G(X, Y) = \underline{G}\mathcal{U}_*(X, Y)$ for the G -space of based maps $X \rightarrow Y$, with G -acting by conjugation, expressing the enrichment of $G\mathcal{T}$ in $G\mathcal{U}_*$.⁸ The notation is meant to emphasize the implied assumption that the object spaces X and Y are nondegenerately based. Thus $\underline{\mathcal{T}}_G(X, Y)$ is a based

⁸We avoid the notation $\underline{G}\mathcal{T}$ since we reserve the underline notation for closed categories.

G -space with fixed point space $G\mathcal{T}(X, Y)$. Viewing \mathcal{T}_G as a G -category such that G acts trivially on objects, we can view $G\mathcal{T}$ as the fixed point category of \mathcal{T}_G .

We have the full subcategory $\mathcal{W}_G \subset \mathcal{T}_G$ of based finite G -CW complexes. It is enriched in $\underline{G\mathcal{U}}_*$, and we can view $G\mathcal{W}$ as the fixed point category of \mathcal{W}_G .

Of course, the category \mathcal{U}_* has two symmetric monoidal products, \wedge with unit S^0 and \times with unit $*$. When enriching in \mathcal{U}_* , we must use \wedge since we must use the closed structure given by the spaces of based maps. However, we occasionally use the forgetful functors $\mathcal{U}_* \rightarrow \mathcal{U}$ and $G\mathcal{U}_* \rightarrow G\mathcal{U}$ to forget basepoints in our enrichments, and implicitly we are then thinking about \times .

Remark 1.16. For a $G\mathcal{U}$ -category \mathcal{E} and a $G\mathcal{U}_*$ -category \mathcal{E}' , we can add disjoint basepoints to the hom objects of \mathcal{E} to form a $G\mathcal{U}_*$ -category \mathcal{E}_+ , and we can forget basepoints to regard \mathcal{E}' as a $G\mathcal{U}$ -category $\mathbb{U}\mathcal{E}'$. Via the adjunction between $(-)_+$ and \mathbb{U} , $G\mathcal{U}$ -functors $\mathcal{E} \rightarrow \mathbb{U}\mathcal{E}'$ can be identified with $G\mathcal{U}_*$ -functors $\mathcal{E}_+ \rightarrow \mathcal{E}'$.

In all variants and generalizations of the Segal machine, we start with a category \mathcal{E} a priori enriched over $G\mathcal{U}$ (although sometimes with G acting trivially on morphisms in \mathcal{E}) such that \mathcal{E} has a zero object 0 , so that there are unique maps $0 \rightarrow n$ and $n \rightarrow 0$ for every object $n \in \mathcal{E}$. Then assigning the map $m \rightarrow 0 \rightarrow n$ as the basepoint of $\mathcal{E}(m, n)$ makes \mathcal{E} into a category enriched over $G\mathcal{U}_*$, as composition in \mathcal{E} descends to a map out of the smash product. We are concerned with $G\mathcal{U}_*$ -functors defined on \mathcal{E} . The following trivial observation has been overlooked since the start of this subject. It depends only on basepoint information, not on the topology or the G -action.

Lemma 1.17. *Let \mathcal{E} be a $G\mathcal{U}$ -category with a zero object, considered as a category enriched over $G\mathcal{U}_*$ as above. Then,*

- (i) *for any $G\mathcal{U}_*$ -functor $X: \mathcal{E} \rightarrow G\mathcal{U}_*$, $X(0) = *$;*
- (ii) *conversely, any $G\mathcal{U}$ -functor $Y: \mathbb{U}\mathcal{E} \rightarrow \underline{G\mathcal{U}}$ such that $Y(0) = *$ can be considered as a $G\mathcal{U}_*$ -functor $\mathcal{E} \rightarrow \underline{G\mathcal{U}}_*$.*

Proof. The unique map $0 \rightarrow 0$ in $\mathcal{E}(0, 0)$ is both the basepoint and the identity; $X: \mathcal{E}(0, 0) \rightarrow \mathcal{U}_*(X(0), X(0))$ must send it to both the trivial map $X(0) \rightarrow X(0)$ that sends all points to the basepoint and to the identity map. This can only happen if $X(0)$ is a point, proving (i).

For (ii), if $Y: \mathbb{U}\mathcal{E} \rightarrow \underline{G\mathcal{U}}$ is such that $Y(0) = *$, the map $Y(0) \rightarrow Y(n)$ induced by $0 \rightarrow n$ gives each $Y(n)$ a basepoint, and these are preserved by the maps induced by maps in \mathcal{E} . Moreover, Y gives a $G\mathcal{U}_*$ -functor $\mathcal{E} \rightarrow \underline{G\mathcal{U}}_*$ since Y then sends the zero map $m \rightarrow 0 \rightarrow n$ to the trivial map

$$Y(m) \rightarrow * = X(0) \rightarrow X(n). \quad \square$$

Remark 1.18. Let \mathcal{E} be a $G\mathcal{U}_*$ -category with a zero object. A $G\mathcal{U}$ -functor $Y: \mathbb{U}\mathcal{E} \rightarrow \underline{G\mathcal{U}}$ is said to be *reduced* if $Y(0)$ is a point. Thus the previous lemma can be interpreted as saying that there is a one-to-one correspondence between reduced $G\mathcal{U}$ -functors $\mathbb{U}\mathcal{E} \rightarrow \underline{G\mathcal{U}}$ and $G\mathcal{U}_*$ -functors $\mathcal{E} \rightarrow \underline{G\mathcal{U}}_*$. If moreover the maps $* = Y(0) \rightarrow Y(n)$ are G -cofibrations, Y will restrict to give a $G\mathcal{U}_*$ -functor $\mathcal{E} \rightarrow \mathcal{T}_G$.

Let \mathcal{E} be a $G\mathcal{U}_*$ -category with a zero object 0 , and let X and Y be covariant and contravariant $G\mathcal{U}_*$ -functors $\mathcal{E} \rightarrow \underline{G\mathcal{U}}_*$, respectively. Then the tensor product

of functors $Y \otimes_{\mathcal{E}} X$ is defined as the coequalizer of the diagram

$$\bigvee_{m,n} Y_n \wedge \mathcal{E}(m,n) \wedge X_m \rightrightarrows \bigvee_n Y_n \wedge X_n,$$

where the arrows are given by the action of \mathcal{E} on X and on Y , respectively.

We obtain $G\mathcal{U}$ -functors $\mathbb{U}X, \mathbb{U}Y: \mathbb{U}\mathcal{E} \rightarrow \underline{G\mathcal{U}}$ by forgetting basepoints. The tensor product of functors $\mathbb{U}Y \otimes_{\mathbb{U}\mathcal{E}} \mathbb{U}X$ is the coequalizer of the diagram

$$\prod_{m,n} Y_n \times \mathcal{E}(m,n) \times X_m \rightrightarrows \prod_n Y_n \times X_n.$$

The following result shows that the difference between these two constructions is only apparent. Later on, we shall sometimes use wedges and smash products and sometimes instead use disjoint unions and products, whichever seems convenient.

Lemma 1.19. *With \mathcal{E} , X , and Y as above, there is a natural isomorphism*

$$\mathbb{U}(Y \otimes_{\mathcal{E}} X) \cong \mathbb{U}Y \otimes_{\mathbb{U}\mathcal{E}} \mathbb{U}X.$$

Proof. We show that the quotient of $\prod_n Y_n \times X_n$ given by the coequalizer encodes the required basepoint identifications. Let $(y, *n) \in Y_n \times X_n$. As noted above, the basepoint in X_n is given by $*n = (0_{0,n})_*(*)_0$, where $*_0$ is the unique point in X_0 and $0_{0,n}$ is the unique map $0 \rightarrow n$ in \mathcal{E} . Then

$$(y, *n) = (y, (0_{0,n})_*(*)_0) \sim (0_{0,n}^*(y), *_0) = (*_0, *_0),$$

the last equation following from the fact that Y_0 is a singleton. A similar argument shows that $(*n, x) \sim (*_0, *_0)$. \square

1.4. Spectrum level preliminaries.

Definition 1.20. A classical G -spectrum,⁹ which we prefer to call a G -prespectrum, is a sequence of based G -spaces $\{T_n\}_{n \geq 0}$ and based G -maps $\sigma_n: \Sigma T_n \rightarrow T_{n+1}$. It is a classical Ω - G -spectrum if the adjoint maps $\tilde{\sigma}_n: T_n \rightarrow \Omega T_{n+1}$ are weak G -equivalences. It is a positive Ω - G -spectrum if the $\tilde{\sigma}_n$ are weak G -equivalences for $n \geq 1$. We let $G\mathcal{P}$ denote the category of G -prespectra. We call zeroth spaces T_0 of classical Ω - G -spectra classical infinite loop G -spaces.

In this paper, our preferred category of (genuine) G -spectra is the category $G\mathcal{S}$ of orthogonal G -spectra. Orthogonal G -spectra and their model structures are studied in [42], to which we refer the reader for details and discussion of the following definition. We use Remark 1.16.

Definition 1.21. Let \mathcal{I}_G be the $G\mathcal{U}$ -category of finite dimensional real G -inner product spaces and linear isometric isomorphisms, with G acting on morphism spaces by conjugation. Note that \mathcal{I}_G is symmetric monoidal under \oplus . An \mathcal{I}_G - G -space is a $G\mathcal{U}$ -functor $\mathcal{I}_G \rightarrow \underline{G\mathcal{U}_*}$ ¹⁰ or, equivalently by Remark 1.16, a $G\mathcal{U}_*$ -functor $\mathcal{I}_{G+} \rightarrow \underline{G\mathcal{U}_*}$. The sphere \mathcal{I}_G - G -space S is given by $S(V) = S^V$. The external smash product

$$X \bar{\wedge} Y: \mathcal{I}_G \times \mathcal{I}_G \rightarrow \mathcal{I}_G$$

⁹We remind the reader that “classical” is often called “naive”.

¹⁰To be consistent with the notation of Remark 1.16 we should write $\mathbb{U}(\underline{G\mathcal{U}_*})$ for the target here, but we have not done so to avoid cluttering the notation.

of \mathcal{I}_G - G -spaces X and Y is the $G\mathcal{U}$ -functor given by

$$(X \bar{\wedge} Y)(V, W) = X(V) \wedge Y(W).$$

A (genuine orthogonal) G -spectrum is an \mathcal{I}_G - G -space $E: \mathcal{I}_G \rightarrow \mathcal{T}_G$ together with a $G\mathcal{U}$ -transformation $E \bar{\wedge} S \rightarrow E \circ \oplus$ between $G\mathcal{U}$ -functors $\mathcal{I}_G \times \mathcal{I}_G \rightarrow \mathcal{T}_G$. Thus we have G -spaces $E(V)$, morphism G -maps

$$\mathcal{I}_G(V, V') \rightarrow \mathcal{U}_G(E(V), E(V'))$$

and structure G -maps

$$\sigma: E(V) \wedge S^W \rightarrow E(V \oplus W).$$

natural in V and W . Note in particular that $\mathcal{I}_G(V, V)$ is the orthogonal group $O(V)$, with G acting by conjugation, so that $E(V)$ is both a G -space and an $O(V)$ -space and σ is a map of both G -spaces and $O(V) \times O(W)$ -spaces. A G -spectrum E is an Ω - G -spectrum if the adjoint maps

$$\tilde{\sigma}: E(V) \rightarrow \Omega^W E(V \oplus W)$$

are weak G -equivalences. It is a positive Ω - G -spectrum if these maps are weak G -equivalences when $V^G \neq 0$. We let $G\mathcal{S}$ denote the category of G -spectra. We call zeroth spaces $E(0)$ of Ω - G -spectra genuine infinite loop G -spaces, or simply infinite loop G -spaces.

Definition 1.22. The forgetful functor $i^*: G\mathcal{S} \rightarrow G\mathcal{P}$ sends a G -spectrum X to the (classical) G -prespectrum with n th space $X_n = X(\mathbb{R}^n)$.

Remark 1.23. If $V^G \neq 0$, we can write $V \cong \mathbb{R} \oplus W$ and thus $S^V \cong S^1 \wedge S^W$ and $\Omega^V \cong \Omega \Omega^W$. Then $\tilde{\sigma}: X_0 \rightarrow \Omega^V X(V)$ factors as the composite

$$X_0 \xrightarrow{\tilde{\sigma}} \Omega X_1 \xrightarrow{\Omega \tilde{\sigma}} \Omega \Omega^W X(\mathbb{R} \oplus W) \cong \Omega^V X(V).$$

If X is a positive Ω - G -spectrum, then the second arrow is a weak G -equivalence. Therefore, if $X_0 \rightarrow \Omega X_1$ is a group completion, then so is $X_0 \rightarrow \Omega^V X(V)$ for all V such that $V^G \neq 0$.

Remark 1.24. We take for granted the basic homotopy theory of orthogonal G -spectra. In particular, the homotopy groups π_q^H of G -spectra are defined in [42, Definition 3.2], and a map of G -spectra is a stable equivalence if it induces an isomorphism of homotopy groups.

Geometric realization of simplicial objects can be carried out in any bicomplete category that is tensored over spaces, using the usual coend definition, and we shall make use of the geometric realization of simplicial orthogonal G -spectra. See [20, Definition X.1.1] for a discussion in the case of EKMM spectra. In the case of orthogonal G -spectra, since colimits and tensors with spaces are constructed levelwise, we see that for a simplicial orthogonal G -spectrum T_* , the V -th level of the geometric realization is given by

$$|T_*|(V) = |T_*(V)|.$$

We shall need the spectrum level analogue of [Theorem 1.12](#). That requires a notion of Reedy cofibrancy for simplicial orthogonal G -spectra, which we now explain. We can define latching objects $L_n T$ for simplicial objects T in any cocomplete category as the evident colimit (see, for example, [34, Definition 15.2.5] or [67]);

Definition 1.10 is a specialization. Since colimits of orthogonal spectra are computed levelwise, we have that $(L_n T)(V) = L_n(T(V))$ for a simplicial orthogonal G -spectrum T .

Definition 1.25. A map of orthogonal G -spectra $A \rightarrow X$ is an *h -cofibration* if it satisfies the homotopy extension property (see [33, §A.5], [42, §I.4], [43, §5]). A simplicial orthogonal G -spectrum is *Reedy h -cofibrant* if the latching map $L_n T \rightarrow T_n$ is an h -cofibration for each n .

Proposition 1.26. *Let $f_*: T_* \rightarrow T'_*$ be a map of Reedy h -cofibrant simplicial orthogonal G -spectra that is a stable equivalence at each simplicial level. Then the map of orthogonal G -spectra $|f_*|$ obtained by geometric realization is a stable equivalence.*

Proof. The proof is the same as for the space level analogue, using the construction of the filtration on geometric realization via pushouts (see [65, Theorem 4.15] and [20, Theorem X.2.4]). The key facts we need about h -cofibrations of orthogonal G -spectra are that they are stable under pushouts [33, Proposition A.62], that they satisfy the analogue of [20, Lemma X.2.3] so that the relevant realizations are constructed by pushouts along h -cofibrations, and that the gluing lemma for h -cofibrations and stable equivalences holds (see [33, Corollary B.21], [42, Theorem I.4.10 (iv)]), and that the filtered colimit along h -cofibrations of a sequence of stable equivalences is a stable equivalence [33, Proposition B.17]. \square

We note that the above result holds generally in any good model category tensored over spaces. A recent treatment is offered in [67, see Corollary 10.6.]. We have not quoted that result because there is no published proof that there is a (mixed) model structure on the category of orthogonal G -spectra in which the cofibrations are the h -cofibrations. However, the methods of [4] (and [3, §6.4]) can be applied to construct one.

2. THE SIMPLICIAL AND CONCEPTUAL VERSIONS OF THE SEGAL MACHINE

There are several variants of the Segal infinite loop space machine, as originally developed by Segal [72] and Woolfson [82]. Later sources include Bousfield and Friedlander [9], working simplicially, and, much later, Mandell, May, Schwede, and Shipley [43]. Equivariant versions appear in Shimada and Shimakawa [74, 75, 76] and, later, Blumberg [6].

We here give a simplicial variant¹¹ and two equivalent conceptual variants, one starting from finite sets and the other starting from finite G -sets. We defer consideration of our preferred homotopical variant to the next section.

The simplicial variant is the equivariant version of Segal's original definition [72]. As far as we know, his paper is the only source in the literature that actually proves the crucial group completion property, and his proof makes essential use of his original simplicial definition.¹² This version does not directly generalize to give genuine G -spectra, and it does not appear in the equivariant literature. Therefore, even at this late date, there is no published account of the equivariant Segal machine that proves the group completion property. Just as nonequivariantly, this property is central to the applications, especially to algebraic K -theory.

¹¹We are referring to simplicial spaces, not simplicial sets, here.

¹²His proof imposes some unnecessary restrictive hypotheses that generally hold in practice.

In fact, we do not know a direct proof of the group completion property starting from the conceptual or homotopical variants treated in [9, 43, 74, 82] and, equivariantly, [6, 75]. Rather, we derive it for the conceptual variants from their equivalence with the simplicial variant. To give the group completion property equivariantly, to fill in other gaps in the published proofs, and to prepare for a comparison with the operadic machine, we give a fully detailed exposition of the Segal machine in all of its forms. This may also be helpful to the modern reader since, even nonequivariantly, the original sources make for hard reading and are sketchy.

2.1. Definitions: the input of the Segal machine. We begin with some standard nonequivariant definitions and observations.

Definition 2.1. Let \mathcal{F} be the opposite of Segal's category Γ .¹³ It is the category of finite based sets $\mathbf{n} = \{0, 1, \dots, n\}$ with 0 as basepoint. The morphisms are the based maps, and the unique morphism that factors through $\mathbf{0}$ is a basepoint for $\mathcal{F}(\mathbf{m}, \mathbf{n})$. Let $\Pi \subset \mathcal{F}$ be the subcategory with the same objects and those morphisms $\phi: \mathbf{m} \rightarrow \mathbf{n}$ such that $\phi^{-1}(j)$ has at most one element for $1 \leq j \leq n$; these are composites of projections, injections, and permutations. Let $\Sigma \subset \Pi$ be the subgroupoid with the same objects and the elements of the symmetric groups Σ_n , regarded as based isomorphisms $\mathbf{n} \rightarrow \mathbf{n}$, as morphisms.

The composition in \mathcal{F} factors through the smash product, and we view \mathcal{F} as a category enriched in $\mathcal{T} \subset \mathcal{U}_*$, with the discrete topology on the based hom sets. We first review the nonequivariant definitions, following [57] for technical details that are needed for homotopical control.

Notation 2.2. Given an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in Π , let Σ_ϕ denote the subgroup of Σ_n consisting of those permutations τ such that $\tau(\text{im } \phi) = \text{im } \phi$. Then Σ_ϕ acts on the set \mathbf{m} and ϕ is a Σ_ϕ -map.

Definition 2.3. A Π -space is a \mathcal{U}_* -functor $X: \mathcal{F} \rightarrow \mathcal{U}_*$, written $\mathbf{n} \mapsto X_n$, such that the map $\phi_*: X_m \rightarrow X_n$ is a Σ_ϕ -cofibration for every injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$. Taking $m = 0$, it follows that X takes values in \mathcal{T} . Similarly, an \mathcal{F} -space is a \mathcal{U}_* -functor whose restriction to Π is a Π -space. A map of Π -spaces or of \mathcal{F} -spaces is a \mathcal{U}_* -natural transformation. We denote the categories of Π -spaces and of \mathcal{F} -spaces by $\Pi[\mathcal{T}]$ and $\mathcal{F}[\mathcal{T}]$.

Remark 2.4. The definition implicitly requires care to check that constructions on \mathcal{F} -spaces that make obvious sense for functors $\mathcal{F} \rightarrow \mathcal{U}_*$ do in fact take values in \mathcal{F} -spaces. The equivariant situation will be similar.

Remark 2.5. We note that it is enough to impose that ϕ_* is a Σ_ϕ -cofibration only for ordered injections ϕ , where ϕ is ordered if $\phi(i) < \phi(i')$ whenever $i < i'$. Indeed, if ϕ is an arbitrary injection, it can be written as the composite $\psi \circ \rho$, where ψ is an ordered injection and $\rho \in \Sigma_m$. One can check that ϕ_* is a Σ_ϕ -cofibration if and only if ψ_* is a Σ_ψ -cofibration by noting that $\Sigma_\phi = \Sigma_\psi$ and, while the actions on X_m are not the same, ρ gives a homeomorphism that intermediates between the two.

Definition 2.6. For a Π -space X , let $\mathbb{L}X = X_1$. For a based space $Y \in \mathcal{T}$, let $\mathbb{R}Y$ denote the Π -space with n th space Y^n . The Π -space structure is given by basepoint inclusions, projections, and permutations.

¹³As in [43] and elsewhere, we use the notation \mathcal{F} to avoid confusion between Γ and $\Gamma^{op} = \mathcal{F}$.

Definition 2.7. The *Segal maps* $\delta_i: \mathbf{n} \rightarrow \mathbf{1}$ in Π send i to 1 and j to 0 for $j \neq i$. For a Π -space X , the *Segal map* $\delta: X_n \rightarrow X_1^n$ has coordinates induced by the δ_i . If $n = 0$, we interpret δ as the terminal map $X_0 \rightarrow *$.

The n th *multiplication map* $\varphi_n: \mathbf{n} \rightarrow \mathbf{1}$ in \mathcal{F} sends j to 1 for $1 \leq j \leq n$. It induces an “ n -fold multiplication” $X_n \rightarrow X_1$ on an \mathcal{F} -space X .

Lemma 2.8. *The pair (\mathbb{L}, \mathbb{R}) is an adjunction. On a based space $Y \in \mathcal{T}$, the counit $\varepsilon: \mathbb{L}\mathbb{R}Y \rightarrow Y$ is the identity map. On a Π -space X , the unit $\delta: X \rightarrow \mathbb{R}\mathbb{L}X$ is given levelwise by the Segal maps.*

Remark 2.9. It is traditional, starting in [72, Definition 1.2], to define a Π -space to be a *not necessarily enriched functor* $\Pi \rightarrow \mathcal{T}$, requiring X_0 to be contractible, and to say that X is *reduced* if X_0 is a point. That led to mistakes and confusion, as explained in [53]. The essential point is that, without enrichment, \mathbb{L} is not left adjoint to \mathbb{R} . As we observed in Lemma 1.17, our requirement that X be a \mathcal{U}_* -functor forces X_0 to be a point for trivial reasons.

Now return to the equivariant context.

Definition 2.10. A Π - G -space X is a \mathcal{U}_* -functor $X: \Pi \rightarrow G\mathcal{U}_*$ such that the map $\phi_*: X_m \rightarrow X_n$ is a $(G \times \Sigma_\phi)$ -cofibration for every injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in Π . It follows that X takes values in $G\mathcal{T}$. Similarly, an \mathcal{F} - G -space is a \mathcal{U}_* -functor $\mathcal{F} \rightarrow G\mathcal{U}_*$ whose restriction to Π is a Π - G -space. A map of Π - G -spaces or of \mathcal{F} - G -spaces is a \mathcal{U}_* -natural transformation. We denote the categories of Π - G -spaces and of \mathcal{F} - G -spaces by $\Pi[G\mathcal{T}]$ and $\mathcal{F}[G\mathcal{T}]$.

As in Remark 2.5, it is enough to check the condition on ordered injections.

Remark 2.11. Since G acts trivially on Π and \mathcal{F} , the notations $\Pi[\mathcal{T}_G]$ and $\mathcal{F}[\mathcal{T}_G]$ carry the same meaning: \mathcal{U}_* -functors $\Pi \rightarrow \mathcal{T}_G$ necessarily land in $G\mathcal{T}$.

While the cofibration condition is always assumed when using the terminology above, it will not be used in this section. The following equivariant generalization of [57, Proposition 1.6] implies that assuming it does not result in any loss of generality. The cited result shows how to “beard” functors $\mathcal{F} \rightarrow \mathcal{U}_*$ to obtain levelwise equivalent functors in $\mathcal{F}[\mathcal{T}]$, and we shall see that equivariance requires only minor changes. We defer the proof to Section 11, where we generalize the result from \mathcal{F} to G -categories of operators over \mathcal{F} , again following [57].

Proposition 2.12. *There is a functor $W: \text{Fun}(\mathcal{F}, G\mathcal{U}_*) \rightarrow \text{Fun}(\mathcal{F}, G\mathcal{U}_*)$ together with a natural transformation $\pi: W \rightarrow \text{Id}$ such that, for any functor $X: \mathcal{F} \rightarrow G\mathcal{U}_*$, WX is in $\mathcal{F}[G\mathcal{T}]$ and $\pi: WX \rightarrow X$ is a levelwise G -homotopy equivalence.*

Remark 2.13. The adjunction (\mathbb{L}, \mathbb{R}) of Definition 2.6 and Lemma 2.8 extends to the equivariant case to give an adjunction between $G\mathcal{T}$ and $\Pi[G\mathcal{T}]$. Indeed, if Y is a nondegenerately based G -space, the functor $\mathbb{R}Y$ that sends n to Y^n satisfies the cofibration condition since, for an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$, $\phi_*: Y^m \rightarrow Y^n$ just inserts the basepoint in the coordinates indexed by those j not in the image of ϕ .

Recall from Definition 1.2 that, for a homomorphism $\alpha: G \rightarrow \Sigma_n$, Λ_α is the subgroup $\{(g, \alpha(g)) \mid g \in G\}$ of $G \times \Sigma_n$. Observe that since $\Sigma \subset \Pi \subset \mathcal{F}$, X_n and X_1^n are $(G \times \Sigma_n)$ -spaces and $\delta: X_n \rightarrow X_1^n$ is a map of $(G \times \Sigma_n)$ -spaces.

Definition 2.14. Let X, Y be Π - G -spaces, and $f: X \rightarrow Y$ be a map of Π - G -spaces.

- (i) The Π -space X is \mathbb{F}_\bullet -special if $\delta: X_n \rightarrow X_1^n$ is an \mathbb{F}_n -equivalence for all $n \geq 0$.
- (ii) Less restrictively, X is special if each $\delta: X_n \rightarrow X_1^n$ is a weak G -equivalence.
- (iii) The map f is an \mathbb{F}_\bullet -level equivalence if each $f_n: X_n \rightarrow Y_n$ is an \mathbb{F}_n -equivalence.
- (iv) Less restrictively, f is a level G -equivalence if each $f_n: X_n \rightarrow Y_n$ is a weak G -equivalence.

An \mathcal{F} - G -space X is \mathbb{F}_\bullet -special or special if its underlying Π - G -space is so. A special \mathcal{F} - G -space X is *grouplike* or, synonymously, *very special* if $\pi_0(X_1^H)$ is a group (necessarily abelian) under the induced product for each $H \subset G$. A map $f: X \rightarrow Y$ of \mathcal{F} - G -spaces is an \mathbb{F}_\bullet -level equivalence or level equivalence if it is so as a map of Π - G -spaces.

We need several technical results about these notions, the first of which is the key to the following two.

Lemma 2.15. *Let X be a G -space, and let $\Lambda_\alpha = \{(h, \alpha(h)) \mid h \in H\} \subset G \times \Sigma_n$, where $H \subset G$ and $\alpha: H \rightarrow \Sigma_n$ is a homomorphism. Then there is a natural homeomorphism*

$$(X^n)^{\Lambda_\alpha} \cong \prod X^{K_i},$$

where the product is taken over the orbits of the H -set \mathbf{n}^α and the $K_i \subset H$ are the stabilizers of chosen elements in the corresponding orbit.

Proof. The Λ_α -action on X^n is given by

$$(h, \alpha(h))(x_1, \dots, x_n) = (hx_{\alpha(h^{-1})(1)}, \dots, hx_{\alpha(h^{-1})(n)}).$$

The partition of \mathbf{n}^α into H -orbits decomposes \mathbf{n} as the wedge of finite subsets, each with a transitive set of shuffled indices, so it is enough to consider each H -orbit separately. Thus we may as well assume that the H -action on \mathbf{n}^α is transitive. Note that this reduction depends only on α and is natural in X .

Let $K \subset H$ be the stabilizer of $1 \in \mathbf{n}$. We claim that projection onto the first coordinate induces the required natural homeomorphism $\pi: (X^n)^{\Lambda_\alpha} \rightarrow X^K$. Let $(x_1, \dots, x_n) \in X^n$ be a Λ_α -fixed point. Then x_1 is a K -fixed point of X since

$$kx_1 = kx_{\alpha(k^{-1})(1)} = x_1$$

for $k \in K$, the second equality holding because (x_1, \dots, x_n) is fixed by Λ_α .

To construct $\pi^{-1}: X^K \rightarrow (X^n)^{\Lambda_\alpha}$, for $1 \leq j \leq n$ choose $h_j \in H$ such that $\alpha(h_j)(1) = j$. This choice amounts to choosing a system of coset representatives for H/K , and the map $j \mapsto [h_j]$ gives a bijection of H -sets between \mathbf{n}^α and H/K . We claim that the map $X \rightarrow X^n$ that sends x to the n -tuple (h_1x, \dots, h_nx) restricts to the required inverse π^{-1} . This map is clearly continuous. We first show that if $x \in X^K$, then (h_1x, \dots, h_nx) is fixed by Λ_α . Let $h \in H$ and note that

$$\alpha(hh_{\alpha(h^{-1})(j)})(1) = \alpha(h)(\alpha(h_{\alpha(h^{-1})(j)})(1)) = \alpha(h)(\alpha(h^{-1})(j)) = j.$$

In view of our bijection between \mathbf{n}^α and H/K , there exists $k \in K$ such that $hh_{\alpha(h^{-1})(j)} = h_jk$. The j th coordinate of $(h, \alpha(h)) \cdot (h_1x, \dots, h_nx)$ is given by

$$hh_{\alpha(h^{-1})(j)}x = h_jkx = h_jx,$$

the second equality holding because $x \in X^K$. Thus (h_1x, \dots, h_nx) is fixed by Λ_α .

Since K is the stabilizer of 1, $h_1 \in K$. Thus the first coordinate of (h_1x, \dots, h_nx) is x itself, and $\pi \circ \pi^{-1} = \text{id}$. If (x_1, \dots, x_n) is fixed by Λ_α , then $x_j = h_jx_1$ for all j . By the definition of h_j , the j th coordinate of $(h_j, \alpha(h_j)) \cdot (x_1, \dots, x_n)$ is $h_jx_{\alpha(h_j^{-1})(j)} = h_jx_1$. Since (x_1, \dots, x_n) is a Λ_α -fixed point, this shows that $x_j = h_jx_1$, hence $\pi^{-1} \circ \pi = \text{id}$. \square

Lemma 2.16. *If $f: X \rightarrow Y$ is a weak equivalence of based G -spaces, then the induced map $\mathbb{R}f: \mathbb{R}X \rightarrow \mathbb{R}Y$ is an \mathbb{F}_\bullet -level equivalence of Π - G -spaces.*

Proof. This is immediate from Lemma 2.15. \square

Lemma 2.17. *Let $f: X \rightarrow Y$ be an \mathbb{F}_\bullet -level equivalence of Π - G -spaces. Then X is \mathbb{F}_\bullet -special if and only if Y is \mathbb{F}_\bullet -special. Similarly, if f is a level G -equivalence, then X is special if and only if Y is special.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \delta \downarrow & & \downarrow \delta \\ X_1^n & \xrightarrow{f_1^n} & Y_1^n \end{array}$$

For the first statement, the horizontal arrows are \mathbb{F}_n -equivalences by assumption and Lemma 2.16, so one of the vertical arrows is a \mathbb{F}_n -equivalence if and only if the other one is. The proof of the second statement is similar but simpler. \square

We defined \mathbb{F}_\bullet -special Π - G -spaces and \mathbb{F}_\bullet -level equivalences in terms of the families \mathbb{F}_n and thus in terms of homomorphisms $\alpha: H \rightarrow \Sigma_n$ for all subgroups H of G , but the following result says that we may restrict to $H = G$.

Lemma 2.18. *A Π - G -space X is \mathbb{F}_\bullet -special if the Segal maps $\delta: X_n \rightarrow X_1^n$ for $n \geq 0$ are weak Λ_α -equivalences for all subgroups Λ_α of $G \times \Sigma_n$ defined by homomorphisms $\alpha: G \rightarrow \Sigma_n$. Similarly, a map $f: X \rightarrow Y$ of Π - G -spaces is an \mathbb{F}_\bullet -level equivalence if the f_n for $n \geq 0$ are weak Λ_α -equivalences for all subgroups Λ_α of $G \times \Sigma_n$ defined by homomorphisms $\alpha: G \rightarrow \Sigma_n$.*

Proof. Given the hypotheses, we must prove that each δ and each f_n is a weak \mathbb{F}_n -equivalence. Thus consider a subgroup $\Lambda_\beta \subset G \times \Sigma_n$, $\beta: H \rightarrow \Sigma_n$. We will show that $\delta: X_n \rightarrow X_1^n$ is a weak Λ_β -equivalence by displaying it as a retract of a suitable weak equivalence. The homomorphism β gives an H -set $B = \mathbf{n}^\beta$. Embed B as a subset of the G -set $A = G_+ \wedge_H B$ and observe that, as an H -set, A splits as $B \vee C$, where $C = (A \setminus B)_+$. Let $p = |A \setminus B|$ and $q = n + p$. Use the given ordering of B and an ordering of C to identify A with some \mathbf{q}^α . Here $\mathbf{q} = \mathbf{n} \vee \mathbf{p}$ and α is a homomorphism $G \rightarrow \Sigma_q$ which when restricted to H is of the form $\beta \vee \gamma$. That is, $\mathbf{q}^{\alpha|_H} = \mathbf{n}^\beta \vee \mathbf{p}^{\gamma|_H}$. Let

$$\iota: \mathbf{n}^\beta \rightarrow \mathbf{q}^{\alpha|_H} \quad \text{and} \quad \pi: \mathbf{q}^{\alpha|_H} \rightarrow \mathbf{n}^\beta$$

be the inclusion that sends i to i for $0 \leq i \leq n$ and the projection that sends i to i for $0 \leq i \leq n$ and i to 0 for $i > n$. Then the following diagram displays a retraction.

Its bottom arrows are the evident inclusion and projection.

$$\begin{array}{ccccc} X_n & \xrightarrow{\iota_*} & X_q & \xrightarrow{\pi_*} & X_n \\ \delta \downarrow & & \downarrow \delta & & \downarrow \delta \\ X_1^n & \longrightarrow & X_1^q & \longrightarrow & X_1^n \end{array}$$

Since X is \mathbb{F}_\bullet -special, the middle vertical arrow δ is a weak Λ_α -equivalence and thus a weak $\Lambda_{\alpha|H}$ -equivalence. Therefore the left arrow δ is a weak Λ_β -equivalence.

Similarly, for f_* , we have a retract diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\iota_*} & X_q & \xrightarrow{\pi_*} & X_n \\ f_n \downarrow & & \downarrow f_q & & \downarrow f_n \\ Y_n & \xrightarrow{\iota_*} & Y_q & \xrightarrow{\pi_*} & Y_n \end{array}$$

in which f_q is a weak Λ_α -equivalence and therefore f_n is a weak Λ_β -equivalence. \square

2.2. The simplicial version of the Segal machine. Let Δ be the usual simplicial category. A simplicial object is a contravariant functor defined on Δ , and a cosimplicial object is a covariant functor. Regarding \mathcal{F} as a full subcategory of the category of based sets, we may regard the simplicial circle $S_s^1 = \Delta[1]/\partial\Delta[1]$ as a contravariant functor $F: \Delta^{op} \rightarrow \mathcal{F}$. By pullback along F , an \mathcal{F} - G -space X can be viewed as a simplicial G -space, and it has a geometric realization $|X| = |X \circ F|$; we use the standard realization, taking degeneracies into account. The evident G -map $X_1 \times I \rightarrow |X|$ factors through a natural G -map $\Sigma X_1 \rightarrow |X|$ with adjoint $\eta: X_1 \rightarrow \Omega|X|$. We shall sketch the proof of the following result in [Section 8.2](#). It is implicit in many early sources; we will follow [\[60, §15\]](#). Here we need our \mathcal{F} - G -spaces to be levelwise nondegenerately based since the argument involves realizations of Reedy cofibrant simplicial G -spaces, but we do not require the stronger cofibration condition of [Definition 2.10](#).

Proposition 2.19. *If X is a special \mathcal{F} - G -space, then the G -map $\eta: X_1 \rightarrow \Omega|X|$ is a group completion of Hopf G -spaces.*

From here, the Segal machine in its first avatar is constructed as follows [\[72, §1\]](#). Working equivariantly, we use a slight reformulation that is given in [\[57\]](#).¹⁴

Remark 2.20. We have the smash product $\wedge: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$. It sends (\mathbf{m}, \mathbf{n}) to $\mathbf{m}\mathbf{n}$ and is strictly associative and unital using lexicographic ordering. The unit is $\mathbf{1}$. We also have the wedge sum $\vee: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ which sends (\mathbf{m}, \mathbf{n}) to $\mathbf{m} + \mathbf{n}$. It is also strictly associative and unital, with unit $\mathbf{0}$, and \mathcal{F} is bipermutative under this sum and product.

Definition 2.21. Let X be a special \mathcal{F} - G -space. We have the functor

$$X \circ \wedge: \mathcal{F} \times \mathcal{F} \rightarrow G\mathcal{F}.$$

For each \mathbf{p} , let $X[\mathbf{p}]$ be the \mathcal{F} - G -space that sends \mathbf{q} to $X(\mathbf{p} \wedge \mathbf{q})$; thus $X[0] = *$ and $X[1] = X$. Following Segal, define the *classifying \mathcal{F} - G -space* $\mathbb{B}X$ to be the

¹⁴Perversely, [\[57\]](#) takes Δ to be the opposite of the category every other reference calls Δ .

\mathcal{F} - G -space whose p th G -space is the realization $|X[p]|$. Observe that $\mathbb{B}X$ is again special since factorizations of the Segal maps for X give commutative diagrams

$$(2.22) \quad \begin{array}{ccc} X[p](\mathbf{q}) & \xrightarrow{\delta} & X[p](\mathbf{1})^q \equiv X(\mathbf{p})^q \\ \parallel & & \downarrow (\delta)^q \\ X(\mathbf{pq}) & \xrightarrow{\delta} & X(\mathbf{1})^{pq}. \end{array}$$

Iterating, with $\mathbb{B}^0 X = X$, define $\mathbb{B}^{n+1} X = \mathbb{B}(\mathbb{B}^n X)$ for $n \geq 0$. The \mathcal{F} - G -spaces $\mathbb{B}^n X$ for $n \geq 1$ are special and, since $(\mathbb{B}^n X)_1$ is G -connected, they are also grouplike.

Definition 2.23. Let $\mathbb{S}_G^C X$ denote the resulting classical G -prespectrum with n th G -space $(\mathbb{S}_G^C X)_n = (\mathbb{B}^n X)_1$ for $n \geq 0$. Thus its 0th G -space is X_1 and, by [Proposition 2.19](#), its structure map $X_1 \rightarrow \Omega(\mathbb{S}_G^C X)_1$ is a group completion and the structure maps $(\mathbb{S}_G^C X)_n \rightarrow \Omega(\mathbb{S}_G^C X)_{n+1}$ for $n \geq 1$ are weak G -equivalences. If we redefine $(\mathbb{S}_G^C X)_0$ to be $\Omega(\mathbb{S}_G^C X)_1$, then $\mathbb{S}_G^C X$ is a classical Ω - G -spectrum with a group completion $\iota: X_1 \rightarrow (\mathbb{S}_G^C X)_0$. This means that \mathbb{S}_G^C is a Segal machine as in [Definition 0.1](#), but with G -action.

When G is trivial, this machine plays a special role: any classical infinite loop space machine as in [Definition 0.1](#) is equivalent to this Segal machine [\[57\]](#). The proof in [\[57\]](#) directly generalizes equivariantly to classical G -spectra. It makes essential use of the fact that the Segal machine produces \mathcal{F} - \mathcal{F} - G -spaces, namely functors from \mathcal{F} to \mathcal{F} - G -spaces. However, this construction does *not* work to construct *genuine* G -spectra from \mathcal{F} - G -spaces: there is no evident way to build in deloopings by non-trivial representations of G .

2.3. The conceptual version of the Segal machine. The more conceptual variants of the nonequivariant Segal machine do generalize to give genuine G -spectra. These variants do *not* make use of the functor $F: \Delta^{op} \rightarrow \mathcal{F}$. That is, underlying simplicial G -spaces play no role in their construction. We follow the nonequivariant exposition of [\[43\]](#). We first introduce notation for the categories of enriched functors that we shall be using. Recall [Notations 1.15](#).

Notation 2.24. For a $G\mathcal{U}_*$ -category \mathcal{E} , let $\text{Fun}(\mathcal{E}, G\mathcal{U}_*)$ denote the category of $G\mathcal{U}_*$ -functors $\mathcal{E} \rightarrow G\mathcal{U}_*$ and $G\mathcal{U}_*$ -natural transformations between them. When G acts trivially on \mathcal{E} , as is the case of \mathcal{F} , a $G\mathcal{U}_*$ -functor defined on \mathcal{E} takes values on morphisms in the fixed point spaces $G\mathcal{U}_*(X, Y)^G = G\mathcal{U}_*(X, Y)$ of based G -maps $X \rightarrow Y$. We often use the alternative notation $\text{Fun}(\mathcal{E}, G\mathcal{U}_*)$ in that case.

Definition 2.25. A \mathcal{W}_G - G -space Y is a $G\mathcal{U}_*$ -functor $\mathcal{W}_G \rightarrow G\mathcal{U}_*$. A map of \mathcal{W}_G - G -spaces is a $G\mathcal{U}_*$ -natural transformation between them. Regarding \mathcal{F} as a G -trivial G -category, it is both a full subcategory of $G\mathcal{W}$ and a G -trivial full G -subcategory of \mathcal{W}_G . We have the functor categories

$$\text{Fun}(\mathcal{F}, G\mathcal{U}_*) = \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}_*})$$

of \mathcal{F} - G -spaces and $\text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}_*})$ of \mathcal{W}_G - G -spaces. The inclusion $\mathcal{F} \subset \mathcal{W}_G$ induces a forgetful functor

$$\mathbb{U}: \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}_*}) \rightarrow \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}_*}).$$

As a matter of elementary category theory (see e.g. [43]), the functor \mathbb{U} has a left adjoint prolongation functor

$$\mathbb{P}: \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*) \longrightarrow \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}}_*).$$

Definition 2.26. Let Y be any \mathcal{W}_G - G -space, such as $Y = \mathbb{P}X$ for an \mathcal{F} - G -space X . For G -spaces $A, B \in G\mathcal{W}$, the adjoint $B \rightarrow \mathcal{W}_G(A, A \wedge B)$ of the identity map on $A \wedge B$ can be composed with Y to obtain a G -map

$$B \longrightarrow \underline{G\mathcal{U}}_*(Y(A), Y(A \wedge B)).$$

Its adjoint is a G -map

$$(2.27) \quad Y(A) \wedge B \longrightarrow Y(A \wedge B).$$

For example, taking $B = I_+$, it follows that the functor Y preserves based homotopies. Letting $A = S^n$ and $B = S^1$ with trivial G -action, these maps give the structure maps

$$\Sigma Y(S^n) \longrightarrow Y(S^{n+1})$$

of a classical G -prespectrum $\mathbb{U}_{G\mathcal{S}}Y$.

Going further, we define an orthogonal G -spectrum given at level V by $Y(S^V)$. The composites

$$\mathcal{I}_G(V, V') \longrightarrow \mathcal{W}_G(S^V, S^{V'}) \xrightarrow{Y} \underline{G\mathcal{U}}_*(Y(S^V), Y(S^{V'}))$$

of Y and the map induced by one-point compactification of maps $V \rightarrow V'$ give a $G\mathcal{U}$ -functor $\mathcal{I}_G \rightarrow G\mathcal{U}_*$ or, equivalently and more sensibly here, a $G\mathcal{U}_*$ -functor $\mathcal{I}_{G+} \rightarrow G\mathcal{U}_*$. Just as in the nonequivariant case [43], letting $A = S^V$ and $B = S^W$ in (2.27) for representations V and W , we obtain the structure G -maps

$$\Sigma^W Y(S^V) \longrightarrow Y(S^{V \oplus W})$$

of an orthogonal G -spectrum $\mathbb{U}_{G\mathcal{S}}Y$ such that $i^*\mathbb{U}_{G\mathcal{S}}Y = \mathbb{U}_{G\mathcal{S}}Y$.

Definition 2.28. The conceptual Segal machine on \mathcal{F} - G -spaces is the restriction of the composite

$$\mathbb{U}_{G\mathcal{S}} \circ \mathbb{P}: \text{Fun}(\mathcal{F}, G\mathcal{U}) \longrightarrow G\mathcal{S}$$

to $\mathcal{F}[G\mathcal{S}]$.

The functor \mathbb{P} is a left Kan extension that is best viewed as a categorical tensor product of functors. For $A \in G\mathcal{W}$, we have the contravariant \mathcal{U}_* -functor $A^\bullet: \mathcal{F} \rightarrow G\mathcal{U}_*$. Conceptually, A^\bullet is the represented functor that sends \mathbf{n} to the function space $\mathcal{W}_G(\mathbf{n}, A) \cong A^n$ with its induced action by G . By definition,

$$(2.29) \quad (\mathbb{P}X)(A) = A^\bullet \otimes_{\mathcal{F}} X.$$

Taking $A = \mathbf{n}$, the unit $\eta: X \rightarrow \mathbb{P}X$ of the adjunction sends $x \in X_n$ to $(\text{id}_{\mathbf{n}}, x)$; by Yoneda, η is a natural isomorphism. For a \mathcal{W}_G - G -space $Y: \mathcal{W}_G \rightarrow \underline{G\mathcal{U}}_*$, the counit $\varepsilon: \mathbb{P}\mathbb{U}Y \rightarrow Y$ is given on $A \in G\mathcal{W}$ by the composites

$$\mathcal{W}_G(\mathbf{n}, A) \wedge Y(\mathbf{n}) \xrightarrow{Y \wedge \text{id}} \underline{G\mathcal{U}}_*(Y(\mathbf{n}), Y(A)) \wedge Y(\mathbf{n}) \xrightarrow{\text{eval}} Y(A).$$

For an \mathcal{F} - G -space $X: \mathcal{F} \rightarrow G\mathcal{S}$, we write X_n for $X(\mathbf{n})$ as before, but we follow the usual convention of abbreviating notation by writing $(\mathbb{P}X)(A) = X(A)$ for general $A \in G\mathcal{W}$. Note that we do not claim that $X(A)$ is nondegenerately based in general. The following result is a variant of Segal's [72, Proposition 3.2 and Lemma 3.7].

Proposition 2.30. *The classical G -prespectrum $\mathbb{U}_{G\mathcal{D}}\mathbb{P}X$ is naturally isomorphic to the classical G -prespectrum $\mathbb{S}_G^C X$ (of [Definition 2.23](#)). Therefore, if X is special, then $\mathbb{U}_{G\mathcal{D}}\mathbb{P}X$ is a positive Ω - G -prespectrum whose bottom structure map is a group completion of X_1 .*

Segal's nonequivariant proof is briefly sketched in [72, §3], in different language. For the reader's convenience, we give a more complete argument in [Section 8.1](#). The following result is the key observation, and it is the crucial point for us. Via [Proposition 2.19](#), it makes the group completion property for $\mathbb{U}_{G\mathcal{D}}\mathbb{P}X$ transparent.

Proposition 2.31. *For \mathcal{F} - G -spaces X , there is a natural G -homeomorphism*

$$|X| \longrightarrow (S^1)^\bullet \otimes_{\mathcal{F}} X = (\mathbb{P}X)(S^1).$$

2.4. \mathcal{F}_G and a factorization of the conceptual Segal machine. We now consider the G -category \mathcal{F}_G of finite based G -sets rather than just the category \mathcal{F} of finite based sets. Use of \mathcal{F}_G in tandem with \mathcal{F} is essential to our work. Input arises most often as \mathcal{F} - G -spaces but, by a result of Shimakawa [76] that we shall reprove, these are interchangeable with \mathcal{F}_G - G -spaces.

Definition 2.32. Let \mathcal{F}_G be the G -category of finite based G -sets and all based functions, with G acting by conjugation on function sets. For convenience and precision, we restrict the objects of \mathcal{F}_G to be the finite G -sets \mathbf{n}^α , as in [Convention 1.4](#). Let Π_G be the G -subcategory with the same objects and those morphisms $\phi: \mathbf{m}^\alpha \longrightarrow \mathbf{n}^\beta$ such that $\phi^{-1}(j)$ has at most one element for $1 \leq j \leq n$. We obtain inclusions $\mathcal{F} \subset \mathcal{F}_G$ and $\Pi \subset \Pi_G$ by restricting to the trivial homomorphisms $\varepsilon_n: G \longrightarrow \Sigma_n$.

As with \mathcal{F} , we view \mathcal{F}_G as a category enriched in $G\mathcal{T} \subset G\mathcal{U}_*$, using the discrete topology on the based hom G -sets $\mathcal{F}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$. The basepoint is the unique map that factors through $\mathbf{0}$, and G acts by conjugation. A finite G -set is evidently a G -CW complex, hence we have an inclusion $\mathcal{F}_G \subset \mathcal{W}_G$.

Definition 2.33. A Π_G - G -space Y is a $G\mathcal{U}_*$ -functor $Y: \Pi_G \longrightarrow G\mathcal{U}_*$ whose underlying functor $\Pi \longrightarrow G\mathcal{U}_*$ is a Π - G -space. We shall see in [Remark 2.39](#) below that Y then takes values in nondegenerately based G -spaces. Similarly, an \mathcal{F}_G - G -space Y is a $G\mathcal{U}_*$ -functor $Y: \Pi_G \longrightarrow \mathcal{T}_G$ whose restriction to Π_G is a Π_G - G -space. Morphisms in both cases are $G\mathcal{U}_*$ -natural transformations, and we write $\Pi_G[\mathcal{T}_G]$ and $\mathcal{F}_G[\mathcal{T}_G]$ for the categories of Π_G - G -spaces and \mathcal{F}_G - G -spaces.

Write $Y(A)$ for the value of Y on $A = \mathbf{n}^\alpha$ and write Y_n for the value of Y on $\mathbf{n}^{\varepsilon_n}$. Recall from [Example 1.5](#) that $(Y_1^n)^\alpha$ denotes Y_1^n with the G -action

$$g(y_1, \dots, y_n) = (gy_{\alpha(g^{-1})(1)}, \dots, gy_{\alpha(g^{-1})(n)})$$

and can be identified with the based G -space $\mathcal{T}_G(\mathbf{n}^\alpha, Y_1) = Y_1^{\mathbf{n}^\alpha}$ when Y_1 is nondegenerately based.

Definition 2.34. For $A = \mathbf{n}^\alpha$ and a Π_G - G -space Y , define a based G -map

$$\varepsilon: A \wedge A \longrightarrow \mathbf{1} = S^0$$

by the Kronecker δ function: $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ if $i \neq j$. Its adjoint is a G -map $A \longrightarrow \mathcal{F}_G(A, \mathbf{1})$. Composing with

$$Y: \mathcal{F}_G(A, \mathbf{1}) \longrightarrow \mathcal{T}_G(Y(A), Y_1)$$

and taking the adjoint, we obtain a G -map $\partial_A: A \wedge Y(A) \rightarrow Y_1$. Thus $\partial_A(j, y) = (\delta_j)_*(y)$ for $1 \leq j \leq n$, where δ_j is induced by the j th projection $\mathbf{n}^\alpha \rightarrow \mathbf{1}^{\varepsilon_1}$. The *Segal map*

$$\delta_A: Y(A) \rightarrow \mathcal{T}_G(A, Y_1) \cong (Y_1^n)^\alpha$$

is the adjoint of ∂_A ; we usually abbreviate δ_A to δ . Note that δ is a G -map, although its components δ_j are usually not.

Definition 2.35. A Π_G - G -space Y is *special* if the δ_A are weak G -equivalences for all $A = \mathbf{n}^\alpha$. A map $f: Y \rightarrow Z$ of Π_G - G -spaces is a *level G -equivalence* if each $f: Y(A) \rightarrow Z(A)$ is a weak G -equivalence. We say that an \mathcal{F}_G - G -space is special if its underlying Π_G - G -space is so and that a map of \mathcal{F}_G - G -spaces is a level G -equivalence if its underlying map of Π_G - G -spaces is so.

The inclusion $\mathcal{F} \hookrightarrow \mathcal{F}_G$ induces a restriction functor

$$\mathbb{U}: \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*) \rightarrow \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*)$$

which has a left adjoint prolongation functor

$$\mathbb{P}: \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*) \rightarrow \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*).$$

The adjunction (\mathbb{P}, \mathbb{U}) of [Definition 2.25](#) factors as the composite of the analogous adjunctions given by the functors

$$\begin{aligned} \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*) &\xrightarrow{\mathbb{P}} \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*) \xrightarrow{\mathbb{P}} \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}}_*), \\ \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}}_*) &\xrightarrow{\mathbb{U}} \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*) \xrightarrow{\mathbb{U}} \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*). \end{aligned}$$

The units of these adjunctions are isomorphisms since the forgetful functors \mathbb{U} are induced by the full and faithful inclusions $\mathcal{F} \hookrightarrow \mathcal{F}_G$ and $\mathcal{F}_G \hookrightarrow \mathcal{W}_G$.

Notation 2.36. For $A \in \mathcal{W}_G$, we now write A^\bullet ambiguously for both the restriction to $\mathcal{F}_G \subset \underline{G\mathcal{U}}_*$ and the restriction to $\mathcal{F} \subset \mathcal{F}_G \subset \underline{G\mathcal{U}}_*$ of the represented functor $\underline{G\mathcal{U}}_*(-, A): \underline{G\mathcal{U}}_*^{\text{op}} \rightarrow \underline{G\mathcal{U}}_*$.

Then the factorization of \mathbb{P} as $\mathbb{P}\mathbb{P}$ takes the explicit form

$$(2.37) \quad A^\bullet \otimes_{\mathcal{F}} X \cong A^\bullet \otimes_{\mathcal{F}_G} (\mathcal{F}_G \otimes_{\mathcal{F}} X) = A^\bullet \otimes_{\mathcal{F}_G} \mathbb{P}X.$$

While our main interest is in \mathcal{F}_G - G -spaces and \mathcal{F} - G -spaces, we will also use the analogous forgetful and prolongation functors relating Π_G - G -spaces and Π - G -spaces.

The following result shows that the notions defined in [Definition 2.35](#) for Π_G - G -spaces correspond to the \mathbb{F}_\bullet -notions for Π - G -spaces. That should help motivate the latter, which may have seemed unnatural at first sight. The analogue for Π_G - G -spaces (or for \mathcal{F}_G - G -spaces) of the dichotomies between \mathbb{F}_\bullet -special and special and between \mathbb{F}_\bullet -level equivalences and level G -equivalences that we have for Π - G -spaces (and thus for \mathcal{F} - G -spaces) will be discussed in a model theoretical context in [Remark 10.3](#).

Theorem 2.38. *The adjoint pairs of functors*

$$\text{Fun}(\Pi, \underline{G\mathcal{U}}_*) \xrightleftharpoons[\mathbb{U}]{\mathbb{P}} \text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$$

and

$$\text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*) \xrightleftharpoons[\mathbb{U}]{\mathbb{P}} \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*)$$

specify equivalences of categories. Remembering the cofibration condition from [Definition 2.10](#), these equivalences of categories restrict to equivalences

$$\Pi[G\mathcal{T}] \begin{array}{c} \xrightarrow{\mathbb{P}} \\ \xleftarrow{\mathbb{U}} \end{array} \Pi_G[\mathcal{T}_G]$$

and

$$\mathcal{F}[G\mathcal{T}] \begin{array}{c} \xrightarrow{\mathbb{P}} \\ \xleftarrow{\mathbb{U}} \end{array} \mathcal{F}_G[\mathcal{T}_G].$$

Moreover, the following statements hold.

- (i) A Π_G - G -space Y is special if and only if the Π - G -space $\mathbb{U}Y$ is \mathbb{F}_\bullet -special.
- (ii) A map $f: Y \rightarrow Z$ of Π_G - G -spaces is a level G -equivalence if and only if the map $\mathbb{U}f: \mathbb{U}Y \rightarrow \mathbb{U}Z$ of Π - G -spaces is an \mathbb{F}_\bullet -level equivalence.
- (iii) A Π - G -space X is \mathbb{F}_\bullet -special if and only if the Π_G - G -space $\mathbb{P}X$ is special.
- (iv) A map f of Π - G -spaces is an \mathbb{F}_\bullet -level equivalence if and only if the map $\mathbb{P}f$ of Π_G - G -spaces is a level G -equivalence.

All of these statements remain true with Π and Π_G replaced by \mathcal{F} and \mathcal{F}_G .

Proof. For a $G\mathcal{U}_*$ -functor $X: \Pi \rightarrow \underline{G\mathcal{U}_*}$ and a finite G -set $A = \mathbf{n}^\alpha$,

$$(\mathbb{P}X)(A) = A^\bullet \otimes_\Pi X,$$

where $A^\bullet: \Pi \rightarrow \underline{G\mathcal{U}_*}$ is the functor that sends \mathbf{m} to $\Pi_G(\mathbf{m}, A)$. Recall that the underlying set of $\Pi_G(\mathbf{m}, A)$ is just $\Pi(\mathbf{m}, \mathbf{n})$ with G -action induced by the action of G on \mathbf{n} given by α . The action of G on $A^\bullet \otimes_\Pi X$ is induced by the diagonal action.

Since the inclusion $\Pi \rightarrow \Pi_G$ is full and faithful, the unit of the adjunction $\eta: X \rightarrow \mathbb{U}\mathbb{P}X$ is an isomorphism. We must show that the counit $\varepsilon: \mathbb{P}\mathbb{U}Y \rightarrow Y$ is an isomorphism for a $G\mathcal{U}_*$ -functor $Y: \Pi_G \rightarrow \underline{G\mathcal{U}_*}$. Again let $A = \mathbf{n}^\alpha \in \Pi_G$. Then

$$\varepsilon: (\mathbb{P}\mathbb{U}Y)(A) = A^\bullet \otimes_\Pi (\mathbb{U}Y) \rightarrow Y(A)$$

is the G -map given by $\varepsilon(\mu, y) = \mu_*y$ for $\mu: \mathbf{m} \rightarrow A$ and $y \in Y_m$, where $\mu_*: Y_m \rightarrow Y(A)$. It is a G -homeomorphism with inverse given by $\varepsilon^{-1}(y) = (\iota^{-1}, \iota_*y)$ for $y \in Y(A)$, where $\iota \in \Pi_G(A, \mathbf{n})$ is the morphism whose underlying function on \mathbf{n} is the identity. Clearly $\varepsilon\varepsilon^{-1} = \text{id}$, and $\varepsilon^{-1}\varepsilon = \text{id}$ since $(\mu, y) \sim (\iota^{-1}, \iota_*\mu_*y)$. The identification uses the morphism $\iota \circ \mu$ in Π . Since ε^{-1} is inverse to a G -map, it is a G -map. The proof with Π and Π_G replaced by \mathcal{F} and \mathcal{F}_G is the same. The restriction of the equivalence to our subcategories of interest follows from the fact that a Π_G - G -space is precisely a functor out of Π_G whose restriction to Π is a Π - G -space.

To prove the rest of the theorem, we use the following explicit description of a $G\mathcal{U}_*$ -functor $Y: \Pi_G \rightarrow \underline{G\mathcal{U}_*}$ in terms of its restriction to Π . Recall from [Definition 1.3](#) that, for a homomorphism $\alpha: G \rightarrow \Sigma_n$, Y_n^α denotes Y_n with a twisted action \cdot_α of G specified in terms of α and the original action of G by $g \cdot_\alpha y = \alpha(g)_*(g \cdot y)$. For a finite G -set \mathbf{n}^α , ε^{-1} identifies $Y(\mathbf{n}^\alpha)$ with the G -space Y_n^α .

To prove (i) and (ii), consider a subgroup $\Lambda_\alpha = \{(h, \alpha(h))\}$ of $G \times \Sigma_n$ such that $\Lambda_\alpha \cap \Pi = \{e\}$. Projection onto the first coordinate gives an isomorphism $\Lambda_\alpha \rightarrow H$. The Λ_α -action on Y_n obtained by restriction of the action of $G \times \Sigma_n$ is given by

$$(h, \alpha(h)) \cdot y = \alpha(h)_*(h \cdot y).$$

Thus it coincides with the restriction to H of the G -action used to define Y_n^α . This immediately implies (ii). Similarly, the Λ_α -action on Y_1^n obtained by restriction of the action of $G \times \Sigma_n$ given by the diagonal action of G and the permutation action of Σ_n is

$$h(y_1, \dots, y_n) = (hy_{\alpha(h^{-1})(1)}, \dots, hy_{\alpha(h^{-1})(n)}).$$

Thus it coincides with the restriction to H of the G -action that we used to define $(Y_1^n)^\alpha$. Therefore ε^{-1} identifies the restriction to H of the Segal G -map $\delta: Y(\mathbf{n}^\alpha) \rightarrow (Y_1^n)^\alpha$ with the Λ_α -map $\delta: Y_n \rightarrow Y_1^n$. This immediately implies (i), and (iii) and (iv) follow formally from (i) and (ii) since $\eta: \text{id} \rightarrow \mathbb{U}\mathbb{P}$ is an isomorphism.

Since statements (i)-(iv) for \mathcal{F} - G -spaces and \mathcal{F}_G - G -spaces depend only on their underlying Π - G -spaces and Π_G - G -spaces, they follow immediately. \square

Remark 2.39. Let Y be a Π_G - G -space. As noted in the proof of [Theorem 2.38](#), for a finite G -set \mathbf{n}^α , we have a G -homeomorphism $Y(\mathbf{n}^\alpha) \cong Y_n^\alpha$. Since $\mathbb{U}Y$ is a Π - G -space, the inclusion of the basepoint $* \rightarrow Y_n$ is a $(G \times \Sigma_n)$ -cofibration, and in particular Y_n^α is in $G\mathcal{F}$. Thus Y takes values in nondegenerately based G -spaces, as claimed in [Definition 2.33](#).

We record analogues for \mathcal{F}_G - G -spaces of [Lemmas 2.16](#) and [2.17](#) for \mathcal{F} - G -spaces. While they could be proven directly, we just observe that the first follows immediately from [Theorem 2.38\(iii\)](#), and the second follows as in the proof of [Lemma 2.17](#).

Definition 2.40. For a based G -space X , let $\mathbb{R}_G X$ denote the Π_G - G -space with \mathbf{n}^α th G -space $(X^n)^\alpha$. Conceptually, it is obtained by prolonging the Π - G -space $\mathbb{R}X$ from [Remark 2.13](#) to a Π_G - G -space.

Lemma 2.41. *If $f: X \rightarrow Y$ is a weak equivalence of based G -spaces, then the induced map $\mathbb{R}_G f: \mathbb{R}_G X \rightarrow \mathbb{R}_G Y$ is a level G -equivalence of Π_G - G -spaces.*

Lemma 2.42. *If $f: X \rightarrow Y$ is a level G -equivalence of Π_G - G -spaces, then X is special if and only if Y is special.*

3. THE HOMOTOPICAL VERSION OF THE SEGAL MACHINE

While $\mathbb{U}_{G,\mathcal{F}}\mathbb{P}(X)$ gives the most conceptually natural equivariant version of the Segal machine on \mathcal{F} - G -spaces X , it is by itself of negligible use since the functor \mathbb{P} does not generally enjoy good homotopical properties before some kind of homotopical approximation of X . We define a classical homotopical Segal machine in [Section 3.3](#) and show its defects. We define a genuine homotopical Segal machine and summarize its homotopical properties in [Section 3.4](#), deferring the longer proofs to [Section 8](#).

The homotopical version of the Segal machine is defined in terms of an appropriate enriched version of the two-sided categorical bar construction. We start in [Section 3.1](#) with a general discussion of that construction. Since it is defined using geometric realization of simplicial G -spaces, it is only well-behaved homotopically when those simplicial G -spaces are Reedy cofibrant, as we will discuss in [Section 3.2](#).

3.1. The categorical bar construction. We here define the variant of the bar construction used in the homotopical Segal machine. We begin with some general definitions that start with a closed symmetric monoidal category \mathcal{V} and a \mathcal{V} -category \mathcal{E} , as in [Section 1.3](#). We could start more generally with \mathcal{V} enriched

over some related category rather than over itself, but we restrict to \mathcal{V} for notational simplicity. Let Y be a contravariant and X be a covariant \mathcal{V} -functor $\mathcal{E} \rightarrow \underline{\mathcal{V}}$. They are given by objects Y_n and X_n in \mathcal{V} and maps

$$Y: \mathcal{E}(m, n) \rightarrow \underline{\mathcal{V}}(Y_n, Y_m) \quad \text{and} \quad X: \mathcal{E}(m, n) \rightarrow \underline{\mathcal{V}}(X_m, X_n)$$

in \mathcal{V} that are compatible with composition and identity. These have adjoint evaluation maps

$$E_Y: \mathcal{E}(m, n) \otimes Y_n \rightarrow Y_m \quad \text{and} \quad E_X: \mathcal{E}(m, n) \otimes X_m \rightarrow X_n.$$

Associated to the triple (Y, \mathcal{E}, X) we have a categorical two-sided bar construction $B_*(Y, \mathcal{E}, X)$. It is a simplicial object in \mathcal{V} . Its q -simplex object in \mathcal{V} is the coproduct

$$(3.1) \quad B_q(Y, \mathcal{E}, X) = \coprod_{(n_0, \dots, n_q)} Y_{n_q} \otimes \mathcal{E}(n_{q-1}, n_q) \otimes \cdots \otimes \mathcal{E}(n_0, n_1) \otimes X_{n_0},$$

where (n_0, \dots, n_q) runs over the $(q+1)$ -tuples of objects of \mathcal{E} . Its faces d_i for $0 \leq i \leq q$ are induced by the evaluation maps of Y and X and by composition in \mathcal{E} , and its degeneracies s_i for $0 \leq i \leq q$ are induced by the unit maps of \mathcal{E} . In more detail, the d_i are induced by

$$\begin{aligned} \mathcal{E}(n_0, n_1) \otimes X_{n_0} &\xrightarrow{E_X} X_{n_1} \quad \text{if } i = 0, \\ \mathcal{E}(n_i, n_{i+1}) \otimes \mathcal{E}(n_{i-1}, n_i) &\xrightarrow{C} \mathcal{E}(n_{i-1}, n_{i+1}) \quad \text{if } 0 < i < q, \\ Y_{n_q} \otimes \mathcal{E}(n_{q-1}, n_q) &\xrightarrow{E_Y} Y_{n_{q-1}} \quad \text{if } i = q. \end{aligned}$$

The s_i are induced by

$$U \xrightarrow{I} \mathcal{E}(n_i, n_i) \quad \text{if } 0 \leq i \leq q.$$

We shall use the following observation in [Section 8.4](#).

Remark 3.2. When \mathcal{V} is cartesian closed, so that $\otimes = \times$, $B_*(Y, \mathcal{E}, X)$ is the nerve of an internal ‘‘Grothendieck category of elements’’ $\mathcal{C}(Y, \mathcal{E}, X)$. The objects and morphisms of this category are both objects of \mathcal{V} . The object of objects is the coproduct

$$\mathcal{C}_0 = \coprod_n Y_n \times X_n,$$

where n runs over the objects of \mathcal{E} . The object of morphisms is the coproduct

$$\mathcal{C}_1 = \coprod_{m, n} Y_n \times \mathcal{E}(m, n) \times X_m,$$

where (m, n) runs over the pairs of objects of \mathcal{E} . We have source, target, and identity maps S , T , and I given as follows: S and T are given on components by the evaluation maps of Y and X .

$$S = E_Y \times \text{id}: Y_n \times \mathcal{E}(m, n) \times X_m \rightarrow Y_m \times X_m$$

$$T = \text{id} \times E_X: Y_n \times \mathcal{E}(m, n) \times X_m \rightarrow Y_n \times X_n;$$

I is induced by the identity maps $U \rightarrow \mathcal{E}(n, n)$ of \mathcal{E} . The composition

$$C: \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$$

is induced by the composition in \mathcal{E} . The point is that when $\otimes = \times$ we have the identification

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \cong \coprod_{(m,n,p)} Y_p \times \mathcal{E}(n,p) \times \mathcal{E}(m,n) \times X_m.$$

where (m, n, p) runs over the triples of objects of \mathcal{E} .

Remark 3.3. The two-sided bar construction goes back to [60, §12]. It is described in the categorical form just given in Shulman [79, Definition 12.1], where more details can be found. The construction in this generality and in this form is central to the study of weighted colimits in enriched category theory.

When \mathcal{V} has a suitable covariant simplex functor $\Delta_* : \Delta \rightarrow \mathcal{V}$ so that we have a “realization” functor from simplicial objects in \mathcal{V} to \mathcal{V} , we define $B(Y, \mathcal{E}, X)$ to be the realization of $B_*(Y, \mathcal{E}, X)$. Formally, it is the categorical tensor product

$$B(Y, \mathcal{E}, X) = B_*(Y, \mathcal{E}, X) \otimes_{\Delta} \Delta_*.$$

Remark 3.4. The target \mathcal{V} could be varied here, as it was in the original homological use of the bar construction (in the 1940’s). There, “realization” went from simplicial R -modules to chain complexes of R -modules.

Example 3.5. One example of Y is the represented contravariant \mathcal{V} -functor $\mathcal{E}_n = \mathcal{E}(-, n)$ for an object n of \mathcal{E} . For an object K of \mathcal{V} , we have the constant simplicial object K_* with q -simplices K for each q . Using composition in \mathcal{E} and the action of \mathcal{E} on X , we obtain a map

$$\varepsilon_n : B_*(\mathcal{E}_n, \mathcal{E}, X) \longrightarrow (X_n)_*$$

of simplicial objects in \mathcal{V} for each n . At each simplicial level, these form a \mathcal{V} -natural transformation of \mathcal{V} -functors out of \mathcal{E} . Thus, upon realization, these give a \mathcal{V} -natural transformation

$$(3.6) \quad \varepsilon : B(\mathcal{E}, \mathcal{E}, X) \longrightarrow X.$$

Using identity morphisms in \mathcal{E} , we also obtain maps

$$\eta_n : (X_n)_* \longrightarrow B_*(\mathcal{E}_n, \mathcal{E}, X)$$

such that $\varepsilon_n \circ \eta_n = \text{id}$. These maps are not \mathcal{V} -natural in \mathcal{E} , as can be readily checked. The standard extra degeneracy argument gives a simplicial homotopy $h : \text{id} \simeq \eta_n \circ \varepsilon_n$. Passing to realization, which in our applications takes simplicial homotopies to homotopies, these induce homotopies $\text{id} \simeq \eta_n \varepsilon_n$, giving homotopy equivalences

$$B(\mathcal{E}_n, \mathcal{E}, X) \simeq X_n$$

We view $B(\mathcal{E}, \mathcal{E}, X)$ as a canonical approximation to X , just as in the original use in homological algebra.

Via the following observation, the general construction gives a natural “derived” approximation to $Y \otimes_{\mathcal{E}} X$.

Lemma 3.7. *There is a natural isomorphism*

$$Y \otimes_{\mathcal{E}} B(\mathcal{E}, \mathcal{E}, X) \cong B(Y, \mathcal{E}, X)$$

of objects in \mathcal{V} and a natural map $B(Y, \mathcal{E}, X) \longrightarrow Y \otimes_{\mathcal{E}} X$ in \mathcal{V} .

Proof. One proof uses a comparison of definitions on the level of q -simplices for each q , but the result is also an application of the categorical Fubini theorem, which implies that we can commute realization and $\otimes_{\mathcal{E}}$. Details are given in [79, Lemma 19.7]. The natural map is obtained by applying $Y \otimes_{\mathcal{E}} -$ to the natural map ε of (3.6). \square

Remark 3.8. More generally, if $\Psi: \mathcal{D} \rightarrow \mathcal{E}$ is a \mathcal{V} -functor, Y is a contravariant \mathcal{V} -functor $\mathcal{E} \rightarrow \mathcal{V}$, and X is a covariant \mathcal{V} -functor $\mathcal{D} \rightarrow \mathcal{V}$, there is an isomorphism

$$B(\Psi^*Y, \mathcal{D}, X) \cong Y \otimes_{\mathcal{E}} B(\mathcal{E}, \mathcal{D}, X),$$

where $\Psi^*Y = Y \circ \Psi$ and each \mathcal{E}_n is viewed as a contravariant \mathcal{V} -functor $\mathcal{D} \rightarrow \mathcal{V}$ by precomposition with Ψ .

3.2. Specializations to spaces and G -spaces. Now let us return to our space level context. The discussion just given applies with $\mathcal{V} = \mathcal{U}$ or $\mathcal{V} = G\mathcal{U}$ for any G , where we take $\otimes = \times$. With no basepoint assumptions, this gives a bar construction $B^\times(Y, \mathcal{E}, X)$. The discussion also applies with $\mathcal{V} = \mathcal{U}_*$ or $\mathcal{V} = G\mathcal{U}_*$, where we take $\otimes = \wedge$. This is our preferred choice and we write

$$(3.9) \quad B(Y, \mathcal{E}, X) = B^\wedge(Y, \mathcal{E}, X).$$

Focusing on $G\mathcal{U}_*$, we make some standing assumptions that will be satisfied in our examples.

Assumption 3.10. As in Section 1.3, we assume that \mathcal{E} has a zero object 0, so that each $\mathcal{E}(0, n)$ and each $\mathcal{E}(n, 0)$ is a point. Then each $\mathcal{E}(m, n)$ has the basepoint $m \rightarrow 0 \rightarrow n$. We assume that the functors Y and X are $G\mathcal{U}_*$ -enriched so that they are given by action G -maps

$$Y_n \wedge \mathcal{E}(m, n) \rightarrow Y_m \quad \text{and} \quad \mathcal{E}(m, n) \wedge X_m \rightarrow X_n.$$

By Lemma 1.17, this is equivalent to requiring $G\mathcal{U}$ -enriched functors such that $Y(0)$ and $X(0)$ are a point. Finally, we assume that the G -spaces Y_n , $\mathcal{E}(m, n)$, and X_m are all nondegenerately based and that the inclusion of the identity map

$$* \rightarrow \mathcal{E}(n, n)$$

is a G -cofibration. This will ensure that our bar constructions are given by the geometric realizations of Reedy cofibrant simplicial G -spaces in $G\mathcal{T}$.

Remark 3.11. When $\mathcal{V} = G\mathcal{U}_*$, we can forget basepoints and consider \mathcal{E} as enriched in $G\mathcal{U}$, and Y and X as landing in $\underline{G}\mathcal{U}$. The variant B^\times suffers from the defect that $B^\times(\mathcal{E}_0, \mathcal{E}, X)$ is not a point, and hence by Lemma 1.17, the functor $B^\times(\mathcal{E}, \mathcal{E}, X)$ is enriched over \mathcal{U} but not over \mathcal{U}_* . Previous work, starting with [75] and going through earlier versions of this paper, focused on an intermediate variant, namely

$$B^\times(Y, \mathcal{E}, X)/B^\times(*, \mathcal{E}, X).$$

There are quotient maps relating the three variants

$$(3.12) \quad B^\times(Y, \mathcal{E}, X) \rightarrow B^\times(Y, \mathcal{E}, X)/B^\times(*, \mathcal{E}, X) \rightarrow B(Y, \mathcal{E}, X).$$

Under our assumption that \mathcal{E} has a zero object, the map ε of (3.6) shows that $B^\times(*, \mathcal{E}, X)$ is contractible, and the nondegeneracy of the basepoints implies that the inclusion

$$B^\times(*, \mathcal{E}, X) \rightarrow B^\times(Y, \mathcal{E}, X)$$

is a G -cofibration, showing that the first map is an equivalence. The second map is not in general an equivalence, but by [28, Theorem 3.19], it gives an equivalence of the associated Segal machines when $\mathcal{E} = \mathcal{F}_G$ and X is special. The choice of (3.9) has distinct technical advantages. Detailed comparisons are given in [28].¹⁵

The following specialization of Example 3.5 makes precise the idea that $B(\mathcal{E}, \mathcal{E}, X)$ is an approximation of X . We write it equivariantly for definiteness, under Assumption 3.10.

Proposition 3.13. *For every object n in \mathcal{E} there are maps*

$$\varepsilon_n: B(\mathcal{E}_n, \mathcal{E}, X) \longrightarrow X_n \quad \text{and} \quad \eta_n: X_n \longrightarrow B(\mathcal{E}_n, \mathcal{E}, X).$$

such that $\varepsilon_n \eta_n = \text{id}$ and $\eta_n \varepsilon_n$ is levelwise G -homotopic to the identity. The maps ε_n assemble to give a $G\mathcal{U}_$ -natural transformation between $G\mathcal{U}_*$ -functors $\mathcal{E} \longrightarrow \mathcal{T}_G$.*

Remark 3.14. The homotopical Segal machine is obtained by using examples of two-sided bar constructions to construct \mathcal{W}_G - G -spaces. They give G -spectra by restricting the variable Y to A^* as in Notation 2.36 for spheres $A = S^V$. The structure maps of these G -spectra come from comparison maps of the form

$$(3.15) \quad B(Y, \mathcal{E}, X) \wedge C \longrightarrow B(Y \wedge C, \mathcal{E}, X).$$

These are immediate with our smash product definition of the bar construction, but we would not have them if we tried to use B^\times . The point is that (2.27) in Definition 2.26 does not apply if we use B^\times due to the difference between unbased and based enrichment. To make this more precise, consider the relationship between smash products and products. For based spaces A , B , and C , there is no natural map

$$(A \times B) \wedge C \longrightarrow (A \wedge C) \times B.$$

Under the natural isomorphism $(A \times B) \times C \cong (A \times C) \times B$, we collapse out different subspaces to construct the source and target, and neither is contained in the other. Explicitly, writing a, b, c for the basepoints of A, B, C and x, y, z for general points of A, B, C , we identify all points (x, y, c) and (a, b, z) with (a, b, c) in the source, but we identify all points (a, z, y) and (x, c, y) with the point (a, c, y) in the target.

Our interest is in the case when \mathcal{E} is \mathcal{F} or \mathcal{F}_G or the more general categories of operators \mathcal{D} and \mathcal{D}_G to be introduced later. Note that although \mathcal{F} is topologically discrete and G -trivial, we still view it as a category enriched in \mathcal{T}_G .

3.3. The classical homotopical Segal machine. Specializing from the previous section, let $Y: \mathcal{F} \longrightarrow G\mathcal{T}$ be a contravariant $G\mathcal{U}_*$ -functor and $X: \mathcal{F} \longrightarrow G\mathcal{T}$ be a covariant $G\mathcal{U}_*$ -functor. We then have the bar construction $B(Y, \mathcal{F}, X)$. The action of G on it is induced diagonally by the actions on the Y_n and X_n . It is important to note that given our assumptions, $B(Y, \mathcal{F}, X)$ is a nondegenerately based G -space.

For $G\mathcal{U}_*$ -functors $Y: \mathcal{F}_G^{op} \longrightarrow \mathcal{T}_G$ and $X: \mathcal{F}_G \longrightarrow \mathcal{T}_G$, we have the analogous two-sided bar construction $B(Y, \mathcal{F}_G, X)$. Just as above, it is a nondegenerately based G -space. For its construction, we must remember the action of G on the finite G -sets $\mathcal{F}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$. While we are interested in general X , in both cases we are only interested in particular Y , namely those of the form $Y = A^\bullet$, as in Notation 2.36.

¹⁵Although published earlier, [28] began as a sequel to this paper.

Nonequivariantly, Woolfson [82] constructed a homotopical Segal machine by restricting $B(A^\bullet, \mathcal{F}, X)$ to spheres $A = S^n$.¹⁶ Equivariantly, we can apply the same construction, taking X to be an \mathcal{F} - G -space, and A to be in $G\mathcal{W}$. For reasons we now explain, this construction *fails* to lead to genuine Ω - G -spectra, even when X is \mathbb{F}_\bullet -special.

When $A = \mathbf{n}$, A^\bullet is the represented functor $\mathcal{F}_n = \mathcal{F}(-, \mathbf{n})$, and as n varies we obtain the \mathcal{F} - G -space $B(\mathcal{F}, \mathcal{F}, X)$ whose n th G -space is $B(\mathcal{F}_n, \mathcal{F}, X)$. We have an implicit and important action of Σ_n on source and target; $\Sigma_n \subset \mathcal{F}(\mathbf{n}, \mathbf{n})$ acts from the left on \mathcal{F}_n by postcomposition in \mathcal{F} , and that induces the action on $B(\mathcal{F}_n, \mathcal{F}, X)$. Observe that $B_*(\mathcal{F}_n, \mathcal{F}, X)$ is a simplicial $(G \times \Sigma_n)$ -space and ε_n is the realization of a map $B_*(\mathcal{F}_n, \mathcal{F}, X) \rightarrow (X_n)_*$ of simplicial $(G \times \Sigma_n)$ -spaces.

Lemma 3.16. *Let X be an \mathcal{F} - G -space. Then $B(\mathcal{F}, \mathcal{F}, X)$ is an \mathcal{F} - G -space.*

Proof. For an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in Π , the induced function

$$\phi_*: \mathcal{F}(\mathbf{q}, \mathbf{m}) \rightarrow \mathcal{F}(\mathbf{q}, \mathbf{n})$$

is a $(G \times \Sigma_\phi)$ -cofibration. Therefore the cofibration condition of [Definition 2.10](#) follows from [Theorem 1.13](#), applied with G replaced by $G \times \Sigma_\phi$. \square

Moreover, [Proposition 3.13](#) specializes to give the following result.

Proposition 3.17. *Let X be an \mathcal{F} - G -space. Then the map*

$$\varepsilon: B(\mathcal{F}, \mathcal{F}, X) \rightarrow X$$

of \mathcal{F} - G -spaces is a level G -equivalence, hence X is special if and only if $B(\mathcal{F}, \mathcal{F}, X)$ is special.

Warning 3.18. Recall from [Example 3.5](#) that while the maps ε_n are natural with respect to maps in \mathcal{F} , the maps η_n are *not*. In particular, while the map ε_n is a map of $(G \times \Sigma_n)$ -spaces, the map

$$\eta_n: X_n \rightarrow B(\mathcal{F}_n, \mathcal{F}, X)$$

is *not* Σ_n -equivariant. Explicitly, the action of Σ_n occurs on $\mathcal{F}(-, \mathbf{n})$ in the target and on X_n in the source, so that $\eta(\sigma x) = (\text{id}_n, \sigma x)$ while $\sigma \eta(x) = (\sigma, x)$. Thus there is no reason to expect ε to be a level \mathbb{F}_\bullet -equivalence and no reason to expect $B(\mathcal{F}, \mathcal{F}, X)$ to be \mathbb{F}_\bullet -special even if X is so.

We now have two very different ways to get around this problem. The solution presented here is to exploit the equivalence between \mathcal{F} - G -spaces and \mathcal{F}_G - G -spaces. The problem does not appear when using \mathcal{F}_G , and it is simple to translate along the adjoint equivalence (\mathbb{P}, \mathbb{U}) . With Guillou, we later [28]¹⁷ found an alternative solution that does not require use of \mathcal{F}_G and which has the advantage of making the Segal machine multiplicative. However, that solution is less convenient for the comparison with the operadic machine that we develop here.

By [Propositions 2.30](#) and [3.17](#), $B(\mathcal{F}, \mathcal{F}, X)$ and X can be used interchangeably when passing to classical G -prespectra. When passing to genuine G -spectra, we must consider the two-sided bar construction obtained by replacing \mathcal{F} with \mathcal{F}_G . Here, for an \mathcal{F}_G - G -space Y , $B(\mathcal{F}_G, \mathcal{F}_G, Y)$ is defined at level \mathbf{n}^α by replacing

¹⁶This is revisionist. He was writing before the two-sided bar construction was formally defined; technically, he worked with B^\times .

¹⁷We repeat that, although published earlier, that paper began as a sequel to this one.

the left variable \mathcal{F}_G by the functor $\mathcal{F}_G(-, \mathbf{n}^\alpha): \mathcal{F}_G^{op} \rightarrow \mathcal{F}_G$ represented by \mathbf{n}^α . [Theorem 1.13](#) implies the analog of [Lemma 3.16](#).

Lemma 3.19. *Let Y be an \mathcal{F}_G - G -space. Then $B(\mathcal{F}_G, \mathcal{F}_G, Y)$ is an \mathcal{F}_G - G -space.*

Moreover, using [Lemma 2.42](#), we have the following more powerful analog of [Proposition 3.17](#).

Proposition 3.20. *Let Y be an \mathcal{F}_G - G -space. Then the map*

$$\varepsilon: B(\mathcal{F}_G, \mathcal{F}_G, Y) \rightarrow Y$$

of \mathcal{F}_G - G -spaces is a level G -equivalence, so Y is special if and only if $B(\mathcal{F}_G, \mathcal{F}_G, Y)$ is special.

We view $\mathbb{U}B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)$ as a genuine homotopical approximation to X in view of the following corollary, which is immediate from [Theorem 2.38](#). Note that we can identify X with $\mathbb{U}PX$ via the unit of the adjunction.

Corollary 3.21. *For any \mathcal{F} - G -space X , the map $\mathbb{U}\varepsilon: \mathbb{U}B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X) \rightarrow X$ is an \mathbb{F}_\bullet -level equivalence, hence X is \mathbb{F}_\bullet -special if and only if $\mathbb{U}B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)$ is \mathbb{F}_\bullet -special.*

Remark 3.22. We compare bar constructions along the adjoint equivalence (\mathbb{P}, \mathbb{U}) between \mathcal{F} - G -spaces and \mathcal{F}_G - G -spaces. For \mathcal{F} - G -spaces X , we have

$$\mathbb{P}B(\mathcal{F}, \mathcal{F}, X) = \mathcal{F}_G \otimes_{\mathcal{F}} B(\mathcal{F}, \mathcal{F}, X) \cong B(\mathcal{F}_G, \mathcal{F}, X).$$

The inclusion $\iota: \mathcal{F} \rightarrow \mathcal{F}_G$ induces a natural map of \mathcal{F}_G - G -spaces

$$\iota_*: \mathbb{P}B(\mathcal{F}, \mathcal{F}, X) \rightarrow B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)$$

such that the following diagrams commute; the second is obtained from the first by applying \mathbb{U} .

$$\begin{array}{ccc} \mathbb{P}B(\mathcal{F}, \mathcal{F}, X) & \xrightarrow{\mathbb{P}\varepsilon} & \mathbb{P}X \\ \downarrow \iota & \nearrow \varepsilon & \\ B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X) & & \end{array} \quad \begin{array}{ccc} B(\mathcal{F}, \mathcal{F}, X) & \xrightarrow{\varepsilon} & X \\ \downarrow \mathbb{U}\iota & \nearrow \mathbb{U}\varepsilon & \\ \mathbb{U}B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X) & & \end{array}$$

In the first, the diagonal arrow ε is a level G -equivalence, but we cannot expect ι and $\mathbb{P}\varepsilon$ to be level G -equivalences since that would imply that all three arrows in the second diagram are \mathbb{F}_\bullet -level equivalences, contradicting [Warning 3.18](#). In the second diagram, $\mathbb{U}\varepsilon$ is an \mathbb{F}_\bullet -level equivalence and the other two arrows are level G -equivalences.

Using [Proposition 2.30](#), we see that this comparison implies the following comparison of classical G -prespectra.

Proposition 3.23. *Let X be a special \mathcal{F} - G -space. Then the positive classical Ω - G -prespectra obtained by prolonging X or $B(\mathcal{F}, \mathcal{F}, X)$ or $\mathbb{U}B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)$ to \mathcal{W}_G - G -spaces and then restricting to spheres S^n are level G -equivalent. Their bottom structural maps are compatible group completions of G -spaces equivalent to X_1 .*

Proof. For any A , functoriality of prolongation applied to the second diagram of [Remark 3.22](#) gives a commutative diagram

$$\begin{array}{ccc} B(A^\bullet, \mathcal{F}, X) & \longrightarrow & X(A). \\ \downarrow & \nearrow & \\ B(A^\bullet, \mathcal{F}_G, \mathbb{P}X) & & \end{array}$$

Here we have used the factorization of prolongation from \mathcal{F} - G -spaces to \mathcal{W}_G - G -spaces through \mathcal{F}_G - G -spaces and the isomorphism $\mathbb{P}\mathbb{U} \cong \text{Id}$, where \mathbb{P} is prolongation from \mathcal{F} - G -spaces to \mathcal{F}_G - G -spaces. By [Lemma 2.17](#) and [Remark 3.22](#), all three of our \mathcal{F} - G -spaces are special. Restricting to spheres S^n , we can apply [Proposition 2.30](#) to each of them. Taking $A = S^0$, we see compatible weak G -equivalences with X_1 , and taking $A = S^1$, we see that the bottom structure maps are compatible group completions. That implies that we have weak G -equivalences at level 1. In turn, since we are comparing Ω - G -prespectra, that implies that we have weak G -equivalences at all levels n . \square

3.4. The genuine homotopical Segal machine. The following definition gives a modernized version of Shimakawa's equivariant Segal machine [\[75\]](#).¹⁸ Recall [Notation 2.36](#).

Definition 3.24. Write $\mathbb{I}Y = B(\mathcal{F}_G, \mathcal{F}_G, Y)$ for an \mathcal{F}_G - G -space Y ; thus \mathbb{I} is a functor $\mathcal{F}_G[\mathcal{T}_G] \rightarrow \mathcal{F}_G[\mathcal{T}_G]$. The Segal machine $\mathbb{S}_G^{\mathcal{F}_G}$ on \mathcal{F}_G - G -spaces is the composite

$$\mathcal{F}_G[\mathcal{T}_G] \xrightarrow{\mathbb{I}} \mathcal{F}_G[\mathcal{T}_G] \xrightarrow{\mathbb{P}} \text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*) \xrightarrow{\mathbb{U}_{G,\mathcal{F}}} G\mathcal{S}.$$

More explicitly, taking $A = S^V$,

$$\mathbb{S}_G(Y)(V) = B(S^{V^\bullet}, \mathcal{F}_G, Y) = S^{V^\bullet} \otimes_{\mathcal{F}_G} \mathbb{I}Y.$$

The Segal machine $\mathbb{S}_G^{\mathcal{F}}$ on \mathcal{F} - G -spaces X is defined by

$$\mathbb{S}_G^{\mathcal{F}} X = \mathbb{S}_G^{\mathcal{F}_G} \mathbb{P}X.$$

The definition makes sense for any X . When X is special, [Proposition 3.23](#) shows that the underlying classical G -prespectrum of $\mathbb{S}_G^{\mathcal{F}} X$ is equivalent to $\mathbb{S}_G^C X$ of [Definition 2.23](#), hence is a positive Ω - G -prespectrum with bottom structural map a group completion of X_1 .

Remark 3.25. The group completion property is not easy to see directly from the definition of $\mathbb{S}_G^{\mathcal{F}_G}$. Shimakawa's strategy [\[77, p 357\]](#), not carried out in detail, was to show that for $H \subset G$, Woolfson's version of the nonequivariant Segal machine $\mathbb{S}(Y^H)$ is equivalent to $(\mathbb{S}_G^{\mathcal{F}_G} Y)^H$, where Y^H is the composite of restriction to \mathcal{F} and the H -fixed point functor, so that $(Y^H)_n = Y(\mathbf{n})^H$, and then to quote the equivalence of Woolfson's version with Segal's original version. In contrast, with our proof, this equivalence on fixed points follows formally from the group completion property, as is shown quite generally in [\[25, Theorem 2.20\]](#).

¹⁸Orthogonal G -spectra had not been developed when [\[75\]](#) was written; he worked with Lewis-May G -spectra.

We are primarily interested in understanding $\mathbb{S}_G^{\mathcal{F}} X$ when X is \mathbb{F}_\bullet -special. Recall that $G\mathcal{W}$ and \mathcal{W}_G are the categories of based G -CW complexes whose respective morphisms are based G -maps and all based maps, with G acting by conjugation. It is standard to say that a functor is linear if it converts homotopy cocartesian squares to homotopy cartesian squares. We will use a variant form of the definition that is convenient for displaying a helpful distinction that is needed in our work.

Definition 3.26. A \mathcal{W}_G - G -space Z is *linear* if for any G -map $f: A \rightarrow B$ in $G\mathcal{W}$ with cofiber $i: B \rightarrow Cf$,

$$Z(A) \xrightarrow{f_*} Z(B) \xrightarrow{i_*} Z(Cf)$$

is a fibration sequence of based G -spaces. That is, the induced map from $Z(A)$ to the homotopy fiber of i_* is a weak G -equivalence. We say that Z is *positive linear* if this condition holds when A is G -connected, but not necessarily in general.

We shall need the following companion definitions.

Definition 3.27. A \mathcal{W}_G - G -space Z *preserves connectivity* if $Z(A)$ is G -connected when A is G -connected.

Definition 3.28. For a G -space A in \mathcal{W}_G , define a \mathcal{W}_G - G -space $Z[A]$ by

$$Z[A](B) = Z(A \wedge B).$$

In [Section 8.3](#), we adapt and extend nonequivariant arguments of Segal and Woolfson [72, 82] to prove the following result. It is perhaps surprising that we only need X to be special, not \mathbb{F}_\bullet -special, for the first statement and that we do not know how to derive either statement from the other. However, we will only make use of the second statement in this paper.

Theorem 3.29. *If X is a special \mathcal{F} - G -space, then the \mathcal{W}_G - G -space that sends A to $B(A^\bullet, \mathcal{F}, X)$ is positive linear and preserves connectivity. If Y is a special \mathcal{F}_G - G -space, such as $\mathbb{P}X$ for an \mathbb{F}_\bullet -special \mathcal{F} - G -space X , then $B(A^\bullet, \mathcal{F}_G, Y)$ is positive linear and preserves connectivity.*

We need the following definition to go from this result to an understanding of the G -spectrum given by Y .

Definition 3.30. Let Z be a \mathcal{W}_G - G -space. We say that Z is *special* if it preserves connectivity and its restriction to \mathcal{F}_G is a special \mathcal{F}_G - G -space.

We defer the proof of the following key result to [Section 8.5](#).

Lemma 3.31. *Let Z be a positively linear and special \mathcal{W}_G - G -space. If A is G -connected, then $Z[A]$ is a linear and special \mathcal{W}_G - G -space.*

The lemma gives input to the following result, which is the technical heart of the Segal machine. For the sake of completeness, we present a proof in [Section 8.5](#). It is a result originally due to Segal [73] and Shimakawa [75]. The proof we present follows the blueprint presented in [73, 75], while providing more details.

Theorem 3.32. *If Z is a linear and special \mathcal{W}_G - G -space, then the structure map*

$$\tilde{\sigma}: Z(S^0) \rightarrow \Omega^V Z(S^V)$$

is a weak G -equivalence for all representations V of G .

The following consequence is originally due to Segal [73], Shimakawa [75], and Blumberg [6] and builds on nonequivariant results in [9, 43, 66, 72, 82].

Theorem 3.33. *Let Z be a positively linear and special \mathcal{W}_G - G -space. Then $\mathbb{U}_{G,\mathcal{F}}Z$ is a positive Ω - G -spectrum.*

Proof. Let V and W be representations of G and assume that $V^G \neq 0$. We must prove that the adjoint structure map

$$\tilde{\sigma}: Z(S^V) \longrightarrow \Omega^W Z(S^{V \oplus W}) = \Omega^W Z(S^V \wedge S^W)$$

is a weak G -equivalence. Since S^V is G -connected, $Z[S^V]$ is linear and special by Lemma 3.31. The conclusion follows by applying Theorem 3.32 with Z there replaced by $Z[S^V]$ here and V there replaced by W here. \square

Here now is the fundamental theorem about the Segal machine.

Theorem 3.34. *Let X be an \mathbb{F}_\bullet -special \mathcal{F} - G -space. Then $\mathbb{S}_G^{\mathcal{F}}X$ is a positive Ω - G -spectrum. Moreover, if $S^V \in \mathcal{W}_G$ and $V^G \neq 0$, then the composite*

$$X_1 \longrightarrow B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)_1 = (\mathbb{S}_G^{\mathcal{F}}X)(S^0) \longrightarrow \Omega^V (\mathbb{S}_G^{\mathcal{F}}X)(S^V)$$

of η_1 and the structure G -map is a group completion.

Proof. Let Z be the \mathcal{W}_G - G -space $B((-)^\bullet, \mathcal{F}_G, \mathbb{P}X)$. Then Z is positive linear and preserves connectivity by Theorem 3.29 and its restriction to \mathcal{F} is an \mathbb{F}_\bullet -special \mathcal{F} - G -space by Corollary 3.21. Therefore Theorem 3.33 implies the first statement, and the second follows from Proposition 3.23 and Remark 1.23. \square

Remark 3.35. Clearly Theorem 3.33 applies to $Z = \mathbb{P}X$ when X is \mathbb{F}_\bullet -special. Despite Theorem 3.29, we have no such conclusion when X is only special. In Section 9.2, we will come back to here to give a partial generalization to compact Lie groups of all results in this section.

4. THE GENERALIZED SEGAL MACHINE

The input of the Segal infinite loop space machine looks nothing like the input of the operadic machine. To compare them, we must generalize the natural input of both to obtain common input to which generalizations of both machines apply. We explain the generalized Segal machine in this section, postponing consideration of operads to the next. We define two equivariant versions of the categories of operators introduced in [57]. One version has finite sets as objects, the other finite G -sets, generalizing \mathcal{F} and \mathcal{F}_G respectively. As in Section 2.4, we show how to construct examples of the second kind from examples of the first kind.

After defining what it means for a G -category of operators to be an E_∞ G -category of operators, we generalize the homotopical version of the Segal machine by generalizing its input from \mathcal{F} - G -spaces to \mathcal{D} - G -spaces, where \mathcal{D} is any E_∞ G -category of operators over \mathcal{F} . We compare the \mathcal{D} and \mathcal{D}_G machines to the \mathcal{F} and \mathcal{F}_G machines by proving that they have equivalent inputs and that they produce equivalent output when fed equivalent input. Thus the increased generality is more apparent than real. The point of the generalization is that operadic data feed naturally into the \mathcal{D} - G -space rather than the \mathcal{F} - G -space machine. We reiterate that the categorical input data of the sequels [26, 44] is intrinsically operadic.

4.1. G -categories of operators \mathcal{D} over \mathcal{F} and \mathcal{D}_G over \mathcal{F}_G .

Definition 4.1. A G -category of operators \mathcal{D} over \mathcal{F} , abbreviated G -CO over \mathcal{F} , is a $G\mathcal{U}_*$ -category with objects the based sets \mathbf{n} for $n \geq 0$, together with $G\mathcal{U}_*$ -functors

$$\Pi \xrightarrow{\iota} \mathcal{D} \xrightarrow{\xi} \mathcal{F}$$

such that ι and ξ are the identity on objects and $\xi \circ \iota$ is the inclusion. Here G acts trivially on Π and \mathcal{F} . We say that \mathcal{D} is reduced if $\mathcal{D}(\mathbf{m}, \mathbf{n})$ is a point if either $m = 0$ or $n = 0$, and we restrict attention to reduced G -COs over \mathcal{F} henceforward. A morphism $\nu: \mathcal{D} \rightarrow \mathcal{E}$ of G -COs over \mathcal{F} is a $G\mathcal{U}_*$ -functor over \mathcal{F} and under Π . In particular, $\xi: \mathcal{D} \rightarrow \mathcal{F}$ is a map of G -COs over \mathcal{F} for any G -CO \mathcal{D} over \mathcal{F} .

Note that we have maps

$$\Pi(\mathbf{m}, \mathbf{n}) \rightarrow \mathcal{D}(\mathbf{m}, \mathbf{n}) \rightarrow \mathcal{F}(\mathbf{m}, \mathbf{n})$$

whose composite is the inclusion. We have the following parallel analogue.

Definition 4.2. A G -category of operators \mathcal{D}_G over \mathcal{F}_G , abbreviated G -CO over \mathcal{F}_G , is a $G\mathcal{U}_*$ -category, with objects the based G -sets \mathbf{n}^α for $n \geq 0$ and $\alpha: G \rightarrow \Sigma_n$, together with $G\mathcal{U}_*$ -functors

$$\Pi_G \xrightarrow{\iota_G} \mathcal{D}_G \xrightarrow{\xi_G} \mathcal{F}_G$$

such that ι_G and ξ_G are the identity on objects and $\xi_G \circ \iota_G$ is the inclusion. We say that \mathcal{D}_G is reduced if $\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$ is a point if either $m = 0$ or $n = 0$, and we restrict attention to reduced G -COs over \mathcal{F}_G henceforward. A morphism $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$ of G -COs over \mathcal{F}_G is a $G\mathcal{U}_*$ -functor over \mathcal{F}_G and under Π_G . In particular, $\xi_G: \mathcal{D}_G \rightarrow \mathcal{F}_G$ is a map of G -COs over \mathcal{F}_G for any G -CO \mathcal{D}_G over \mathcal{F}_G .

We similarly have G -maps

$$\Pi_G(\mathbf{m}^\alpha, \mathbf{n}^\beta) \rightarrow \mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta) \rightarrow \mathcal{F}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$$

whose composite is the inclusion.

Following [57, Addendum 1.7], cofibration conditions will be added to the previous two definitions in [Addendum 4.6](#).

The full subcategory $\mathcal{D} \subset \mathcal{D}_G$ with objects \mathbf{n} is a G -category of operators over \mathcal{F} . Conversely, just as we constructed Π_G and \mathcal{F}_G from Π and \mathcal{F} , we can construct a G -CO \mathcal{D}_G over \mathcal{F}_G from any G -CO \mathcal{D} over \mathcal{F} . As noted in [26, Proposition 6.9], up to isomorphism all \mathcal{D}_G can be constructed in this fashion.

Construction 4.3. Let \mathcal{D} be a G -category of operators over \mathcal{F} . We define a G -category of operators \mathcal{D}_G over \mathcal{F}_G whose full subcategory of objects \mathbf{n} is \mathcal{D} . The morphism G -space $\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$ is the space $\mathcal{D}(\mathbf{m}, \mathbf{n})$, with G -action induced by conjugation and the original G -action on $\mathcal{D}(\mathbf{m}, \mathbf{n})$. Explicitly, for $f \in \mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$,

$$g \cdot f = \beta(g) \circ (gf) \circ \alpha(g^{-1});$$

We check that $g \cdot (h \cdot f) = (gh) \cdot f$ using that G acts trivially on permutations since they are in the image of Π . Composition and identity maps are inherited from \mathcal{D} and are appropriately equivariant.

A routine verification shows the following.

Lemma 4.4. *The inclusion $\mathcal{D} \hookrightarrow \mathcal{D}_G$ makes the following diagram of $G\mathcal{U}_*$ -categories commute.*

$$(4.5) \quad \begin{array}{ccccc} \Pi & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_G & \longrightarrow & \mathcal{D}_G & \longrightarrow & \mathcal{F}_G. \end{array}$$

Moreover, a map $\nu: \mathcal{D} \rightarrow \mathcal{E}$ of G -COs over \mathcal{F} induces a map

$$\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$$

of G -COs over \mathcal{F}_G that is compatible with the inclusions.

The following cofibration conditions will hold automatically for categories of operators constructed from operads.

Addendum 4.6. We add the following cofibration conditions to the definition of a G -CO \mathcal{D}_G over \mathcal{F}_G .

- (i) The morphism G -spaces $\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$ are nondegenerately based.
- (ii) The inclusions $* \rightarrow \mathcal{D}_G(\mathbf{n}^\alpha, \mathbf{n}^\alpha)$ of identity maps are G -cofibrations.
- (iii) The map $\mathcal{D}_G(\mathbf{q}^\alpha, \mathbf{m}) \rightarrow \mathcal{D}_G(\mathbf{q}^\alpha, \mathbf{n})$ induced by an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in $\Pi \subset \Pi_G$ is a $(G \times \Sigma_\phi)$ -cofibration, where Σ_ϕ is as defined in [Notation 2.2](#).

We add to the definition of a G -CO \mathcal{D} over \mathcal{F} the requirement that its prolonged category of operators \mathcal{D}_G satisfies the conditions just specified. Note that these conditions imply in particular that \mathcal{D} satisfies the following.

- (i) The morphism G -spaces $\mathcal{D}(\mathbf{m}, \mathbf{n})$ are nondegenerately based.
- (ii) The inclusions $* \rightarrow \mathcal{D}(\mathbf{n}, \mathbf{n})$ of identity maps are G -cofibrations.
- (iii) The map $\mathcal{D}(\mathbf{q}, \mathbf{m}) \rightarrow \mathcal{D}(\mathbf{q}, \mathbf{n})$ induced by an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ is a $(G \times \Sigma_\phi)$ -cofibration.

We have the following generalizations of [Definitions 2.10, 2.14, 2.33](#) and [2.35](#) from \mathcal{F} and \mathcal{F}_G to G -categories of operators \mathcal{D} and \mathcal{D}_G .

Definition 4.7. A \mathcal{D} - G -space X is a $G\mathcal{U}_*$ -functor $X: \mathcal{D} \rightarrow G\mathcal{U}_*$ whose restriction to Π is a Π - G -space. A map of \mathcal{D} - G -spaces is a $G\mathcal{U}_*$ -natural transformation. We say that X is \mathbb{F}_\bullet -special if its underlying Π - G -space is \mathbb{F}_\bullet -special. We say that a map of \mathcal{D} - G -spaces is an \mathbb{F}_\bullet -level equivalence if its underlying map of Π - G -spaces is an \mathbb{F}_\bullet -level equivalence. Let $\mathcal{D}[\mathcal{T}_G]$ denote the category of \mathcal{D} - G -spaces.

Definition 4.8. A \mathcal{D}_G - G -space Y is a $G\mathcal{U}_*$ -functor $Y: \mathcal{D}_G \rightarrow G\mathcal{U}_*$ whose restriction to Π_G is a Π_G - G -space. A map of \mathcal{D}_G - G -spaces is a $G\mathcal{U}_*$ -natural transformation. We say that Y is special if its underlying Π_G - G -space is special. We say that a map of \mathcal{D}_G - G -spaces is a level G -equivalence if its underlying map of Π_G - G -spaces is a level G -equivalence. Let $\mathcal{D}_G[\mathcal{T}_G]$ denote the category of \mathcal{D}_G - G -spaces.

Generalizing [Proposition 2.12](#), we show in [Section 11](#) that the implied cofibration assumptions result in no loss of generality. Condition (iii) in [Addendum 4.6](#) ensures that the following analogues of [Lemma 3.16](#) hold. In the context of maps $\nu: \mathcal{D} \rightarrow \mathcal{F}$ or $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$, they have evident generalizations to bar constructions $B(\mathcal{E}, \mathcal{D}, X)$ and $B(\mathcal{E}_G, \mathcal{D}_G, Y)$.

Lemma 4.9. *Let X be a \mathcal{D} - G -space. Then $B(\mathcal{D}, \mathcal{D}, X)$ is a \mathcal{D} - G -space.*

Lemma 4.10. *Let Y be a \mathcal{D}_G - G -space. Then $B(\mathcal{D}_G, \mathcal{D}_G, Y)$ is a \mathcal{D}_G - G -space.*

4.2. **The equivalence between $\mathcal{D}[\mathcal{T}_G]$ and $\mathcal{D}_G[\mathcal{T}_G]$.** We can now generalize Section 2.4 to a comparison between \mathcal{D} - G -spaces and \mathcal{D}_G - G -spaces. The forgetful functor

$$\mathbb{U}: \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*) \longrightarrow \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*)$$

has a left adjoint prolongation functor

$$\mathbb{P}: \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*) \longrightarrow \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*).$$

Explicitly,

$$(\mathbb{P}X)(\mathbf{n}^\alpha) = \mathcal{D}_G(-, \mathbf{n}^\alpha) \otimes_{\mathcal{D}} X = \bigvee_m \mathcal{D}_G(\mathbf{m}, \mathbf{n}^\alpha) \wedge X_m / \sim,$$

where $(f, \phi_*x) \sim (f\phi, x)$ for a map $\phi: \mathbf{k} \rightarrow \mathbf{m}$ in \mathcal{D} , an element $x \in X_k$, and a map $f: \mathbf{m} \rightarrow \mathbf{n}^\alpha$ in $\mathcal{D}_G(\mathbf{m}, \mathbf{n}^\alpha)$. (We have written out this coequalizer of G -spaces explicitly to facilitate checks of details). The following result generalizes Theorem 2.38 from \mathcal{F} to an arbitrary G - CO over \mathcal{F} .

Theorem 4.11. *The adjoint pair of functors*

$$\text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*) \begin{array}{c} \xrightarrow{\mathbb{P}} \\ \xleftarrow{\mathbb{U}} \end{array} \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*)$$

specifies an equivalence of categories. Remembering the cofibration condition from Definitions 4.7 and 4.8, this equivalence restricts to an equivalence

$$\mathcal{D}[\mathcal{T}_G] \begin{array}{c} \xrightarrow{\mathbb{P}} \\ \xleftarrow{\mathbb{U}} \end{array} \mathcal{D}_G[\mathcal{T}_G].$$

The proof is very similar to that of the special case $\mathcal{D} = \mathcal{F}$ dealt with in Theorem 2.38. Only points of equivariance require comment, and the following lemma is the key to understanding the relevant G -actions. It identifies the G -space $(\mathbb{P}X)(\mathbf{n}^\alpha)$ with the G -space X_n^α of Definition 1.3.

Lemma 4.12. *For a $G\mathcal{U}_*$ -functor $X: \mathcal{D} \rightarrow \underline{G\mathcal{U}}_*$, the G -space $(\mathbb{P}X)(\mathbf{n}^\alpha)$ is G -homeomorphic to the G -space X_n^α , namely X_n with the G -action \cdot_α specified by $g \cdot_\alpha x = \alpha(g)_*(gx)$. Via this homeomorphism, the evaluation maps*

$$\mathcal{D}_G(\mathbf{n}^\alpha, \mathbf{p}^\beta) \wedge (\mathbb{P}X)(\mathbf{n}^\alpha) \longrightarrow (\mathbb{P}X)(\mathbf{p}^\beta)$$

are given on the underlying spaces by the corresponding maps for X ,

$$\mathcal{D}(\mathbf{n}, \mathbf{p}) \wedge X_n \longrightarrow X_p.$$

Proof. Modulo equivariance, this is an application of the Yoneda lemma. Write $\text{id}_\alpha: \mathbf{n} \rightarrow \mathbf{n}^\alpha$ for $\text{id}: \mathbf{n} \rightarrow \mathbf{n}$ regarded as an element of $\mathcal{D}_G(\mathbf{n}, \mathbf{n}^\alpha)$. Define

$$F: X_n^\alpha \longrightarrow \mathcal{D}_G(-, \mathbf{n}^\alpha) \otimes_{\mathcal{D}} X$$

by sending $x \in X_n$ to the equivalence class of (id_α, x) . Then F is a G -map since

$$F(g \cdot_\alpha x) = (\text{id}_\alpha, g \cdot_\alpha x) = (\text{id}_\alpha, \alpha(g)_*(gx)) \sim (\text{id}_\alpha \circ \alpha(g), gx) = (g \text{id}_\alpha, gx).$$

Define an inverse map

$$F^{-1}: \mathcal{D}_G(-, \mathbf{n}^\alpha) \otimes_{\mathcal{D}} X \longrightarrow X_n^\alpha$$

by sending the equivalence class of $(f, x) \in \mathcal{D}_G(\mathbf{m}, \mathbf{n}^\alpha) \wedge X_m$ to $f_*(x)$, where we think of f as a map $\mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{D} and interpret $f_*(x)$ to mean $X(f)(x) \in X_n$. Note that F^{-1} is well defined. We have

$$F^{-1}F(x) = F^{-1}(\text{id}_\alpha, x) = \text{id}_*x = x$$

and

$$FF^{-1}(f, x) = F(f_*x) = (\text{id}_\alpha, f_*x) \sim (f, x),$$

hence F and F^{-1} are inverse homeomorphisms. Note that F^{-1} is automatically a G -map since it is inverse to the G -map F . The compatibility with the action of \mathcal{D}_G is clear. \square

Using this, we mimic the proof of [Theorem 2.38](#) to prove the equivalence of the categories of \mathcal{D} - G -spaces and \mathcal{D}_G - G -spaces.

Proof of [Theorem 4.11](#). Since the inclusion $\mathcal{D} \rightarrow \mathcal{D}_G$ is full and faithful, the unit $X \rightarrow \mathbb{U}PX$ of the adjunction is an isomorphism for any $G\mathcal{U}_*$ -functor $X: \mathcal{D} \rightarrow \underline{G\mathcal{U}_*}$. Let Y be a $G\mathcal{U}_*$ -functor $\mathcal{D}_G \rightarrow \underline{G\mathcal{U}_*}$. We must show that the counit $\mathbb{P}UY \rightarrow Y$ of the adjunction is an isomorphism. A check of definitions shows that the counit G -map $(\mathbb{P}UY)(\mathbf{n}^\alpha) \rightarrow Y(\mathbf{n}^\alpha)$ agrees under the isomorphism of [Lemma 4.12](#) with the map, necessarily a G -map,

$$\text{id}_{\alpha_*}: Y_n^\alpha \rightarrow Y(\mathbf{n}^\alpha)$$

induced by the morphism $\text{id}_\alpha \in \mathcal{D}_G(\mathbf{n}, \mathbf{n}^\alpha)$. Writing ${}_\alpha\text{id}: \mathbf{n}^\alpha \rightarrow \mathbf{n}$ for $\text{id}: \mathbf{n} \rightarrow \mathbf{n}$ regarded as an element of $\mathcal{D}_G(\mathbf{n}^\alpha, \mathbf{n})$, we see that ${}_\alpha\text{id}$ induces the inverse homeomorphism

$${}_\alpha\text{id}_*: Y(\mathbf{n}^\alpha) \rightarrow Y_n^\alpha$$

to id_{α_*} . Again, ${}_\alpha\text{id}_*$ is automatically a G -map since it is inverse to a G -map.

The restriction of the equivalence to the subcategories of \mathcal{D} - G -spaces and \mathcal{D}_G - G -spaces follows directly from the definitions of our cofibration conditions, since these depend only on the underlying Π - G -spaces. \square

Just as for \mathcal{F} - G -spaces in [Section 2.4](#), a \mathcal{D} - G -space X has two Π_G - G -spaces associated to it. We can either apply \mathbb{P} to its underlying Π - G -space or we can apply \mathbb{P} to X and take its underlying Π_G - G -space. [Lemma 4.12](#) implies that these two Π_G - G -spaces coincide. Therefore the four statements about Π and Π_G that are listed in [Theorem 2.38](#) also hold for \mathcal{D} and \mathcal{D}_G . We record them in the following two corollaries.

Corollary 4.13. *A \mathcal{D} - G -space X is \mathbb{F}_\bullet -special if and only if the \mathcal{D}_G - G -space $\mathbb{P}X$ is special. A \mathcal{D}_G - G -space Y is special if and only if the \mathcal{D} - G -space $\mathbb{U}Y$ is \mathbb{F}_\bullet -special.*

Corollary 4.14. *A map f of \mathcal{D} - G -spaces is an \mathbb{F}_\bullet -level equivalence if and only if $\mathbb{P}f$ is a level G -equivalence of \mathcal{D}_G - G -spaces. A map f of \mathcal{D}_G - G -spaces is a level G -equivalence if and only if $\mathbb{U}f$ is an \mathbb{F}_\bullet -level equivalence of \mathcal{D} - G -spaces.*

4.3. Comparisons of \mathcal{D} - G -spaces and \mathcal{E} - G -spaces for $\nu: \mathcal{D} \rightarrow \mathcal{E}$. Precomposition with $\xi: \mathcal{D} \rightarrow \mathcal{F}$ induces a functor $\xi^*: \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}_*}) \rightarrow \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}_*})$, and the parallel conditions in [Definitions 2.10](#) and [4.7](#) imply that ξ^* restricts to give a functor $\mathcal{F}[G\mathcal{T}] \rightarrow \mathcal{D}[\mathcal{T}_G]$. Moreover, an \mathcal{F} - G -space X and the \mathcal{D} - G -space $\xi^*X = X \circ \xi$ have the same underlying Π - G -space, hence one is \mathbb{F}_\bullet -special or special if and only if the other is so. The functor ξ^* has a left adjoint given by the left Kan extension along ξ , but that is not well behaved homotopically. Instead, following [\[57, Theorem 1.8\]](#) nonequivariantly, we expect the bar construction to give a variant that is homotopically well-behaved. With \mathcal{F} replaced by \mathcal{D} , the analogue of [Proposition 3.17](#) holds and admits the same proof.

To implement this strategy, we start with an \mathbb{F}_\bullet -special \mathcal{D} - G -space Y and construct from it an \mathbb{F}_\bullet -special \mathcal{F} - G -space ξ_*Y together with a zigzag of \mathbb{F}_\bullet -equivalences

between Y and $\xi^*\xi_*Y$. We shall use this to construct a Segal machine whose input is an \mathbb{F}_\bullet -special \mathcal{D} - G -space Y and whose output is equivalent to $\mathbb{S}_G^{\mathcal{F}}\xi_*Y$.

As in [57], we work more generally here, starting from a map $\nu: \mathcal{D} \rightarrow \mathcal{E}$ of G -COs over \mathcal{F} and comparing \mathcal{D} - G -spaces and \mathcal{E} - G -spaces. We are mainly interested in the case $\nu = \xi$. We write $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$ for the induced map of G -COs over \mathcal{F}_G . Focus on ν_G rather than ν allows us to focus on G -equivalence rather than \mathbb{F}_\bullet -equivalence. For clarity, we sometimes write $\mathbb{U}_{\mathcal{D}}$ and $\mathbb{P}_{\mathcal{D}}$ instead of \mathbb{U} and \mathbb{P} for the adjunction between $\text{Fun}(\mathcal{D}, \underline{G}\mathcal{U}_*)$ and $\text{Fun}(\mathcal{D}_G, \underline{G}\mathcal{U}_*)$, and similarly for \mathcal{E} .

Definition 4.15. For $Z \in \text{Fun}(\mathcal{D}_G, \underline{G}\mathcal{U}_*)$, define $\nu_*^G Z \in \text{Fun}(\mathcal{E}_G, \underline{G}\mathcal{U}_*)$ by

$$\nu_*^G Z = B(\mathcal{E}_G, \mathcal{D}_G, Z).$$

Here the target is defined levelwise by replacing \mathcal{E}_G by the composite

$$\mathcal{E}_G(-, \mathbf{n}^\alpha) \circ \nu_G: \mathcal{D}^{op} \rightarrow \mathcal{I}_G,$$

of ν_G and the $\underline{G}\mathcal{U}_*$ -functor $\mathcal{E}_G^{op} \rightarrow \mathcal{I}_G$ represented by \mathbf{n}^α . If Z is moreover a \mathcal{D}_G - G -space, so that each $Z(\mathbf{n}^\alpha)$ is nondegenerately based, then all of the conditions in [Assumption 3.10](#) are satisfied, and the bar construction is the geometric realization of a Reedy cofibrant simplicial G -space. Moreover, as in [Lemma 3.16](#), condition (iii) of [Addendum 4.6](#) implies that $\nu_*^G Z$ is indeed a \mathcal{E}_G - G -space. We thus obtain a functor

$$\nu_*^G: \mathcal{D}_G[\mathcal{I}_G] \rightarrow \mathcal{E}_G[\mathcal{I}_G].$$

For $Y \in \text{Fun}(\mathcal{D}, \underline{G}\mathcal{U}_*)$, define $\nu_* Y \in \text{Fun}(\mathcal{E}, \underline{G}\mathcal{U}_*)$ by

$$\nu_* Y = \mathbb{U}_{\mathcal{E}} \nu_*^G \mathbb{P}_{\mathcal{D}} Y = \mathbb{U}_{\mathcal{E}} B(\mathcal{E}_G, \mathcal{D}_G, \mathbb{P}_{\mathcal{D}} Y).$$

If Y is moreover a \mathcal{D} - G -space, we are again in the situation of [Assumption 3.10](#). We thus obtain a functor

$$\nu_*: \mathcal{D}[\mathcal{I}_G] \rightarrow \mathcal{E}[\mathcal{I}_G].$$

Definition 4.16. A map $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$ of G -COs over \mathcal{F}_G is a G -equivalence if each map

$$\nu_G: \mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta) \rightarrow \mathcal{E}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$$

is a weak G -equivalence. A map $\nu: \mathcal{D} \rightarrow \mathcal{E}$ of G -COs over \mathcal{F} is an \mathbb{F}_\bullet -equivalence if the associated map $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$ of G -COs over \mathcal{F}_G is a G -equivalence.

Recall the notion of an \mathbb{F}_\bullet -level equivalence of Π - G -spaces from [Definition 2.14](#) and [Lemma 2.18](#). Recall too that a map of Π - G -spaces is an \mathbb{F}_\bullet -level equivalence if and only if its associated map of Π_G - G -spaces is a level G -equivalence; see [Definition 2.35](#) and [Theorem 2.38\(iv\)](#). These definitions and results are inherited by \mathcal{D} and \mathcal{D}_G (as in [Corollary 4.14](#)). The following definition recalls notation from [Section 3.1](#).

Definition 4.17. For a G -CO \mathcal{D} over \mathcal{F} and a fixed n , let \mathcal{D}_n be the corepresented \mathcal{D} - G -space specified by $\mathcal{D}_n(\mathbf{p}) = \mathcal{D}(\mathbf{n}, \mathbf{p})$, with the action of \mathcal{D} given by composition. Analogously, we have corepresented \mathcal{D}_G - G -spaces $(\mathcal{D}_G)_{\mathbf{n}^\alpha}$.

Lemma 4.18. *If $\nu: \mathcal{D} \rightarrow \mathcal{E}$ is an \mathbb{F}_\bullet -equivalence of G -COs over \mathcal{F} , then for each n , ν restricts to an \mathbb{F}_\bullet -level equivalence of \mathcal{D} - G -spaces $\mathcal{D}_n \rightarrow \nu^* \mathcal{E}_n$.*

Proof. One can check that the cited restriction is a map of \mathcal{D} - G -spaces. Moreover, an easy comparison of definitions shows that $\mathbb{P}\mathcal{D}_n$ can be identified with $(\mathcal{D}_G)_{\mathbf{n}}$. Therefore the conclusion follows from [Corollary 4.14](#) and our definition of an \mathbb{F}_\bullet -equivalence of G -COs over \mathcal{F} , which of course was chosen in order to make this and cognate results true. \square

Theorem 4.19. *Let $\nu_G: \mathcal{D}_G \rightarrow \mathcal{E}_G$ be a G -equivalence of G -COs over \mathcal{F}_G .*

- (i) *Let X be an \mathcal{E}_G - G -space and Y a \mathcal{D}_G - G -space. Then there are natural zigzags of level G -equivalences between $\nu_*^G \nu_*^G X$ and X and between $\nu_*^G \nu_*^G Y$ and Y .*
- (ii) *Y is a special \mathcal{D}_G - G -space if and only if $\nu_*^G Y$ is a special \mathcal{E}_G - G -space.*
- (iii) *A map f of \mathcal{D}_G - G -spaces is a level G -equivalence if and only if $\nu_*^G f$ is a level G -equivalence of \mathcal{E}_G - G -spaces.*

Proof. Abbreviate notation here by writing α for a finite G -set \mathbf{n}^α and X_α for $X(\mathbf{n}^\alpha)$ when X is an \mathcal{E}_G -space. Starting with X , we have the natural maps

$$(4.20) \quad (\nu_*^G \nu_*^G X)_\alpha = B(\mathcal{E}_G, \mathcal{D}_G, \nu_*^G X)_\alpha \xrightarrow{B(\text{id}, \nu_G, \text{id})} B(\mathcal{E}_G, \mathcal{E}_G, X)_\alpha \xrightarrow{\varepsilon} X_\alpha.$$

Starting with Y we have the natural maps

$$(4.21) \quad (\nu_*^G \nu_*^G Y)_\alpha = \nu_*^G B(\mathcal{E}_G, \mathcal{D}_G, Y)_\alpha \xleftarrow{B(\nu_G, \text{id}, \text{id})} B(\mathcal{D}_G, \mathcal{D}_G, Y)_\alpha \xrightarrow{\varepsilon} Y_\alpha.$$

The maps ε with targets X_α and Y_α are G -equivalences, with the usual inverse equivalences η , as in the proof of [Proposition 3.17](#). At each level α , the other two maps are realizations of levelwise simplicial G -equivalences, and the Reedy cofibrancy of the simplicial bar construction ensures that these realizations are themselves G -equivalences, by [Theorem 1.12](#). Note that we do not need X or Y to be special to prove (i).

By [Lemma 2.42](#), (i) implies (ii). Indeed, Y is special if and only if $\nu_*^G \nu_*^G Y$ is special and, since $\nu_*^G \nu_*^G Y$ and $\nu_*^G Y$ have the same underlying Π - G -space, one is special if and only if the other is so. Part (iii) follows from the naturality of the G -equivalences in (4.21). \square

As in [Lemma 4.4](#), the following diagram of $G\mathcal{U}_*$ -categories commutes.

$$\begin{array}{ccccccc} \Pi & \longrightarrow & \mathcal{D} & \xrightarrow{\nu} & \mathcal{E} & \xrightarrow{\xi} & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Pi_G & \longrightarrow & \mathcal{D}_G & \xrightarrow{\nu_G} & \mathcal{E}_G & \xrightarrow{\xi_G} & \mathcal{F}_G \end{array}$$

Therefore $\nu^* \mathbb{U}_\mathcal{E} = \mathbb{U}_\mathcal{D} \nu^*$, as we shall use in the proof of the following analogue of [Theorem 4.19](#) for G -COs over \mathcal{F} .

Theorem 4.22. *Let $\nu: \mathcal{D} \rightarrow \mathcal{E}$ be an \mathbb{F}_\bullet -equivalence of G -COs over \mathcal{F} .*

- (i) *Let X be an \mathcal{E} - G -space and Y be a \mathcal{D} - G -space. Then there are natural zigzags of \mathbb{F}_\bullet -equivalences between $\nu_* \nu^* X$ and X and between $\nu_* \nu^* Y$ and Y .*
- (ii) *Y is an \mathbb{F}_\bullet -special \mathcal{D} - G -space if and only if $\nu_* Y$ is an \mathbb{F}_\bullet -special \mathcal{E} - G -space.*
- (iii) *A map f of \mathcal{D} - G -spaces is an \mathbb{F}_\bullet -level equivalence if and only if $\nu_* f$ is an \mathbb{F}_\bullet -level equivalence of \mathcal{E} - G -spaces.*

Proof. Recall from [Theorem 4.11](#) that $(\mathbb{P}_{\mathcal{D}}, \mathbb{U}_{\mathcal{D}})$ is an adjoint equivalence of categories, and similarly for \mathcal{E} . Note that we have the following sequence of natural isomorphisms of \mathcal{D}_G - G -spaces

$$\mathbb{P}_{\mathcal{D}}\nu^*X \cong \mathbb{P}_{\mathcal{D}}\nu^*\mathbb{U}_{\mathcal{E}}\mathbb{P}_{\mathcal{E}}X = \mathbb{P}_{\mathcal{D}}\mathbb{U}_{\mathcal{D}}\nu_G^*\mathbb{P}_{\mathcal{E}}X \cong \nu_G^*\mathbb{P}_{\mathcal{E}}X.$$

To prove (i), write \simeq to indicate a zigzag of \mathbb{F}_{\bullet} -level equivalences. Recall from [Corollary 4.14](#) that \mathbb{P} takes \mathbb{F}_{\bullet} -level equivalences to level G -equivalences and \mathbb{U} takes level G -equivalences to \mathbb{F}_{\bullet} -level equivalences, while ν_*^G preserves level G -equivalences by [Theorem 4.19\(iii\)](#). Therefore, by [Theorem 4.19\(i\)](#), we have the zigzags

$$\nu_*\nu^*X = \mathbb{U}_{\mathcal{E}}\nu_*^G\mathbb{P}_{\mathcal{D}}\nu^*X \cong \mathbb{U}_{\mathcal{E}}\nu_*^G\nu_G^*\mathbb{P}_{\mathcal{E}}X \simeq \mathbb{U}_{\mathcal{E}}\mathbb{P}_{\mathcal{E}}X \cong X$$

and

$$\nu^*\nu_*Y = \nu^*\mathbb{U}_{\mathcal{E}}\nu_*^G\mathbb{P}_{\mathcal{D}}Y = \mathbb{U}_{\mathcal{D}}\nu_G^*\nu_*^G\mathbb{P}_{\mathcal{D}}Y \simeq \mathbb{U}_{\mathcal{D}}\mathbb{P}_{\mathcal{D}}Y \cong Y$$

of level G -equivalences. Using [Corollary 4.14](#), (ii) and (iii) follow as in the proof of [Theorem 4.19](#). \square

4.4. Comparisons of inputs and outputs of the generalized Segal machine.

Definition 4.23. We say that a G - CO \mathcal{D}_G over \mathcal{F}_G is an E_{∞} G - CO if the map $\xi_G: \mathcal{D}_G \rightarrow \mathcal{F}_G$ is a G -equivalence. We say that a G - CO \mathcal{D} over \mathcal{F} is an E_{∞} G - CO if its associated \mathcal{D}_G is an E_{∞} G - CO over \mathcal{F}_G .

The term “ E_{∞} ” is convenient, but it is a little misleading, as will become clear when we turn to operads.

We assume throughout this section that \mathcal{D} is an E_{∞} G - CO over \mathcal{F} , and we specialize the results of the previous section to $\xi: \mathcal{D} \rightarrow \mathcal{F}$ and $\xi_G: \mathcal{D}_G \rightarrow \mathcal{F}_G$. The following results are just specializations of [Theorems 4.19](#) and [4.22](#).

Theorem 4.24. *The following conclusions hold.*

- (i) *Let X be an \mathcal{F}_G - G -space and Y a \mathcal{D}_G - G -space. Then there are natural zigzags of level G -equivalences between $\xi_*^G\xi^*X$ and X and between $\xi_G^*\xi_*^GY$ and Y .*
- (ii) *Y is a special \mathcal{D}_G - G -space if and only if ξ_*^GY is a special \mathcal{F}_G - G -space.*
- (iii) *A map f of \mathcal{D}_G - G -spaces is a level G -equivalence if and only if ξ_*^Gf is a level G -equivalence of \mathcal{F}_G - G -spaces.*

Theorem 4.25. *The following conclusions hold.*

- (i) *Let X be an \mathcal{F} - G -space and Y a \mathcal{D} - G -space. Then there are natural zigzags of \mathbb{F}_{\bullet} -equivalences between $\xi_*\xi^*X$ and X and between $\xi^*\xi_*Y$ and Y .*
- (ii) *Y is an \mathbb{F}_{\bullet} -special \mathcal{D} - G -space if and only if ξ_*Y is an \mathbb{F}_{\bullet} -special \mathcal{F} - G -space.*
- (iii) *A map f of \mathcal{D} - G -spaces is an \mathbb{F}_{\bullet} -level equivalence if and only if ξ_*f is an \mathbb{F}_{\bullet} -level equivalence of \mathcal{F} - G -spaces.*

Therefore the pair of functors

$$\xi_*^G: \mathcal{F}_G[\mathcal{T}_G] \rightarrow \mathcal{D}_G[\mathcal{T}_G] \quad \text{and} \quad \xi_*^G: \mathcal{D}_G[\mathcal{T}_G] \rightarrow \mathcal{F}_G[\mathcal{T}_G]$$

induce inverse equivalences between the homotopy categories obtained by inverting the respective level G -equivalences, and this remains true if we restrict to special objects. Similarly, the pair of functors

$$\xi^*: \mathcal{F}[G\mathcal{T}] \rightarrow \mathcal{D}[\mathcal{T}_G] \quad \text{and} \quad \xi^*: \mathcal{D}[\mathcal{T}_G] \rightarrow \mathcal{F}[G\mathcal{T}]$$

induce inverse equivalences between the homotopy categories obtained by inverting the respective \mathbb{F}_{\bullet} -level equivalences, and this remains true if we restrict to \mathbb{F}_{\bullet} -special

objects. We conclude that the four input categories for Segal machines displayed in the square of the following diagram are essentially equivalent. Formally, as we shall see in [Section 10.2](#), we can reinterpret this comparison in terms of Quillen equivalences of model categories.

$$(4.26) \quad \begin{array}{ccc} \mathcal{F}[G\mathcal{T}] & \xleftarrow{\mathbb{U}} & \mathcal{F}_G[\mathcal{T}_G] \\ \xi_* \downarrow & & \downarrow \xi_G^* \\ \mathcal{D}[\mathcal{T}_G] & \xleftarrow{\mathbb{U}} & \mathcal{D}_G[\mathcal{T}_G] \end{array}$$

Here $\xi_*\mathbb{U} = \mathbb{U}\xi_G^*$. We regard the four categories in the square as possible domain categories for generalized Segal infinite loop space machines.

By [Theorems 2.38](#) and [4.11](#), the inclusions of \mathcal{F} in \mathcal{F}_G and \mathcal{D} in \mathcal{D}_G induce equivalences of categories, denoted \mathbb{U} in the diagram. Their left adjoints \mathbb{P} give inverses, and the functors \mathbb{U} and \mathbb{P} preserve the relevant special objects and levelwise equivalences. [Theorems 4.24](#) and [4.25](#) show that the vertical arrows ξ_* and ξ_G^* become equivalences of homotopy categories with inverses ξ_*^G and ξ_G^* (not the left adjoints) after inverting the relevant equivalences. Consider the following diagram of functors.

$$(4.27) \quad \begin{array}{ccccc} \mathcal{F}[G\mathcal{T}] & \xrightarrow{\mathbb{P}} & \mathcal{F}_G[\mathcal{T}_G] & \xrightarrow{\mathbb{P}} & \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}}_*) \xrightarrow{\mathbb{U}_{G,\mathcal{F}}} G\mathcal{S} \\ \xi_* \uparrow & & \uparrow \xi_*^G & & \\ \mathcal{D}[\mathcal{T}_G] & \xrightarrow{\mathbb{P}} & \mathcal{D}_G[\mathcal{T}_G] & & \end{array}$$

The isomorphism $\mathbb{P}\mathbb{U} \cong \text{id}$ on \mathcal{F}_G - G -spaces and the definitions of ξ_* and ξ_*^G imply that the square commutes up to natural isomorphism. The composite in the top row is our original conceptual Segal machine. We can specialize by letting $\mathcal{D} = \mathcal{F}$, $\xi = \text{id}: \mathcal{F} \rightarrow \mathcal{F}$ and $\xi_G = \text{id}: \mathcal{F}_G \rightarrow \mathcal{F}_G$. According to [Definition 3.24](#), writing \mathbb{I} for id_* , the composite $\mathbb{U}_{G,\mathcal{F}}\mathbb{P}\mathbb{I}$ is the Segal machine $\mathbb{S}_G^{\mathcal{F}_G}$ on \mathcal{F}_G - G -spaces Y and the composite $\mathbb{U}_{G,\mathcal{F}}\mathbb{P}\mathbb{I}\mathbb{P}$ is the Segal machine $\mathbb{S}_G^{\mathcal{F}}$ on \mathcal{F} - G -spaces X . That is

$$(4.28) \quad \mathbb{S}_G^{\mathcal{F}_G} Y = \mathbb{U}_{G,\mathcal{F}}\mathbb{P}\mathbb{I}Y \quad \text{and} \quad \mathbb{S}_G^{\mathcal{F}} X = \mathbb{U}_{G,\mathcal{F}}\mathbb{P}\mathbb{I}\mathbb{P}X \cong \mathbb{U}_{G,\mathcal{F}}\mathbb{P}\mathbb{I}X.$$

We define the composites obtained by replacing the functors $\mathbb{I} = \text{id}_*$ with ξ_* and ξ_*^G as generalized Segal machines $\mathbb{S}_G^{\mathcal{D}}$ defined on \mathcal{D} - G -spaces and $\mathbb{S}_G^{\mathcal{D}_G}$ defined on \mathcal{D}_G - G -spaces.

Definition 4.29. Let X be a \mathcal{D} - G -space and Y a \mathcal{D}_G - G -space. The generalized Segal machine is defined as

$$\mathbb{S}_G^{\mathcal{D}_G} Y = \mathbb{U}_{G,\mathcal{D}}\mathbb{P}\xi_*^G Y \quad \text{and} \quad \mathbb{S}_G^{\mathcal{D}} X = \mathbb{U}_{G,\mathcal{D}}\mathbb{P}\xi_*\mathbb{P}X \cong \mathbb{U}_{G,\mathcal{D}}\mathbb{P}\mathbb{P}\xi_* X.$$

Clearly the machines starting with \mathcal{F} - G -spaces or \mathcal{F}_G - G -spaces are equivalent and similarly for \mathcal{D} and \mathcal{D}_G . [Theorem 4.31](#) below will show that the machines starting with \mathcal{F} - G -spaces or \mathcal{D} - G -spaces and the machines starting with \mathcal{F}_G - G -spaces or \mathcal{D}_G - G -spaces are equivalent. Thus the four machines in sight are equivalent under our equivalences of input data. That is, we obtain equivalent output by starting at any of the four vertices of the square, converting input data from the other three vertices to that one, and taking the relevant machine \mathbb{S}_G . We use the following invariance principle in the proof of [Theorem 4.31](#).

Proposition 4.30. *The following conclusions about homotopy invariance hold.*

(i) *If $f: X \rightarrow Y$ is a level G -equivalence of \mathcal{D} - G -spaces, then*

$$f: B(A^\bullet, \mathcal{D}, X) \rightarrow B(A^\bullet, \mathcal{D}, Y)$$

is a weak G -equivalence for all $A \in G\mathcal{W}$.

(ii) *If $f: X \rightarrow Y$ is a level G -equivalence of \mathcal{D}_G - G -spaces, then the induced map*

$$f: B(A^\bullet, \mathcal{D}_G, X) \rightarrow B(A^\bullet, \mathcal{D}_G, Y)$$

is a weak G -equivalence for all $A \in G\mathcal{W}$.

(iii) *If $f: X \rightarrow Y$ is an \mathbb{F}_\bullet -level equivalence of \mathcal{D} - G -spaces, then the induced map*

$$f: B(A^\bullet, \mathcal{D}_G, \mathbb{P}X) \rightarrow B(A^\bullet, \mathcal{D}_G, \mathbb{P}Y)$$

is a weak G -equivalence for all $A \in G\mathcal{W}$.

Proof. Our bar constructions are all geometric realizations of Reedy cofibrant simplicial G -spaces, hence [Theorem 1.12](#) gives the first conclusion. The second statement follows similarly. The third follows from the second using that $\mathbb{P}f$ is a level G -equivalence of \mathcal{D}_G - G -spaces by [Corollary 4.14](#). \square

The limitations of the first part and need for the second are clear from the fact that $B(\mathcal{F}, \mathcal{D}, X)$ is only level G -equivalent, not \mathbb{F}_\bullet -level equivalent, to X .

Theorem 4.31. *The following four equivalences of outputs hold.*

(i) *If X is a \mathcal{D} - G -space, then the G -spectra $\mathbb{S}_G^{\mathcal{D}}X$ and $\mathbb{S}_G^{\mathcal{F}}\xi_*X$ are equivalent.*

(ii) *If Y is a \mathcal{D}_G - G -space, then the G -spectra $\mathbb{S}_G^{\mathcal{D}_G}Y$ and $\mathbb{S}_G^{\mathcal{F}_G}\xi_*^G Y$ are equivalent.*

(iii) *If X is an \mathcal{F} - G -space, then the G -spectra $\mathbb{S}_G^{\mathcal{F}}X$ and $\mathbb{S}_G^{\mathcal{D}}\xi^*X$ are equivalent.*

(iv) *If Y is an \mathcal{F}_G - G -space, then the G -spectra $\mathbb{S}_G^{\mathcal{F}_G}Y$ and $\mathbb{S}_G^{\mathcal{D}_G}\xi_G^*Y$ are equivalent.*

Proof. We first prove (ii), which is the hardest part, and then show the rest. Thus let Y be a \mathcal{D}_G - G -space. We claim that the \mathcal{W}_G - G -spaces $\mathbb{P}\xi_*^G Y$ and $\mathbb{P}\mathbb{I}\xi_*^G Y$ are level equivalent. Applying $\mathbb{U}_{G,\mathcal{D}}$, this will give (ii). Thus let $A \in \mathcal{W}_G$. Then

$$(\mathbb{P}\xi_*^G Y)(A) = A^\bullet \otimes_{\mathcal{F}_G} B(\mathcal{F}_G, \mathcal{D}_G, Y) \cong B(A^\bullet, \mathcal{D}_G, Y)$$

and

$$(\mathbb{P}\mathbb{I}\xi_*^G Y)(A) = A^\bullet \otimes_{\mathcal{F}_G} B(\mathcal{F}_G, \mathcal{F}_G, \xi_*^G Y) \cong B(A^\bullet, \mathcal{F}_G, \xi_*^G Y).$$

In both cases, the isomorphism comes from [Lemma 3.7](#). By [Theorem 4.24](#)(i) there is a natural zigzag of level G -equivalences between $\xi_G^* \xi_*^G Y$ and Y . By [Proposition 4.30](#)(ii), this gives a zigzag of level G -equivalences of \mathcal{D}_G - G -spaces

$$B(A^\bullet, \mathcal{D}_G, Y) \simeq B(A^\bullet, \mathcal{D}_G, \xi_G^* \xi_*^G Y).$$

Since $\xi_G: \mathcal{D}_G \rightarrow \mathcal{F}_G$ is a G -equivalence of G -categories of operators, $B(\text{id}, \xi_G, \text{id})$ induces an equivalence at the level of q -simplices of the bar constructions

$$B(A^\bullet, \mathcal{D}_G, \xi_G^* \xi_*^G Y) \rightarrow B(A^\bullet, \mathcal{F}_G, \xi_*^G Y).$$

Again, since these bar constructions are geometric realizations of Reedy cofibrant simplicial G -spaces, we get a weak G -equivalence on geometric realizations by [Theorem 1.12](#). This proves the claim and thus proves (ii).

To prove (i), let X be a \mathcal{D} - G -space. Applying (ii) to $Y = \mathbb{P}X$ and using [\(4.28\)](#) and [Definition 4.29](#), we see that

$$\mathbb{S}_G^{\mathcal{F}} X = \mathbb{S}_G^{\mathcal{F}_G} \mathbb{P}X \simeq \mathbb{S}_G^{\mathcal{D}_G} \xi_*^G \mathbb{P}X \cong \mathbb{S}_G^{\mathcal{D}_G} \mathbb{P}\xi_* X = \mathbb{S}_G^{\mathcal{D}} \xi_* X.$$

To prove (iv), let Y be an \mathcal{F}_G - G -space. We claim that the \mathcal{W}_G - G -spaces $\mathbb{P}Y$ and $\mathbb{P}\xi_*^G \xi_G^* Y$ are level equivalent. Here

$$(\mathbb{P}Y)(A) = A^\bullet \otimes_{\mathcal{F}_G} B(\mathcal{F}_G, \mathcal{F}_G, Y) \cong B(A^\bullet, \mathcal{F}_G, Y)$$

and

$$(\mathbb{P}\xi_*^G \xi_G^* Y)(A) = A^\bullet \otimes_{\mathcal{F}_G} B(\mathcal{F}_G, \mathcal{D}_G, \xi_G^* Y) \cong B(A^\bullet, \mathcal{D}_G, \xi_G^* Y).$$

The isomorphisms again come from [Lemma 3.7](#). Just as before, $B(\text{id}, \xi_G, \text{id})$ induces a weak G -equivalence

$$B(A^\bullet, \mathcal{D}_G, \xi_G^* Y) \longrightarrow B(A^\bullet, \mathcal{F}_G, Y).$$

Finally, (iii) follows by application of (iv) to $Y = \mathbb{P}X$. \square

Use of the generalized homotopical Segal machine will be convenient when we compare the Segal and operadic machines, but it is logically unnecessary. We could just as well replace \mathcal{D} - G -spaces Y by the \mathcal{F} - G -spaces $\xi_* Y$ and apply the Segal machine $\mathbb{S}_G^{\mathcal{F}_G}$ on them. We have just shown that we obtain equivalent outputs from these two homotopical variants of the Segal machine. We conclude that all homotopical Segal machines in sight produce equivalent output when fed equivalent input. The resulting G -spectra are equivalent via compatible natural zigzags. We conclude that all of our machines are essentially equivalent, and they are all equivalent to our preferred machine $\mathbb{S}_G^{\mathcal{F}}$ on \mathbb{F}_\bullet -special \mathcal{F} - G -spaces.

5. THE GENERALIZED OPERADIC MACHINE

Having redeveloped the Segal infinite loop space machine equivariantly, we now review and generalize the operadic equivariant infinite loop space machine. We assume that the reader is familiar with operads, as originally defined in [\[45\]](#). More recent expositions can be found in [\[49, 50\]](#). Operads make sense in any symmetric monoidal category. Ours will be in the cartesian monoidal category $G\mathcal{U}$, and we use the notation \mathcal{C}_G for an operad in $G\mathcal{U}$, or G -operad for short.

Assumption 5.1. We assume once and for all that our operads \mathcal{C}_G are reduced, meaning that $\mathcal{C}_G(0)$ is a point, and that the inclusion $*$ \longrightarrow $\mathcal{C}_G(1)$ of the identity element is a G -cofibration.

This holds in all of our examples.

We recall the Steiner operads in [Section 5.1](#) and the classical operadic machine in [Section 5.2](#). Since the prequel [\[25\]](#) of Guillou and the first author develops this machine in detail and the equivariant generalizations of the basic definitions of [\[45\]](#) are straightforward, we shall be brief.

Our focus is on material that is needed here and is not treated in [\[25\]](#). We show how the operadic machine, like Segal machine, can be generalized to G -categories of operators. This is again based on the nonequivariant theory developed in [\[57\]](#), but considerations of equivariance require some work.

In [Section 5.3](#), we construct a G -category of operators \mathcal{D} over \mathcal{F} and a G -category of operators \mathcal{D}_G over \mathcal{F}_G from a G -operad \mathcal{C}_G .¹⁹ We show there that the construction takes E_∞ G -operads to E_∞ G -categories of operators, which is not obvious equivariantly. We construct monads associated to G -categories of operators in [Section 5.4](#). Finally, in [Section 5.5](#), we compare the inputs and outputs of the classical and generalized machines and show that they are equivalent. When \mathcal{C}_G is

¹⁹To avoid a clash of notation, we choose not to use the original notation \mathcal{C} from [\[57\]](#).

an E_∞ G -operad, the generalized input of \mathcal{D} and \mathcal{D}_G -algebras is the same as the generalized input to the Segal machine.

5.1. The Steiner operads. The advantages of the Steiner operads over the little cubes or little discs operads are explained in detail in [54, §3]. The little cubes operads \mathcal{C}_n work well nonequivariantly, but are too square for multiplicative and equivariant purposes. The little discs operads are too round to allow maps of operads $\mathcal{D}_n \rightarrow \mathcal{D}_{n+1}$ that are compatible with the natural map $\Omega^n X \rightarrow \Omega^{n+1} \Sigma X$. The Steiner operads are more complicated to define, but they enjoy all of the good properties of both the little cubes and little discs operads. Some such family of operads must play a central role in any version of the operadic machine, but the special features of the Steiner operads play an entirely new and unexpected role in our comparison of the operadic and Segal machines.

We review the definition and salient features of the equivariant Steiner operads from [25, §1.1] and [80], referring to those sources for more detailed treatments. Let V be a finite dimensional real inner product space and let G act on V through linear isometries. Define a *Steiner path* to be a continuous map h from I to the space of distance-reducing embeddings $V \rightarrow V$ such that $h(1)$ is the identity map; distance reducing means that $|h(t)(v) - h(t)(w)| \leq |v - w|$ for all $v, w \in V$ and $t \in I$. Define $\pi(h): V \rightarrow V$ by $\pi(h) = h(0)$ and define the “center point” of h to be the value of $0 \in V$ under the embedding $h(0)$, that is $c(h) = \pi(h)(0) \in V$. Crossing embeddings $V \rightarrow V$ with id_W sends Steiner paths in V to Steiner paths in $V \oplus W$.

For $j \geq 0$, define $\mathcal{K}_V(j)$ to be the G -space of j -tuples (h_1, \dots, h_j) of Steiner paths such that the embeddings $\pi(h_r) = h_r(0)$ for $1 \leq r \leq j$ have disjoint images; G acts by conjugation on embeddings and thus on Steiner paths and on j -tuples thereof. Pictorially, one can think of a point in the Steiner operad as a continuous deformation of V into a point in the little disks operad. The symmetric group Σ_j permutes j -tuples. We take $\mathcal{K}_V(0) = *$ and let $\text{id} \in \mathcal{K}_V(1)$ be the constant path at the identity $V \rightarrow V$. Compose Steiner paths pointwise, $(\ell \circ h)(t) = \ell(t) \circ h(t): V \rightarrow V$. Define the structure maps

$$\gamma: \mathcal{K}_V(k) \times \mathcal{K}_V(j_1) \times \cdots \times \mathcal{K}_V(j_k) \longrightarrow \mathcal{K}_V(j_1 + \cdots + j_k)$$

by sending

$$((\ell_1, \dots, \ell_k); \langle h_{1,1}, \dots, h_{1,j_1} \rangle, \dots, \langle h_{k,1}, \dots, h_{k,j_k} \rangle)$$

to

$$\langle \ell_1 \circ h_{1,1}, \dots, \ell_1 \circ h_{1,j_1}, \dots, \ell_k \circ h_{k,1}, \dots, \ell_k \circ h_{k,j_k} \rangle.$$

Note that \mathcal{K}_0 is the trivial operad, $\mathcal{K}_0(0) = *$, $\mathcal{K}_0(1) = \{\text{id}\}$ and $\mathcal{K}_0(j) = \emptyset$ for $j \geq 2$. Via π , the operad \mathcal{K}_V acts on $\Omega^V X$ for any G -space X in the same way that the little cubes operad acts on n -fold loop spaces or the little discs operad acts on V -fold loop spaces.

Define $\zeta: \mathcal{K}_V(j) \rightarrow \mathbf{Conf}(V, j)$, where $\mathbf{Conf}(V, j)$ is the configuration G -space of ordered j -tuples of distinct points of V , by sending (h_1, \dots, h_j) to $(c(h_1), \dots, c(h_j))$. The original argument of Steiner [80] generalizes without change equivariantly to prove that ζ is a $(G \times \Sigma_j)$ -deformation retraction. We may take the colimit over V in a complete G -universe U to obtain an E_∞ G -operad \mathcal{K}_U .

5.2. The operadic machine.

Definition 5.2. We say that a G -operad \mathcal{C}_G is an E_∞ G -operad if $\mathcal{C}_G(n)$ is a universal principal (G, Σ_n) -bundle for each n . This means that $\mathcal{C}_G(n)$ is a Σ_n -free $(G \times \Sigma_n)$ -space such that

$$\mathcal{C}_G(n)^\Lambda \simeq * \quad \text{if } \Lambda \subset G \times \Sigma_n \text{ and } \Lambda \cap \Sigma_n = e \text{ (that is, if } \Lambda \in \mathbb{F}_n).$$

Since $\mathcal{C}_G(n)$ is Σ_n -free, $\mathcal{C}_G(n)^\Lambda = \emptyset$ if $\Lambda \notin \mathbb{F}_n$.

A \mathcal{C}_G -algebra (X, θ) is a G -space X together with $(G \times \Sigma_j)$ -maps

$$\theta_j: \mathcal{C}_G(j) \times X^j \rightarrow X$$

for $j \geq 0$ such that the diagrams specified in [45, §1] or [49, 50] commute. We call an algebra over \mathcal{C}_G a \mathcal{C}_G -space. Since $\mathcal{C}_G(0) = \{*\}$, the action determines a basepoint in X , and we assume that it is nondegenerate. Several examples of E_∞ operads \mathcal{C}_G and \mathcal{C}_G -spaces appear in [25, 30].

A G -space with an action by some E_∞ operad is called an E_∞ G -space. The operadic infinite loop space machine sends an E_∞ G -space X to a genuine positive Ω - G -spectrum $\mathbb{E}_G X$ with a group completion $X \rightarrow \Omega(\mathbb{E}_G X)_1$. As explained in [54, §8], the operadic machine is a homotopical adaptation of Beck's categorical monadicity theorem. We summarize the construction of \mathbb{E}_G , following [25]. Let \mathcal{C}_G be a fixed chosen E_∞ G -operad throughout this section.

The operad \mathcal{C}_G determines a monad \mathbb{C}_G on based G -spaces such that the category $\mathcal{C}_G[G\mathcal{T}]$ of \mathcal{C}_G -algebras is isomorphic to the category $\mathbb{C}_G[G\mathcal{T}]$ of \mathbb{C}_G -algebras. Nonequivariantly, this motivated the definition of operads [45, 49, 50], and it is proven equivariantly in [25]. Intuitively, $\mathbb{C}_G X$ is constructed as the quotient of $\coprod_{j \geq 0} \mathcal{C}_G(j) \times_{\Sigma_j} X^j$ by basepoint identifications. The proof of [45, Proposition 2.6] directly generalizes equivariantly to describe a cofibered filtration of \mathbb{C}_G with easily understood subquotients and, in particular, to show that $\mathbb{C}_G X$ is nondegenerately based since X is so.

Formally, $\mathbb{C}_G X$ is the categorical tensor product of functors $\mathcal{C}_G \otimes_\Lambda X^\bullet$, where Λ is the category of based finite sets $\mathbf{j} = \{0, 1, \dots, j\}$ with basepoint 0 and based injections. To be explicit, for $1 \leq i \leq j$, let $\sigma_i: \mathbf{j} - \mathbf{1} \rightarrow \mathbf{j}$ be the ordered based injection that skips i in its image. Note that the morphisms in Λ are generated by the maps σ_i and the permutations. Then \mathcal{C}_G is regarded as a functor $\Lambda^{op} \rightarrow G\mathcal{U}$ via the right action of Σ_j and the “degeneracy maps” $\sigma_i: \mathcal{C}_G(j) \rightarrow \mathcal{C}_G(j-1)$ specified in terms of the structure map γ of \mathcal{C}_G by

$$(5.3) \quad \sigma_i(c) = \gamma(c; \text{id}^{i-1} \times 0 \times \text{id}^{j-i})$$

where $0 \in \mathcal{C}_G(0)$ and $\text{id} \in \mathcal{C}_G(1)$; X^\bullet is the covariant functor $\Lambda \rightarrow G\mathcal{U}$ that sends \mathbf{j} to X^j and uses the left action of Σ_j and the injections $\sigma_i: X^{j-1} \rightarrow X^j$ given by insertion of the basepoint in the i th position.

Remark 5.4. Assumption 5.1 implies that the monad unit $\eta: X \rightarrow \mathbb{C}_G X$ is a G -cofibration for all $X \in G\mathcal{T}$. This ensures that the bar constructions below are realizations of Reedy cofibrant simplicial G -spaces.

There are several choices that can be made in the construction of the machine \mathbb{E}_G , as discussed in [25]. We use the construction landing in orthogonal G -spectra. It is more natural topologically to land in Lewis-May or EKMM G -spectra [20, 38, 42] since all such G -spectra are fibrant and the relationship of \mathbb{E}_G to the equivariant

Barratt-Priddy-Quillen theorem is best explained using them. That variant of the operadic machine is discussed and applied in [25], and a reinterpretation with better formal properties will be given in [37].

Regardless of such choices, the construction of \mathbb{E}_G is based on the two-sided monadic bar construction defined in [45, §9]. In our context, that specializes to give a based G -space $B(F, \mathbb{C}_G, X)$ for a monad \mathbb{C}_G in $G\mathcal{T}$, a \mathbb{C}_G -algebra X , and a \mathbb{C}_G -functor $F = (F, \lambda)$. Here $F: G\mathcal{T} \rightarrow G\mathcal{T}$ is a functor and $\lambda: F\mathbb{C}_G \rightarrow F$ is a natural transformation such that $\lambda \circ F\eta = \text{Id}: F \rightarrow F$ and

$$\lambda \circ F\mu = \lambda \circ \lambda\mathbb{C}_G: F\mathbb{C}_G\mathbb{C}_G \rightarrow F.$$

There results a simplicial G -space $B_*(F, \mathbb{C}_G, X)$ with q -simplices $F\mathbb{C}_G^q X$. Its geometric realization is $B(F, \mathbb{C}_G, X)$. We emphasize that while the bar construction can be specified in sufficiently all-embracing generality that both the categorical version used in the Segal machine and the monadic version used in the operadic machine are special cases [61, 62, 63, 79], these constructions look very different.

To incorporate the relationship to loop spaces encoded in the Steiner operads, we define $\mathcal{C}_V = \mathcal{C}_G \times \mathcal{X}_V$. We view \mathcal{C}_G -spaces as \mathcal{C}_V -spaces via the projection to \mathcal{C}_G , and we view G -spaces $\Omega^V X$ as \mathcal{C}_V -spaces via the projection to \mathcal{X}_V . We write \mathbb{C}_V for the monad on $G\mathcal{T}$ associated to \mathcal{C}_V .

Theorem 5.5. *The composite of the map $\mathbb{C}_V X \rightarrow \mathbb{C}_V \Omega^V \Sigma^V X$ induced by the unit of the adjunction (Σ^V, Ω^V) and the action map $\mathbb{C}_V \Omega^V \Sigma^V X \rightarrow \Omega^V \Sigma^V X$ specifies a natural map $\alpha: \mathbb{C}_V X \rightarrow \Omega^V \Sigma^V X$ which is a weak group completion if V contains \mathbb{R} . These maps specify a map of monads $\mathbb{C}_V \rightarrow \Omega^V \Sigma^V$.*

The second statement is proven by the same formal argument as in [45, Theorem 5.2]. The first statement is part of [25, Theorem 1.14]. Weak group completions of G -spaces are defined in [25, Definitions 1.9 and 1.10], but all we need to know is that they are group completions if their sources and targets are homotopy commutative [25, Theorem 1.8], as holds in our applications. The adjoint $\tilde{\alpha}: \Sigma^V \mathbb{C}_V \rightarrow \Sigma^V$ of α is an action of the monad \mathbb{C}_V on the functor Σ^V and we have the monadic bar construction

$$(\mathbb{E}_G X)(V) = B(\Sigma^V, \mathbb{C}_V, X)$$

for \mathbb{C}_G -spaces X . In later comparisons, we write $\mathbb{E}_G X = \mathbb{E}_G^{\mathcal{C}} X$.

An isometric isomorphism $V \rightarrow V'$ in \mathcal{S}_G induces natural transformations $\Sigma^V \rightarrow \Sigma^{V'}$ and $\mathbb{C}_V \rightarrow \mathbb{C}_{V'}$, which in turn induce a map

$$B(\Sigma^V, \mathbb{C}_V, X) \rightarrow B(\Sigma^{V'}, \mathbb{C}_{V'}, X).$$

These maps assemble to make $\mathbb{E}_G X$ into an \mathcal{S}_G - G -space. Smashing with G -spaces commutes with realization of based simplicial G -spaces, by the same proof as in the nonequivariant case [45, Proposition 12.1], and inclusions $V \rightarrow W$ induce maps of monads $\mathbb{C}_V \rightarrow \mathbb{C}_W$. This gives the structural maps

$$\sigma: \Sigma^{W-V} \mathbb{E}_G X(V) \cong B(\Sigma^W, \mathbb{C}_V, X) \rightarrow B(\Sigma^W, \mathbb{C}_W, X) = \mathbb{E}_G Y(W)$$

that make $\mathbb{E}_G X$ into an orthogonal G -spectrum. The structure maps of $\mathbb{E}_G X$ and their adjoints are closed inclusions. As explained in [25], and as goes back to [45] nonequivariantly and to Costenoble and Waner equivariantly [16], we have the following theorem, which gives the basic homotopical property of the infinite loop space machine \mathbb{E}_G .

Theorem 5.6. *There are natural maps*

$$X \xleftarrow{\varepsilon} B(\mathbb{C}_V, \mathbb{C}_V, X) \xrightarrow{B(\alpha, \text{id}, \text{id})} B(\Omega^V \Sigma^V, \mathbb{C}_V, X) \xrightarrow{\zeta} \Omega^V B(\Sigma^V, \mathbb{C}_V, X)$$

of \mathcal{C}_V -spaces. The map ε is a G -homotopy equivalence with a natural G -homotopy inverse ν (which is not a \mathcal{C}_V -map), the map $B(\alpha, \text{id}, \text{id})$ is a group completion when V contains \mathbb{R} , and the map ζ is a weak G -equivalence.

Proof. The first statement is a standard property of the bar construction that works just as well equivariantly as nonequivariantly [45, Proposition 9.8] or [79, Lemma 9.9]. The second statement is deduced from Theorem 5.5 by passage to fixed point spaces and use of the same argument as in the nonequivariant case [46, Theorem 2.3(ii)]. The last statement is an equivariant generalization of [45, Theorem 12.7] that is proven carefully in [16, Lemmas 5.4, 5.5]. See [25, 54] for further discussion and variants of the construction. \square

Define

$$\xi = \zeta \circ B(\alpha, \text{id}, \text{id}) \circ \nu: X \longrightarrow \Omega^V \mathbb{E}_G X(V).$$

Then ξ is a natural group completion when $V \supset \mathbb{R}$ and is thus a weak G -equivalence when X is grouplike. The following diagram commutes, where $\tilde{\sigma}$ is adjoint to σ .

$$\begin{array}{ccc} & X & \\ \xi \swarrow & & \searrow \xi \\ \Omega^V \mathbb{E}_G X(V) & \xrightarrow{\Omega^V \tilde{\sigma}} & \Omega^{V \oplus W} \mathbb{E}_G X(V \oplus W). \end{array}$$

Therefore $\Omega^V \tilde{\sigma}$ is a weak equivalence if V contains \mathbb{R} .

5.3. G -categories of operators associated to a G -operad \mathcal{C}_G .

Definition 5.7. Let \mathcal{C}_G be an operad of G -spaces. We construct a G -CO over \mathcal{F} , which we denote by $\mathcal{D}(\mathcal{C}_G)$, abbreviated \mathcal{D} when there is no risk of confusion. Similarly, we write $\mathcal{D}_G = \mathcal{D}_G(\mathcal{C}_G)$ for the associated G -CO over \mathcal{F}_G . The morphism G -spaces of \mathcal{D} are

$$\mathcal{D}(\mathbf{m}, \mathbf{n}) = \prod_{\phi \in \mathcal{F}(\mathbf{m}, \mathbf{n})} \prod_{1 \leq j \leq n} \mathcal{C}_G(\phi_j)$$

with G -action induced by the G -actions on the $\mathcal{C}_G(n)$. Here $\phi_j = |\phi^{-1}(j)|$. Write elements in the form (ϕ, c) , where $c = (c_1, \dots, c_n)$. For $(\phi, c): \mathbf{m} \rightarrow \mathbf{n}$ and $(\psi, d): \mathbf{k} \rightarrow \mathbf{m}$, define

$$(\phi, c) \circ (\psi, d) = (\phi \circ \psi, \prod_{1 \leq j \leq n} \gamma(c_j; \prod_{\phi(i)=j} d_i) \sigma_j).$$

Here γ denotes the structural maps of the operad. The d_i with $\phi(i) = j$ are ordered by the natural order on their indices i and σ_j is that permutation of $(\phi \circ \psi)_j$ letters which converts the natural ordering of $(\phi \circ \psi)^{-1}(j)$ as a subset of $\{1, \dots, k\}$ to its ordering obtained by regarding it as $\prod_{\phi(i)=j} \psi^{-1}(i)$, so ordered that elements of $\psi^{-1}(i)$ precede elements of $\psi^{-1}(i')$ if $i < i'$ and each $\psi^{-1}(i)$ has its natural ordering as a subset of $\{1, \dots, k\}$.

The identity element in $\mathcal{D}(\mathbf{n}, \mathbf{n})$ is (id, id^n) , where id on the right is the unit element in $\mathcal{C}_G(1)$. The map $\xi: \mathcal{D} \rightarrow \mathcal{F}$ sends (ϕ, c) to ϕ . The inclusion $\iota: \Pi \rightarrow \mathcal{D}$

sends $\phi: \mathbf{m} \rightarrow \mathbf{n}$ to (ϕ, c) , where $c_i = \text{id} \in \mathcal{C}_G(1)$ if $\phi(i) = 1$ and $c_i = * \in \mathcal{C}_G(0)$ if $\phi(i) = 0$. This makes sense since Π is the subcategory of \mathcal{F} with morphisms ϕ such that $\phi_j \leq 1$ for $1 \leq j \leq n$.

Observe that \mathcal{D} is reduced as a G -CO over \mathcal{F} since \mathcal{C}_G is reduced as an operad. [Assumption 5.1](#) implies the following result, which says that \mathcal{D}_G satisfies [Definition 4.2\(ii\)](#). It ensures that the bar constructions used in the Segal machine are Reedy cofibrant, and it will also ensure that the bar constructions used in the operadic machine are Reedy cofibrant.

Lemma 5.8. *The map $* \rightarrow \mathcal{D}_G(\mathbf{n}^\alpha, \mathbf{n}^\alpha)$ given by the inclusion of the identity is a G -cofibration for all finite G -sets \mathbf{n}^α .*

Observation 5.9. Since we will need it later and it illustrates the definition, we describe explicitly how composition behaves when the point c or d in one of the maps is of the form $(\text{id}, \dots, \text{id}) \in \mathcal{C}(1) \times \dots \times \mathcal{C}(1)$.

For $\phi: \mathbf{m} \rightarrow \mathbf{n}$ and a permutation $\tau: \mathbf{m} \rightarrow \mathbf{m}$,

$$\begin{aligned} (\phi, c_1, \dots, c_n) \circ (\tau, \text{id}, \dots, \text{id}) &= (\phi \circ \tau, \prod_j (\gamma(c_j, \text{id}, \dots, \text{id}) \sigma_j)) \\ &= (\phi \circ \tau, c_1 \sigma_1, \dots, c_n \sigma_n), \end{aligned}$$

where $c_j \in \mathcal{C}(|\phi^{-1}(j)|)$, and $\sigma_j \in \Sigma_{(|(\phi \circ \tau)^{-1}(j)|)} = \Sigma_{|\phi^{-1}(j)|}$. Note that σ_j depends only on ϕ and τ and not on c .

For $\psi: \mathbf{m} \rightarrow \mathbf{n}$ and a permutation $\tau: \mathbf{n} \rightarrow \mathbf{n}$,

$$\begin{aligned} (\tau, \text{id}, \dots, \text{id}) \circ (\psi, d_1, \dots, d_n) &= (\tau \circ \psi, \prod_j (\gamma(\text{id}, d_{\tau^{-1}(j)}) \sigma_j)) \\ &= (\tau \circ \psi, d_{\tau^{-1}(1)}, \dots, d_{\tau^{-1}(n)}) \end{aligned}$$

since each σ_j is the identity (because there is only one i such that $\tau(i) = j$).

Example 5.10. The commutativity operad \mathcal{N} has n th space a point for all n . We think of it as a G -trivial G -operad. Then $\mathcal{F} = \mathcal{D}(\mathcal{N})$, again regarded as G -trivial.

For any operad \mathcal{C}_G , an \mathcal{F} - G -space Y can be viewed as the $\mathcal{D}(\mathcal{C}_G)$ - G -space ξ^*Y . Thus $\mathcal{D}(\mathcal{C}_G)$ - G -spaces give a generalized choice of input to the Segal machine. As we shall discuss in [Section 5.5](#), they also give generalized input to the operadic machine. Indeed, an action of the operad \mathcal{C}_G on a G -space X gives rise to an action of $\mathcal{D}(\mathcal{C}_G)$ on the Π - G -space $\mathbb{R}X$ with $(\mathbb{R}X)_n = X^n$.

Now recall the definitions of E_∞ operads and E_∞ G -categories of operators from [Definitions 5.2](#) and [4.23](#). We prove the following expected theorem.

Theorem 5.11. *If \mathcal{C}_G is an E_∞ G -operad, then $\mathcal{D} = \mathcal{D}(\mathcal{C}_G)$ is an E_∞ G -CO over \mathcal{F} or, equivalently, \mathcal{D}_G is an E_∞ G -CO over \mathcal{F}_G .*

Consider the trivial map of G -operads $\xi: \mathcal{C}_G \rightarrow \mathcal{N}$ that sends each $\mathcal{C}_G(n)$ to the point $\mathcal{N}(n)$. Of course, \mathcal{N} is not an E_∞ G -operad, but it is clear from the definitions that $\mathcal{F} = \mathcal{D}(\mathcal{N})$ is an E_∞ G -CO over \mathcal{F} . The map $\xi: \mathcal{D} \rightarrow \mathcal{F}$ can be thought of as $\mathcal{D}(\xi)$. The following result with $\mathcal{C}' = \mathcal{N}$ has [Theorem 5.11](#) as an immediate corollary. The proof requires pedantic details of equivariance and is therefore deferred to [Section 7.1](#), but this is the crux of the comparison of inputs of the Segal and operadic machines.

Theorem 5.12. *Let $\nu: \mathcal{C}_G \rightarrow \mathcal{C}'_G$ be a map of G -operads such that the fixed point map $\nu^\Lambda: \mathcal{C}_G(n)^\Lambda \rightarrow \mathcal{C}'_G(n)^\Lambda$ is a weak equivalence for all n and all $\Lambda \in \mathbb{F}_n$. Then the induced map $\nu_G: \mathcal{D}_G \rightarrow \mathcal{D}'_G$ of G -COs over \mathcal{F}_G is a G -equivalence.*

5.4. The monads \mathbb{D} and \mathbb{D}_G associated to the G -categories \mathcal{D} and \mathcal{D}_G .

Recall that the category $\mathcal{C}_G[G\mathcal{F}]$ of algebras over an operad \mathcal{C}_G is isomorphic to the category $\mathbb{C}_G[G\mathcal{F}]$ of algebras over the associated monad \mathbb{C}_G . Let $\mathcal{D} = \mathcal{D}(\mathcal{C}_G)$ be the G -CO over \mathcal{F} associated to \mathcal{C}_G and let \mathcal{D}_G be the associated G -CO over \mathcal{F}_G . As worked out nonequivariantly in [57, §5], we define monads \mathbb{D} on the category of Π - G -spaces and \mathbb{D}_G on the category of Π_G - G -spaces. The commutative diagram of inclusions of categories

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi_G \\ \downarrow i & & \downarrow i_G \\ \mathcal{D} & \longrightarrow & \mathcal{D}_G \end{array}$$

gives rise to a commutative diagram of forgetful functors

$$(5.13) \quad \begin{array}{ccc} \text{Fun}(\Pi, \underline{G\mathcal{U}}_*) & \xleftarrow{\mathbb{U}} & \text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*) \\ \uparrow i^* & & \uparrow i_G^* \\ \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*) & \xleftarrow{\mathbb{U}} & \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*) \end{array}$$

Categorical tensor products then give left adjoints making the following diagram commute up to natural isomorphism.

$$(5.14) \quad \begin{array}{ccc} \text{Fun}(\Pi, \underline{G\mathcal{U}}_*) & \xrightarrow{\mathbb{P}} & \text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*) \\ \mathbb{D} \downarrow & & \mathbb{D}_G \downarrow \\ \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*) & \xrightarrow{\mathbb{P}} & \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*) \end{array}$$

Here \mathbb{D} and \mathbb{D}_G are the left adjoints of i^* and i_G^* , respectively. By a standard abuse of notation, we write \mathbb{D} and \mathbb{D}_G for the resulting endofunctors $i^*\mathbb{D}$ on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ and $i_G^*\mathbb{D}_G$ on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$. Explicitly, the monads \mathbb{D} and \mathbb{D}_G are defined as

$$(5.15) \quad (\mathbb{D}X)(\mathbf{n}) = \mathcal{D}(-, \mathbf{n}) \otimes_{\Pi} X$$

for a Π - G -space X , where $\mathcal{D}(-, \mathbf{n})$ is the represented functor induced by i , and

$$(5.16) \quad (\mathbb{D}_G Y)(\mathbf{n}^\alpha) = \mathcal{D}_G(-, \mathbf{n}^\alpha) \otimes_{\Pi_G} Y$$

for a Π_G - G -space Y , where $\mathcal{D}_G(-, \mathbf{n}^\alpha)$ is the represented functor induced by i_G .

The units η of the adjunctions (\mathbb{D}, i^*) and (\mathbb{D}_G, i_G^*) give the unit maps of the monads, and the action maps of the \mathcal{D} - G -spaces $\mathbb{D}X$ and \mathcal{D}_G - G -spaces $\mathbb{D}_G Y$ give the products μ . More concretely, $\mu: \mathbb{D}\mathbb{D} \rightarrow \mathbb{D}$ and $\mu: \mathbb{D}_G\mathbb{D}_G \rightarrow \mathbb{D}_G$ are derived from the compositions in \mathcal{D} and \mathcal{D}_G , respectively, and thus from the structure maps γ of \mathcal{C}_G . The unit maps η are derived from the identity morphisms in these categories and thus from the unit element $\text{id} \in \mathcal{C}_G(1)$.

Moreover, we have the following standard categorical result.

Proposition 5.17. *The category of \mathbb{D} -algebras in $\text{Fun}(\Pi, \underline{G}\mathcal{U}_*)$ is isomorphic to the category $\text{Fun}(\mathcal{D}, \underline{G}\mathcal{U}_*)$. Similarly, the category of \mathbb{D}_G -algebras in $\text{Fun}(\Pi_G, \underline{G}\mathcal{U}_*)$ is isomorphic to the category $\text{Fun}(\mathcal{D}_G, \underline{G}\mathcal{U}_*)$.*

The above discussion used the existence of left adjoints that are constructed formally. Our cofibrancy conditions in the definition of Π - G -spaces ensure that these constructions are well behaved homotopically.

Proposition 5.18. *The functors \mathbb{D} and \mathbb{D}_G restrict to give functors*

$$\mathbb{D}: \Pi[G\mathcal{T}] \longrightarrow \mathcal{D}[\mathcal{T}_G] \quad \text{and} \quad \mathbb{D}_G: \Pi_G[\mathcal{T}_G] \longrightarrow \mathcal{D}_G[\mathcal{T}_G].$$

Restricting appropriately, we obtain analogues of the diagrams (5.13) and (5.14).

$$\begin{array}{ccc} \Pi[G\mathcal{T}] & \xleftarrow{\mathbb{U}} & \Pi_G[\mathcal{T}_G] & & \Pi[G\mathcal{T}] & \xrightarrow{\mathbb{P}} & \Pi_G[\mathcal{T}_G] \\ \uparrow i^* & & \uparrow i_G^* & & \downarrow \mathbb{D} & & \downarrow \mathbb{D}_G \\ \mathcal{D}[\mathcal{T}_G] & \xleftarrow{\mathbb{U}} & \mathcal{D}_G[\mathcal{T}_G] & & \mathcal{D}[\mathcal{T}_G] & \xrightarrow{\mathbb{P}} & \mathcal{D}_G[\mathcal{T}_G]. \end{array}$$

The monads \mathbb{D} and \mathbb{D}_G restrict to give monads on the categories $\Pi[G\mathcal{T}]$ and $\Pi_G[\mathcal{T}_G]$, respectively.

Proof. By definition, the functors \mathbb{U} and \mathbb{P} restrict appropriately to $\mathcal{D}[\mathcal{T}_G]$ and $\mathcal{D}_G[\mathcal{T}_G]$ (and their Π analogues), and they give equivalences of categories by Theorems 2.38 and 4.11. Thus, it is enough to prove that i^* and \mathbb{D} preserve the cofibrancy conditions required of Π and \mathcal{D} - G -spaces. This is certainly true for i^* , since the condition is specified precisely on the underlying Π - G -space. The verification for \mathbb{D} requires detailed combinatorial analysis and is relegated to Section 7.3. \square

Denote by $\mathbb{D}[\Pi[G\mathcal{T}]]$ the category of algebras over the monad \mathbb{D} in $\Pi[G\mathcal{T}]$, and similarly, denote by $\mathbb{D}_G[\Pi_G[\mathcal{T}_G]]$ the category of algebras over the monad \mathbb{D}_G in $\Pi_G[\mathcal{T}_G]$. The previous result implies the following refinement of Proposition 5.17.

Proposition 5.19. *The categories $\mathcal{D}[\mathcal{T}_G]$ and $\mathbb{D}[\Pi[G\mathcal{T}]]$ are isomorphic and the categories $\mathcal{D}_G[\mathcal{T}_G]$ and $\mathbb{D}_G[\Pi_G[\mathcal{T}_G]]$ are isomorphic. Therefore the categories $\mathbb{D}[\Pi[G\mathcal{T}]]$ and $\mathbb{D}_G[\Pi_G[\mathcal{T}_G]]$ are equivalent.*

Therefore the monads \mathbb{D} and \mathbb{D}_G can be used interchangeably. This contrasts markedly with the Segal machine, where considerations of specialness led us to focus on \mathcal{D}_G rather than \mathcal{D} . We shall give a conceptual explanation of the difference in Section 10.2.

We need some homotopical and some formal properties of the monads \mathbb{D} and \mathbb{D}_G , following [57]. We first establish the formal properties, whose proofs are identical to those in [57]. We write the following results in terms of \mathcal{D} and \mathbb{D} for simplicity. With attention to enrichment, the parallel results for \mathcal{D}_G work in the same way. They can also be derived from the results for \mathbb{D} using $\mathbb{P}\mathbb{D} \cong \mathbb{D}_G\mathbb{P}$ and Proposition 5.19.

Recall from Section 2.1 that the evident functors $\mathbb{L}: \Pi[G\mathcal{T}] \longrightarrow G\mathcal{T}$ and $\mathbb{R}: G\mathcal{T} \longrightarrow \Pi[G\mathcal{T}]$ give an adjunction such that $\mathbb{L}\mathbb{R} = \text{Id}$ and the unit of the adjunction is given by the Segal maps. It induces an operadic variant. Let $\mathcal{C}_G[G\mathcal{T}]$ denote the category of \mathcal{C}_G -spaces.

Proposition 5.20. *The adjunction (\mathbb{L}, \mathbb{R}) between $\Pi[G\mathcal{T}]$ and $G\mathcal{T}$ induces an adjunction between $\mathcal{D}[\mathcal{T}_G]$ and $\mathcal{C}_G[G\mathcal{T}]$ such that $\mathbb{L}\mathbb{R} = \text{Id}$ and the unit $\delta: \text{Id} \longrightarrow \mathbb{R}\mathbb{L}$*

is given by the Segal maps. A \mathcal{D} - G -space with underlying Π - G -space $\mathbb{R}X$ determines and is determined by a \mathcal{C}_G -space structure on $\mathbb{L}\mathbb{R}X = X$.

Proof. The nonequivariant proof of [57, Lemma 4.2] applies verbatim. \square

We require an analysis of the behavior of the monad \mathbb{D} with respect to the adjunction (\mathbb{L}, \mathbb{R}) . The following two results are equivariant generalizations of results in [57, Section 6] and the equivariance adds no complications. The proofs are inspections of definitions and straightforward diagram chases.²⁰

Proposition 5.21. *Let X be a G -space and Y be a Π - G -space.*

- (i) *The G -space $\mathbb{L}\mathbb{D}\mathbb{R}X = (\mathbb{D}\mathbb{R}X)_1$ is naturally G -homeomorphic to $\mathbb{C}_G X$.*
- (ii) *The Π - G -space $\mathbb{D}\mathbb{R}X$ is naturally isomorphic to the Π - G -space $\mathbb{R}\mathbb{C}_G X$.*
- (iii) *The following diagram is commutative for each n .*

$$\begin{array}{ccc} (\mathbb{D}Y)_n & \xrightarrow{(\mathbb{D}\delta)_n} & (\mathbb{D}\mathbb{R}LY)_n \cong (\mathbb{C}_G \mathbb{L}Y)^n \\ \delta \downarrow & & \delta \downarrow \cong \\ (\mathbb{D}Y)_1^n & \xrightarrow{(\mathbb{D}\delta)_1^n} & (\mathbb{D}\mathbb{R}LY)_1^n \cong (\mathbb{C}_G \mathbb{L}Y)^n \end{array}$$

- (iv) *The functor $\mathbb{R}\mathbb{C}_G \mathbb{L}$ on Π - G -spaces is a monad with product and unit induced from those of \mathbb{C}_G via the composites*

$$\mathbb{R}\mathbb{C}_G \mathbb{L}\mathbb{R}\mathbb{C}_G \mathbb{L} = \mathbb{R}\mathbb{C}_G \mathbb{C}_G \mathbb{L} \xrightarrow{\mathbb{R}\mu \mathbb{L}} \mathbb{R}\mathbb{C}_G \mathbb{L} \quad \text{and} \quad \text{Id} \xrightarrow{\delta} \mathbb{R}\mathbb{L} \xrightarrow{\mathbb{R}\eta \mathbb{L}} \mathbb{R}\mathbb{C}_G \mathbb{L}.$$

- (v) *The natural transformation $\mathbb{D}\delta: \mathbb{D} \rightarrow \mathbb{D}\mathbb{R}\mathbb{L} \cong \mathbb{R}\mathbb{C}_G \mathbb{L}$ is a morphism of monads in the category $\Pi[G\mathcal{T}]$.*
- (vi) *If (F, λ) is a \mathbb{C}_G -functor in \mathcal{V} , then $FL: \text{Fun}(\Pi, G\mathcal{T}) \rightarrow \mathcal{V}$ is an $\mathbb{R}\mathbb{C}_G \mathbb{L}$ -functor in \mathcal{V} with action $\lambda \mathbb{L}: FL\mathbb{R}\mathbb{C}_G \mathbb{L} = FC_G \mathbb{L} \rightarrow FL$. Therefore, by pullback, FL is a \mathbb{D} -functor in \mathcal{V} with action the composite*

$$FL\mathbb{D} \xrightarrow{FL\mathbb{D}\delta} FL\mathbb{D}\mathbb{R}\mathbb{L} \cong FL\mathbb{R}\mathbb{C}_G \mathbb{L} = FC_G \mathbb{L} \xrightarrow{\lambda \mathbb{L}} FL.$$

Proof. Nonequivariantly, these results are given in [57, §6] and the equivariance adds no complications. The proofs are inspections of definitions and straightforward diagram chases. \square

Using the adjunction (\mathbb{P}, \mathbb{U}) and the isomorphism $\mathbb{D}_G \mathbb{P} \cong \mathbb{P}\mathbb{D}$, we derive the analogue for \mathbb{D}_G . As in Definition 2.40, we write

$$\mathbb{R}_G = \mathbb{P}\mathbb{R}: \mathcal{C}_G[G\mathcal{T}] \rightarrow \mathcal{D}_G[\mathcal{T}_G]$$

and

$$\mathbb{L}_G = \mathbb{L}\mathbb{U}: \mathcal{D}_G[\mathcal{T}_G] \rightarrow \mathcal{C}_G[G\mathcal{T}].$$

With \mathbb{D} , \mathbb{R} , and \mathbb{L} replaced by \mathbb{D}_G , \mathbb{R}_G , and \mathbb{L}_G , we then have the following analogue of Proposition 5.21.

Proposition 5.22. *Let X be a G -space and Y be a Π_G - G -space.*

- (i) *The G -space $\mathbb{L}_G \mathbb{D}_G \mathbb{R}_G X$ is naturally G -homeomorphic to $\mathbb{C}_G X$.*
- (ii) *The Π_G - G -space $\mathbb{D}_G \mathbb{R}_G X$ is naturally isomorphic to $\mathbb{R}_G \mathbb{C}_G X$.*

²⁰These results are specializations of a more general one with other applications [37].

(iii) The following diagram is commutative for each \mathbf{n}^α .

$$\begin{array}{ccc} (\mathbb{D}_G Y) \mathbf{n}^\alpha & \xrightarrow{\mathbb{D}_G \delta} & (\mathbb{D}_G \mathbb{R}_G \mathbb{L}_G Y) \mathbf{n}^\alpha \cong (\mathbb{C}_G \mathbb{L}_G Y) \mathbf{n}^\alpha \\ \delta \downarrow & & \delta \downarrow \cong \\ (\mathbb{D}_G Y)_1 \mathbf{n}^\alpha & \xrightarrow{(\mathbb{D}_G \delta)_1 \mathbf{n}^\alpha} & (\mathbb{D}_G \mathbb{R}_G \mathbb{L}_G Y)_1 \mathbf{n}^\alpha \cong (\mathbb{C}_G \mathbb{L}_G Y) \mathbf{n}^\alpha \end{array}$$

(iv) The functor $\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G$ on Π_G - G -spaces is a monad with product and unit induced from those of \mathbb{C}_G via the composites

$$\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G = \mathbb{R}_G \mathbb{C}_G \mathbb{C}_G \mathbb{L}_G \xrightarrow{\mathbb{R}_G \mu \mathbb{L}_G} \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G$$

and

$$\text{Id} \xrightarrow{\delta} \mathbb{R}_G \mathbb{L}_G \xrightarrow{\mathbb{R}_G \eta \mathbb{L}_G} \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G.$$

- (v) The natural transformation $\mathbb{D}_G \delta: \mathbb{D}_G \rightarrow \mathbb{D}_G \mathbb{R}_G \mathbb{L}_G \cong \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G$ is a morphism of monads in the category $\Pi_G[\mathcal{T}_G]$.
- (vi) If (F, λ) is a \mathbb{C}_G -functor in \mathcal{V} , then $F \mathbb{L}_G: \text{Fun}(\Pi_G, \mathcal{T}_G) \rightarrow \mathcal{V}$ is an $\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G$ -functor in \mathcal{V} with action $\lambda \mathbb{L}_G: F \mathbb{L}_G \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G = F \mathbb{C}_G \mathbb{L}_G \rightarrow F \mathbb{L}_G$. Therefore, by pullback, $F \mathbb{L}_G$ is a \mathbb{D}_G -functor in \mathcal{V} with action the composite

$$F \mathbb{L}_G \mathbb{D}_G \xrightarrow{F \mathbb{L}_G \mathbb{D}_G \delta} F \mathbb{L}_G \mathbb{D}_G \mathbb{R}_G \mathbb{L}_G \cong F \mathbb{L}_G \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G = F \mathbb{C}_G \mathbb{L}_G \xrightarrow{\lambda \mathbb{L}_G} F \mathbb{L}_G.$$

We now turn to the homotopical properties of the monads \mathbb{D} and \mathbb{D}_G and their algebras. The proofs of the homotopical properties are similar to those in [57], but considerably more difficult, so some will be deferred to Section 7.4. In contrast with the Segal machine, we start with \mathbb{D} rather than \mathbb{D}_G . Our interest is in E_∞ operads, but we allow more general operads until otherwise indicated.

Following [45] nonequivariantly, we say that a G -operad \mathcal{C}_G is Σ -free if the action of Σ_j on $\mathcal{C}_G(j)$ is free for each j . Surprisingly, we only need that much structure to prove the following result. It is the equivariant generalization of [57, Lemma 5.6].

Theorem 5.23. *Let \mathcal{C}_G be a Σ -free G -operad.*

- (i) *If $f: X \rightarrow Y$ is an \mathbb{F}_\bullet -level equivalence of Π - G -spaces, then $\mathbb{D}f: \mathbb{D}X \rightarrow \mathbb{D}Y$ is an \mathbb{F}_\bullet -level equivalence.*
- (ii) *If X is an \mathbb{F}_\bullet -special Π - G -space, then $\mathbb{D}X$ is \mathbb{F}_\bullet -special.*

Proof. The proof of (i) requires quite lengthy combinatorics about the structure of $\mathbb{D}X$ and its fixed point subspaces, hence we defer it to Section 7.4. By (i) applied to the Segal map $\delta: X \rightarrow \mathbb{R} \mathbb{L} X$, the map $\mathbb{D} \delta$ is an \mathbb{F}_\bullet -level equivalence. Since its target $\mathbb{D} \mathbb{R} \mathbb{L} X \cong \mathbb{R} \mathbb{C}_G \mathbb{L} X$ is \mathbb{F}_\bullet -special, Lemma 2.17 implies that $\mathbb{D}X$ is also \mathbb{F}_\bullet -special, giving (ii). \square

By Theorem 4.11 and Corollaries 4.13 and 4.14, the isomorphism $\mathbb{P} \mathbb{D} \cong \mathbb{D}_G \mathbb{P}$ and Theorem 5.23 imply the following analogue of that result.

Proposition 5.24. *Assume that each $\mathcal{C}_G(j)$ is Σ_j -free.*

- (i) *If $f: X \rightarrow Y$ is an \mathbb{F}_\bullet -level equivalence of Π - G -spaces, then $\mathbb{D}_G \mathbb{P} f: \mathbb{D}_G \mathbb{P} X \rightarrow \mathbb{D}_G \mathbb{P} Y$, is a level G -equivalence.*
- (ii) *If X is a \mathbb{F}_\bullet -special, then $\mathbb{D}_G \mathbb{P} X$ is a special Π_G - G -space.*

5.5. Comparisons of inputs and outputs of the operadic machine. We give three ways to construct G -spectra from \mathcal{D}_G - G -spaces. We can convert \mathcal{D}_G - G -spaces to \mathcal{D} - G -spaces via [Proposition 5.19](#), and we can convert those to \mathcal{C}_G -spaces by [Proposition 5.27](#) below. We can then apply the original operadic machine, or we can generalize the machine to both \mathcal{D} - G -spaces and \mathcal{D}_G - G -spaces. All three make use of the two-sided monadic bar construction of [\[45\]](#), starting from the formalities of [Propositions 5.21](#) and [5.22](#). We show that all three are equivalent. In fact, we really only have two machines in view of the following result. We emphasize how different this is from the Segal machine, where the \mathcal{D} and \mathcal{D}_G bar constructions are not even equivalent, let alone isomorphic.

Proposition 5.25. *Let \mathcal{C}_G be a G -operad with category of operators \mathcal{D} . For any \mathcal{D} -space X and any \mathbb{C}_G -functor $F: G\mathcal{T} \rightarrow G\mathcal{T}$, there is a natural isomorphism of G -spaces*

$$B(\mathbb{F}\mathbb{L}, \mathbb{D}, X) \cong B(\mathbb{F}\mathbb{L}_G, \mathbb{D}_G, \mathbb{P}X).$$

Proof. Since $\text{Id} \cong \text{UP}$, $\mathbb{P}\mathbb{D} \cong \mathbb{D}_G\mathbb{P}$ and $\mathbb{L}_G = \mathbb{L}\mathbb{U}$, we have isomorphisms of q -simplices

$$\mathbb{F}\mathbb{L}\mathbb{D}^q X \cong \mathbb{F}\mathbb{L}_G\mathbb{D}_G^q \mathbb{P}X.$$

Formal checks show that these isomorphisms commute with the face and degeneracy operators. The conclusion follows on passage to geometric realization. \square

For variety, and because that is what we shall use in the next section, we focus on \mathcal{D}_G rather than \mathcal{D} in this section. The following result compares the inputs to machines given by \mathcal{D}_G - G -spaces and \mathcal{C}_G -spaces.

Definition 5.26. For a \mathcal{D}_G - G -space Y , define a \mathcal{C}_G -space $\mathbb{X}(Y)$ by

$$\mathbb{X}(Y) = B(\mathbb{C}_G\mathbb{L}_G, \mathbb{D}_G, Y).$$

Here \mathbb{C}_G is regarded as a functor $G\mathcal{T} \rightarrow \mathcal{C}_G[G\mathcal{T}]$, and the construction makes sense since the realization of a simplicial \mathcal{C}_G -space is a \mathcal{C}_G -space, exactly as nonequivariantly [\[45, Theorem 12.2\]](#).

Proposition 5.27. *For special \mathcal{D}_G - G -spaces Y , there is a zigzag of natural level G -equivalences of \mathcal{D}_G - G -spaces between Y and $\mathbb{R}_G\mathbb{X}(Y)$. For \mathcal{C}_G -spaces X , there is a natural G -equivalence of \mathcal{C}_G -spaces from $\mathbb{X}(\mathbb{R}_G X)$ to X .*

Proof. [Proposition 5.24](#) implies that $\mathbb{D}_G\delta: \mathbb{D}_G Y \rightarrow \mathbb{D}_G \mathbb{R}_G \mathbb{L}_G Y \cong \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G Y$ is a level G -equivalence of \mathcal{D}_G - G -spaces. The realization of simplicial \mathbb{I}_G - G -spaces is defined levelwise, and since realization commutes with products of G -spaces, we have the indicated isomorphism in the diagram

$$Y \xleftarrow{\varepsilon} B(\mathbb{D}_G, \mathbb{D}_G, Y) \xrightarrow{B(\mathbb{D}_G\delta, \text{id}, \text{id})} B(\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G, \mathbb{D}_G, Y) \cong \mathbb{R}_G B(\mathbb{C}_G \mathbb{L}_G, \mathbb{D}_G, Y) = \mathbb{R}_G \mathbb{X}(Y).$$

By standard properties of the bar construction, as in [\[45\]](#) nonequivariantly, ε is a level G -equivalence of \mathcal{D}_G - G -spaces. Since the bar constructions are geometric realizations of Reedy cofibrant simplicial G -spaces (see [Remark 1.14](#)), it follows from [Theorem 1.12](#) that $B(\mathbb{D}_G\delta, \text{id}, \text{id})$ is a level G -equivalence of \mathcal{D}_G - G -spaces. For the second statement, we apply \mathbb{L}_G to the level G -equivalence of \mathcal{D}_G - G -spaces

$$\mathbb{R}_G \mathbb{X}(\mathbb{R}_G X) \cong B(\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G, \mathbb{D}_G, \mathbb{R}_G X) \cong B(\mathbb{R}_G \mathbb{C}_G \mathbb{L}_G, \mathbb{R}_G \mathbb{C}_G \mathbb{L}_G, \mathbb{R}_G X) \xrightarrow{\varepsilon} \mathbb{R}_G X,$$

where the second isomorphism follows from [Proposition 5.22](#) and inspection. \square

Therefore, after inverting the respective equivalences, the functors \mathbb{R}_G and \mathbb{X} induce an equivalence of categories between \mathcal{C}_G -spaces and special \mathcal{D}_G - G -spaces whose underlying Π_G - G -spaces are Reedy cofibrant. We conclude that, for an E_∞ -operad \mathcal{C}_G , the input categories for operadic machines given by \mathcal{C}_G -spaces and by \mathcal{D}_G - G -spaces are essentially equivalent.

To generalize the machine from \mathcal{C}_G -spaces to \mathcal{D}_G - G -spaces, we again use the product operads $\mathcal{C}_V = \mathcal{C}_G \times \mathcal{K}_V$, where \mathcal{K}_V is the V th Steiner operad. We write $\mathbb{D}_{G,V}$ for the monad associated to the resulting category of operators $\mathcal{D}_{G,V} = \mathcal{D}_G(\mathcal{C}_V)$ over \mathcal{F}_G . Then a \mathcal{D}_G -space is a $\mathcal{D}_{G,V}$ -space for any representation V by pullback along the projection $\mathcal{D}_{G,V} \rightarrow \mathcal{D}_G$.

Definition 5.28. For a \mathcal{D}_G - G -space Y , define the V th space of the orthogonal G -spectrum $\mathbb{E}_G^{\mathcal{D}_G} Y$ to be the monadic two-sided bar construction

$$(5.29) \quad \mathbb{E}_G^{\mathcal{D}_G}(Y)(V) = B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y).$$

The right action of $\mathbb{D}_{G,V}$ on $\Sigma^V \mathbb{L}_G$ is obtained from the projection $\mathbb{D}_{G,V} \rightarrow \mathbb{K}_V$ and the action of \mathbb{K}_V on Σ^V , via [Proposition 5.21](#)(vi). The \mathcal{D}_G - G -space structure is given as follows. For an isometric isomorphism $V \rightarrow V'$ in \mathcal{I}_G , the map

$$B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y) \rightarrow B(\Sigma^{V'} \mathbb{L}_G, \mathbb{D}_{G,V'}, Y)$$

is the geometric realization of maps induced at all simplicial levels by $S^V \rightarrow S^{V'}$ and $\mathcal{K}_V \rightarrow \mathcal{K}_{V'}$. Similarly, since smashing commutes with geometric realization, the structure maps

$$B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y) \wedge S^W \rightarrow B(\Sigma^{V \oplus W} \mathbb{L}_G, \mathbb{D}_{G,V \oplus W}, Y)$$

are induced from the maps of monads $\mathbb{D}_{G,V} \rightarrow \mathbb{D}_{G,V \oplus W}$.

The machine $\mathbb{E}_G^{\mathcal{D}_G}$ on \mathcal{D}_G - G -spaces Y generalizes the machine $\mathbb{E}_G^{\mathcal{C}}$ on \mathcal{C}_G -spaces X . To see that, observe that [Proposition 5.22](#)(ii) implies that we have a natural isomorphism $\mathbb{D}_{G,V} \mathbb{R}_G X \cong \mathbb{R}_G \mathbb{C}_V X$. Since $\mathbb{L}_G \mathbb{R}_G = \text{Id}$, that gives us a natural isomorphism

$$(5.30) \quad B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, \mathbb{R}_G X) \cong B(\Sigma^V, \mathbb{C}_V, X),$$

where we regard $\mathbb{R}_G X$ as a \mathcal{D}_G - G -space via [Proposition 5.20](#).²¹ Together with [Proposition 5.27](#), this gives the following comparison of outputs of our machines.

Corollary 5.31. *For \mathcal{C}_G -spaces X , $\mathbb{E}_G^{\mathcal{C}} X$ is naturally isomorphic to $\mathbb{E}_G^{\mathcal{D}_G} \mathbb{R}_G X$. For special \mathcal{D}_G - G -spaces Y whose underlying Π_G - G -space is Reedy cofibrant, there is a zigzag of natural equivalences connecting $\mathbb{E}_G^{\mathcal{D}_G} Y$ to $\mathbb{E}_G^{\mathcal{C}} \mathbb{R}_G \mathbb{X}(Y) \cong \mathbb{E}_G^{\mathcal{C}} \mathbb{X}(Y)$.*

Thus the machines \mathbb{E}_G on \mathcal{C}_G -spaces and on special \mathcal{D}_G - G -spaces are essentially equivalent. Properties of the machine on special \mathcal{D}_G - G -spaces are essentially the same as properties of the machine on \mathcal{C}_G -spaces, as can either be proven directly or read off from the equivalence of machines.

²¹As in [\[25, Remark 2.9\]](#), checks of definitions when $V = 0$ give $\Sigma^0 = \text{Id}$, $\mathbb{K}_0 = \text{Id}$, $\mathbb{C}_0 X \cong \mathcal{C}_G(1)_+ \wedge X$ for a G -space X , and $\mathbb{D}_{G,0} Y \cong \mathbb{R}_G \mathbb{C}_0 \mathbb{L}_G Y$ for a Π_G - G -space Y .

6. THE EQUIVALENCE BETWEEN THE SEGAL AND OPERADIC MACHINES

We give an explicit comparison between the generalized Segal and generalized operadic infinite loop space machines. The comparison is needed for consistency and because each machine has significant advantages over the other. That was already clear nonequivariantly, and it seems even more true equivariantly. As in [25, 54], in the previous section we used the Steiner operads rather than the little cubes operads that were used in [45, 57]. That change made equivariant generalization easy, and [54] gave other good reasons for the change. However, nothing like the present comparison was envisioned in earlier work. As we have recalled, the Steiner operad is built from paths of embeddings. We shall see that these paths give rise to a homotopy that at one end relates to the generalized Segal machine and at the other end relates to the generalized operadic machine. That truly seems uncanny.

6.1. The statement of the comparison theorem. To set the stage, we recapitulate some of what we have done. We fix an E_∞ operad \mathcal{C}_G of G -spaces. We then have an E_∞ G -CO $\mathcal{D} = \mathcal{D}(\mathcal{C}_G)$ over \mathcal{F} and an E_∞ G -CO \mathcal{D}_G over \mathcal{F}_G . Our primary interest here is in infinite loop space machines defined either on special \mathcal{F}_G - G -spaces or on \mathcal{C}_G -spaces. The Segal machine is defined on the former and the operadic machine is defined on the latter. We have generalized both machines so that they accept special \mathcal{D}_G - G -spaces as input. Moreover, we have compared inputs and shown that both special \mathcal{F}_G - G -spaces and \mathcal{C}_G -spaces are equivalent to special \mathcal{D}_G - G -spaces and therefore to each other. Further, we have compared outputs. We have shown that application of the generalized Segal machine to \mathcal{D}_G - G -spaces is equivalent to application of the original homotopical Segal machine to \mathcal{F}_G - G -spaces, and that application of the generalized operadic machine to \mathcal{D}_G - G -spaces is equivalent to application of the original operadic machine to \mathcal{C}_G -spaces.

In more detail, \mathbb{F}_\bullet -special \mathcal{F} - G -spaces, \mathbb{F}_\bullet -special \mathcal{D} - G -spaces, special \mathcal{F}_G - G -spaces, and special \mathcal{D}_G - G -spaces are all equivalent by Theorems 4.24 and 4.25, and the Segal machines on all four equivalent inputs give equivalent output by Theorem 4.31. We may therefore focus on the Segal machine $\mathbb{S}_G = \mathbb{S}_G^{\mathcal{D}_G}$ defined on special \mathcal{D}_G - G -spaces Y . Similarly, \mathcal{C}_G -spaces, \mathbb{F}_\bullet -special \mathcal{D} - G -spaces, and special \mathcal{D}_G - G -spaces are equivalent by Propositions 5.20 and 5.27, and the operadic machine on \mathcal{C}_G -spaces is a special case of the operadic machine on \mathcal{D}_G - G -spaces by Corollary 5.31. Thus we may again focus on the operadic machine $\mathbb{E}_G = \mathbb{E}_G^{\mathcal{D}_G}$ defined on special \mathcal{D}_G - G -spaces Y . Thus, fixing an E_∞ operad \mathcal{C}_G with associated category of operators \mathcal{D}_G over \mathcal{F}_G , we consider special \mathcal{D}_G - G -spaces Y . Our goal is to give a constructive proof of the following comparison theorem.

Theorem 6.1. *For special \mathcal{D}_G - G -spaces Y , there is a natural zigzag of equivalences of orthogonal G -spectra between $\mathbb{S}_G Y$ and $\mathbb{E}_G Y$.*

6.2. The proof of the comparison theorem. Here we display the zigzag and then fill in the required constructions and proofs in subsequent sections. We first introduce generic notations that may enhance readability.

Notation 6.2. We shall use \star as a place holder for representations V and we shall use \bullet as a placeholder for finite G -sets \mathbf{n}^α . For fixed V , $S^{V\bullet}$ denotes the contravariant functor on \mathcal{F}_G that sends the object \mathbf{n}^α to $(S^V)^{\mathbf{n}^\alpha} = \mathcal{I}_G(\mathbf{n}^\alpha, S^V)$. Precomposing with ξ_G , we view $S^{V\bullet}$ as a contravariant functor $\mathcal{D}_G \rightarrow \mathcal{I}_G$. We write $S^{\star\bullet}$ for the functor obtained by letting V vary.

Since wedges taken over G -sets \mathbf{n}^α play a significant role in our argument, we introduce the following convenient notation.

Notation 6.3. For a based space A , let ${}^n A$ denote the wedge sum of n copies of A . Similarly, for a G -set \mathbf{n}^α and a based G -space A , let ${}^{\mathbf{n}^\alpha} A$ denote the wedge sum of n copies of A with G -acting on A , but also interchanging the wedge summands. We write $(j, a) \in {}^{\mathbf{n}^\alpha} A$ for the element a in the j th summand. The G -action is given explicitly by $g \cdot (j, a) = (\alpha(g)(j), g \cdot a)$.

Recall from [Definition 4.29](#) that the V th G -space of $\mathbb{S}_G Y$ is

$$(\mathbb{S}_G Y)(V) = B(S^{V^\bullet}, \mathcal{D}_G, Y).$$

We thus adopt the notation

$$\mathbb{S}_G Y = B(S^{\star\bullet}, \mathcal{D}_G, Y).$$

Let \mathcal{C}_V be the product operad $\mathcal{C}_G \times \mathcal{K}_V$ and let $\mathcal{D}_{G,V} = \mathcal{D}_G(\mathcal{C}_V)$ with associated monad $\mathbb{D}_{G,V}$ on the category of Π_G - G -spaces. As in [Definition 5.28](#), the V th G -space of $\mathbb{E}_G Y$ is

$$(\mathbb{E}_G Y)(V) = B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y).$$

We thus adopt the notation

$$\mathbb{E}_G Y = B(\Sigma^\star \mathbb{L}_G, \mathbb{D}_{G,\star}, Y).$$

Note the different uses of the \star notation. In both machines, it is a placeholder for representations V . However, in the Segal machine, we are using cartesian powers of G -spheres S^V to obtain functors $S^{V^\bullet}: \mathcal{F}_G^{op} \rightarrow \mathcal{T}_G$, whereas in the operadic machine we are using the suspension functor Σ^V associated to S^V together with the Steiner operad \mathcal{K}_V . While a two-sided bar construction is used in both machines, the similarity of notation hides how different these bar constructions really are: the use of categories and contravariant and covariant functors in one is quite different from the use of monads, (right) actions on functors, and (left) actions on objects in the other.

The zigzag of stable equivalences relating the machines has the following shape. All of the constructions are two sided categorical bar constructions except the operadic machine at the last step, which is a monadic bar construction.

$$(6.4) \quad \mathbb{S}_G Y \xlongequal{\quad} B(S^{*\bullet}, \mathcal{D}_G, Y)$$

$$\begin{array}{c} \uparrow \pi \\ B(S^{*\bullet}, \mathcal{D}_{G,*}, Y) \\ \downarrow i_1 \\ B(I_+ \wedge S^{*\bullet}, \mathcal{D}_{G,*}, Y) \\ \uparrow i_0 \\ B(S_0^{*\bullet}, \mathcal{D}_{G,*}, Y) \\ \uparrow \iota \\ B(\bullet S_0^*, \mathcal{D}_{G,*}, Y) \\ \downarrow \omega \\ B(\Sigma^* \mathbb{L}_G, \mathbb{D}_{G,*}, Y) \xlongequal{\quad} \mathbb{E}_G Y. \end{array}$$

Remark 6.5. The heart of the construction is to use the Steiner operads to define a certain homotopy H that specifies the required contravariant action of $\mathcal{D}_{G,*}$ on $I_+ \wedge S^{*\bullet}$; see (6.9). At $t = 1$, this action restricts to $S_1^{*\bullet} = S^{*\bullet}$. At $t = 0$, it restricts to a new contravariant action of $\mathcal{D}_{G,*}$ on $S^{*\bullet}$, denoted $S_0^{*\bullet}$. Crucially, this action restricts to an action, denoted $\bullet S_0^*$, on $\bullet S^*$.

We shall shortly construct the intermediate orthogonal G -spectra and maps in this zigzag and prove that all of the maps except ω are stable equivalences. As we now explain, this will imply that ω is also a stable equivalence. Recall that the homotopy groups of a pointed G -space X are $\pi_q^H(X) = \pi_q(X^H)$ and the homotopy groups of an orthogonal G -spectrum T are

$$(6.6) \quad \pi_q^H(T) = \operatorname{colim}_V \pi_q^H(\Omega^V T(V)),$$

where the colimits are formed using the adjoint structure maps of T ; our G -spectra are all connective, so that their negative homotopy groups are zero. A map $T \rightarrow T'$ is a stable equivalence if its induced maps of homotopy groups are isomorphisms. That depends only on large V . Thus we may focus on those V that contain \mathbb{R} , so that the group completions of Theorems 3.34 and 5.5 are available.

Applying Ω^V to the V th spaces implicit in the diagram Theorem 9.8, we obtain a diagram of G -spaces under Y_1 . By completely different proofs, both maps

$$Y_1 \rightarrow \Omega^V \mathbb{S}_G Y(V) \quad \text{and} \quad Y_1 \rightarrow \Omega^V \mathbb{E}_G Y(V)$$

are group completions. Therefore, once we prove that the arrows other than ω are stable equivalences, it will follow that ω is also a stable equivalence. Indeed, arranging as we may that our outputs are Ω - G -spectra and using that they are connective, ω is a stable equivalence if and only if the map ω_0 it induces on 0th G -spaces is a weak G -equivalence. The displayed group completions imply that ω_0 induces a homology isomorphism on fixed point spaces. Since these spaces are Hopf spaces, hence simple, it follows that ω_0 induces an isomorphism on homotopy groups, so that ω_0 is a weak G -equivalence.

Remark 6.7. In constructing the diagram, we shall encounter an annoying but minor clash of conventions. There is a dichotomy in how one chooses to define the faces and degeneracies of the categorical bar construction. We made one choice in [Section 3.1](#), but to mesh with the monadic bar construction as defined in [\[45, Construction 9.6\]](#), we must now make the other. Therefore, on q -simplices, we agree to replace the previous d_i and s_i by d_{q-i} and s_{q-i} , respectively. With the new convention, d_0 is given by the evaluation map of the left (contravariant) variable in the categorical two-sided bar construction, rather than the right variable.

6.3. Construction and analysis of the map π . Turning to the diagram [Theorem 9.8](#), we first define the top map π . We start by defining its source orthogonal G -spectrum $B(S^{\star\bullet}, \mathcal{D}_{G,\star}, Y)$. The V th space, as the notation indicates, is defined by plugging in V for \star ; it is the bar construction $B(S^{V\bullet}, \mathcal{D}_{G,V}, Y)$, as defined in [Section 3.1](#), namely it is the geometric realization of the simplicial space with q -simplices given by the wedge over all sequences $(\mathbf{n}_q^{\alpha_q}, \dots, \mathbf{n}_0^{\alpha_0})$ of the G -spaces

$$(S^V)^{\mathbf{n}_q^{\alpha_q}} \wedge \mathcal{D}_{G,V}(\mathbf{n}_{q-1}^{\alpha_{q-1}}, \mathbf{n}_q^{\alpha_q}) \wedge \dots \wedge \mathcal{D}_{G,V}(\mathbf{n}_0^{\alpha_0}, \mathbf{n}_1^{\alpha_1}) \wedge Y(\mathbf{n}_0^{\alpha_0}).$$

We have implicitly composed Y with the evident projections $\mathcal{D}_{G,V} \rightarrow \mathcal{D}_G$ to regard Y as a $G\mathcal{U}_*$ -functor defined on each $\mathcal{D}_{G,V}$, and we have composed the $S^{V\bullet}$ with the composite $\mathcal{D}_{G,V} \rightarrow \mathcal{D}_G \rightarrow \mathcal{F}_G$ to regard the $S^{V\bullet}$ as functors defined on $\mathcal{D}_{G,V}$. Note that $B(S^{\star\bullet}, \mathcal{D}_{G,\star}, Y)$ is not the restriction of a \mathcal{W}_G - G -space, but it is an \mathcal{I}_G - G -space. For an isometric isomorphism $V \rightarrow V'$ in \mathcal{I}_G , the map

$$B(S^{V\bullet}, \mathcal{D}_{G,V}, Y) \rightarrow B(S^{V'\bullet}, \mathcal{D}_{G,V'}, Y)$$

is the geometric realization of the map induced at each simplicial level by the maps $\mathcal{K}_V \rightarrow \mathcal{K}_{V'}$ and $S^V \rightarrow S^{V'}$.

Geometric realization commutes with \wedge , and the structure maps of the orthogonal G -spectrum $B(S^{\star\bullet}, \mathcal{D}_{G,\star}, Y)$ are geometric realizations of levelwise simplicial maps given by the maps $j: \mathcal{D}_{G,V} \rightarrow \mathcal{D}_{G,V \oplus W}$ induced by the inclusions $\mathcal{K}_V \rightarrow \mathcal{K}_{V \oplus W}$ and the maps

$$(6.8) \quad i: (S^V)^{\mathbf{n}^\alpha} \wedge S^W \rightarrow (S^{V \oplus W})^{\mathbf{n}^\alpha}$$

defined by

$$i((v_1, \dots, v_n) \wedge w) = (v_1 \wedge w, \dots, v_n \wedge w).$$

An alternative characterization of this construction is to use [Definition 2.26](#) together with [\(3.15\)](#) in [Remark 3.14](#) to obtain G -maps

$$B(S^{V\bullet}, \mathcal{D}_{G,V \oplus W}, Y) \wedge S^W \rightarrow B(S^{V \oplus W\bullet}, \mathcal{D}_{G,V \oplus W}, Y)$$

and to precompose with the G -map

$$B(S^{V\bullet}, \mathcal{D}_{G,V}, Y) \rightarrow B(S^{V\bullet}, \mathcal{D}_{G,V \oplus W}, Y)$$

induced by $j: \mathcal{D}_{G,V} \rightarrow \mathcal{D}_{G,V \oplus W}$. One can easily check that these maps do indeed give maps of bar constructions that specify the structure maps for an orthogonal G -spectrum. The projections $\mathcal{D}_{G,V} \rightarrow \mathcal{D}_G$ induce the top map π of orthogonal G -spectra in [Theorem 9.8](#).

Recall that $\text{colim}_V \mathcal{K}_V(j) = \mathcal{K}_U(j)$, so that $\text{colim}_V (\mathcal{C}_G \times \mathcal{K}_V)$ is the product $\mathcal{C}_G \times \mathcal{K}_U$, which is an E_∞ G -operad since it is the product of two such operads. Therefore the projection $(\mathcal{C}_G \times \mathcal{K}_U)(j) \rightarrow \mathcal{C}_G(j)$ is a Λ -equivalence for all $\Lambda \in \mathbb{F}_j$

and, by [Theorem 5.12](#), the map $\mathcal{D}_G(\mathcal{C}_G \times \mathcal{K}_U) \longrightarrow \mathcal{D}_G(\mathcal{C}_G)$ is a G -equivalence of G -COs over \mathcal{F}_G . The projection map

$$\pi: B(S^{\star\bullet}, \mathcal{D}_{G,\star}, Y) \longrightarrow B(S^{\star\bullet}, \mathcal{D}_G, Y)$$

is not a level G -equivalence, but a direct comparison of colimits shows that π is a stable equivalence. In more detail, in computing π on homotopy groups, we start from the commutative diagrams

$$\begin{array}{ccc} \Omega^V B(S^{V\bullet}, \mathcal{D}_{G,V}, Y) & \longrightarrow & \Omega^W B(S^{W\bullet}, \mathcal{D}_{G,W}, Y) \\ \pi \downarrow & & \downarrow \pi \\ \Omega^V B(S^{V\bullet}, \mathcal{D}_G, Y) & \longrightarrow & \Omega^W B(S^{W\bullet}, \mathcal{D}_G, Y), \end{array}$$

where $V \subset W$. We then take H -fixed points and pass to homotopy groups. Since the inclusions $\mathcal{D}_{G,V} \longrightarrow \mathcal{D}_{G,W}$ become isomorphisms on homotopy groups in increasing ranges of dimensions, by inspection of the homotopy types of the G -spaces comprising the Steiner operads in [\[25, §1.1\]](#), we see that π is a stable equivalence.

6.4. The contravariant functors $I_+ \wedge S^{V\bullet}$ on $\mathcal{D}_{G,V}$. In the notation $I_+ \wedge S^{\star\bullet}$ in [Theorem 9.8](#), \star is again a place holder for V , and the notation stands for $G\mathcal{U}_\star$ -functors

$$I_+ \wedge S^{V\bullet}: (\mathcal{D}_{G,V})^{op} \longrightarrow \mathcal{F}_G$$

that are given on objects by sending \mathbf{n}^α to $I_+ \wedge (S^V)^{\mathbf{n}^\alpha}$, where I is the unit interval; we have adjoined a disjoint basepoint and taken the smash product in order to have domains for based homotopies. The crux of our comparison is to specify the functors on $I_+ \wedge S^{V\bullet}$ on morphisms in terms of homotopies that are deduced from the paths that comprise the Steiner operads.

Recall from [Example 1.5](#) that $(S^V)^{\mathbf{n}^\alpha}$ is just $(S^V)^n$ with the G -action \cdot_α specified by

$$g \cdot_\alpha (x_1, \dots, x_n) = (gx_{\alpha(g)^{-1}(1)}, \dots, gx_{\alpha(g)^{-1}(n)}) = \alpha(g)_*(gx_1, \dots, gx_n).$$

Therefore, by [Theorem 4.11](#) and [Lemma 4.12](#), it is enough to instead define $G\mathcal{U}_\star$ -functors

$$I_+ \wedge S^{V\bullet}: (\mathcal{D}_V)^{op} \longrightarrow \mathcal{F}_G$$

given on objects by sending \mathbf{n} to $I_+ \wedge (S^V)^n$ and then apply the functor \mathbb{P} defined in [Section 4.2](#) to obtain the desired functors on $\mathcal{D}_{G,V}$. We choose to do this in order to make the definitions a little less cumbersome.

We construct the required maps on hom objects as composites

$$\mathcal{D}_V(\mathbf{m}, \mathbf{n}) \longrightarrow \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n}) \xrightarrow{\tilde{H}} \mathcal{F}_G(I_+ \wedge (S^V)^n, I_+ \wedge (S^V)^m).$$

The first map is the evident projection, and we shall use the same letter for maps and their composites with that projection. To define \tilde{H} , we shall construct a homotopy

$$(6.9) \quad H: I_+ \wedge (S^V)^n \wedge \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n}) \longrightarrow (S^V)^m$$

and then set

$$(6.10) \quad \tilde{H}(f)(t, v) = (t, H(t, v, f)),$$

where $t \in I$, $v \in (S^V)^n$, and $f \in \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n})$. We have written variables in the order appropriate to thinking of the homotopies H as the core of the evaluation maps of the contravariant functor $I_+ \wedge S^{V\bullet}: \mathcal{D}_V \longrightarrow \mathcal{F}_G$. Note that such evaluation

maps, after prolongation to $\mathcal{D}_{G,V}$, give the zeroth face operation d_0 in the simplicial G -spaces whose realizations give the central bar constructions in [Theorem 9.8](#).

Writing H_t for H at time t , H_1 will relate to the evaluation maps of the represented functor $S^{V\bullet}$ used in the left variable of the Segal machine and H_0 will relate to the maps that define the action of the monad \mathbb{K}_V on the functor Σ^V that is used in the left variable of the operadic machine.

The following construction is the heart of the matter. Recall that a Steiner path in V is a map $h: I \rightarrow R_V$ such that $h(1) = \text{id}$, where R_V is the space of distance reducing embeddings $V \rightarrow V$. The space $\mathcal{K}_V(s)$ of the Steiner operad is the space of s -tuples of Steiner paths h_r such that the $h_r(0)$ have disjoint images. We define a homotopy

$$\gamma: I \times S^V \times \mathcal{K}_V(s) \rightarrow (S^V)^s$$

with coordinates γ_r , $1 \leq r \leq s$, by letting

$$\gamma_r(t, v, \langle h_1, \dots, h_s \rangle) = \begin{cases} w & \text{if } h_r(t)(w) = v \\ * & \text{if } v \notin \text{im}(h_r(t)). \end{cases}$$

If $t = 1$, this is just the diagonal map $S^V \rightarrow (S^V)^s$, which is relevant to the Segal machine. If $t = 0$, this map lands in the s -fold wedge ${}^s(S^V)$ of copies of S^V since the conditions $v \in \text{im}(h_r(0))$ as r varies are mutually exclusive; that is, there is at most one r such that $v \in \text{im}(h_r(0))$. This is relevant to the operadic machine since the action map

$$\tilde{\alpha}: \Sigma^V K_V X = K_V X \wedge S^V \rightarrow X \wedge S^V = \Sigma^V X$$

is given by

$$\tilde{\alpha}(\langle \langle h_1, \dots, h_s \rangle, x_1, \dots, x_s \rangle, v) = \begin{cases} (x_r, w_r) & \text{if } h_r(0)(w_r) = v \\ * & \text{if } v \notin \text{im}(h_r(0)) \text{ for } 1 \leq r \leq s. \end{cases}$$

Remember that we understand $\Sigma^V A$ to be $A \wedge S^V$ for a based G -space A , but we write the V coordinate on the left when looking at the evaluation maps of the functor $I_+ \wedge S^{V\bullet}$.

We now define the homotopy H of [\(6.9\)](#). Recall from [Definition 5.7](#) that

$$\mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n}) = \coprod_{\phi: \mathbf{m} \rightarrow \mathbf{n}} \prod_{1 \leq j \leq n} \mathcal{K}_V(\phi_j),$$

where $\phi_j = |\phi^{-1}(j)|$. Let $f = (\phi; k_1, \dots, k_n) \in \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n})$, where $\phi \in \mathcal{F}(\mathbf{m}, \mathbf{n})$ and $k_j \in \mathcal{K}_V(\phi_j)$. For $1 \leq i \leq m$, define the i th coordinate of H as follows. If $\phi(i) = j$, $1 \leq j \leq n$, and i is the r th element of $\phi^{-1}(j)$ with its natural ordering as a subset of \mathbf{m} , then

$$H(t, v_1, \dots, v_n, f)_i = \gamma_r(t, v_j, k_j),$$

where γ_r is the r th coordinate of

$$\gamma: I \times S^V \times \mathcal{K}_V(\phi_j) \rightarrow (S^V)^{\phi_j}.$$

If $\phi(i) = 0$, then the i th coordinate of H is the trivial map.

It requires some combinatorial inspection to check that these maps do indeed specify a $G\mathcal{U}_*$ -functor $I_+ \wedge S^{V\bullet}: \mathcal{D}_V^{op} \rightarrow \mathcal{T}_G$, but we leave that to the reader. Prolonging these functors using [Lemma 4.12](#), we obtain the $G\mathcal{U}_*$ -functors

$$I_+ \wedge S^{V\bullet}: \mathcal{D}_{G,V}^{op} \rightarrow \mathcal{T}_G, \quad \mathbf{n}^\alpha \mapsto I_+ \wedge (S^V)^{\mathbf{n}^\alpha},$$

needed to define the two-sided bar constructions $B(I_+ \wedge S^{V\bullet}, \mathcal{D}_{G,V}, Y)$. Just as in [Section 6.3](#), the assignment $V \mapsto B(I_+ \wedge S^{V\bullet}, \mathcal{D}_{G,V}, Y)$ gives an \mathcal{I}_G - G -space, and we can construct structure maps that make it into an orthogonal G -spectrum.

We denote by $S_1^{V\bullet}$ the restrictions of the functors $I_+ \wedge S^{V\bullet}$ to $t = 1$. The $S_1^{V\bullet}$ are just the functors $S^{V\bullet}$ used to define $B(S^{V\bullet}, \mathcal{D}_{G,V}, Y)$. Similarly, we denote by $S_0^{V\bullet}$ the restrictions of the functors $I_+ \wedge S^{V\bullet}$ to $t = 0$. For any based G -space A , let i_0 and i_1 denote the inclusions of the top and bottom copy of A into the cylinder $I_+ \wedge A$, where G acts trivially on the interval I . Note that i_0 and i_1 are G -homotopy equivalences.

The functors $S_0^{V\bullet}$ and $S_1^{V\bullet}$ from $\mathcal{D}_{G,V}^{op}$ to \mathcal{T}_G are restrictions of $I_+ \wedge S^{V\bullet}$, hence they commute with the face d_0 , which is given by the evaluation maps of these functors. It is clear that the maps i_0 and i_1 commute with all other faces and degeneracies. Since they are levelwise G -equivalences of Reedy cofibrant simplicial G -spaces, their realizations

$$(6.11) \quad B(S_0^{V\bullet}, \mathcal{D}_{G,V}, Y) \xrightarrow{i_0} B(I_+ \wedge S^{V\bullet}, \mathcal{D}_{G,V}, Y) \xleftarrow{i_1} B(S_1^{V\bullet}, \mathcal{D}_{G,V}, Y),$$

are weak G -equivalences. Therefore the maps i_0 and i_1 in [Theorem 9.8](#) are level equivalences of orthogonal G -spectra.

6.5. Construction and analysis of the map ι . To define the map of orthogonal G -spectra labeled ι in [Theorem 9.8](#), we must look more closely at $S_0^{V\bullet}$. Note that H_0 sends an element indexed on $\phi: \mathbf{m} \rightarrow \mathbf{n}$ to an element of the product over $1 \leq j \leq n$ of the wedge sums $\phi_j(S^V)$, where $|\phi^{-1}(j)| = \phi_j$. If we restrict the domain of H_0 to ${}^n(S^V) \wedge \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n}) \subset (S^V)^n \wedge \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n})$, we land in ${}^m(S^V)$ since for any element (v_1, \dots, v_n, f) in the domain of H_0 such that all but one of the factors v_j is $*$, only those i such that $\phi(i) = j$ can contribute a non-basepoint image. Therefore \tilde{H}_0 restricts to a $G\mathcal{U}_*$ -functor $\bullet S_0^V: (\mathcal{D}_V)^{op} \rightarrow \mathcal{T}_G$ that on objects sends \mathbf{n} to ${}^n(S^V)$ and on morphism spaces is the adjoint of the restriction of H_0 from products to wedges. On wedges, we have the composite

$${}^n(S^V) \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{n}) \rightarrow {}^n(S^V) \wedge \mathcal{D}(\mathcal{K}_V)(\mathbf{m}, \mathbf{n}) \rightarrow {}^m(S^V)$$

of projection and the map obtained by unravelling the definition of H . Upon applying prolongation \mathbb{P} , we obtain a $G\mathcal{U}_*$ -functor

$$\bullet S_0^V: (\mathcal{D}_{G,V})^{op} \rightarrow \mathcal{T}_G.$$

It is defined on objects by sending \mathbf{n}^α to $\mathbf{n}^\alpha(S^V)$, and it is a subfunctor of the functor $S_0^{V\bullet}: (\mathcal{D}_{G,V})^{op} \rightarrow \mathcal{T}_G$. The map $i: (S^V)^n \wedge S^W \rightarrow (S^{V \oplus W})^n$ of [\(6.8\)](#) restricts to the canonical identification of ${}^n(S^V) \wedge S^W$ with ${}^n(S^{V \oplus W})$, and this works just as well when the twisted action of α is taken into account. Just as in [Section 6.3](#), by [\(3.15\)](#) in [Remark 3.14](#) these maps give rise to the structure maps of the G -spectrum $B(\bullet S_0^*, \mathcal{D}_{G,*}, Y)$. The inclusions of wedges into products give the inclusions of bar constructions that together specify the map of G -spectra labeled ι in [Theorem 9.8](#).

It is worth pausing to say what is going on philosophically before showing that ι is a stable equivalence of orthogonal G -spectra. The contravariant functor $S^{V\bullet}$ from $\mathcal{D}_{G,V}$ to \mathcal{T}_G is purely categorical since it factors through \mathcal{F}_G and applies just as well to give a functor A^\bullet for any A . The action of \mathcal{F}_G on $S^{V\bullet}$ does *not* restrict to an action on the system of subspaces $\bullet S^V$. Use of the Steiner operad in effect gives a new and more geometric functor $S_0^{V\bullet}$. It is again defined on products, but

it depends on the geometry encoded in the Steiner operads and it *does* restrict to a functor defined on $\bullet S_0^V$. That is, we have commutative diagrams

$$\begin{array}{ccc} \mathbf{n}^\beta(S^V) \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{n}^\beta) & \longrightarrow & \mathbf{m}^\alpha(S^V) \\ \downarrow \iota \wedge \text{id} & & \downarrow \iota \\ (S^V)^{\mathbf{n}^\beta} \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{n}^\beta) & \longrightarrow & (S^V)^{\mathbf{m}^\alpha} \end{array}$$

where the horizontal arrows are evaluation maps of the functors $\bullet S_0^V$, and $S_0^V \bullet$, respectively, and the vertical arrows are given by inclusions of wedges in products. The diagram displays the essential part of the map d_0 in the simplicial bar constructions that ι compares; ι is the identity on all factors in $\mathcal{D}_{G,V}$ or Y of the G -spaces of q -simplices in these bar constructions. From here, it is routine to check that these maps of bar constructions specify a map of orthogonal G -spectra.

We claim that ι is a stable equivalence. Note that both the source and target of ι are orthogonal G -spectra which at level V are geometric realizations of simplicial G -spaces. Before passage to geometric realization, we have functors $\Delta^{op} \times \mathcal{I}_G \rightarrow \mathcal{I}_G$. It is not hard to see that the simplicial structure commutes with the spectrum structure maps, so that we can view these as simplicial orthogonal G -spectra, as discussed in [Section 1.4](#). Therefore, we can view the source and target of ι as geometric realizations of simplicial orthogonal G -spectra and we can view ι as the geometric realization of a map ι_* of simplicial orthogonal G -spectra. By the following two lemmas, [Proposition 1.26](#) applies to prove the claim.

Lemma 6.12. *The map ι_q of q -simplices is a stable equivalence of orthogonal G -spectra for each q .*

Proof. This means that ι_q induces isomorphisms

$$\text{colim}_V \pi_*^H(\Omega^V B_q(\bullet S_0^V, \mathcal{D}_{G,V}, Y)) \longrightarrow \text{colim}_V \pi_*^H(\Omega^V B_q(S_0^V \bullet, \mathcal{D}_{G,V}, Y)).$$

To prove this, recall that it is standard that finite wedges are finite products in the stable category [[1](#), Proposition III.3.11]; the same proof works equivariantly. The maps $j: \mathcal{D}_{G,V} \rightarrow \mathcal{D}_{G,V \oplus W}$ of Steiner operads also induce isomorphisms on homotopy groups in increasing dimensions. The result follows by inspection of colimits. \square

Lemma 6.13. *The source and target simplicial orthogonal G -spectra of ι_* are Reedy h -cofibrant.*

Proof. Since for each V , the simplicial G -spaces are bar constructions, they are Reedy cofibrant, so it remains to show that the homotopy extensions can be made compatibly with the orthogonal G -spectrum structure. To see this, note that the latching maps are constructed from the G -cofibration $* \rightarrow \mathcal{C}_V(1) = \mathcal{C}_G(1) \times \mathcal{K}_V(1)$ given by the inclusion of the identity element in $\mathcal{C}_V(1)$. That map is a G -cofibration since $* \rightarrow \mathcal{C}_G(1)$ is a G -cofibration by [Assumption 5.1](#) and $\{\text{id}\} \hookrightarrow \mathcal{K}_V(1)$ is the inclusion of a G -deformation retract. The explicit retraction from $\mathcal{K}_V(1)$ onto $\{\text{id}\}$ is easily seen to be compatible with the \mathcal{I}_G - G -space structure and with the inclusions $j: \mathcal{K}_V(1) \rightarrow \mathcal{K}_{V \oplus W}(1)$, so it is compatible with the structure maps. This in turn implies the compatibility of the retracts for the latching maps of the bar constructions as V varies. \square

6.6. Construction of the map ω . To construct the map ω in [Theorem 9.8](#) and thus to complete the proof of [Theorem 6.1](#), we must define maps

$$B(\bullet S_0^V, \mathcal{D}_{G,V}, Y) \longrightarrow B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y).$$

Remember that $\mathbb{L}_G Y = Y_1$. Both source and target are realizations of simplicial (based) G -spaces, and we define ω as the realization of a map of simplicial G -spaces. On the spaces of 0-simplices we define

$$\omega_0: B_0(\bullet S_0^V, \mathcal{D}_{G,V}, Y) = \bigvee_{\mathbf{n}^\alpha} \mathbf{n}^\alpha S^V \wedge Y(\mathbf{n}^\alpha) \longrightarrow \Sigma^V Y_1 = B_0(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y)$$

to be given on each wedge summand by the composite

$$\mathbf{n}^\alpha S^V \wedge Y(\mathbf{n}^\alpha) \xrightarrow{\text{id} \wedge \delta} \mathbf{n}^\alpha S^V \wedge Y_1^{\mathbf{n}^\alpha} \xrightarrow{\nu} S^V \wedge Y_1 \xrightarrow{\tau} Y_1 \wedge S^V = \Sigma^V Y_1.$$

Here, for based spaces A and B , define $\nu: {}^n A \wedge B^n \longrightarrow A \wedge B$ by

$$\nu((i, a), (b_1, \dots, b_n)) = (a, b_i)$$

where (i, a) denotes the element $a \in A$ of the i th wedge summand of ${}^n A$ and $b_j \in B$, $1 \leq j \leq n$. We check explicitly that ν is G -equivariant:

$$g \cdot ((i, a), (b_1, \dots, b_n)) = ((\alpha(g)(i), g \cdot a), (g \cdot b_{\alpha(g)^{-1}(1)}, \dots, g \cdot b_{\alpha(g)^{-1}(n)})) \mapsto (g \cdot a, g \cdot b_i)$$

since the $\alpha(g)(i)$ position is $\alpha(g)^{-1} \alpha(g)(i) = i$. The map $\tau: A \wedge B \longrightarrow B \wedge A$ is the usual twist. Then all of the maps in the definition of ω_0 are equivariant.

Notation 6.14. For $q > 0$, we may write the space of q -simplices of $B(\bullet S_0^V, \mathcal{D}_{G,V}, Y)$ as the wedge over pairs $(\mathbf{m}^\alpha, \mathbf{n}^\beta)$ of the spaces

$$\mathbf{n}^\beta S^V \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{n}^\beta) \wedge Z(\mathbf{m}^\alpha),$$

where $Z(\mathbf{m}^\alpha)$ denotes the wedge over sequences $(\mathbf{m}_0^{\alpha_0}, \dots, \mathbf{m}_{q-2}^{\alpha_{q-2}})$ of the spaces

$$\mathcal{D}_{G,V}(\mathbf{m}_{q-2}^{\alpha_{q-2}}, \mathbf{m}^\alpha) \wedge \dots \wedge \mathcal{D}_{G,V}(\mathbf{m}_0^{\alpha_0}, \mathbf{m}_1^{\alpha_1}) \wedge Y(\mathbf{m}_0^{\alpha_0}).$$

Recall the definition of $\mathbb{D}_{G,V} Y$ from [\(5.16\)](#). The G -space $(\mathbb{D}_{G,V} Y)(\mathbf{n}^\beta)$ is a quotient of the wedge over all \mathbf{m}^α of the G -spaces

$$\mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{n}^\beta) \wedge Y(\mathbf{m}^\alpha).$$

Therefore the space $\Sigma^V \mathbb{L}_G \mathbb{D}_{G,V}^q Y$ of q -simplices of $B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y)$ is a quotient of the wedge over all \mathbf{m}^α of the spaces

$$\mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{1}) \wedge Z(\mathbf{m}^\alpha) \wedge S^V.$$

Define ω_q by passage to wedges over \mathbf{m}^α from the composites

$$\begin{array}{c} \mathbf{n}^\beta S^V \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{n}^\beta) \wedge Z(\mathbf{m}^\alpha) \\ \downarrow \text{id} \wedge \delta \wedge \text{id} \\ \mathbf{n}^\beta S^V \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{1})^{\mathbf{n}^\beta} \wedge Z(\mathbf{m}^\alpha) \\ \downarrow \nu \wedge \text{id} \\ S^V \wedge \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{1}) \wedge Z(\mathbf{m}^\alpha) \\ \downarrow \tau \\ \mathcal{D}_{G,V}(\mathbf{m}^\alpha, \mathbf{1}) \wedge Z(\mathbf{m}^\alpha) \wedge S^V. \end{array}$$

Since we know that these maps are G -equivariant, to check commutative diagrams we may drop the α 's and β 's from the notation and only consider the underlying nonequivariant spaces. On underlying spaces, the composite above is

$$\begin{array}{c}
{}^n S^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{n}) \wedge Z_m \\
\downarrow \text{id} \wedge \delta \wedge \text{id} \\
{}^n S^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1})^n \wedge Z_m \\
\downarrow \nu \wedge \text{id} \\
S^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1}) \wedge Z_m \\
\downarrow \tau \\
\mathcal{D}_V(\mathbf{m}, \mathbf{1}) \wedge Z_m \wedge S^V.
\end{array}$$

We must show that these maps ω_q specify a map of simplicial spaces. Again recall [Remark 6.7](#). In both the categorical and monadic bar constructions, the face maps d_i for $i > 1$ are induced by composition in \mathcal{D}_V and the action of \mathcal{D}_V on Y . Commutation of ω with these face maps is evident. Similarly, commutation with the degeneracy maps s_i for $i > 0$ is evident. We must show commutation with s_0 , d_0 , and d_1 . An essential point is that the components of the Segal maps are in Π_G , and we are taking the categorical tensor product over Π_G in the target. First consider s_0 on zero simplices. For (i, v) in ${}^n S^V$, that is, v in the i th summand, and $y \in Y_n$,

$$\begin{aligned}
\omega s_0((i, v), y) &= \omega((i, v), \text{id}_n, y) \\
&= (\delta_i, y) \wedge v \\
&= (\text{id}_1, \delta_i(y)) \wedge v \\
&= s_0 \omega((i, v), y).
\end{aligned}$$

Here $\text{id}_n \in \mathcal{D}_V(\mathbf{n}, \mathbf{n})$, the third equation uses $\delta_i = \text{id}_1 \circ \delta_i$ and the equivalence relation defining $\mathbb{D}_V Y$, and the last equation uses that $\omega((i, v), y) = \delta_i(y) \wedge v$. The commutation of ω and s_0 on q -simplices for $q > 0$ is similar. The following diagrams prove the commutation of ω with d_0 and d_1 on 1-simplices, and the argument for q -simplices for $q > 1$ is similar. Again, we only write the maps of underlying spaces, dropping from the notation the indices that indicate G -actions since the G -action is not relevant to checking that the diagrams commute.

$$\begin{array}{ccccc}
nS^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{n}) \wedge Y_m & \xrightarrow{H_0 \wedge \text{id}} & mS^V \wedge Y_m & & \\
\downarrow \text{id} \wedge \delta \wedge \text{id} & & \downarrow \text{id} \wedge \delta & & \\
nS^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1})^n \wedge Y_m & \xrightarrow{H_0 \wedge \text{id}} & mS^V \wedge Y_1^m & & \\
\downarrow \nu \wedge \text{id} & & \downarrow \nu & & \\
S^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1}) \wedge Y_m & & S^V \wedge Y_1 & & \\
\downarrow & & \downarrow & & \\
\Sigma^V \mathbb{L} \mathbb{D}_V Y & \xrightarrow{\cong} \Sigma^V \mathbb{C}_V \mathbb{L} Y & \xrightarrow{\hat{\alpha}} & \Sigma^V Y_1 & \\
\end{array}$$

$$\begin{array}{ccc}
nS^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{n}) \wedge Y_m & \xrightarrow{\text{id} \wedge \theta} & nS^V \wedge Y_n \\
\downarrow \text{id} \wedge \delta \wedge \text{id} & & \downarrow \text{id} \wedge \delta \\
nS^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1})^n \wedge Y_m & & nS^V \wedge Y_1^n \\
\downarrow \nu \wedge \text{id} & & \downarrow \nu \\
S^V \wedge \mathcal{D}_V(\mathbf{m}, \mathbf{1}) \wedge Y_m & \xrightarrow{\text{id} \wedge \theta} & S^V \wedge Y_1 \\
\downarrow & & \downarrow \\
\Sigma^V \mathbb{L} \mathbb{D}_V Y & \xrightarrow{\Sigma^V \mathbb{L} \theta} & \Sigma^V Y_1
\end{array}$$

Here θ denotes the action of \mathbb{D}_V on Y , which is given by the adjoints of the G -maps $Y: \mathcal{D}_V(\mathbf{m}, \mathbf{n}) \rightarrow \mathcal{I}_G(Y_m, Y_n)$. Both top pieces of the diagrams commute by formal inspection, the trapezoid in the first diagram commutes by the definitions of H_0 and $\hat{\alpha}$, as recalled in the previous section, and the lower rectangle in the second diagram commutes by definition. It is not hard to check that the maps ω are maps of \mathcal{I}_G - G -spaces and that they are compatible with the structure maps, so that they give a map of orthogonal G -spectra.

7. PROOFS OF TECHNICAL RESULTS ABOUT THE OPERADIC MACHINE

We prove Theorems 5.12 and 5.23 in this section.

7.1. Levelwise G -equivalences of categories of operators over \mathcal{F}_G . With the notations and hypotheses of Theorem 5.12, we must prove that the map

$$\nu_G: \mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta) \rightarrow \mathcal{D}'_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$$

is a weak G -equivalence for all \mathbf{m}^α and \mathbf{n}^β . Recall that $\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)$ is just $\mathcal{D}(\mathbf{m}, \mathbf{n})$ with the G -action given by $g \cdot f = \beta(g) \circ (gf) \circ \alpha(g^{-1})$.

Let H be a subgroup of G . We claim that there is a homeomorphism

$$[\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)]^H \cong \coprod_{\phi} \prod_i \mathcal{C}(d_i)^{\Lambda_i},$$

where ϕ runs over the H -equivariant maps $\mathbf{m}^\alpha \rightarrow \mathbf{n}^\beta$, i runs over the H -orbits of \mathbf{n}^β , and d_i and Λ_i depend only on ϕ and H , with Λ_i in \mathbb{F}_{d_i} . This homeomorphism

is moreover compatible with the map ν_G . Thus our hypothesis gives that the map

$$(\nu_G)^H: [\mathcal{D}_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)]^H \longrightarrow [\mathcal{D}'_G(\mathbf{m}^\alpha, \mathbf{n}^\beta)]^H$$

is a weak equivalence for all subgroups H , as wanted.

To prove the claim, recall that

$$\mathcal{D}(\mathbf{m}, \mathbf{n}) = \prod_{\phi: \mathbf{m} \rightarrow \mathbf{n}} \prod_{1 \leq j \leq n} \mathcal{C}(\phi_j),$$

where $\phi_j = |\phi^{-1}(j)|$. Using [Definition 5.7](#) and in particular [Observation 5.9](#), the new action on $\mathcal{D}(\mathbf{m}, \mathbf{n})$ is given by

$$g \cdot (\phi; x_1, \dots, x_n) = (\beta(g)\phi\alpha(g^{-1}); gx_{\beta(g^{-1})(1)}\sigma_{\beta(g^{-1})(1)}(g^{-1}), \dots, gx_{\beta(g^{-1})(n)}\sigma_{\beta(g^{-1})(n)}(g^{-1})).$$

Here $\sigma_j(g^{-1}) \in \Sigma_{(\phi \circ \alpha(g^{-1}))_j} = \Sigma_{\phi_j}$ is that permutation of $(\phi \circ \alpha(g^{-1}))_j$ letters which converts the natural ordering of $(\phi \circ \alpha(g^{-1}))^{-1}(j)$ as a subset of $\{1, \dots, m\}$ to its ordering obtained by regarding it as $\coprod_{\phi(i)=j} \alpha(g)(i)$, so ordered that $\alpha(g)(i)$ precedes $\alpha(g)(i')$ if $i < i'$.

Note that the component corresponding to ϕ is nonempty in the H -fixed points if and only if ϕ is H -equivariant. In what follows we fix such a ϕ . Careful analysis of the definition of $\sigma_j(h)$ shows that for $j \in \{1, \dots, n\}$, and $h, k \in H$, we have

$$(7.1) \quad \sigma_j(hk) = \sigma_j(h)\sigma_{\beta(h^{-1})(j)}(k).$$

The H -action shuffles the indices within each H -orbit of $\mathbf{n}^{\beta|H}$, so it is enough to consider each H -orbit separately. We can assume then that the H -action on $\mathbf{n}^{\beta|H}$ is transitive. The rest of the proof is analogous to the proof of [Lemma 2.15](#).

Since ϕ is H -equivariant and the action is transitive, all the sets $\phi^{-1}(j)$ have the same cardinality, say d . Let K be the stabilizer of $1 \in \mathbf{n}$ under the action of H . By [\(7.1\)](#), σ_1 restricted to K is a homomorphism, and thus

$$\Lambda = \{(k, \sigma_1(k)) \mid k \in K\} \subseteq G \times \Sigma_d$$

is a subgroup that belongs to \mathbb{F}_d . To complete the proof of the claim, we note that the projection to the first coordinate induces a homeomorphism

$$(\mathcal{C}(d) \times \dots \times \mathcal{C}(d))^H \longrightarrow \mathcal{C}(d)^\Lambda.$$

One can easily check that if $(x_1, \dots, x_n) \in \mathcal{C}(d)^n$ is an H -fixed point, then x_1 is a Λ -fixed point, since for all $k \in K$ we have that

$$(k, \sigma_1(k)) \cdot x_1 = kx_1\sigma_1(k^{-1}) = kx_{\beta(k^{-1})(1)}\sigma_{\beta(k^{-1})(1)}(k^{-1}) = x_1,$$

the last equality being true by the assumption that (x_1, \dots, x_n) is fixed by H . To construct an inverse, for every j choose $h_j \in H$ such that $\beta(h_j)(1) = j$. Note that choosing these amounts to choosing a system of coset representatives for H/K . Consider the map $\mathcal{C}(d) \rightarrow \mathcal{C}(d)^n$ that sends x to the n -tuple with j th coordinate

$$x_j = h_j x \sigma_1(h_j^{-1}).$$

We leave it to the reader to check that this map restricts to the fixed points and is inverse to the projection.

7.2. The structure of $\mathbb{D}X$. In the rest of this section, let \mathcal{C}_G be a Σ -free G -operad and let \mathbb{D} be the monad on Π - G -spaces associated to the category of operators $\mathcal{D} = \mathcal{D}(\mathcal{C}_G)$. Part (i) of [Theorem 5.23](#) asserts that \mathbb{D} preserves \mathbb{F}_\bullet -equivalences, and its proof is the hardest equivariant work we face. It involves a detailed combinatorial analysis of the structure of $\mathbb{D}X$ for a Π - G -space X . This entails combinatorial analysis of Π and \mathcal{F} that will also be relevant to the technical proofs for the Segal machine in [Section 8.1](#).

Fix n . Then the definition of $(\mathbb{D}X)_n$ given in [\(5.15\)](#) implies that it is the quotient

$$\left(\bigvee_q \mathcal{D}(\mathbf{q}, \mathbf{n}) \wedge X_q \right) / (\sim)$$

where \sim is the equivalence relation specified by

$$(\psi^*d; x) \sim (d; \psi_*x)$$

for $d \in \mathcal{D}(\mathbf{q}, \mathbf{n})$, $\psi \in \Pi(\mathbf{p}, \mathbf{q})$ and $x \in X_p$. By [Lemma 1.19](#), we can replace wedges and smash products by disjoint unions and products, that is, $(\mathbb{D}X)_n$ is the quotient

$$\left(\prod_q \mathcal{D}(\mathbf{q}, \mathbf{n}) \times X_q \right) / (\sim).$$

Recall that the morphism space $\mathcal{D}(\mathbf{q}, \mathbf{n})$ is given by the disjoint union of components indexed on all $\phi: \mathbf{q} \rightarrow \mathbf{n}$ in \mathcal{F}

$$\mathcal{D}(\mathbf{q}, \mathbf{n}) = \prod_{\phi \in \mathcal{F}(\mathbf{q}, \mathbf{n})} \prod_{1 \leq j \leq n} \mathcal{C}_G(\phi_j),$$

where $\phi_j = |\phi^{-1}(j)|$. The basepoint is the component indexed on $\phi = 0_{q,n}$. We write a non-basepoint morphism as $(\phi; c)$, where $c = (c_1, \dots, c_n)$ with $c_j \in \mathcal{C}_G(\phi_j)$. For a morphism $\psi: \mathbf{p} \rightarrow \mathbf{q}$ in Π , write

$$\psi^*: \prod_{1 \leq j \leq n} \mathcal{C}_G(\phi_j) \rightarrow \prod_{1 \leq j \leq n} \mathcal{C}_G((\phi \circ \psi)_j)$$

for the map $\mathcal{D}(\mathbf{q}, \mathbf{n}) \rightarrow \mathcal{D}(\mathbf{p}, \mathbf{n})$ induced by ψ from the component of ϕ to the component of $\phi \circ \psi$, and write $\psi_*: X_p \rightarrow X_q$ for the induced morphism giving the covariant functoriality of X . Then the equivalence relation \sim takes the form

$$(7.2) \quad (\phi \circ \psi; \psi^*c; x) \sim (\phi; c; \psi_*x)$$

for $(\phi; c) \in \mathcal{D}(\mathbf{q}, \mathbf{n})$ and $x \in X_p$. We shall use the identifications induced by \sim to cut down on the number of components that need be considered. To this end, we first describe the structure of Π and \mathcal{F} , partially following [\[57, §5\]](#).

Definition 7.3. Recall that Π is the subcategory of \mathcal{F} with the same objects and those maps $\phi: \mathbf{m} \rightarrow \mathbf{n}$ such that $\phi_j \leq 1$ for $1 \leq j \leq n$. A map $\pi \in \Pi$ is a *projection* if $\pi_j = 1$ for $1 \leq j \leq n$. A map $\iota \in \Pi$ is an *injection* if $\iota^{-1}(0) = \{0\}$. The permutations are the maps in Π that are both injections and projections. A projection or injection is *proper* if it is not a permutation. Recall that Λ is the subcategory of injections in Π .

Definition 7.4. A map $\phi \in \mathcal{F}$ is *ordered* (or more accurately monotonic) if $i < j$ implies $\phi(i) \leq \phi(j)$; note that this does not restrict the ordering of those i such that $\phi(i) = k$ for some fixed k . A map $\varepsilon \in \mathcal{F}$ is *effective* if $\varepsilon^{-1}(0) = \{0\}$, and an effective map ε is *essential* if it is surjective, that is, if $\varepsilon_j \geq 1$ for $1 \leq j \leq n$.

Observe that every morphism of Π is a composite of proper projections, proper injections, and permutations, and that \mathcal{F} is generated under wedge sum and composition by Π and the single product morphism $\varphi_2: \mathbf{2} \rightarrow \mathbf{1}$.

Lemma 7.5. *A map $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{F} factors as the composite $\iota \circ \varepsilon \circ \pi$ of a projection π , an essential map ε and an injection ι , uniquely up to permutation. That is, given two such decompositions of ϕ , there are permutations σ and τ making the following diagram commute.*

$$\begin{array}{ccccc}
 & & \mathbf{q} & \xrightarrow{\varepsilon} & \mathbf{r} \\
 & \nearrow \pi & \downarrow \sigma & & \downarrow \iota \\
 \mathbf{m} & & & & \mathbf{n} \\
 & \searrow \pi' & \downarrow \tau & & \downarrow \iota' \\
 & & \mathbf{q} & \xrightarrow{\varepsilon'} & \mathbf{r}
 \end{array}$$

Proof. The projection π is determined up to order by which $i \geq 1$ in \mathbf{m} are mapped to 0 in \mathbf{n} . The injection ι is determined up to order by which $j \geq 1$ in \mathbf{n} are not in the image of ϕ . Up to order, ε is the wedge sum in \mathcal{F} of the product maps $\varphi_{\phi_j}: \phi_j \rightarrow \mathbf{1}$, where $\phi_j = |\phi^{-1}(j)|$ for those j such that $1 \leq j \leq n$ and $\phi^{-1}(j)$ is nonempty. Up to permutation, these ϕ_j run through the numbers ε_j , $1 \leq j \leq r$. \square

Remark 7.6. We remark that Π and \mathcal{F} are dualizable Reedy categories, as defined by Berger and Moerdijk [5, Definition 1.1 and Example 1.9(b)], although we have chosen not to use that language. They write \mathcal{F}^+ for the monomorphisms and \mathcal{F}_- for the epimorphisms in \mathcal{F} . We have factored epimorphisms into composites of projections and essential maps to make the structure clearer.

A map $\phi: \mathbf{q} \rightarrow \mathbf{n}$ in \mathcal{F} is ineffective if and only if it factors as a composite

$$\mathbf{q} \xrightarrow{\pi} \mathbf{p} \xrightarrow{\zeta} \mathbf{n},$$

where $p = q - \phi_0$, π is the proper ordered projection such that $\pi(k) = 0$ if and only if $\phi(k) = 0$, and ζ is an effective morphism. Then $\zeta_j = \phi_j$ for $j \geq 1$ and, as in [Observation 5.9](#), $\pi^*(c) = c$ for any $c \in \prod_j \mathcal{C}_G(\zeta_j)$. Therefore

$$(\phi; c; x) = (\zeta \circ \pi; \pi^*(c); x) \sim (\zeta; c; \pi_*(x)).$$

This says both that we may restrict to those wedge summands that are indexed on the effective morphisms of \mathcal{F} , ignoring the ineffective ones, and that we can ignore the proper projections in Π , restricting further analysis to \sim applied to morphisms of $\Lambda \subset \Pi$. Here we must start paying attention to permutations.

Lemma 7.7. *If $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ is an effective morphism in \mathcal{F} , there is a permutation $\nu \in \Sigma_p$ such that $\varepsilon \circ \nu$ is ordered; ν is not unique, but the ordered morphism $\varepsilon \circ \nu$ is independent of the choice.*

Applying \sim to the permutations ν , we can further restrict to components indexed on ordered effective morphisms. We abbreviate notation.

Notation 7.8. We say that an ordered effective morphism in \mathcal{F} is an *OE-function*. If ε is effective and $\varepsilon \circ \nu$ is ordered, we call it the *OE-function* associated to ε . We let $\mathcal{E}(\mathbf{p}, \mathbf{n})$ denote the set of all *OE-functions* $\mathbf{p} \rightarrow \mathbf{n}$.

Definition 7.9. Let $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ be an *OE*-function, with $\varepsilon_j = |\varepsilon^{-1}(j)|$. The sum of the ε_j is p and we define

$$\Sigma(\varepsilon) = \Sigma_{\varepsilon_1} \times \cdots \times \Sigma_{\varepsilon_n} \subset \Sigma_p,$$

where the inclusion is determined by identifying \mathbf{p} with $\varepsilon_1 \vee \cdots \vee \varepsilon_n$. In other words, we partition $\{1, \dots, p\}$ into n blocks of letters, as dictated by ε .

Lemma 7.10. *If $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ is an *OE*-function and $\nu \in \Sigma_p$, then $\varepsilon \circ \nu$ is ordered (and hence equal to ε) if and only if ν is in the subgroup $\Sigma(\varepsilon)$.*

The equivalence relation \sim is defined in terms of precomposition of morphisms of \mathcal{F} with morphisms of Π , while the action of Π on $\mathbb{D}X$ is defined in terms of postcomposition of morphisms of \mathcal{F} with morphisms of Π . Especially for permutations, these are related. We discuss composition with permutations on both sides in the following three remarks.

Remark 7.11. Let $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ be an *OE*-function and let $\sigma \in \Sigma_n$. Define $\tau(\sigma) \in \Sigma_p$ to be the permutation that permutes the n blocks of size $\{\varepsilon_1, \dots, \varepsilon_n\}$ as σ permutes n letters. Then $\sigma \circ \varepsilon \circ \tau(\sigma)^{-1}$ is again ordered, and it is the *OE*-function associated to $\sigma \circ \varepsilon$. Moreover, the function $\tau = \tau_\varepsilon: \Sigma_n \rightarrow \Sigma_p$ is a homomorphism. Inspecting our identifications and using [Observation 5.9](#), we see that postcomposition with σ sends a point with representative $(\varepsilon; c_1, \dots, c_n; x)$ to the point with representative

$$(7.12) \quad (\sigma \varepsilon \tau(\sigma)^{-1}; c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; \tau(\sigma)_* x).$$

Remark 7.13. One can think of an *OE*-function $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ as an ordered partition of the set $\{1, \dots, p\}$ into n ordered subsets. Observe that ε is essential if and only if $\varepsilon_j > 0$ for all $1 \leq j \leq n$, so that our n subsets are all nonempty. Define the signature of ε to be the unordered set of numbers $\{\varepsilon_1, \dots, \varepsilon_n\}$. The action of Σ_n permutes partitions with the same signature (and thus the same p). The *OE*-functions ε and $\sigma \circ \varepsilon \circ \tau(\sigma)^{-1}$ have the same signature, and any two ordered partitions with the same signature are connected this way.

Remark 7.14. If $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ is an *OE*-function and $\rho \in \Sigma_p$, then

$$(\varepsilon \circ \rho; c; x) \sim (\varepsilon; (\rho^{-1})^* c; \rho_*^{-1} x).$$

We have now accounted for \sim applied to all proper projections and to all permutations. It remains to consider proper injections ι . For any such $\iota: \mathbf{p} \rightarrow \mathbf{q}$, there is a permutation $\nu \in \Sigma_p$ such that $\iota \circ \nu$ is ordered. Recall that we have the ordered injections $\sigma_i: \mathbf{p} - \mathbf{1} \rightarrow \mathbf{p}$ that skip i , $1 \leq i \leq p$. Every proper ordered injection is a composite of such σ_i , so it remains to account for \sim applied to the σ_i . These give an equivalence relation on

$$\coprod_q \coprod_{\varepsilon \in \mathcal{E}(\mathbf{q}, \mathbf{n})} \left(\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} X_q$$

whose quotient is $(\mathbb{D}X)_n$. The component with $q = 0$ gives the basepoint.

The monad \mathbb{D} , like the monad \mathbb{C}_G , is filtered. Its p th filtration at level n , denoted $F_p(\mathbb{D}X)_n$, is the image of the components indexed on $q \leq p$. We can think of the quotient as given by filtration-lowering “basepoint identifications”, namely

$$(7.15) \quad (\varepsilon; c_1, \dots, c_n; (\sigma_i)_* x) \sim (\varepsilon \circ \sigma_i; c_1, \dots, c_{i-1}, \sigma_r c_i, c_{i+1}, \dots, c_n; x),$$

for some $i = 1, \dots, q$. Here r is the position of i within its block of $\varepsilon_j = |\varepsilon^{-1}(j)|$ letters, where $j = \varepsilon(i)$, and $\sigma_r: \mathcal{C}_G(\varepsilon_j) \rightarrow \mathcal{C}_G(\varepsilon_j - 1)$ is the map from (5.3).²² In equivalent abbreviated notation, we write this as

$$(7.16) \quad (\varepsilon; c; (\sigma_i)_* x) \sim (\varepsilon \circ \sigma_i; \sigma_i^* c; x)$$

We give a definition of *latching spaces* for functors $\Pi \rightarrow \underline{G}\mathcal{U}_*$, in the spirit of the usual definition for simplicial spaces. This definition coincides with the one in [5] when considering Π as a dualizable Reedy category (see Remark 7.6). Via Lemma 7.18 below, this concept will allow us to make inductive arguments using the pushout diagram (7.21). This is similar to classical arguments involving simplicial spaces.

Definition 7.17. For a functor $X: \Pi \rightarrow \underline{G}\mathcal{U}_*$, the ordered injections $\sigma_i: \mathbf{n} - \mathbf{1} \rightarrow \mathbf{n}$ induce maps $(\sigma_i)_*: X_{n-1} \rightarrow X_n$. A point of X_n in the image of some $(\sigma_i)_*$ is said to be degenerate. We define the n th latching space of X to be the set of degenerate points in X_n :

$$L_n X = \bigcup_{i=1}^n (\sigma_i)_*(X_{n-1}).$$

It is a $(G \times \Sigma_n)$ -space, and the inclusion $L_n X \rightarrow X_n$ is a $(G \times \Sigma_n)$ -map. For a G -space X , $L_n \mathbb{R}X$ is the subspace of X^n consisting of those points at least one coordinate of which is the basepoint.

Lemma 7.18. *If X is a Π - G -space, then the inclusion $L_n X \rightarrow X_n$ is a $(G \times \Sigma_n)$ -cofibration.*

Proof. This is a direct application of [8, Theorem A.2.7]. \square

Following up Definition 7.17, it is helpful to be more explicit about degenerate elements.

Definition 7.19. Fixing an $\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})$, write $[c; x]$ for an element of

$$\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j) \times_{\Sigma(\varepsilon)} X_p,$$

meaning that $(c; x)$ is a representative element for an orbit $[c; x]$ under the action of $\Sigma(\varepsilon)$. The element $x \in X_p$ is degenerate if $x \in L_p X$, that is, if $x = (\sigma_i)_* y$ for some $y \in X_{p-1}$ and some i , and we say that $(\varepsilon; [c; x])$ is degenerate if x is degenerate. Since $\tau \circ \sigma_i$ is a proper injection for any $\tau \in \Sigma(\varepsilon)$ and any i , the condition of being degenerate is independent of the orbit representative $(c; x)$.

By use of (7.16), we reach the following description of the elements of $(\mathbb{D}X)_n$.

Lemma 7.20. *A point of $(\mathbb{D}X)_n$ has a unique nondegenerate representative $(\varepsilon; [c; x])$.*

Just as nonequivariantly ([57, p. 218]), we have pushouts of $(G \times \Sigma_n)$ -spaces

$$(7.21) \quad \begin{array}{ccc} \prod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})} \left(\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} L_p X & \xrightarrow{\nu} & F_{p-1}(\mathbb{D}X)_n \\ \downarrow & & \downarrow \\ \prod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})} \left(\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} X_p & \longrightarrow & F_p(\mathbb{D}X)_n \end{array}$$

²²For consistency of notation here, we might have written σ_r^* instead of σ_r , as appears in (5.3).

With notation as in (7.16), the map ν sends a point with orbit representative $(\varepsilon; c; (\sigma_i)_*x)$ to the point with orbit representative $(\varepsilon \circ \sigma_i; \sigma_i^*c; x)$.

By Lemma 7.18, when X is a Π - G -space the inclusion $L_p X \rightarrow X_p$ is a $(G \times \Sigma_p)$ -cofibration for each p . Then each component of the left vertical map is a $(G \times \Sigma_p)$ -cofibration before passing to orbits under $\Sigma(\varepsilon)$, so the map on orbits is a G -cofibration by [8, Lemma A.2.3]. Since Σ_n acts by permuting components and the factors of the displayed products, it follows that the left vertical map is a $(G \times \Sigma_n)$ -cofibration, hence so is the right vertical arrow.

7.3. The proof that \mathbb{D} preserves Π - G -spaces. We here complete the proof of Proposition 5.18 by proving that \mathbb{D} preserves the cofibration condition in Definition 2.10. By Remark 2.5, we must prove that for every ordered injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in Π , the induced map $\phi_*: (\mathbb{D}X)_m \rightarrow (\mathbb{D}X)_n$ is a $(G \times \Sigma_\phi)$ -cofibration. To do so, we adopt the following notation.

Let $T \subset \mathbf{n} \setminus \{0\}$. We write Σ_T for the subgroup of Σ_n of those permutations τ such that $\tau(T) = T$. Note that Σ_T consists of those permutations that act separately within T and its complement. As a group, Σ_T is isomorphic to $\Sigma_{|T|} \times \Sigma_{n-|T|}$.

Notation 7.22. We denote by $\sigma_T: \mathbf{n} - |T| \rightarrow \mathbf{n}$ the ordered injection that misses the elements of T . It can be written as $\sigma_T = \sigma_{i_k} \cdots \sigma_{i_1}$ where $i_1 < \cdots < i_k$ are the elements of T . If T is empty we use the convention that $\sigma_T = \text{id}$ (which makes sense as the empty composition). For a Π - G -space X , there is an induced map $(\sigma_T)_*: X_{n-|T|} \rightarrow X_n$, which for the sake of readability we denote by σ_T . Then, for a Π - G -space X ,

$$\sigma_T X_{n-|T|} = \bigcap_{i \in T} \sigma_i X_{n-1}.$$

For the case $T = \emptyset$, we interpret this as $\sigma_T X_n = X_n$.

Note that any ordered injection ϕ is of the form σ_T where $T = \mathbf{n} \setminus \text{im}(\phi)$, and in that case, $\Sigma_\phi = \Sigma_T$. Thus, it suffices to show that for all subsets T , the maps

$$(7.23) \quad \sigma_T: (\mathbb{D}X)_{n-|T|} \rightarrow (\mathbb{D}X)_n$$

are $(G \times \Sigma_T)$ -cofibrations. Note that the action of Σ_T on $(\mathbb{D}X)_{n-|T|}$ is given by restricting to the action on the block $\mathbf{n} \setminus T$ and identifying it with the set $\mathbf{n} - |T|$.

Consider the cube obtained by mapping the pushout square of (7.21) for $(\mathbb{D}X)_{n-|T|}$ to the pushout square (7.21) for $(\mathbb{D}X)_n$. Write σ_T for the maps from the four corners of the first square to the four corners of the second square. We will prove by induction on p that the map

$$(7.24) \quad \sigma_T: F_p(\mathbb{D}X)_{n-|T|} \rightarrow F_p(\mathbb{D}X)_n$$

is a $(G \times \Sigma_T)$ -cofibration, and we assume this for the map with p replaced by $p-1$. The vertical maps in the diagram (7.21) for $\mathbb{D}X_{n-|T|}$ are $(G \times \Sigma_{n-|T|})$ -cofibrations, so they are $(G \times \Sigma_T)$ -cofibrations via the action described above. The vertical maps in the diagram (7.21) for $\mathbb{D}X_n$ are $(G \times \Sigma_n)$ -cofibrations, so in particular they are also $(G \times \Sigma_T)$ -cofibrations.

The map σ_T on the left corners of the diagram is given by

$$(\varepsilon, [(c_1, \dots, c_{n-|T|}), x]) \mapsto (\sigma_T \circ \varepsilon, [(d, x)]),$$

where d is the n -tuple given by

$$d_j = \begin{cases} 0 \in \mathcal{C}_G(0) & \text{if } j \in T \\ c_{\sigma_T^{-1}(j)} & \text{if } j \notin T \end{cases}$$

It is not hard to see that for the left entries of the pushout diagram, the map σ_T is the inclusion of those components labeled by maps $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ that miss the elements of T . The groups $\Sigma_{n-|T|}$ and Σ_n act on the source and target, respectively, as stated in [Remark 7.11](#). In particular, both actions shuffle components, hence the inclusion of components is a $(G \times \Sigma_T)$ -cofibration. By the induction hypothesis, the map connecting the top right corners of the cube is also a $(G \times \Sigma_T)$ -cofibration.

[Proposition 12.4](#) below, which concerns the preservation of cofibrations by pushouts, implies that the map [\(7.24\)](#) connecting the bottom right corners of the cube is a $(G \times \Sigma_T)$ -cofibration, as claimed, noting that

$$\sigma_T \left(\prod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n}-|T|)} \left(\prod_{j=1}^{n-|T|} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} X_p \right) \cap \left(\prod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})} \left(\prod_{j=1}^n \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} L_p X \right)$$

is equal to

$$\sigma_T \left(\prod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n}-|T|)} \left(\prod_{j=1}^{n-|T|} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} L_p X \right).$$

[Proposition 12.5](#) below, which concerns the preservation of cofibrations by sequential colimits implies that the map of colimits

$$\sigma_T: (\mathbb{D}X)_{n-|T|} = \operatorname{colim}_p F_p(\mathbb{D}X)_{n-|T|} \longrightarrow \operatorname{colim}_p F_p(\mathbb{D}X)_n = (\mathbb{D}X)_n$$

is a $(G \times \Sigma_T)$ -cofibration. To see this, we must check that the intersection condition

$$\sigma_T(F_p(\mathbb{D}X)_{n-|T|}) \cap F_{p-1}(\mathbb{D}X)_n = \sigma_T(F_{p-1}(\mathbb{D}X)_{n-|T|})$$

of [Proposition 12.5](#) is satisfied. One inclusion is obvious. For the other, take an element $(\varepsilon, [c, x]) \in F_p(\mathbb{D}X)_{n-|T|} \setminus F_{p-1}(\mathbb{D}X)_{n-|T|}$; in particular, $x \in X_p \setminus L_p X$. Then $\sigma_T(\varepsilon, [c, x]) = (\sigma_T \varepsilon, [\sigma_T^* c, x]) \in F_p(\mathbb{D}X)_n \setminus F_{p-1}(\mathbb{D}X)_n$, as required.

This completes the proof that the map [\(7.23\)](#) is a $(G \times \Sigma_T)$ -cofibration.

7.4. The proof that \mathbb{D} preserves \mathbb{F}_\bullet -equivalences. We assume given an \mathbb{F}_\bullet -equivalence $f: X \rightarrow Y$ between Π - G -spaces. [Theorem 5.23\(i\)](#) asserts that the map $\mathbb{D}f: \mathbb{D}X \rightarrow \mathbb{D}Y$ is an \mathbb{F}_\bullet -equivalence. We shall prove it by showing by induction on p that f induces an \mathbb{F}_n -equivalence $F_p(\mathbb{D}X)_n \rightarrow F_p(\mathbb{D}Y)_n$ for each n and each $p \geq 0$, there being nothing to prove when $p = 0$. By the usual gluing lemma on pushouts, proven equivariantly in [\[7, Theorem A.4.4\]](#) (but also a model theoretic formality), it suffices to prove that the maps induced by f on the source and target of the left vertical arrow in [\(7.21\)](#) induce equivalences on Λ -fixed point spaces, where $\Lambda \subset G \times \Sigma_n$ and $\Lambda \cap \Sigma_n = \{e\}$. We have $\Lambda = \{(h, \alpha(h)) \mid h \in H\}$ for some subgroup H of G and homomorphism $\alpha: H \rightarrow \Sigma_n$, and we regard \mathbf{n} as a based H -set via α . Fixing Λ for the rest of the section, we shall prove [Theorem 5.23\(i\)](#) by analyzing Λ -fixed points.

We first consider the target, that is the lower left corner of the diagram. To clarify the argument, we separate out some of its combinatorics before proceeding.

Definition 7.25. Let $\varepsilon: \mathbf{p} \rightarrow \mathbf{n}$ be an *OE*-function, let $\tau: \Sigma_n \rightarrow \Sigma_p$ be the homomorphism determined by ε as defined in [Remark 7.11](#), and define $\beta: H \rightarrow \Sigma_p$ to be the composite homomorphism $\tau\alpha$. Say that ε is Λ -fixed if $\alpha(h)\varepsilon = \varepsilon\beta(h)$ for all $h \in H$. Note that this implies that $\varepsilon_j = \varepsilon_k$ if j and k are in the same H -orbit.

Define $\mathcal{E}(\mathbf{p}, \mathbf{n})^\Lambda$ to be the set of all Λ -fixed *OE*-functions $\mathbf{p} \rightarrow \mathbf{n}$. Fix $\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})^\Lambda$. Say that a function

$$\gamma = (\gamma_1, \dots, \gamma_n): H \rightarrow \Sigma(\varepsilon)$$

is *admissible*, or admissible with respect to α , if

$$(7.26) \quad \gamma_j(hk) = \gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(k)$$

for $h, k \in H$ and $1 \leq j \leq n$. For any function $\gamma: H \rightarrow \Sigma(\varepsilon)$, define a function²³ $\gamma \cdot \beta: H \rightarrow \Sigma_p$ by $(\gamma \cdot \beta)(h) = \gamma(h)\beta(h)$.

We leave the combinatorial proof of the following lemma to the reader. When the action of H on $\mathbf{n} \setminus \{0\}$ has a single orbit, there is a conceptual rather than combinatorial proof using wreath products.

Lemma 7.27. Fix $\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})^\Lambda$. A function $\gamma: H \rightarrow \Sigma(\varepsilon)$ is admissible with respect to α if and only if $\gamma \cdot \beta$ is a homomorphism $H \rightarrow \Sigma_p$.

The following result is the central step of the proof of [Theorem 5.23\(i\)](#). It identifies the Λ -fixed points of the bottom left corner of the pushout diagram [\(7.21\)](#).

Proposition 7.28. Let $\Lambda = \{(h, \alpha(h))\}$ and assume that the action of H on $\mathbf{n} \setminus \{0\}$ defined by α is transitive. Then there is a natural homeomorphism

$$\begin{array}{c} \left(\coprod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})} \left(\left(\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j) \right) \times_{\Sigma(\varepsilon)} X_p \right) \right)^\Lambda \\ \downarrow \omega \\ \coprod_{\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})^\Lambda} \left(\prod_{\gamma: H \rightarrow \Sigma(\varepsilon)} \mathcal{C}_G(\varepsilon_1)^{\Lambda_\gamma^1} \times X_p^{\Lambda_\gamma} \right) / \Sigma(\varepsilon). \end{array}$$

In the target, the second wedge runs over all admissible functions

$$\gamma = (\gamma_1, \dots, \gamma_n): H \rightarrow \Sigma(\varepsilon);$$

the groups Λ_γ and Λ_γ^1 are specified by

$$\Lambda_\gamma = \{(h, (\gamma \cdot \beta)(h)) \mid h \in H\} \subset G \times \Sigma_p$$

and

$$\Lambda_\gamma^1 = \{(k, \gamma_1(k)) \mid k \in K\} \subset G \times \Sigma_{\varepsilon_1},$$

where $K \subset H$ is the isotropy group of 1 under the action of H on \mathbf{n} given by α .

Proof. Since $\alpha(k)(1) = 1$ for $k \in K$, γ_1 is a homomorphism $K \rightarrow \Sigma_{\varepsilon_1}$ by specialization of [\(7.26\)](#). In the target, we pass to orbits from the $\Sigma(\varepsilon)$ -action defined on the term in parentheses by

$$\rho(\gamma; c; x) = (\rho * \gamma; c\rho_1^{-1}; \rho_*x),$$

Here $\rho = (\rho_1, \dots, \rho_n)$ is in $\Sigma(\varepsilon)$, γ is admissible, $c \in \mathcal{C}_G(\varepsilon_1)^{\Lambda_\gamma^1}$, and $x \in X_p^{\Lambda_\gamma}$. The j th coordinate of $\rho * \gamma$ is defined by

$$(7.29) \quad (\rho * \gamma)_j(h) = \rho_j \gamma_j(h) \rho_{\alpha(h)^{-1}(j)}^{-1}.$$

²³We use the notation \cdot since we often use juxtaposition to mean composition in this section.

A quick check of definitions shows that

$$(\rho * \gamma) \cdot \beta = \rho(\gamma \cdot \beta)\rho^{-1},$$

which also implies that $\rho * \gamma$ is admissible since γ is admissible. Similarly, $c\rho_1^{-1}$ is fixed by $\Lambda_{\rho*\gamma}^1$ since c is fixed by Λ_γ^1 and $\rho_*(x)$ is fixed by $\Lambda_{\rho*\gamma}$ since x is fixed by Λ_γ . Thus the action makes sense. Moreover, as we shall need later, this action is free. If $\rho(\gamma; c; x) = (\gamma; c; x)$, then $c\rho_1^{-1} = c$ and thus $\rho_1 = 1$ since Σ_{ε_1} acts freely on $\mathcal{C}_G(\varepsilon_1)$. Also, $\rho * \gamma = \gamma$ and thus $\rho_j^{-1}\gamma_j(h)\rho_{\alpha(h)^{-1}(j)} = \gamma_j(h)$ for all h . Taking $j = 1$, this implies that $\rho_{\alpha(h)^{-1}(1)} = 1$ for all h . Since we are assuming the action of H induced by α on $\mathfrak{n} \setminus \{0\}$ is transitive, this implies that $\rho = 1 \in \Sigma(\varepsilon)$.

We turn to the promised homeomorphism. By (7.12), for a point z represented by $(\varepsilon; c_1, \dots, c_n; x)$, $c_j \in \mathcal{C}_G(\varepsilon_j)$ and $x \in X_p$, $(h, \alpha(h))z$ is represented by

$$(\alpha(h)\varepsilon\beta(h)^{-1}; hc_{\alpha(h)^{-1}(1)}, \dots, hc_{\alpha(h)^{-1}(n)}; \beta(h)_*(hx))$$

where, as before, $\beta = \tau\alpha$. Assume that z is fixed by Λ . Then we must have $\alpha(h)\varepsilon\beta(h)^{-1} = \varepsilon$, so that $\alpha(h)\varepsilon = \varepsilon\beta(h)$ and thus $\varepsilon \in \mathcal{E}(\mathfrak{p}, \mathfrak{n})^\Lambda$. We must also have

$$(c_1, \dots, c_n; x) \sim (hc_{\alpha(h)^{-1}(1)}, \dots, hc_{\alpha(h)^{-1}(n)}; \beta(h)_*(hx)),$$

so that for each $h \in H$ there exists $\gamma(h) = \gamma_1(h) \times \dots \times \gamma_n(h) \in \Sigma(\varepsilon)$ such that

$$(7.30) \quad c_j\gamma_j(h) = hc_{\alpha(h)^{-1}(j)} \quad \text{and} \quad x = \gamma(h)_*\beta(h)_*(hx).$$

Note that for any given n -tuple (c_1, \dots, c_n) , $\gamma(h)$ is unique since the action of $\Sigma(\varepsilon)$ on $\prod_{1 \leq j \leq n} \mathcal{C}_G(\varepsilon_j)$ is free. For $h, k \in H$ and $1 \leq j \leq n$,

$$\begin{aligned} c_j\gamma_j(hk) &= (hk)c_{\alpha(hk)^{-1}(j)} \\ &= h(kc_{\alpha(k)^{-1}(\alpha(h)^{-1}(j))}) \\ &= hc_{\alpha(h)^{-1}(j)}\gamma_{\alpha(h)^{-1}(j)}(k) \\ &= c_j\gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(k). \end{aligned}$$

Since the action of Σ_{ε_j} on $\mathcal{C}_G(\varepsilon_j)$ is free, this implies that (7.26) holds, so that γ is admissible. Note that we have not yet used that the action given by α is transitive.

Now the map ω is defined by

$$\omega(\varepsilon; c_1, \dots, c_n; x) = (\varepsilon; \gamma; c_1; x).$$

We see from (7.30) that x is in $X_p^{\Lambda_\gamma}$ and that c_1 is in $\mathcal{C}_G(\varepsilon_1)^{\Lambda_\gamma^1}$, the latter using the fact that K is the isotropy group of $1 \in \mathfrak{n} \setminus \{0\}$. We must check that ω is well-defined. Thus suppose that

$$(\varepsilon; c_1, \dots, c_n; x) \sim (\varepsilon; d_1, \dots, d_n; y).$$

Then there exists $\rho \in \Sigma(\varepsilon)$ such that $c_j = d_j\rho_j$ and $y = \rho_*x$. Using (7.29) and (7.30), we see that

$$\begin{aligned} hd_{\alpha(h)^{-1}(j)} &= hc_{\alpha(h)^{-1}(j)}\rho_{\alpha(h)^{-1}(j)}^{-1} \\ &= c_j\gamma_j(h)\rho_{\alpha(h)^{-1}(j)}^{-1} \\ &= d_j\rho_j\gamma_j(h)\rho_{\alpha(h)^{-1}(j)}^{-1} \\ &= d_j(\rho * \gamma(h))_j. \end{aligned}$$

Thus, comparing with (7.30) for $(d_1, \dots, d_n; y)$, and using the freeness of the action, we see that

$$\omega(\varepsilon; d_1, \dots, d_n; y) = (\varepsilon; \rho * \gamma, d_1, y) = (\varepsilon; \rho * \gamma, c_1\rho_1^{-1}, \rho_*x) = \rho(\varepsilon; \gamma; c_1; x),$$

so that the targets of our equivalent elements are equivalent.

Clearly ω is continuous since it is obtained by passage to orbits from a (disconnected) cover by restriction to subspaces of the projection that forgets the coordinates (c_2, \dots, c_n) .

To define ω^{-1} , first choose coset representatives for H/K where K is the isotropy group of 1, that is, choose $h_j \in H$ such that $\alpha(h_j)(1) = j$ for $1 \leq j \leq n$, taking $h_1 = e$. Then define ω^{-1} by

$$\omega^{-1}(\varepsilon; \gamma; c; x) = (\varepsilon; c_1, \dots, c_n; x)$$

where $c_j = h_j c \gamma_j (h_j)^{-1}$. Note that $c_1 = c$ and that ω^{-1} does not depend on the choice of coset representatives. Here ε is Λ -fixed, $\gamma: H \rightarrow \Sigma(\varepsilon)$ is admissible, $c \in \mathcal{C}_G(\varepsilon_1)^{\Lambda^\dagger}$ and $x \in X_p^{\Lambda^\dagger}$. We must show that $\omega^{-1}(\varepsilon; \gamma; c; x)$ is fixed by Λ . First, note that $(h, \alpha(h))$ sends ε to $\alpha(h)\varepsilon\beta(h)^{-1} = \varepsilon$, since $\varepsilon \in \mathcal{E}(\mathbf{p}, \mathbf{n})^\Lambda$. Omitting ε from the notation for readability,

$$\begin{aligned} (h, \alpha(h))(c_1, \dots, c_n, x) &= (hc_{\alpha(h)^{-1}(1)}, \dots, hc_{\alpha(h)^{-1}(n)}, \beta(h)_*(hx)) \\ &= (hc_{\alpha(h)^{-1}(1)}, \dots, hc_{\alpha(h)^{-1}(n)}, \gamma(h)_*^{-1}(x)) \\ &\sim (hc_{\alpha(h)^{-1}(1)}\gamma_1(h)^{-1}, \dots, hc_{\alpha(h)^{-1}(n)}\gamma_n(h)^{-1}, x). \end{aligned}$$

We claim that $hc_{\alpha(h)^{-1}(j)}\gamma_j(h)^{-1} = c_j$. The definition of $c_{\alpha(h)^{-1}(j)}$ gives us the identification

$$hc_{\alpha(h)^{-1}(j)}\gamma_j(h)^{-1} = h(h_{\alpha(h)^{-1}(j)}c\gamma_{\alpha(h)^{-1}(j)}(h_{\alpha(h)^{-1}(j)})^{-1})\gamma_j(h)^{-1}.$$

Now note that

$$\alpha(hh_{\alpha(h)^{-1}(j)})(1) = \alpha(h)(\alpha(h)^{-1}(j)) = j,$$

hence $hh_{\alpha(h)^{-1}(j)}$ is in the coset represented by h_j . Thus there exists a $k \in K$ such that $hh_{\alpha(h)^{-1}(j)} = h_j k$. Since γ satisfies equation (7.26),

$$\begin{aligned} \gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(h_{\alpha(h)^{-1}(j)}) &= \gamma_j(hh_{\alpha(h)^{-1}(j)}) \\ &= \gamma_j(h_j k) \\ &= \gamma_j(h_j)\gamma_{\alpha(h_j)^{-1}(j)}(k) \\ &= \gamma_j(h_j)\gamma_1(k). \end{aligned}$$

Thus

$$hc_{\alpha(h)^{-1}(j)}\gamma_j(h)^{-1} = h_j k c \gamma_1(k)^{-1} \gamma_j(h_j)^{-1} = h_j c \gamma_j(h_j)^{-1} = c_j$$

as claimed. Therefore the map ω^{-1} really does land in the Λ -fixed points.

To show that ω^{-1} is well-defined, note that if $(\varepsilon; \gamma; c; x) \sim (\varepsilon; \rho * \gamma; c\rho_1^{-1}, \rho_* x)$ for some $\rho \in \Sigma(\varepsilon)$, then ω^{-1} sends the latter to $(\varepsilon; d_1, \dots, d_n; \rho_* x)$, where

$$\begin{aligned} d_j &= h_j c \rho_1^{-1} (\rho_j \gamma_j (h_j) \rho_{\alpha(h_j)^{-1}(j)}^{-1})^{-1} \\ &= h_j c \rho_1^{-1} (\rho_j \gamma_j (h_j) \rho_1^{-1})^{-1} \\ &= h_j c \rho_1^{-1} \rho_1 \gamma_j (h_j)^{-1} \rho_j^{-1} \\ &= c_j \rho_j^{-1}. \end{aligned}$$

Thus

$$(d_1, \dots, d_n, \rho_* x) = \rho \cdot (c_1, \dots, c_n, x),$$

so that ω^{-1} is well-defined. This map is clearly continuous.

It is easy to see that the map forward and the map backward composed in either order are the identity, hence we have the claimed homeomorphism. \square

The restriction to transitive action by α in the previous result serves only to simplify the combinatorics. The following remark indicates the changes that are needed to deal with the general case.

Remark 7.31. When the action of H on $\mathbf{n} \setminus \{0\}$ is not transitive, we argue analogously to [Lemma 2.15](#) and [Theorem 5.12](#) to obtain an analogous homeomorphism. We break the H -set $\mathbf{n} \setminus \{0\}$ given by α into a disjoint union of orbits H/K_a of size $n_a = |H/K_a|$, where $\sum_a n_a = n$ and K_a is the isotropy group of the initial element, denoted 1_a , in its orbit in $\mathbf{n} \setminus \{0\}$. That breaks \mathbf{n} into the wedge of subsets \mathbf{n}_a and breaks $\Sigma(\varepsilon)$ into a product of subgroups $\Sigma(\varepsilon(a)) = \prod_{j \in H/K_a} \Sigma(\varepsilon_j)$. Paying attention to the ordering, the product of the $\mathcal{C}_G(\varepsilon_j)$ in the source of ω breaks into the product over a of those $\mathcal{C}_G(\varepsilon_j)$ such that j is in the a th orbit of $\mathbf{n} \setminus \{0\}$. To generalize the target of ω accordingly, define subgroups

$$\Lambda_\gamma^{1_a} = \{(k, \gamma_{1_a}(k)) \mid k \in K_a\} \subset G \times \Sigma_{\varepsilon_{1_a}}$$

and replace $\mathcal{C}_G(\varepsilon_1)^{\Lambda_\gamma^1}$ by the product over a of the $\mathcal{C}_G(\varepsilon_{1_a})^{\Lambda_\gamma^{1_a}}$. With these changes of source and target and just a bit of extra bookkeeping, it is straightforward to state and prove the general analogue of [Proposition 7.28](#).

In the single orbit case, observe that if $f: X_p \rightarrow Y_p$ is a Λ_γ -equivalence, it induces an equivalence on the target of ω before passage to quotients under the action of $\Sigma(\varepsilon)$. Since the $\Sigma(\varepsilon)$ action is free, the equivalence passes to the quotients. Generalizing to the multi-orbit case, this concludes our proof that we have a Λ -equivalence in the lower left corner of the pushout diagram [\(7.21\)](#).

We next consider the upper left corner of [\(7.21\)](#). Precisely the same argument as that just given, but with X_p replaced by $L_p X$, identifies the Λ -fixed subspace of the upper left corner in terms of appropriate fixed point subspaces of $L_p X$. Therefore the same argument as that just given shows that the following result implies that f induces an equivalence on the upper left corner, as needed to complete the proof of [Theorem 5.23\(i\)](#).

Proposition 7.32. *If $f: X \rightarrow Y$ is an \mathbb{F}_\bullet -equivalence of Π - G -spaces, then $f: L_n X \rightarrow L_n Y$ is an \mathbb{F}_n -equivalence for each n .*

To prove this, we need more combinatorics to identify $(L_n X)^\Lambda$, where $\Lambda \in \mathbb{F}_n$. We again take $\Lambda = \{(h, \alpha(h)) \mid \alpha: H \rightarrow \Sigma_n\} \subset G \times \Sigma_n$ and again view $\mathbf{n} \setminus \{0\}$ as an H -set via α . For a subset U of $\mathbf{n} \setminus \{0\}$ with u elements, recall from [Notation 7.22](#) that $\sigma_U: \mathbf{n} - \mathbf{u} \rightarrow \mathbf{n}$ denotes the ordered injection that skips the elements in U . It is a composite of degeneracies $\sigma_U = \sigma_{i_k} \cdots \sigma_{i_1}$ where $i_1 < \cdots < i_k$ are the elements of U . Given $1 \leq i \leq n$, let $\pi_i: \mathbf{n} \rightarrow \mathbf{n} - \mathbf{1}$ be the ordered projection that sends i to 0. Similarly, define $\pi_U: \mathbf{n} \rightarrow \mathbf{n} - \mathbf{u}$ to be the ordered projection that sends the elements of U to 0; explicitly, $\pi_U = \pi_{i_1} \cdots \pi_{i_k}$. Note that $\pi_U \sigma_U$ is the identity, and

$$\sigma_U \pi_U(i) = \begin{cases} i & \text{if } i \notin U \\ 0 & \text{if } i \in U. \end{cases}$$

Remark 7.33. The maps σ_i correspond to the degeneracies in Δ^{op} via the inclusion $F: \Delta^{op} \rightarrow \mathcal{F}$, except there is a shift since we are indexing on the non-zero elements of \mathbf{n} . The maps π_i are mostly invisible to Δ . The collection of maps $\{\sigma, \pi\}$ satisfies

the following subset of the simplicial relations, as can be easily checked.

$$\begin{aligned}\pi_i\pi_j &= \pi_{j-1}\pi_i & \text{if } i < j \\ \sigma_j\sigma_i &= \sigma_i\sigma_{j-1} & \text{if } i < j \\ \pi_i\sigma_j &= \sigma_{j-1}\pi_i & \text{if } i < j \\ \pi_i\sigma_i &= \text{id}.\end{aligned}$$

Now assume that $U \subset \mathbf{n} \setminus \{0\}$ is a H -subset of $\mathbf{n} \setminus \{0\}$ and note that its complement is also a H -subset of $\mathbf{n} \setminus \{0\}$. For $h \in H$, define

$$\alpha_U(h) = \pi_U \alpha(h) \sigma_U : \mathbf{n} - \mathbf{u} \longrightarrow \mathbf{n} - \mathbf{u}.$$

This is essentially the restriction of $\alpha(h)$ to $\mathbf{n} \setminus U$, but using the ordered inclusion σ_U to identify that set with $\mathbf{n} - \mathbf{u}$. Note that $\alpha_U(e) = \text{id}$ and that

$$(7.34) \quad \alpha(h)\sigma_U = \sigma_U\pi_U\alpha(h)\sigma_U = \sigma_U\alpha_U(h)$$

since $\sigma_U\pi_U$ is the identity on $\mathbf{n} \setminus U$ and 0 on U and since $\alpha(h)\sigma_U$ is 0 on U and takes $\mathbf{n} \setminus U$ to itself. This implies that α_U is a homomorphism $H \longrightarrow \Sigma_{n-u}$ since

$$\alpha_U(h)\alpha_U(k) = \pi_U\alpha(h)\sigma_U\pi_U\alpha(k)\sigma_U = \pi_U\alpha(h)\alpha(k)\sigma_U = \pi_U\alpha(hk)\sigma_U = \alpha_U(hk),$$

where the second equality uses (7.34) with h replaced by k . Thus we can define

$$\Lambda_U = \{(h, \alpha_U(h)) \mid h \in H\}.$$

We have the following identification of $(L_n X)^\Lambda$. Henceforward we abbreviate notation for the action of \mathcal{F} on X , writing σ_U for σ_{U*} and so forth.

Lemma 7.35.

$$(L_n X)^\Lambda = \bigcup \sigma_U \left((X_{n-u})^{\Lambda_U} \right)$$

where the union runs over the H -orbits $U \subset \mathbf{n} \setminus \{0\}$.

Proof. For $U \subset \mathbf{n} \setminus \{0\}$ and $z \in (X_{n-u})^{\Lambda_U}$, $\sigma_U z$ is a Λ -fixed point since

$$(h, \alpha(h)) \cdot (\sigma_U z) = \alpha(h)(h\sigma_U z) = \alpha(h)\sigma_U(hz) = \sigma_U\alpha_U(h)(hz) = \sigma_U z,$$

for $h \in H$; the next to last equality holds by (7.34) and the last holds since z is a Λ_U -fixed point. This gives one inclusion.

For the other inclusion, we first note that the action of Σ_n on $L_n X$ can be expressed as follows. Let $\rho \in \Sigma_n$ and $x \in L_n X$, so that $x = \sigma_i y$ for some i , $1 \leq i \leq n$, and some $y \in X_{n-1}$. Then

$$\rho x = \rho\sigma_i y = \sigma_{\rho(i)} \tilde{\rho} y$$

where $\tilde{\rho} = \pi_{\rho(i)} \rho \sigma_i$ is a permutation in Σ_{n-1} , as is easily checked. Now suppose that x is a Λ -fixed point and let U be the H -orbit of i in $\mathbf{n} \setminus \{0\}$. Then $x \in \sigma_j(X_{n-1})$ for all $j \in U$ since

$$x = (h, \alpha(h)) \cdot x = \alpha(h)(hx) = \alpha(h)(h\sigma_i y) = \alpha(h)\sigma_i(hy) = \sigma_{\alpha(h)(i)} \widetilde{\alpha(h)}(hy)$$

for $h \in H$. It follows that $x = \sigma_U z$, where $z = \pi_U x \in X_{n-u}$. We claim that z is a Λ_U -fixed point. Indeed

$$\sigma_U z = x = \alpha(h)(hx) = \alpha(h)\sigma_U(hz) = \sigma_U\alpha_U(h)(hz),$$

and the claim follows since σ_U is injective. This gives the other inclusion. \square

Next we consider the intersection of the subspaces corresponding to two such subsets U .

Lemma 7.36. *Let U and V be disjoint H -subsets of the action of H on $\mathbf{n} \setminus \{0\}$ given by α . Let U have u elements and V have v elements. Then*

$$\begin{aligned} \sigma_U \left((X_{n-u})^{\Lambda_U} \right) \cap \sigma_V \left((X_{n-v})^{\Lambda_V} \right) &= \sigma_{U \cup V} \left((X_{n-u-v})^{\Lambda_{U \cup V}} \right) \\ &= \sigma_U \sigma_{\tilde{V}} \left((X_{n-u-v})^{(\Lambda_U)_{\tilde{V}}} \right), \end{aligned}$$

where \tilde{V} , also with v elements, is the subset of $\mathbf{n} \setminus (U \cup \{0\})$ that σ_U maps onto the subset V of $\mathbf{n} \setminus \{0\}$.

Proof. The notation obscures the fact that \tilde{V} depends on U , but we always use it directly following the U to which it pertains. Note that the last equality follows from the facts that $\sigma_{U \cup V} = \sigma_U \sigma_{\tilde{V}}$ and $\Lambda_{U \cup V} = (\Lambda_U)_{\tilde{V}}$. Suppose that x is in the intersection. Then $x = \sigma_i y_i$ for all $i \in U$ and also for all $i \in V$, hence $x = \sigma_{U \cup V} z$, where $z = \pi_{U \cup V} x \in X_{n-|U \cup V|}$. By the same argument as in the proof of the previous lemma, z is a $\Lambda_{U \cup V}$ -fixed point.

For the opposite inclusion, let $x = \sigma_{U \cup V} z$, where $z \in (X_{n-|U \cup V|})^{\Lambda_{U \cup V}}$. Then $x = \sigma_U y_U$ where $y_U = \sigma_{\tilde{V}} z$ is a Λ_U -fixed point by the same argument as in the previous proof. Indeed, its first part works for all H -subsets, not just orbits, to give that x is a Λ -fixed point, and then its second part gives that y_U is a Λ_U -fixed point. The symmetric argument gives that $x = \sigma_V y_V$ where y_V is a Λ_V -fixed point. \square

Finally, we use these lemmas to prove [Proposition 7.32](#).

Proof of Proposition 7.32. We first observe that an argument similar to the proof of [Lemma 7.35](#) shows that for all orbits (in fact, all H -subsets) U of \mathbf{n} ,

$$\sigma_U \left((X_{n-u})^{\Lambda_U} \right) = \left(\sigma_U (X_{n-u}) \right)^\Lambda = \sigma_U (X_{n-u}) \cap X_n^\Lambda.$$

Since σ_U is a closed inclusion, this shows that $\sigma_U \left((X_{n-u})^{\Lambda_U} \right)$ is closed in X_n^Λ . Let U_1, \dots, U_m be the orbits of the H -set $\mathbf{n} \setminus \{0\}$, with corresponding cardinalities u_1, \dots, u_m . For $1 \leq k \leq m$, we have

$$\begin{aligned} \sigma_{U_k} \left((X_{n-u_k})^{\Lambda_{U_k}} \right) \cap \left(\bigcup_{i=1}^{k-1} \sigma_{U_i} \left((X_{n-u_i})^{\Lambda_{U_i}} \right) \right) &= \bigcup_{i=1}^{k-1} \sigma_{U_k} \left((X_{n-u_k})^{\Lambda_{U_k}} \right) \cap \sigma_{U_i} \left((X_{n-u_i})^{\Lambda_{U_i}} \right) \\ &= \bigcup_{i=1}^{k-1} \sigma_{U_k} \sigma_{\tilde{U}_i} \left((X_{n-u_k-u_i})^{(\Lambda_{U_k})_{\tilde{U}_i}} \right) \\ &= \sigma_{U_k} \left(\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i} \left((X_{n-u_k-u_i})^{(\Lambda_{U_k})_{\tilde{U}_i}} \right) \right) \end{aligned}$$

where the next to last equality holds by [Lemma 7.36](#) and the others are standard set manipulations. We therefore have inclusions which give the following pushout diagram.

$$(7.37) \quad \begin{array}{ccc} \sigma_{U_k} \left(\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i} \left((X_{n-u_k-u_i})^{(\Lambda_{U_k})_{\tilde{U}_i}} \right) \right) & \longrightarrow & \bigcup_{i=1}^{k-1} \sigma_{U_i} \left((X_{n-u_i})^{\Lambda_{U_i}} \right) \\ \downarrow & & \downarrow \\ \sigma_{U_k} \left((X_{n-u_k})^{\Lambda_{U_k}} \right) & \longrightarrow & \bigcup_{i=1}^k \sigma_{U_i} \left((X_{n-u_i})^{\Lambda_{U_i}} \right) \end{array}$$

This diagram is clearly a pushout of sets. It is a pushout of spaces since the lower horizontal and the right vertical arrows are closed inclusions by our first observation, so that their target has the topology of the union. By [Lemma 7.35](#), the lower right corner is $(L_n X)^\Lambda$ when $k = m$.

We claim that the left vertical arrow and therefore the right vertical arrow is a cofibration for each $k \leq m$, the assertion being vacuous if $k = 1$. We prove this by induction on n . Thus suppose it holds for all values less than n . In particular, assume that it holds for each $n - u_k$. Note that the orbits of $\mathbf{n} \setminus (U_k \cup \{0\})$ are $\tilde{U}_1, \dots, \tilde{U}_{k-1}, \tilde{U}_{k+1}, \dots, \tilde{U}_m$. Since σ_{U_k} (or any restriction of it to a subspace) is a homeomorphism onto its image, it suffices to prove that the left vertical map is a cofibration before application of σ_{U_k} . With n replaced by $n - u_k$ and with each \tilde{U}_i referring to U_k , the induction hypothesis applied to right vertical arrows gives that

$$\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i}(X_{n-u_k-u_i}^{(\Lambda_{U_k})\tilde{U}_i}) \longrightarrow \bigcup_{i=1, \dots, k-1, k+1} \sigma_{\tilde{U}_i}(X_{n-u_k-u_i}^{(\Lambda_{U_k})\tilde{U}_i})$$

and each map

$$\bigcup_{i=1, \dots, k-1, k+1, \dots, k+j-1} \sigma_{\tilde{U}_i}(X_{n-u_k-u_i}^{(\Lambda_{U_k})\tilde{U}_i}) \longrightarrow \bigcup_{i=1, \dots, k-1, k+1, \dots, k+j} \sigma_{\tilde{U}_i}(X_{n-u_k-u_i}^{(\Lambda_{U_k})\tilde{U}_i}),$$

$2 \leq j \leq m$, is a cofibration. When $j = m$, the target of the last map is $(L_{n-u_k} X)^{\Lambda_{U_k}}$, and

$$(L_{n-u_k} X)^{\Lambda_{U_k}} \longrightarrow (X_{n-u_k})^{\Lambda_{U_k}}$$

is a cofibration by [Lemma 7.18](#). Application of σ_{U_k} to the composite of these cofibrations gives the left vertical arrow, completing the proof of our claim.

This allows us to prove by induction on k that we have a weak equivalence

$$\bigcup_{i=1}^k \sigma_{U_i}(X_{n-u_i}^{\Lambda_{U_i}}) \longrightarrow \bigcup_{i=1}^k \sigma_{U_i}(Y_{n-u_i}^{\Lambda_{U_i}})$$

for any $k \leq m$. Since the σ_{U_i} are homeomorphisms onto their images, the base case $k = 1$ holds by our assumption on f . The inductive step is an application of the gluing lemma to the pushout diagram [\(7.37\)](#). For $k = m$, this gives that

$$(L_n X)^\Lambda \longrightarrow (L_n Y)^\Lambda$$

is a weak equivalence, as required. \square

8. PROOFS OF TECHNICAL RESULTS ABOUT THE SEGAL MACHINE

We prove [Propositions 2.30](#) and [2.31](#) in [Section 8.1](#), focusing on the combinatorial analysis of the simplicial, conceptual, and homotopical versions of the Segal machine. We prove the group completion property, [Proposition 2.19](#), in [Section 8.2](#). We prove the positive linearity property, [Theorem 3.29](#), in [Section 8.3](#). The main step is to verify that the relevant \mathscr{W}_G - G -spaces satisfy the wedge axiom formulated in [Definition 8.12](#), and we prove that in [Section 8.4](#). We prove [Theorem 3.32](#) in [Section 8.5](#).

8.1. Combinatorial analysis of $A^\bullet \otimes_{\mathcal{F}} X$. Let X be an \mathcal{F} - G -space. We first compare the functor $A^\bullet \otimes_{\mathcal{F}} X$ with geometric realization. Recall that the objects of Δ are the ordered finite sets $[n] = \{0, 1, \dots, n\}$ and its morphisms are the nondecreasing functions. As in Section 2.2, let F denote the simplicial circle $S_s^1 = \Delta[1]/\partial\Delta[1]$ viewed as a functor $\Delta^{op} \rightarrow \mathcal{F}$. Take the topological circle to be $S^1 = I/\partial I$.

Remark 8.1. The functor F sends the ordered set $[n]$ to the based set \mathbf{n} . For a map $\phi: [n] \rightarrow [m]$ in Δ , $F\phi: \mathbf{m} \rightarrow \mathbf{n}$ sends i to j if $\phi(j-1) < i \leq \phi(j)$ and sends i to 0 if there is no such j . Thus

$$(F\phi)^{-1}(j) = \{i \mid \phi(j-1) < i \leq \phi(j)\} \text{ for } 1 \leq j \leq n.$$

If $\delta_i: [n-1] \rightarrow [n]$ and $\sigma_i: [n+1] \rightarrow [n]$, $0 \leq i \leq n$, are the standard face and degeneracy maps that skip or repeat i in the target, then $F\delta_i = d_i: \mathbf{n} \rightarrow \mathbf{n-1}$ is the ordered surjection that repeats i for $i < n$ but sends n to 0 if $i = n$, and $F\sigma_i = s_i: \mathbf{n} \rightarrow \mathbf{n+1}$ is the ordered injection²⁴ that skips $i+1$. Note in particular that $F\delta_1 = \varphi_2: \mathbf{2} \rightarrow \mathbf{1}$, which sends 1 and 2 to 1. In \mathcal{F} , we also have ordered projections $\pi_i: \mathbf{n} \rightarrow \mathbf{n-1}$, used in Section 7.4, that are mostly invisible to Δ . The map π_i sends i to 0 and it sends j to j if $j < i$ and to $j-1$ if $j > i$.

To prove Proposition 2.31, we must compare

$$|X| = X \otimes_{\Delta} \Delta \text{ with } X(A) := \mathbb{P}(X)(A) = A^\bullet \otimes_{\mathcal{F}} X$$

when $A = S^1$. To aid in the comparison, we rewrite $|X|$ as $\Delta \otimes_{\Delta^{op}} X$. Here Δ on the left is the covariant functor $\Delta \rightarrow \mathcal{U}$ that sends $[n]$ to the topological simplex

$$\Delta_n = \{(t_1, \dots, t_n) \mid 0 \leq t_1 \leq \dots \leq t_n \leq 1\}.$$

Nowadays it is more usual to use tuples (s_0, s_1, \dots, s_n) such that $0 \leq s_i \leq 1$ and $\sum_i s_i = 1$, but the formulae $s_i = t_{i+1} - t_i$ and $t_i = s_0 + \dots + s_{i-1}$ translate between the two descriptions. For $0 \leq i \leq n$, the face map $\delta_i: \Delta_{n-1} \rightarrow \Delta_n$ and the degeneracy map $\sigma_i: \Delta_{n+1} \rightarrow \Delta_n$ are given by

$$\delta_i(t_1, \dots, t_{n-1}) = \begin{cases} (0, t_1, \dots, t_{n-1}) & \text{if } i = 0 \\ (t_1, \dots, t_{i-1}, t_i, t_i, t_{i+1}, \dots, t_{n-1}) & \text{if } 0 < i < n \\ (t_1, \dots, t_{n-1}, 1) & \text{if } i = n \end{cases}$$

$$\sigma_i(t_1, \dots, t_{n+1}) = (t_1, \dots, t_i, t_{i+2}, \dots, t_{n+1}).$$

Map Δ_n to $(S^1)^n$ by sending $(t_1, \dots, t_n) \in \Delta_n$ to $(t_1, \dots, t_n) \in (S^1)^n$. Looking at the definition of the functor F , we see that this defines a map $\xi: \Delta \rightarrow (S^1)^\bullet$ of cosimplicial spaces,²⁵ where $(S^1)^\bullet$ is a cosimplicial space by pullback along F . Therefore ξ induces a natural map

$$\xi_*: |X| = \Delta \otimes_{\Delta^{op}} X \rightarrow (S^1)^\bullet \otimes_{\mathcal{F}} X = X(S^1).$$

Recall that every point of $|X|$ is represented by a unique point (u, x) such that $u \in \Delta_p$ is an interior point and $x \in X_p$ is a nondegenerate point [59, Lemma 14.2]. Said another way, $|X|$ is filtered with strata

$$F_p |X| \setminus F_{p-1} |X| = (\Delta_p \setminus \partial\Delta_p) \times (X_p \setminus L_p X),$$

²⁴In Section 5.2 and Section 7 it was denoted σ_{i+1} as a map in the category Λ of finite sets and injections.

²⁵Warning: we are thinking of both source and target as cosimplicial *unbased* spaces.

where $L_p X$, the p th latching space, is the union of the subspaces $s_i(X_{p-1})$ (see [Definition 1.10](#)). The construction of $F_p|X|$ from $F_{p-1}|X|$ is summarized by the concatenated pushout diagrams

$$(8.2) \quad \begin{array}{ccccc} \partial\Delta_p \times L_p X & \longrightarrow & \Delta_p \times L_p X & & \\ \downarrow & & \downarrow & & \\ \partial\Delta_p \times X_p & \longrightarrow & \Delta_p \times L_p X \cup \partial\Delta_p \times X_p & \longrightarrow & F_{p-1}|X| \\ & & \downarrow & & \downarrow \\ & & \Delta_p \times X_p & \longrightarrow & F_p|X| \end{array}$$

We shall describe $X(A)$ similarly for all $A \in G\mathcal{W}$, and we shall specialize to $A = S^1$ to see that ξ_* is a natural homeomorphism, using results about the structure of \mathcal{F} recorded in [Section 7.2](#).

Remark 8.3. The functor F is a map of generalized Reedy categories in the sense of [\[5\]](#). Recall that the latching G -space $L_p X \subset X_p$ of an \mathcal{F} - G -space X is defined to be the latching space of its underlying Π - G -space, as defined in [Definition 7.17](#). The \mathcal{F} - G -space X also has a latching space when regarded as a simplicial G -space via F . Direct comparison of definitions shows that these two latching spaces are the same.

By [Lemma 1.19](#), the G -space $X(A)$ is the quotient of $\coprod_{n \geq 0} A^n \times X_n$ obtained by identifying $(\phi^*(a), x)$ with $(a, \phi_*(x))$ for all $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{F} , $a \in A^n$, and $x \in X_m$. Here $\phi^*(a_1, \dots, a_n) = (b_1, \dots, b_m)$ where $b_i = a_{\phi(i)}$, with $b_i = *$ if $\phi(i) = 0$, and $\phi_*(x)$ is given by the covariant functoriality of X . The image of $\coprod_{n \leq p} A^n \times X_n$ is topologized as a quotient and denoted $F_p X(A)$, and $X(A)$ is given the topology of the union of the $F_p X(A)$.

Notation 8.4. For an unbased G -space U , the configuration space $\mathbf{Conf}(U, p)$ is the G -subspace of X^p of points (u_1, \dots, u_p) such that $u_i \neq u_j$ for $i \neq j$. For a based G -space A , the *based fat diagonal* $\delta A^p \subset A^p$ is the G -subspace of points (a_1, \dots, a_p) such that either some a_i is the basepoint or $a_i = a_j$ for some $i \neq j$. Observe that

$$A^p \setminus \delta A^p = \mathbf{Conf}(A \setminus \{*\}, p).$$

Lemma 8.5. $F_0 X(A) = * \times X_0$. For $p \geq 1$, the stratum $F_p X(A) \setminus F_{p-1} X(A)$ is

$$\mathbf{Conf}(A \setminus \{*\}, p) \times_{\Sigma_p} (X_p \setminus L_p X).$$

The construction of $F_p X(A)$ from $F_{p-1} X(A)$ is summarized by the concatenated pushout diagrams

$$\begin{array}{ccccc} \delta A^p \times_{\Sigma_p} L_p X & \longrightarrow & A^p \times_{\Sigma_p} L_p X & & \\ \downarrow & & \downarrow & & \\ \delta A^p \times_{\Sigma_p} X_p & \longrightarrow & A^p \times_{\Sigma_p} L_p X \cup \delta A^p \times_{\Sigma_p} X_p & \longrightarrow & F_{p-1} X(A) \\ & & \downarrow & & \downarrow \\ & & A^p \times_{\Sigma_p} X_p & \longrightarrow & F_p X(A) \end{array}$$

Proof. Using projections in \mathcal{F} , every point of $\coprod_{n \geq 1} A^n \times X_n$ is equivalent to a point (a, x) such that either $n = 0$ or no coordinate of a is the basepoint of A . Using permutations and multiplication maps $\varphi_i: \mathbf{i} \rightarrow \mathbf{1}$ when i coordinates of a are equal, every point is equivalent to a point (a, x) such that a has no repeated coordinates. We must take orbits under the action of Σ_p as stated to avoid double counting of elements. Using injections, every point is equivalent to a point (a, x) such that x is nondegenerate. Taking care of the order in which the cited operations are taken, using [Lemma 7.5](#), the conclusion follows. \square

It is now easy to see that $\xi: |X| \rightarrow X(S^1)$ is a homeomorphism.

Proof of [Proposition 2.31](#). As noted in [Remark 8.3](#), the latching subspaces $L_p X$ for X as a Δ^{op} - G -space and as an \mathcal{F} - G -space agree. We consider the strata on the filtrations for $|X|$ and $X(S^1)$ and find that ξ defines homeomorphisms

$$(\Delta_p \setminus \partial \Delta_p) \times (X_p \setminus L_p X) \longrightarrow \mathbf{Conf}(S^1 \setminus \{*\}, p) \times_{\Sigma_p} (X_p \setminus L_p X).$$

To see this, identify $\mathbf{Conf}(S^1 \setminus \{*\}, p)$ with $\mathbf{Conf}(I \setminus \{0, 1\}, p)$. Then ξ sends a point $((t_1, \dots, t_p), x)$ in the domain with $0 < t_1 < \dots < t_p < 1$ to the equivalence class of $((t_1, \dots, t_p), x)$ in the target. Note that $((t_1, \dots, t_p), x)$ is the unique representative of its class such that the coordinates t_i are in increasing order. \square

Proof of [Proposition 2.30](#). Recall from [Definition 2.21](#) that we have the classifying \mathcal{F} - G -space $\mathbb{B}X$ with p th space $|X[p]|$ and that $(\mathbb{S}_G^C X)_n = (\mathbb{B}^n X)_1$. We must prove that $(\mathbb{B}^n X)_1$ is homeomorphic to $X(S^n)$, and [Proposition 2.31](#) shows that $|X[p]| \cong X[p](S^1)$. For any $A \in G\mathcal{W}$, we have an \mathcal{F} - G -space $A \otimes X$ with p -th space $(X[p])(A)$. Thus $\mathbb{B}X \cong S^1 \otimes X$. We claim that $S^n \otimes X$ is isomorphic to $\mathbb{B}^n X$; evaluating at $p = 1$, this gives the desired homeomorphism. Since $S^n = S^1 \wedge S^{n-1}$, the claim is an immediate induction from the following result, which is essentially Segal's [[72](#), Lemma 3.7]. \square

Proposition 8.6. *For $A, B \in G\mathcal{W}$, $A \otimes (B \otimes X)$ and $(A \wedge B) \otimes X$ are naturally isomorphic.*

Proof. Recall that $X[p]$ has j th space X_{pj} , that is $X(\mathbf{p} \wedge \mathbf{j})$. Thus $X[p](B)$ is a quotient of $\coprod_j B^j \times X(\mathbf{p} \wedge \mathbf{j})$. We can write it schematically as $B^\bullet \otimes_{\mathcal{F}} X(\mathbf{p} \wedge \bullet)$. Writing \star for another schematic variable, we can write the p -th space of $A \otimes (B \otimes X)$ as

$$A^\star \otimes_{\mathcal{F}} (B^\bullet \otimes_{\mathcal{F}} X(\mathbf{p} \wedge \star \wedge \bullet)).$$

It is a quotient of $\coprod_{i,j} A^i \times B^j \times X_{pji}$. We define a map

$$A \otimes (B \otimes X) \longrightarrow (A \wedge B) \otimes X$$

by passage to coequalizers from the maps that send

$$((a_1, \dots, a_i), (b_1, \dots, b_j), x) \quad \text{to} \quad ((a_r \wedge b_s), x)$$

where the a_r 's are in A , the b_s 's are in B , and $x \in X_{pij}$. Here $(a_r \wedge b_s)$ means the set of $a_r \wedge b_s$ in lexicographic order. Indeed, since $\mathbf{i} \wedge \mathbf{j}$ is ordered lexicographically and X_{pij} is defined in terms of $\mathbf{i} \wedge \mathbf{j}$, we must order the set $\{a_r \wedge b_s\}$ to match. In the other direction, given

$a_t \wedge b_t$ for $1 \leq t \leq k$ and $x \in X_{pk}$ we map

$$((a_1 \wedge b_1, \dots, a_k \wedge b_k), x) \quad \text{to} \quad ((a_1, \dots, a_k), (b_1, \dots, b_k), \Delta_*(x)),$$

where $\Delta: \mathbf{p} \wedge \mathbf{k} \rightarrow \mathbf{p} \wedge \mathbf{k} \wedge \mathbf{k}$ is $\text{id} \wedge \Delta$. Following Segal [72, p. 304], “we shall omit the verification that the two maps are well-defined and inverse to each other”. It can be seen in terms of the explicit description of the filtration strata in Lemma 8.5. \square

8.2. The proof of the group completion property. We sketch the proof of Proposition 2.19 by reducing it to details in [60, Section 15]. We also give some relevant background and references. We first reduce the proof to that of a nonequivariant generalization, Theorem 8.7, of the result for classifying spaces of monoids that is proven in [60]. We then explain how the details in [60] actually apply to prove Theorem 8.7, even though its statement does not seem to appear in the literature.

Let X be a special \mathcal{F} - G -space. Then the Segal maps

$$\delta^H: X_n^H \rightarrow (X_1^n)^H = (X_1^H)^n$$

are weak equivalences and X^H is a nonequivariant special \mathcal{F} -space. We emphasize that we only need this classical condition: we do not require X to be \mathbb{F}_\bullet -special.

It is convenient but not essential to modify the definition of a special \mathcal{F} - G -space by requiring the Segal maps δ to be G -homotopy equivalences rather than just weak G -equivalences, and then their fixed point maps δ^H are homotopy equivalences. We can make this assumption without loss of generality since we can replace given special \mathcal{F} - G -spaces by cofibrant approximations, for which the assumption holds.

We give X_1 a Hopf G -space structure by choosing a G -homotopy inverse to δ when $n = 2$ and using φ_2 . Then X_1 and each X_1^H are homotopy associative and commutative, as in our standing conventions about Hopf G -spaces in Section 1.1. We could instead work with weak Hopf G -spaces, but doing so explicitly only obscures the exposition.

We must prove that the canonical map $\eta: X_1 \rightarrow \Omega|X|$ is a group completion in the sense of Definition 1.8. Passage to H -fixed point spaces commutes with realization, as it is a finite limit. It also commutes with taking loops since G acts trivially on S^1 . Thus the equivariant case of Proposition 2.19 follows directly from the nonequivariant case. We therefore take $G = e$ and ignore equivariance in the rest of this section.

If M is a topological monoid, we use its product to define a simplicial space B_*M with $B_nM = M^n$. Then $|B_*M|$ is just the classical classifying space BM . When M is commutative, B_*M is the simplicial space obtained by pullback of the evident special \mathcal{F} -space with n th space M^n . When X is a special \mathcal{F} -space its first space X_1 plays the role of M . Since $X_0 = *$, X_1 has a specified unit element e . Spaces of the form X_1 for a special \mathcal{F} -space X give the Segal version of an E_∞ -space.

It makes sense to ask that a simplicial space X be reduced and special, since we can use iterated face maps to define Segal maps $X_n \rightarrow X_1^n$. The Segal maps of \mathcal{F} -spaces pull back along the map $F: \Delta^{op} \rightarrow \mathcal{F}$ of Section 2.2 to examples of these more general Segal maps. Then X_1 is a Hopf space with product induced by a homotopy inverse to the second Segal map and the map $d_1: X_2 \rightarrow X_1$. Since $\varphi_2 = Fd_1$, this product on reduced special simplicial spaces generalizes the product on the space X_1 of a special \mathcal{F} -space X . Spaces of the form X_1 for a reduced special simplicial space X give the Segal version of an A_∞ -space. The group completion theorem for (reduced) special \mathcal{F} -spaces is a special case of the following more general result.

Theorem 8.7. *If X is a reduced special simplicial space such that X_1 and $\Omega|X|$ are homotopy commutative, then $\eta: X_1 \rightarrow \Omega|X|$ is a group completion.*

Rather than give full details, we describe the conceptual framework for this nonequivariant result. Just as there is a uniqueness result for infinite loop space machines, there is a uniqueness result for 1-fold delooping machines. It was first developed by Thomason [81] and Fiedorowicz [21]. A later paper by Dunn [17] combined features of [21, 81] to give a treatment closely parallel to that in [57], and it gives a similar uniqueness theorem for n -fold delooping machines.

Theorem 8.7 is a result about Segal's 1-fold delooping machine. The papers just cited make clear that X_1 is naturally equivalent to a topological monoid M and that η is equivalent to the natural inclusion $\zeta: M \rightarrow \Omega BM$. The proof makes use of the Moore loop space functor. See [81, Theorem 2.5 and Lemma 2.8] or [17, Example 2.8]. These sources also show that η and ζ induce group completions on π_0 ; an easy direct argument proves that when $\pi_0(M)$ is abelian [60, Lemma 15.2].

The sources [81, 21, 17] do not consider the homological group completion property. That would not hold in the generality in which they work since it requires some homotopy commutativity. It was proven for $\zeta: M \rightarrow \Omega BM$ with homotopy commutativity weakened to the assumption that left and right multiplication by any element are homotopic in [60, §15]. However, the proof is actually given in the context of the Segal machine, and it can easily be adapted to prove **Theorem 8.7** directly for η .

Indeed, its starting point is Segal's based variant of the standard adjunction between topological spaces and simplicial sets. That gives an adjunction (T, S) relating the category of reduced special simplicial spaces, denoted $\mathcal{S}^+ \mathcal{T}$ in [60, Section 15], to the category \mathcal{T} . Here the functor $T = |-|$ is geometric realization. The functor S is a reduced version of the total singular complex. For a based space K , $S_p K$ is the set of p -simplices $\Delta_p \rightarrow K$ that map all vertices to the basepoint. In particular, $S_1 K = \Omega K$. Let $\phi: TSK \rightarrow K$ and $\psi: X \rightarrow STX$ be the counit and unit of the adjunction. Then [60, Proposition 15.5] gives the following result.

Proposition 8.8. *If $K \in \mathcal{T}$ is path connected, then $\phi: TSK \rightarrow K$ is a weak equivalence. For any $X \in \mathcal{S}^+ \mathcal{T}$, $T\psi: TX \rightarrow TSTX$ is a weak equivalence.*

From here, the main tool is a comparison of the standard homology spectral sequence of the filtered space $TX = |X|$ to the analogous spectral sequence for $TSTX$, where X is as in **Theorem 8.7**. The map ψ induces a map of spectral sequences for homology with coefficients in a field k that converges to the homology isomorphism induced by the weak equivalence $TX \rightarrow TSTX$. It is clearly an isomorphism on E^∞ , and its map of E_2 -terms is

$$Tor^{\eta^*}(\text{id}, \text{id}): Tor^{H^*(X_1)}(k, k) \rightarrow Tor^{H^*(\Omega|X|)}(k, k).$$

A version of the comparison theorem for spectral sequences gives sufficient information about E^2 to deduce **Theorem 8.7** from a classical result of Cartan and Eilenberg [10, Proposition VI.4.1.1] to the effect that Tor commutes with localization. We refer the reader to [60, Section 15] for the details in the case of $\zeta: M \rightarrow \Omega BM$, but the details are exactly the same for X as in **Theorem 8.7**.

8.3. The positive linearity theorem. We prove **Theorem 3.29** in this section and the next. In both, let G be a group and let X be an \mathcal{F} - G -space. For a based G -CW complex A , write $Y(A)$ ambiguously for either $B(A^\bullet, \mathcal{F}, X)$, where X is special, or $B(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$, where X is \mathbb{F}_\bullet -special; here $\mathbb{P}X$ is a special \mathcal{F}_G - G -space, and up to isomorphism all special \mathcal{F}_G - G -spaces are of this form. Under either

assumption, we write $Y_*(A)$ for the simplicial bar construction whose realization is $Y(A)$. These notations remain fixed throughout these two subsections. Recall [Section 1.2](#). As asserted in [Remark 1.14](#) and is easily checked using that the degeneracy maps are given by identity maps of \mathcal{F} and \mathcal{F}_G , $Y_*(A)$ is Reedy cofibrant in the simplicial sense. Remember that we are assuming that given based G -spaces are nondegenerately based.

We start with the proof of the easier part of [Theorem 3.29](#).

Lemma 8.9. *The \mathcal{W}_G - G -space Y preserves connectivity.*

Proof. Let $A \in G\mathcal{W}$ be G -connected, so that A^H is connected for all H . We must show that $Y(A)$ is G -connected. Using that passage to fixed points commutes with pushouts one leg of which is a closed inclusion, it is easily checked that geometric realization and the bar construction commute with passage to H -fixed points. Using this, we see that $Y(A)^H$ is the geometric realization of a simplicial space with 0-simplices given by

$$(8.10) \quad \bigvee_n (A^H)^n \wedge X_n^H$$

or

$$(8.11) \quad \bigvee_{\mathbf{n}^\alpha} (A^{\mathbf{n}^\alpha})^H \wedge ((\mathbb{P}X)(\mathbf{n}^\alpha))^H,$$

depending on whether $Y(A)$ is $B(A^\bullet, \mathcal{F}, X)$ or $B(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$ for an \mathcal{F} - G -space X . We claim that the space of 0-simplices is connected in either case, so that the geometric realization $Y(A)^H$ is also connected. In the first case, it is clear that the space [\(8.10\)](#) is connected since we assume that A^H is connected, and in the second case, the space [\(8.11\)](#) is connected because $(A^{\mathbf{n}^\alpha})^H$ is connected. To see that, note that $(A^{\mathbf{n}^\alpha})^H \cong (A^n)^\Lambda$ where $\Lambda = \{(h, \alpha(h)) \mid h \in H\} \subset G \times \Sigma_n$. Thus by [Lemma 2.15](#),

$$(A^{\mathbf{n}^\alpha})^H \cong \prod A^{K_i},$$

where the product is taken over the orbits of the H -set $\mathbf{n}^{\alpha|H}$ and the $K_i \subset H$ are the stabilizers of elements in the corresponding orbit. Again, by our assumption that the A^{K_i} are connected it follows that $(A^{\mathbf{n}^\alpha})^H$ is connected. \square

To prove that the \mathcal{W}_G - G -space Y is positive linear, as defined in [Definition 3.26](#), we first isolate properties that, together with preservation of connectivity, imply positive linearity.

Definition 8.12. A \mathcal{W}_G - G -space Z satisfies the wedge axiom if for all A and B in $G\mathcal{W}$ the natural map

$$\pi: Z(A \vee B) \longrightarrow Z(A) \times Z(B)$$

induced by the canonical G -maps $A \vee B \longrightarrow A$ and $A \vee B \longrightarrow B$ is a weak G -equivalence.

Definition 8.13. Let Z be a \mathcal{W}_G - G -space and consider simplicial based G -CW complexes A_* .

- (i) Z commutes with geometric realization if the natural G -map

$$|Z(A_*)| \longrightarrow Z(|A_*|)$$

is a homeomorphism.

- (ii) Z preserves Reedy cofibrancy if the simplicial G -space $Z(A_*)$ is Reedy cofibrant when A_* is Reedy cofibrant.

Our \mathcal{W}_G - G -space Y satisfies all of these properties. We record the results here, with the proofs delayed to later.

Proposition 8.14. *The \mathcal{W}_G - G -space Y satisfies the wedge axiom.*

Proposition 8.15. *The \mathcal{W}_G - G -space Y commutes with realization and preserves Reedy cofibrancy.*

Granting these results, [Theorem 3.29](#) is an application of the following result.

Theorem 8.16. *Let Z be a \mathcal{W}_G - G -space that satisfies the wedge axiom, commutes with geometric realization, preserves Reedy cofibrancy, and preserves connectivity. Then Z is positive linear.*

The proof centers around the following construction of cofiber sequences in terms of wedges and geometric realizations; it is a corrected version and equivariant generalization of a construction due to Woolfson [\[82\]](#).

Construction 8.17. Let $f: A \rightarrow B$ be a map in $G\mathcal{W}$ with cofiber $i: B \rightarrow Cf$, where A is G -connected. We give an elementary simplicial description of Cf in terms of wedges. Let ${}^q A$ denote the wedge of q copies of A , labelling the i th wedge summand as A_i and setting ${}^0 A = *$. We define a simplicial G -space $W_*(B, A)$ whose space of q -simplices $W_q(B, A)$ is $B \vee {}^q A$. Define face and degeneracy operators d_i and s_i with domain $W_q(B, A)$ for $0 \leq i \leq q$ as follows.

- All d_i and s_i map B onto B by the identity map.
- d_0 maps A_1 to B via f and maps A_j to A_{j-1} by the identity map if $j > 1$.
- d_i , $0 < i < q$, maps A_j to A_j by the identity map if $j < i$ and maps A_j to A_{j-1} by the identity map if $j > i$.
- d_q maps A_j to A_j by the identity map if $j < q$ and maps A_q to $*$.
- s_i maps A_j to A_j by the identity map if $j \leq i$ and maps A_j to A_{j+1} by the identity map if $j > i$.

The simplicial identities are easily checked.²⁶ Note that there is not much choice: the s_i must be inclusions with $s_i s_j = s_{j+1} s_i$ for $i \leq j$ and they must satisfy $d_i s_i = \text{id} = d_{i+1} s_i$. If we specify the s_i as stated, then the d_i must be as stated except in the exceptional cases noted for d_0 and d_q .

Lemma 8.18. *The realization $|W_*(B, A)|$ is homeomorphic to Cf . In particular, the realization $|W_*(*, A)|$ is homeomorphic to ΣA .*

Proof. Clearly every point of $W_q(B, A)$ for $q \geq 2$ is degenerate. Therefore, identifying Δ_1 with I , the realization is the quotient of $B \times \{*\} \amalg (B \vee A) \times I$ obtained by the identifications

$$\begin{aligned} (b, t) &\sim (b, *) \text{ for } b \in B \text{ and } t \in I \text{ since } s_0 b = b \\ (a, 0) &\sim (f(a), *) \text{ for } a \in A \text{ since } d_0(a) = f(a) \\ (a, 1) &\sim (*, *) \text{ for } a \in A \text{ since } d_1(a) = * \end{aligned}$$

It is simple to verify that the result is homeomorphic to $Cf = B \cup_f (A \wedge I)$. \square

²⁶Our specification of faces is a correction of the specification in [\[82, Lemma 1.10\]](#).

We record the following result, which is the equivariant generalization of [45, Theorem 12.7]. It is proven by applying that result to H -fixed point simplicial spaces for all subgroups H of G . A G -map f is a G -quasifibration if each f^H is a quasifibration. Similarly, a map p of simplicial based G -spaces is a simplicial based Hurewicz G -fibration if each p^H is a simplicial based Hurewicz fibration in the sense of [45, Definition 12.5]).

Theorem 8.19. *Let E_* and B_* be simplicial based G -spaces and let $p_*: E_* \rightarrow B_*$ be a simplicial based Hurewicz G -fibration with fiber $F_* = p_*^{-1}(*)$. If each B_q is G -connected and B_* is Reedy cofibrant, then the realization $|p_*|: |E_*| \rightarrow |B_*|$ is a G -quasifibration with fiber $|F_*|$.*

Proof of Theorem 8.16. Let $f: A \rightarrow B$ be a map in $G\mathcal{W}$, where A is G -connected. We must show that application of Z to the cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{i} Cf$$

gives a fiber sequence.

Let $p_*: W_*(B, A) \rightarrow W_*(*, A)$ be the map of simplicial G -spaces given by sending B to the basepoint and let $i_*: B_* \rightarrow W_*(B, A)$ be the inclusion, where B_* denotes the constant simplicial space at B . On passage to realization, these give the canonical maps

$$B \xrightarrow{i} Cf \xrightarrow{p} \Sigma A.$$

We apply our given \mathcal{W}_G - G -space Z to these maps to obtain the sequence

$$Z(B) \xrightarrow{Z(i)} Z(Cf) \xrightarrow{Z(p)} Z(\Sigma A).$$

We claim that it is a fiber sequence. Since Z commutes with realization and since $Z(B) \cong |Z(B_*)|$, this sequence is G -homeomorphic to the sequence

$$|Z(B_*)| \xrightarrow{|Z(i_*)|} |Z(W_*(B, A))| \xrightarrow{|Z(p_*)|} |Z(W_*(*, A))|.$$

On q -simplices, before realization, we have the commutative diagram

$$\begin{array}{ccccc} Z(B) & \longrightarrow & Z(B \vee {}^q A) & \longrightarrow & Z({}^q A) \\ \parallel & & \downarrow & & \downarrow \\ Z(B) & \longrightarrow & Z(B) \times Z(A)^q & \longrightarrow & Z(A)^q \end{array}$$

where the horizontal arrows are given by the evident inclusions and projections. The vertical arrows are the canonical maps and are weak G -equivalences by the wedge axiom. The simplicial G -spaces $W_*(B, A)$ and $W_*(*, A)$ are trivially Reedy cofibrant, hence so are $Z(W_*(B, A))$ and $Z(W_*(*, A))$. The nondegeneracy of basepoints implies that $Z(B) \times Z(A)^\bullet$ and $Z(A)^\bullet$ are also Reedy cofibrant. Therefore the vertical arrows become weak G -equivalences after passage to realization. Since A and therefore $Z(A)$ is G -connected, Theorem 8.19 applies to show that the realization of the bottom row is a fibration sequence up to homotopy. Thus we have the fiber sequence

$$Z(B) \rightarrow Z(Cf) \rightarrow Z(\Sigma A)$$

and therefore also the fiber sequence

$$\Omega Z(\Sigma A) \rightarrow Z(B) \rightarrow Z(Cf).$$

Specializing to the map $\text{id}: A \rightarrow A$, we also have a fiber sequence

$$Z(A) \rightarrow Z(CA) \rightarrow Z(\Sigma A).$$

Since $Z(CA)$ is G -contractible, this implies that we have a weak G -equivalence $Z(A) \rightarrow \Omega Z(\Sigma A)$ and therefore the desired fiber sequence

$$Z(A) \rightarrow Z(B) \rightarrow Z(Cf). \quad \square$$

The following result will apply to show that Y commutes with realization.

Lemma 8.20. *Let Z be a \mathcal{W}_G - G -space such that $Z(A)$ is naturally isomorphic to $|Z_*(A)|$ for some functor Z_* from \mathcal{W}_G to simplicial based G -spaces. If the natural G -map*

$$|Z_q(A_*)| \rightarrow Z_q(|A_*|)$$

is a G -homeomorphism for all simplicial based G -spaces A_ and all $q \geq 0$, then the natural map*

$$|Z(A_*)| \rightarrow Z(|A_*|)$$

is a G -homeomorphism.

Proof. For a bisimplicial G -space, realizing first in one direction and then the other gives a space that is G -homeomorphic to the one obtained by realizing in the opposite order. Let A_* be a simplicial $G\mathcal{W}$ -space. Then $Z(A_p) = |q \mapsto Z_q(A_p)|$, so

$$|Z(A_*)| = |p \mapsto |q \mapsto Z_q(A_p)||.$$

By the assumption on Z_q ,

$$Z(|A_*|) = |q \mapsto Z_q(|p \mapsto A_p|)| \cong |q \mapsto |p \mapsto Z_q(A_p)||.$$

The result follows. \square

Proof of Proposition 8.15. We first prove that Y commutes with realization by checking that the natural map

$$|Y_q(A_\bullet)| \rightarrow Y_q(|A_\bullet|)$$

is a G -homeomorphism for all $q \geq 0$, so that Lemma 8.20 applies. Writing out the simplicial bar construction, we see that $Y_q(A_*)$ is a wedge of simplicial G -spaces of the form $A_*^n \wedge K$ for some $n \geq 0$ and some G -space K . Since realization commutes with wedges and smash products with G -spaces, this gives the proof.

Now assume that A_* is Reedy cofibrant. We must prove that $Y(A_*)$ is Reedy cofibrant. Let $Y = B((-)^\bullet, \mathcal{F}, X)$; the proof in the case $Y = B((-)^\bullet, \mathcal{F}_G, \mathbb{P}X)$ is the same. By Lemma 1.11, it suffices to show that all of the degeneracy maps

$$S_i: B((A_{n-1})^\bullet, \mathcal{F}, X) \xrightarrow{B((s_i)^\bullet, \text{id}, \text{id})} B((A_n)^\bullet, \mathcal{F}, X),$$

are G -cofibrations. Using a shorthand notation, the maps S_i are geometric realizations of maps of Reedy cofibrant simplicial G -spaces that are given on q -simplices by the G -cofibrations

$$(s_i)^{n_q} \wedge (\text{id}): \bigvee_{n_0, \dots, n_q} (A_{n-1})^{n_q} \wedge (-) \rightarrow \bigvee_{n_0, \dots, n_q} (A_n)^{n_q} \wedge (-).$$

By Theorem 1.13, the S_i are therefore G -cofibrations. \square

8.4. The proof that the wedge axiom holds. In this section we prove [Proposition 8.14](#), which says that the \mathcal{W}_G - G -space Y satisfies the wedge axiom.

Proof of Proposition 8.14. We write the proof for $Y(A) = B(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$, where X is \mathbb{F}_\bullet -special. The proof for $Y(A) = B(A^\bullet, \mathcal{F}, X)$, where X is (classically) special, is essentially the same, but simpler since it is simpler to keep track of equivariance in that case. For convenience of notation, we write a, b, c, d, e, f for based finite G -sets, that is objects of \mathcal{F}_G . We write $X(a)$ for the value of $\mathbb{P}X$ on a and A^a for the based G -space $\mathcal{F}_G(a, A)$, with G acting by conjugation.

Recall from [\(3.9\)](#) that $B(A^\bullet, \mathcal{F}_G, \mathbb{P}X) = B^\wedge(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$. As in [\[28, Theorems 2.5 and 3.15\]](#), the quotient map

$$B^\times(A^\bullet, \mathcal{F}_G, \mathbb{P}X) \longrightarrow B(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$$

is a G -equivalence. Thus it suffices to prove the result for $Y(A) = B^\times(A^\bullet, \mathcal{F}_G, \mathbb{P}X)$.²⁷

To do so, we shall construct a G -homotopy commutative diagram of G -spaces

$$(8.21) \quad \begin{array}{ccc} & Z(A, B) & \\ F \swarrow & & \searrow Q \\ Y(A \vee B) & \xrightarrow{P} & Y(A) \times Y(B) \end{array}$$

in which F and Q are weak G -equivalences and P is the canonical map induced by the projections $\pi_A: A \vee B \rightarrow A$ and $\pi_B: A \vee B \rightarrow B$. This will prove that P is a weak G -equivalence.

In this subsection, we abbreviate notation by writing $\mathcal{C}(A; X)$ for the category internal to $G\mathcal{U}$ whose nerve is $B_*^\times(A, \mathcal{F}_G, \mathbb{P}X)$ and whose classifying G -space is therefore $Y(A)$. Recall from [Remark 3.2](#) that the object and morphism G -spaces of $\mathcal{C}(A; X)$ are

$$\coprod_a A^a \times X(a)$$

and

$$\coprod_{a,c} A^c \times \mathcal{F}_G(a, c) \times X(a).$$

Its source and target G -maps S and T are induced from the evaluation maps of the contravariant $G\mathcal{U}_*$ -functor A^\bullet and the covariant $G\mathcal{U}_*$ -functor X from \mathcal{F}_G to \mathcal{F}_G . The identity and composition G -maps I and C are induced from identity morphisms and composition in \mathcal{F}_G .

Analogously, we define $Z(A, B)$ to be the classifying G -space of the category $\mathcal{C}(A, B; X)$ internal to $G\mathcal{U}$ whose nerve is the simplicial bar construction

$$B_*^\times(A^\bullet \times B^\bullet; \mathcal{F}_G \times \mathcal{F}_G; (\mathbb{P}X)(\bullet \vee \star)).$$

Its object and morphism G -spaces are

$$\coprod_{a,b} (A^a \times B^b) \times X(a \vee b)$$

and

$$\coprod_{a,b,c,d} (A^c \times B^d) \times \mathcal{F}_G(a, c) \times \mathcal{F}_G(b, d) \times X(a \vee b).$$

²⁷The defects of B^\times pointed out in [Remark 3.14](#) are irrelevant here.

Here S and T are induced from the evaluation maps of $A^\bullet \times B^\bullet$ and the composite of $\vee: \mathcal{F}_G \times \mathcal{F}_G \rightarrow \mathcal{F}_G$ with the evaluation maps of $\mathbb{P}X$. Again, identity and composition maps are induced from identity morphisms and composition in \mathcal{F}_G .

Using categories, functors, and natural transformations to mean these notions internal to $G\mathcal{U}$ in what follows, we shall define functors giving the following diagram of categories and shall prove that it is commutative up to natural transformation.

$$(8.22) \quad \begin{array}{ccc} & \mathcal{C}(A, B; X) & \\ & \swarrow F & \searrow Q \\ \mathcal{C}(A \vee B; X) & \xrightarrow{P} & \mathcal{C}(A; X) \times \mathcal{C}(B; X) \end{array}$$

Passing to classifying G -spaces, this will give the diagram (8.21).

To define F and Q , it is convenient to write elements of A^a as based maps $\mu: a \rightarrow A$, with G acting by conjugation on maps. The functor F sends an object (μ, ν, x) to $(\mu \vee \nu, x)$ and sends a morphism $(\mu, \nu, \phi, \psi, x)$ to $(\mu \vee \nu, \phi \vee \psi, x)$. The functor Q sends an object (μ, ν, x) to $(\mu, x_a) \times (\nu, x_b)$ where x_a and x_b are obtained from $x \in X(a \vee b)$ by using the G -maps induced by the projections $\pi_a: a \vee b \rightarrow a$ and $\pi_b: a \vee b \rightarrow b$. It sends a morphism $(\mu, \nu, \phi, \psi, x)$ to the morphism $(\mu, \phi, x_a) \times (\nu, \psi, x_b)$. As in (8.21), P is induced by the projections π_A and π_B . Noting that π_a and π_b are G -fixed morphisms of \mathcal{F}_G and that $\pi_A \circ (\mu \vee \nu) = \mu \circ \pi_a$ and $\pi_B \circ (\mu \vee \nu) = \nu \circ \pi_b$, we see that the morphisms

$$(\mu, \pi_a, x) \times (\nu, \pi_b, x): (\pi_A \circ (\mu \vee \nu), x) \times (\pi_B \circ (\mu \vee \nu), x) \rightarrow (\mu, x_a) \times (\nu, x_b)$$

give a natural transformation $P \circ F \rightarrow Q$ in diagram (8.22) that induces a G -homotopy $P \circ F \rightarrow Q$ in diagram (8.21).

While Q need not be an equivalence of categories of any sort, we see from the assumption that $\mathbb{P}X$ is special and our use of the projections π_a and π_b that Q gives a level weak equivalence of simplicial G -spaces on passage to nerves. Reedy cofibrancy of the bar constructions then implies that the induced map Q in (8.21) is a weak equivalence of classifying G -spaces. To complete the proof, we shall construct a functor $F^{-1}: \mathcal{C}(A \vee B; X) \rightarrow \mathcal{C}(A, B; X)$ and natural transformations $\text{Id} \rightarrow F^{-1} \circ F$ and $\text{Id} \rightarrow F \circ F^{-1}$. Passing to classifying G -spaces, this will imply that F in (8.21) is a G -homotopy equivalence.

For $\omega: f \rightarrow A \vee B$, define $\omega_A = \pi_A \circ \omega$ and $\omega_B = \pi_B \circ \omega$. Define $\sigma_\omega: f \rightarrow f \vee f$, called the splitting of ω , by

$$\sigma_\omega(j) = \begin{cases} j & \text{in the first copy of } f \text{ if } \omega(j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } \omega(j) \in B \setminus * \\ * & \text{if } \omega(j) = * \end{cases}$$

Observe that ω factors as the composite

$$f \xrightarrow{\sigma_\omega} f \vee f \xrightarrow{\omega_A \vee \omega_B} A \vee B.$$

Define F^{-1} on objects by

$$F^{-1}(\omega, x) = (\omega_A, \omega_B, (\sigma_\omega)_*(x)).$$

For a morphism (ω, ϕ, x) , $\omega: f \rightarrow A \vee B$, $\phi: e \rightarrow f$, and $x \in X(e)$, observe that

$$(\omega \circ \phi)_A = \omega_A \circ \phi \quad \text{and} \quad (\omega \circ \phi)_B = \omega_B \circ \phi.$$

Define F^{-1} on morphisms by

$$F^{-1}(\omega, \phi, x) = (\omega_A, \omega_B, \phi, \phi, (\sigma_{\omega \circ \phi})_*(x)).$$

A check of definitions using the commutative diagram

$$\begin{array}{ccccc} e & \xrightarrow{\phi} & f & \xrightarrow{\omega} & A \vee B \\ \sigma_{\omega \circ \phi} \downarrow & & \sigma_{\omega} \downarrow & \nearrow \omega_A \vee \omega_B & \\ e \vee e & \xrightarrow{\phi \vee \phi} & f \vee f & & \end{array}$$

shows that $S \circ F^{-1} = F^{-1} \circ S$ and $T \circ F^{-1} = F^{-1} \circ T$, and F^{-1} is clearly compatible with composition and identities. It is easily checked that F^{-1} is continuous on object and morphism G -spaces, but equivariance is a little more subtle. We first claim that $\sigma_{g \cdot \omega} = g \cdot \sigma_{\omega}$. Since $(g \cdot \omega)(j) = g\omega(g^{-1}j)$,

$$\sigma_{g \cdot \omega}(j) = \begin{cases} j & \text{in the first copy of } f \text{ if } g\omega(g^{-1}j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } g\omega(g^{-1}j) \in B \setminus * \\ * & \text{if } g\omega(g^{-1}j) = *. \end{cases}$$

On the other hand, using the definition of σ_{ω} and the fact that $gg^{-1} = 1$,

$$g \cdot \sigma_{\omega}(j) = g\sigma_{\omega}(g^{-1}j) = \begin{cases} j & \text{in the first copy of } f \text{ if } \omega(g^{-1}j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } \omega(g^{-1}j) \in B \setminus * \\ g* = * & \text{if } \omega(g^{-1}j) = *. \end{cases}$$

Observing that $gz \in A \setminus *$ if and only if $z \in A \setminus *$ and similarly for B , we see that these agree, proving the claim. Since π_A is a G -map, we also have

$$g \cdot \omega_A = g \cdot (\pi_A \circ \omega) = g\pi_A \omega g^{-1} = \pi_A g \omega g^{-1} = \pi_A \circ g \cdot \omega = (g \cdot \omega)_A,$$

and similarly for B . Putting these together gives

$$\begin{aligned} F^{-1}(g \cdot (\omega, x)) &= F^{-1}(g \cdot \omega, gx) = ((g \cdot \omega)_A, (g \cdot \omega)_B, (\sigma_{g \cdot \omega})_*(gx)) \\ &= (g \cdot \omega_A, g \cdot \omega_B, (g \cdot \sigma_{\omega})_*(gx)). \end{aligned}$$

Since the evaluation map $\mathcal{F}_G(a, b) \wedge X(a) \rightarrow X(b)$ is a G -map, this is equal to

$$(g \cdot \omega_A, g \cdot \omega_B, g \cdot ((\sigma_{\omega})_*x)) = g \cdot F^{-1}(\omega, x).$$

This shows that F^{-1} is a G -map on objects. Following the same argument and using that $g \cdot (\omega \circ \phi) = (g \cdot \omega) \circ (g \cdot \phi)$, we see that F^{-1} is a G -map on morphisms.

Now consider the composite $F \circ F^{-1}$. It sends the object (ω, x) in $\mathcal{C}(A \vee B; X)$ to the object $(\omega_A \vee \omega_B, (\sigma_{\omega})_*x)$. Here $(\omega_A \vee \omega_B, \sigma_{\omega}, x)$ is a morphism from (ω, x) to $(\omega_A \vee \omega_B, (\sigma_{\omega})_*x)$. We claim that this morphism is the component at (ω, x) of a natural transformation $\text{Id} \rightarrow F \circ F^{-1}$. These morphisms clearly give a continuous map from the object G -space of $\mathcal{C}(A \vee B; X)$ to its morphism G -space, and it is not hard to check naturality using the diagram just above. To show equivariance, if $g \in G$ then

$$g \cdot (\omega_A \vee \omega_B, \sigma_{\omega}, x) = (g \cdot (\omega_A \vee \omega_B), g \cdot \sigma_{\omega}, gx) = ((g \cdot \omega)_A \vee (g \cdot \omega)_B, \sigma_{g \cdot \omega}, gx),$$

which is precisely the component at $g \cdot (\omega, x) = (g\omega, gx)$.

The composite $F^{-1} \circ F$ sends an object (μ, ν, x) in $\mathcal{C}(A, B; X)$ to the object

$$(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), (\sigma_{\mu \vee \nu})_*x) = (\mu \circ \pi_a, \nu \circ \pi_b, (\sigma_{\mu \vee \nu})_*x).$$

Note that $\sigma_{\mu \vee \nu}$ is given by $\tilde{\sigma}_\mu \vee \tilde{\sigma}_\nu$, where $\tilde{\sigma}_\mu: a \rightarrow a \vee b$ is the inclusion, except that if $\mu(j) = *$, then $\tilde{\sigma}_\mu(j) = *$, and similarly for $\tilde{\sigma}_\nu: b \rightarrow a \vee b$. Here

$$(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), \tilde{\sigma}_\mu, \tilde{\sigma}_\nu, x)$$

is a morphism in $\mathcal{C}(A, B; X)$ from (μ, ν, x) to $(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), (\sigma_{\mu \vee \nu})_* x)$. To see that the source of this morphism is as claimed, observe that $\mu \circ \pi_a \circ \tilde{\sigma}_\mu = \mu$ since $\pi_a \circ \tilde{\sigma}_\mu = \text{id}$ except on those j such that $\mu(j) = *$, and similarly for ν . The naturality follows from the equations

$$\tilde{\sigma}_\mu \circ \phi = (\phi \vee \psi) \circ \tilde{\sigma}_{\mu \circ \phi} \quad \text{and} \quad \tilde{\sigma}_\nu \circ \psi = (\phi \vee \psi) \circ \tilde{\sigma}_{\nu \circ \psi},$$

which are easily checked. The continuity of the assignment is also easily verified. For $g \in G$, a verification similar to that for $F \circ F^{-1}$ shows that g acting on the component of our natural transformation at (μ, ν, x) is the component of the transformation at $g \cdot (\mu, \nu, x) = (g\mu, g\nu, gx)$. \square

8.5. The heart of the Segal machine. We prove [Lemma 3.31](#) and [Theorem 3.32](#) in this subsection. Recall from [Definition 3.30](#) that Z is a special \mathcal{W}_G - G -space if it preserves connectivity and its restriction to \mathcal{F}_G is a special \mathcal{F}_G - G -space. Recall also that $Z[A](B) = Z(A \wedge B)$ for $A, B \in \mathcal{W}_G$. Since $A \wedge B$ is G -connected if either A or B is G -connected and since smashing preserves cofiber sequences, the proof of the following lemma is immediate from the definitions.

Lemma 8.23. *Suppose that Z is positive linear and preserves connectivity. Then $Z[A]$ is positive linear and preserves connectivity for any A . If A is G -connected, then $Z[A]$ is linear rather than just positive linear.*

Given [Lemma 8.23](#), to prove [Lemma 3.31](#) it remains to prove that the restriction of a \mathcal{W}_G - G -space $Z[A]$ to a \mathcal{F}_G - G -space is special when A is G -connected. After the preliminary [Lemma 8.24](#), we prove this as [Lemma 8.25](#).

Until otherwise specified, we assume that Z is a positively linear and special \mathcal{W}_G - G -space.

Lemma 8.24. *Let α be a finite G -set and let $Y[\alpha]$ denote the restriction of $Z[\alpha]$ to \mathcal{F}_G . Then $Y[\alpha]$ is a special \mathcal{F}_G - G -space.*

Proof. We are given that $Y[\mathbf{1}]$ is a special \mathcal{F}_G - G -space. For a finite G -set β , we have the analogue for \mathcal{F}_G of the diagram [\(2.22\)](#):

$$\begin{array}{ccc} Y[\alpha](\beta) & \xrightarrow{\delta} & Y[\alpha](\mathbf{1})^\beta \quad \text{=====} & Y(\alpha)^\beta \\ \parallel & & & \downarrow \delta^\beta \\ Y(\alpha \wedge \beta) & \xrightarrow{\delta} & Y(\mathbf{1})^{\alpha \wedge \beta} \quad \text{=====} & (Y(\mathbf{1})^\alpha)^\beta \end{array}$$

The right and bottom arrows δ are weak G -equivalences, hence so is the top arrow δ . Thus $Y[\alpha]$ is special. \square

Lemma 8.25. *Let $A \in \mathcal{W}_G$ be G -connected and let $Y[A]$ denote the restriction of $Z[A]$ to \mathcal{F}_G . Then $Y[A]$ is a special \mathcal{F}_G - G -space.*

Proof. Let S_s^1 again be the simplicial circle. Its q -simplices $(S_s^1)_q$ are finite based sets, hence are finite based G -sets with trivial G -action. By [Lemma 8.24](#), for each finite G -set α we have the simplicial special \mathcal{F}_G -space with q -simplices $Y[(S_s^1)_q \wedge \alpha]$.

Its geometric realization gives a special \mathcal{F}_G - G -space isomorphic to $Y[S^1 \wedge \alpha]$. In particular, we can take α to be a finite wedge of based G -sets $(G/H)_+$.

By positive linearity, for a cofiber sequence

$$A \longrightarrow B \longrightarrow C$$

in \mathcal{W}_G where A is G -connected and a finite based G -set α we have an induced map of fiber sequences

$$\begin{array}{ccccc} Z(A \wedge \alpha) & \longrightarrow & Z(B \wedge \alpha) & \longrightarrow & Z(C \wedge \alpha) \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ Z(A)^\alpha & \longrightarrow & Z(B)^\alpha & \longrightarrow & Z(C)^\alpha. \end{array}$$

Therefore, if $Y[A]$ and $Y[B]$ are special, then so is $Y[C]$. By definition, any $A \in \mathcal{W}_G$ is a finite based G -CW complex. When A is G -connected, we can replace A by a G -homotopy equivalent based G -CW complex whose 0-skeleton is a point and whose attaching maps are based G -maps defined on G -spheres $G/H_+ \wedge S^n$ for $n \geq 1$. By induction on n , using the cofiber sequences $S^{n-1} \rightarrow D^n \rightarrow S^n$, each $Y[G/H_+ \wedge S^n]$ is special. From here, we see that $Y[A]$ is special by induction on the number of cells in A . \square

The rest of this section is devoted to the proof of [Theorem 3.32](#). From now on, we assume that Z is special and linear (not just positively linear). We have in mind the $Z[A]$ for a G -connected $A \in \mathcal{W}_G$. The proof of [Lemma 8.25](#) simplifies to give the following analog.

Lemma 8.26. *When Z is special and linear, the restriction $Y[A]$ of $Z[A]$ to \mathcal{F}_G is a special \mathcal{F}_G - G -space for any $A \in \mathcal{W}_G$.*

We restate [Theorem 3.32](#). The rest of this subsection is devoted to its proof, following [\[73, 75\]](#).

Theorem 8.27. *If Z is a special and linear \mathcal{W}_G - G -space, then the structure map*

$$\tilde{\sigma}: Z(S^0) \longrightarrow \Omega^V Z(S^V)$$

is a weak G -equivalence.

We first recall from [Definition 2.26](#) the general definition of such structure maps. For based G -spaces A, B, C in \mathcal{W}_G and a based G -map

$$\mu: B \longrightarrow \mathcal{W}_G(A, C),$$

composition with an application of Z gives a based G -map

$$B \longrightarrow \underline{G}\mathcal{U}_*(Z(A), Z(C)).$$

In turn, by adjunction, this gives a based G -map

$$\hat{\mu}: Z(A) \longrightarrow \underline{G}\mathcal{U}_*(B, Z(C)).$$

Explicitly, for $z \in Z(A)$ and $b \in B$,

$$\hat{\mu}(z)(b) = Z(\mu(b))(z).$$

When $\mu: B \longrightarrow \mathcal{W}_G(A, A \wedge B)$ is the adjoint of the identity map of $A \wedge B$, this is a natural map

$$\tilde{\sigma}: Z(A) \longrightarrow \underline{G}\mathcal{U}_*(B, Z(A \wedge B)).$$

Taking $A = S^0$ and $B = S^V$, this is the adjoint structure map

$$\tilde{\sigma}: Z(S^0) \longrightarrow \Omega^V Z(S^V).$$

For a subspace M of V , let M_ε be the open ε -neighborhood of M in V . Explicitly,

$$M_\varepsilon = \{v \in V \mid \text{there exists } m \in M \text{ such that } |v - m| < \varepsilon\} \subset V.$$

Write $(-)^c$ for one-point compactification, which we recall is functorial on proper maps between locally compact spaces, and denote points at infinity by ∞ . Recall that $(-)^c$ converts cartesian products to smash products and that $A^c = A_+$ if A is compact. For a small ε , let 0_ε be the ε -neighborhood of the origin in V . Then 0_ε^c is isomorphic to the one-point compactification S^V of V and we treat the evident isomorphism as an identification from now on.

For any compact (unbased) G -subspace M of V we have a G -map

$$(8.28) \quad M \longrightarrow \underline{G\mathcal{U}}(0_\varepsilon, M_\varepsilon), \quad x \longmapsto (v \mapsto x + v).$$

Applying a Pontryagin-Thom construction gives a based G -map

$$\mu: M_+ \longrightarrow \mathcal{W}_G(M_\varepsilon^c, 0_\varepsilon^c).$$

Explicitly, for $x \in M$ and $y \in M_\varepsilon$, $\mu(x)(y) = y - x$ if $|y - x| < \varepsilon$ and $\mu(x)(y) = \infty$ otherwise. As described above, this gives rise to a map

$$(8.29) \quad \hat{\mu}_M: Z(M_\varepsilon^c) \longrightarrow \underline{G\mathcal{U}}_*(M_+, Z(S^V)).$$

Remark 8.30. We may take $M = \{0\}$. Then $\mu: S^0 = M_+ \longrightarrow \mathcal{W}_G(0_\varepsilon^c, 0_\varepsilon^c)$ sends 0 to the identity map and thus $\hat{\mu}_{\{0\}}$ can be identified with the identity map of $Z(S^V)$.

We will prove that when $M = S$ is the unit sphere in V , the map $\hat{\mu}_S$ is an equivalence. Before we prove this, we set up a little bit more notation. Consider the radial projection

$$\pi: S_\varepsilon = \{v \in V \mid 1 - \varepsilon < |v| < 1 + \varepsilon\} \longrightarrow S$$

given by $\pi(v) = v/|v|$. For a subset X of S , let

$$\hat{X} = \pi^{-1}(X) = \{v \in V \mid 1 - \varepsilon < |v| < 1 + \varepsilon \text{ and } \pi(v) \in X\}.$$

Thus \hat{X} is the subset X on the sphere, thickened by ε in the radial direction. Note that $\hat{X} \subseteq X_\varepsilon$, but these are in general not equal. They are equal in the case of $X = S$. Finally, again for X a subset of S , let

$$\tilde{X} = X - (S - X)_{2\varepsilon} = \{x \in X \mid |x - v| \geq 2\varepsilon \text{ for all } v \in S - X\}^{28}.$$

Thus \tilde{X} is the subset X on the sphere, with a 2ε -neighborhood cropped out in the tangent direction. Note that $\tilde{S} = S$. With this definition, for any subset X of S the G -map of (8.28) with $M = S$ restricts to a G -map

$$(8.31) \quad \alpha_X: \tilde{X} \longrightarrow \underline{G\mathcal{U}}(0_\varepsilon, \hat{X}).$$

Again using the Pontryagin-Thom construction we obtain a map

$$\mu_X: \tilde{X}_+ \longrightarrow \mathcal{W}_G(\hat{X}^c, 0_\varepsilon^c) = \mathcal{W}_G(\hat{X}^c, S^V),$$

²⁸The original sources [72, 75] use ε in this definition, but it seems to us that we need 2ε in order to obtain the restricted map (8.31) for subspaces X of S

which again gives rise to a map

$$\hat{\mu}_X: Z(\widehat{X}^c) \longrightarrow \underline{G\mathcal{U}}_*(\widetilde{X}_+, Z(S^V))$$

for any subset X of S . When $X = S$, this is the map $\hat{\mu}_S$ that we constructed in (8.29) above (with $M = S$).

Theorem 8.32. *Let Z be a special \mathcal{W}_G - G -space. Then the map*

$$\hat{\mu}_S: Z(S_\varepsilon^c) \longrightarrow \underline{G\mathcal{U}}_*(S_+, Z(S^V))$$

is a weak G -equivalence.

Proof. By Illman [36], there exists an equivariant triangulation of S . Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be the collection of open stars of open simplexes. This is an open cover of S . We index the stars by the barycenter of the simplex, that is, λ is the barycenter of the simplex for which C_λ is the star. Taking a refinement if necessary, we can assume that for all $g \in G$, either $g \cdot C_\lambda = C_\lambda$, or $g \cdot C_\lambda$ and C_λ are disjoint. From now on, we let 2ε be small enough compared to all the radii of all the simplexes in the triangulation, so that X deformation retracts onto \widetilde{X} for any X which is a G -stable union of C_λ 's.

Let $X = \bigcup_{\lambda \in T} C_\lambda$ be a G -stable union of C_λ 's. We claim that the map $\hat{\mu}_X$, and in particular $\hat{\mu}_S$, is a G -equivalence. We proceed by induction on the number of G -orbits contained in the indexing set T for the open stars in X , which we recall is a set of barycenters. When T is a single orbit, the inclusion $T \rightarrow X$ induces G -equivalences $T_+ \rightarrow \widetilde{X}_+$ and $T_+ \wedge S^V \rightarrow \widehat{X}^c$. Via these equivalences, $\hat{\mu}_X$ can be identified with

$$Z(T_+ \wedge S^V) \longrightarrow \underline{G\mathcal{U}}_*(T_+, Z(S^V)),$$

which is a G -equivalence since $Z[S^V]$ is special by Lemma 3.31. For the inductive step, we take G -stable unions of stars X and Y , assume the conclusion for X , Y and $X \cap Y$, and prove the conclusion for $X \cup Y$. The open stars of a triangulation form a covering in the sense that the intersection of any two stars is a star, so that $X \cap Y$ is indeed also a G -stable union of stars, possibly empty. The conclusion that $\hat{\mu}_{X \cup Y}$ is an equivalence follows by induction from the following diagram, using the claim that its front and back squares are homotopy cartesian since we are assuming that $\hat{\mu}_X$, $\hat{\mu}_Y$ and $\hat{\mu}_{X \cap Y}$ are equivalences.

$$\begin{array}{ccccc}
Z((\widehat{X \cup Y})^c) & \xrightarrow{\quad} & Z(\widehat{X}^c) & \xrightarrow{\quad} & Z(\widehat{X}^c) \\
\downarrow & \searrow^{\hat{\mu}_{X \cup Y}} & \downarrow & \searrow^{\hat{\mu}_X} & \downarrow \\
& & \underline{G\mathcal{U}}_*(\widetilde{X \cup Y}_+, Z(S^V)) & \xrightarrow{\quad} & \underline{G\mathcal{U}}_*(\widetilde{X}_+, Z(S^V)) \\
\downarrow & & \downarrow & & \downarrow \\
Z(\widehat{Y}^c) & \xrightarrow{\quad} & Z((\widehat{X \cap Y})^c) & \xrightarrow{\quad} & Z((\widehat{X \cap Y})^c) \\
\downarrow & \searrow^{\hat{\mu}_Y} & \downarrow & \searrow^{\hat{\mu}_{X \cap Y}} & \downarrow \\
& & \underline{G\mathcal{U}}_*(\widetilde{Y}_+, Z(S^V)) & \xrightarrow{\quad} & \underline{G\mathcal{U}}_*(\widetilde{X \cap Y}_+, Z(S^V))
\end{array}$$

It thus remains to show that the front and back squares are homotopy cartesian. The front square is homotopy cartesian because the square

$$\begin{array}{ccc} \widetilde{X \cap Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \widetilde{Y} & \longrightarrow & \widetilde{X \cup Y} \end{array}$$

is homotopy cocartesian, since X, Y are open in $X \cup Y$, and ε was chosen small enough so that there is a G -deformation retraction $\widetilde{W} \simeq W$ for all G -stable unions of open stars W in our triangulation. For the back square, note that

$$(\widehat{X \cup Y} - \widehat{X})^c \longrightarrow \widehat{X \cup Y}^c \text{ and } (\widehat{Y} - \widehat{X \cap Y})^c \longrightarrow \widehat{Y}^c$$

are G -cofibrations. Moreover, we have that

$$\widehat{X \cup Y} - \widehat{X} = \widehat{Y} - \widehat{X \cap Y}$$

since both sets consist of those $v \in S_\varepsilon$ such that $\pi(v)$ is in Y but not in X .²⁹ Identifying the cofibers, we then get a diagram of homotopy cofiber sequences:

$$\begin{array}{ccccc} (\widehat{X \cup Y} - \widehat{X})^c & \longrightarrow & (\widehat{X \cup Y})^c & \longrightarrow & \widehat{X}^c \\ \parallel & & \downarrow & & \downarrow \\ (\widehat{Y} - \widehat{X \cap Y})^c & \longrightarrow & \widehat{Y}^c & \longrightarrow & \widehat{X \cap Y}^c. \end{array}$$

Since Z is linear, after applying Z , we get a diagram of fiber sequences

$$\begin{array}{ccccc} Z((\widehat{X \cup Y} - \widehat{X})^c) & \longrightarrow & Z((\widehat{X \cup Y})^c) & \longrightarrow & Z(\widehat{X}^c) \\ \parallel & & \downarrow & & \downarrow \\ Z((\widehat{Y} - \widehat{X \cap Y})^c) & \longrightarrow & Z(\widehat{Y}^c) & \longrightarrow & Z(\widehat{X \cap Y}^c). \end{array}$$

In particular, the right square in the diagram is homotopy cartesian. \square

We now give the proof of [Theorem 8.27](#).

Proof of Theorem 8.27. Let S and D be the unit sphere and disk in V , and let $i: S \rightarrow D$ be the inclusion.

Define $p: D_\varepsilon^c \rightarrow S_\varepsilon^c$ by $p(x) = x$ if $x \in S_\varepsilon$ and $p(x) = \infty$ otherwise. Then the following diagram commutes.

$$\begin{array}{ccc} S_+ & \xrightarrow{\mu} & \mathcal{W}_G(S_\varepsilon^c, 0_\varepsilon^c) \\ \downarrow i & & \downarrow p^* \\ D_+ & \xrightarrow{\mu} & \mathcal{W}_G(D_\varepsilon^c, 0_\varepsilon^c) \end{array}$$

Note that $D_\varepsilon \setminus S_\varepsilon \rightarrow D_\varepsilon$ is proper, let $j: (D_\varepsilon \setminus S_\varepsilon)^c \rightarrow D_\varepsilon^c$ be the inclusion, and observe that the quotient of j is the G -space S_ε^c with quotient map p .

²⁹Note that this equality would not be true if instead of taking the inverse images under the radial projections $S_\varepsilon \rightarrow S$ we took actual ε neighborhoods X_ε of subsets of the sphere.

Define

$$(8.33) \quad \nu: S^V \cong D/S \longrightarrow \mathscr{W}_G((D_\varepsilon \setminus S_\varepsilon)^c, 0_\varepsilon^c)$$

by applying a relative Pontryagin-Thom construction. Explicitly, ν is the based G -map such that if $x \in D \setminus S$ and $y \in D_\varepsilon \setminus S_\varepsilon$, then $\nu(x)(y) = y - x$ if $|y - x| < \varepsilon$ and $\nu(x)(y) = \infty$ otherwise. Then the following diagram commutes, where q is the quotient map.

$$\begin{array}{ccc} D_+ & \xrightarrow{\mu} & \mathscr{W}_G(D_\varepsilon^c, 0_\varepsilon^c) \\ q \downarrow & & \downarrow j^* \\ S^V \cong D/S & \xrightarrow{\nu} & \mathscr{W}_G((D_\varepsilon \setminus S_\varepsilon)^c, 0_\varepsilon^c) \end{array}$$

Just as μ gives rise to $\hat{\mu}$, ν gives rise to the map $\hat{\nu}$ in the following diagram. By easy diagram chases, writing S^V for 0_ε^c , the squares above imply that the squares in the diagram commute.

$$(8.34) \quad \begin{array}{ccc} Z((D_\varepsilon \setminus S_\varepsilon)^c) & \xrightarrow{\hat{\nu}} & \underline{G}\mathscr{U}_*(D/S, Z(S^V)) \\ Z(i) \downarrow & & \downarrow q^* \\ Z(D_\varepsilon^c) & \xrightarrow{\hat{\mu}} & \underline{G}\mathscr{U}_*(D_+, Z(S^V)) \\ Z(p) \downarrow & & \downarrow j^* \\ Z(S_\varepsilon^c) & \xrightarrow{\hat{\mu}} & \underline{G}\mathscr{U}_*(S_+, Z(S^V)) \end{array}$$

The right column is clearly a fiber sequence. We claim that

$$(8.35) \quad (D_\varepsilon \setminus S_\varepsilon)^c \xrightarrow{j} D_\varepsilon^c \xrightarrow{p} S_\varepsilon^c.$$

is a cofiber sequence so that, since Z is linear, the left hand column is also a fiber sequence.

Observe that $D_\varepsilon \setminus S_\varepsilon$ is the closed disk $D(1 - \varepsilon)$ of radius $1 - \varepsilon$ in V . Applying $(-)^c$ adds a disjoint basepoint. Therefore the inclusion

$$S^0 = \{0\} \amalg \{\infty\} \longrightarrow (D_\varepsilon \setminus S_\varepsilon)^c \cong D(1 - \varepsilon)_+$$

is a G -homotopy equivalence. The inclusion

$$j: D(1 - \varepsilon)_+ \longrightarrow D_\varepsilon^c \cong S^V,$$

which sends the disjoint basepoint to the north pole (point at ∞) and sends the disk of radius $1 - \varepsilon$ to its image around the south pole (the point $0 \in V$), is a G -cofibration. We conclude that (8.35) is G -equivalent to the cofiber sequence

$$S^0 \longrightarrow S^V \longrightarrow S^V/S^0.$$

In the diagram (8.34) the bottom horizontal arrow is a weak G -equivalences by [Theorem 8.32](#), and the middle horizontal arrow is easily seen to be a weak equivalence since $D_\varepsilon^c \cong S^V$ and $D_+ \simeq S^0$. Hence the top horizontal arrow $\hat{\nu}$ is also a weak equivalence. Its domain is weak G -equivalent to $Z(S^0)$ and its target is weak G -equivalent to $\Omega^V Z(S^V)$. Under these equivalences, $\hat{\nu}$ agrees with $\hat{\sigma}$, at least up to sign. On the one hand, for $z \in Z(S^0)$ and $y \in S^V$,

$$\tilde{\sigma}(z)(y) = Z(\text{id}(y))(z),$$

where $\text{id}(y): S^V \rightarrow \mathscr{W}_G(S^0, S^V)$ is given by $\text{id}(y)(0) = y$. On the other hand, for $x \in D/S \cong D(1 - \varepsilon)/S(1 - \varepsilon)$, we have

$$\hat{\nu}(z)(x) = Z(\nu(x))(z),$$

where $\nu: D(1 - \varepsilon)_+ \rightarrow \mathscr{W}_G(S^0, S^V)$ satisfies $\nu(x)(0) = x \in U_\varepsilon^c$. Therefore $\tilde{\sigma}$ and $\hat{\nu}$ agree modulo identifications of spheres. The standard choice of homeomorphism $S^V \cong D/S$ would introduce a sign [51, Section 13.2], but that is irrelevant to the conclusion that $\tilde{\sigma}$ is a weak G -equivalence since $\hat{\nu}$ is a weak G -equivalence. \square

9. GENERAL TOPOLOGICAL GROUPS AND COMPACT LIE GROUPS

As said in the introduction, we have so far focused on finite groups G . This should seem strange since the basic foundations for equivariant stable homotopy theory, for example in [38], deal with compact Lie groups and do not greatly simplify when specialized to finite groups. However, that is not true of equivariant infinite loop space theory. We explain here what is and is not true for infinite groups.

We consider general topological groups and classical G -spectra in Section 9.1. To avoid pathology, we assume throughout that the identity element of G is a nondegenerate basepoint. Subgroups of G are understood to be closed, and homomorphisms are understood to be continuous. We consider the Segal and operadic machines for compact Lie groups and genuine G -spectra in Section 9.2 and Section 9.3 and their comparison in Section 9.4.

In brief, the group G could be any topological group in all of our formal theory about classical G -spectra. When considering genuine G -spectra, our formal theory makes sense for general compact Lie groups and everything we do remains valid provided that we restrict attention to G -representations whose isotropy groups are of finite index in G . A quick explanation is that orbit G -spectra of compact Lie groups are self-dual only for such subgroups. As examples, it is often sensible in applications to remember that maximal tori have finite index in their normalizers. For example, multiplicative norms were first defined for subgroups of finite index in compact Lie groups, in [24, Definition 3.6]. They were used there to give a partial analogue in equivariant complex cobordism of the Atiyah-Segal completion theorem in equivariant K -theory. The special role played by finite index subgroups suggests generalization to profinite groups, but we have not pursued that idea.

9.1. Classical G -spectra for topological groups G . It is straightforward to generalize infinite loop space theory so as to accept equivariant input and to deliver classical equivariant spectra as output. The input can be \mathscr{F} - G -spaces or it can be G -spaces with actions by classical E_∞ G -operads, as defined below. Moreover, the nonequivariant comparison of infinite loop space machines of [57], recalled in Theorem 0.3, generalizes effortlessly to this context. Any infinite loop space machine landing in classical G -spectra is equivalent to the classical Segal machine. Modifying what needs to be modified, our new explicit comparison of the Segal and operadic machines also works for general topological groups and classical G -spectra.

Although this all works for any G , caveats appear immediately. We started work by defining families \mathbb{F}_n in Definition 1.2 and considering finite G -sets in Convention 1.4. The definitions and conventions there make sense in general, but they force focus on the subgroups H of finite index in G .

Remark 9.1. Since our homomorphisms are continuous, any $\alpha: G \rightarrow F$, where F is a finite group, factors through a homomorphism $\pi_0(G) \rightarrow F$. In particular, there

are no non-trivial α if G is connected. There are also infinite discrete groups G that admit no non-trivial α and have no non-trivial finite dimensional representations. Thus although the families \mathbb{F}_n can be defined in general, they are mainly of interest when G is finite. Note that the kernel of any homomorphism $\alpha: G \rightarrow F$ must have finite index in G and, if G acts on a finite set S , then the isotropy group of any element of S must have finite index in G .

Remark 9.2. Since the objects of \mathcal{F}_G are finite G -sets, the category \mathcal{F}_G sees only those orbits G/H such that H has finite index in G . Provided we restrict attention to such subgroups, we can use \mathcal{F}_G for general topological groups G , but there seems little point in doing so.

For the classical Segal machine for general topological groups, we ignore \mathcal{F}_G and restrict attention to \mathcal{F} and the notions of specialness and of level G -equivalence that are defined in (ii) and (iv) of [Definition 2.14](#). When there are no non-trivial homomorphisms $G \rightarrow \Sigma_n$, for example when G is connected, a Π - G -space is \mathbb{F}_\bullet -special if and only if it is special. The notion of an \mathbb{F}_\bullet -special Π - G -space is only of substantial interest when G is finite, although [Section 2.4](#) gives some motivation that applies in general.

In [Section 1](#), the discussion of orthogonal G -spectra applies verbatim to compact Lie groups G and everything else applies verbatim to general G . In [Section 2](#) and much of [Section 3](#), everything applies verbatim to general G , except that everything referring to the families \mathbb{F}_n and to \mathcal{F}_G is best ignored. The classical Segal machine appears in [Definition 2.23](#) and in [Propositions 2.30](#) and [3.23](#), as discussed in [Section 3.3](#). We reiterate that this works precisely as written for any topological group G . We reformulate [Theorem 3.29](#) to emphasize its perhaps surprising generality.

Theorem 9.3. *Let G be a topological group and let X be a special \mathcal{F} - G -space. Then the \mathcal{W}_G - G -space that sends A to $B(A^\bullet, \mathcal{F}, X)$ is positive linear and preserves connectivity.*

Although unrestricted, this result is the starting point towards understanding the genuine Segal machine. Subsections [8.1](#) through [8.4](#) apply to general topological groups and are largely devoted to the proof of [Theorem 9.3](#). Focusing on categories of operators over \mathcal{F} rather than over \mathcal{F}_G , the generalization of the Segal machine in [Section 4](#), the generalization of the operadic machine in [Section 5](#), and the comparison of machines in [Section 6](#) work just as written in the generality of classical infinite loop space machines for general topological groups.

9.2. The Segal machine for compact Lie groups. We summarize our conclusions for the Segal machine. For any topological group G , we have a functor \mathbb{S}_G^C that takes \mathcal{F} - G -spaces to G -prespectra. It has four variants. The first uses Segal's original simplicially defined inductive machine. The second is the conceptual machine, and the third and fourth are composites that first take \mathcal{F} - G -spaces to \mathcal{W}_G - G -spaces by a choice of bar constructions and then take \mathcal{W}_G - G -spaces to G -prespectra, exactly as explained in [Proposition 3.23](#). Restricting attention to special \mathcal{F} - G -spaces X , the four choices are equivalent and the functor \mathbb{S}_G^C assigns a positive classical Ω - G -spectrum to X , together with a group completion $\eta: X_1 \simeq (\mathbb{S}_G^C X)_0 \rightarrow \Omega(\mathbb{S}_G^C X)_1$.

If H is a subgroup of G , with inclusion denoted $\iota: H \rightarrow G$, then we have various functors ι^* that restrict given G actions to H actions. By inspection, these functors commute with all constructions in sight. For example, $\iota^* \mathbb{S}_G^C X \cong \mathbb{S}_H^C \iota^* X$. These

functors also commute with the group completion maps η , so that

$$\eta\iota^* = \iota^*\eta: \iota^*X_1 \longrightarrow \Omega(\mathbb{S}_H^C\iota^*X)_1.$$

For finite groups G , if X is \mathbb{F}_\bullet -special rather than just special, we have a genuine positive Ω - G -spectrum $\mathbb{S}_G X$ with underlying classical positive Ω - G -spectrum $\mathbb{S}_G^C X$. Formally we are using a forgetful functor from genuine orthogonal G -spectra indexed on a complete G -universe to classical G -spectra indexed on the trivial universe. Restriction to subgroups works the same way on the level of genuine G -spectra as it does on the level of classical G -spectra. Note that if H is a finite subgroup of a topological group G and X is an \mathcal{F} - G -space which is \mathbb{F}_\bullet -special as an \mathcal{F} - H -space, then $\iota^*\mathbb{S}_G^C X \cong \mathbb{S}_H^C\iota^*X$ is the underlying classical Ω - H -spectrum of a genuine Ω - H -spectrum.

Now let G be a compact Lie group. One might hope to apply the Segal machine to somehow construct genuine Ω G -spectra from functors defined on \mathcal{F} - G -spaces. However, Blumberg shows in [6, Appendix C] that the Segal machine does not give such an infinite loop space machine even when $G = S^1$, and in fact there is no condition on an \mathcal{F} - S^1 -space that can imply that it gives a positive Ω - S^1 -spectrum. It does not help to switch from \mathcal{F} to functors defined on \mathcal{F}_G , and this illustrates the problem. We cannot expect functors defined on the category of finite G -sets to give homotopical information about representations that have isotropy groups of infinite index in G . Nevertheless, as said before, when we restrict to G -representations with isotropy groups of finite index, everything we say about finite groups applies equally well for compact Lie groups. We explain in outline how this works. In effect, the following starting point describes restriction to those based finite G -CW complexes that are built up from finite G -sets.

Definition 9.4. Define \mathcal{W}_G^f to be the full $G\mathcal{U}_*$ -subcategory of \mathcal{W}_G whose objects are the based finite G -CW complexes whose isotropy groups have finite index in G . Note that $\mathcal{F}_G \subset \mathcal{W}_G$. Define a \mathcal{W}_G^f - G -space to be a $G\mathcal{U}_*$ -functor $\mathcal{W}_G^f \rightarrow G\mathcal{U}_*$. By results of Illman [36], S^V is in \mathcal{W}_G^f for all G -representations V . Define \mathcal{I}_G^f to be the full subcategory of \mathcal{I}_G that is obtained by restricting to those G -representations such that S^V is in \mathcal{W}_G^f , and define $G\mathcal{S}^f$ to be the analogue of the category of orthogonal G -spectra $G\mathcal{S}$ (see Definition 1.21) defined by replacing \mathcal{I}_G with \mathcal{I}_G^f . Define $\mathbb{U}_G^f: \text{Fun}(\mathcal{W}_G^f, \mathcal{I}_G) \rightarrow G\mathcal{S}^f$ to be the evident forgetful functor. It fits into the commutative diagram of forgetful functors

$$\begin{array}{ccc} \text{Fun}(\mathcal{W}_G, G\mathcal{U}_*) & \longrightarrow & \text{Fun}(\mathcal{W}_G^f, G\mathcal{U}_*) \\ \mathbb{U}_G \downarrow & & \downarrow \mathbb{U}_G^f \\ G\mathcal{S} & \longrightarrow & G\mathcal{S}^f. \end{array}$$

We say that a G -spectrum E is a *positive finite index Ω - G -spectrum* if the maps $\tilde{\sigma}: E(V) \rightarrow \Omega^W E(V \oplus W)$ are weak G -equivalences when the isotropy groups of V and W have finite index in G and $V^G \neq 0$.

Our constructions work to obtain an orthogonal G -spectrum $\mathbb{S}_G X$ from a \mathbb{F}_\bullet -special \mathcal{F} - G -space X . Equivalently, we can construct $\mathbb{S}_G Y$ from a special \mathcal{F}_G - G -space Y and start from there. We think of $Y = \mathbb{P}X$ or, equivalently, $X = \mathbb{U}Y$. This works without restriction to give Ω - G -spectra when G is finite. In that case, $\mathbb{S}_G X$ is a (genuine) positive Ω - G -spectrum whose bottom structural G -map is a group

completion of X_1 . Prolongation of $B(\mathcal{F}, \mathcal{F}, X)$ to a \mathcal{W}_G - G -space does not give a positive Ω - G -spectrum; prolongation of $B(\mathcal{F}_G, \mathcal{F}_G, Y)$ to a \mathcal{W}_G - G -space does. This remains true for compact Lie groups if we restrict attention to \mathcal{W}_G^f and \mathbb{U}_G^f . Said differently, prolongation of $B(\mathcal{F}_G, \mathcal{F}_G, Y)$ gives rise to a positive finite index Ω - G -spectrum.

Therefore, we now modify the definition, [Definition 0.6](#), of an infinite loop space machine by replacing $G\mathbf{Sp}_{\text{cp}}$ with the category $G\mathbf{Sp}_{\text{cp}}^f$ of connective orthogonal G -spectra $\{E_V\}$ which restrict to positive Ω - G -spectra on the subuniverse obtained by allowing only those representations V whose isotropy subgroups H have finite index in G .

Modifying everything by restricting isotropy subgroups of representations to have finite index in G , everything in subsections [3.4](#) and [8.5](#) works with no changes other than notation to prove the following generalization of the fundamental theorem, [Theorem 3.34](#), about the Segal machine.

Theorem 9.5. *Let G be a compact Lie group and let X be an \mathbb{F}_\bullet -special \mathcal{F} - G -space. Then $\mathbb{S}_G X$ is a positive finite index Ω - G -spectrum. Moreover, if $S^V \in \mathcal{W}_G^f$ and $V^G \neq 0$, then the composite*

$$X_1 \longrightarrow B(\mathcal{F}_G, \mathcal{F}_G, \mathbb{P}X)_1 = (\mathbb{S}_G X)(S^0) \longrightarrow \Omega^V(\mathbb{S}_G X)(S^V)$$

of η_1 and the structure G -map is a group completion.

The restriction to finite index isotropy groups is illuminated by consideration of duality and the Wirthmüller isomorphism. The suspension G -spectrum $\Sigma_+^\infty G/H$ is self dual if and only if H has finite index in G . In general, its dual is $\Sigma_+^{-L} \Sigma_+^\infty G/H$, where L is the tangent representation of G/H at the identity coset. Similarly, the Wirthmüller isomorphism says that, for an H -spectrum D , the left adjoint $G_+ \wedge_H D$ of the forgetful functor from G -spectra to H -spectra is equivalent to its right adjoint $F_H(G_+, D)$ when H has finite index in G , but is equivalent to $F_H(G_+, \Sigma^L D)$ in general. A quick proof is given in [\[52\]](#). It is the appearance of L in these results that prevents \mathbb{S}_G from giving positive Ω - G -spectra and thus prevents us from having a fully satisfactory Segal machine for compact Lie groups.

Blumberg's paper [\[6\]](#) focuses on finding necessary and sufficient conditions for a \mathcal{W}_G - G -space to restrict to a positive Ω - G -spectrum for a compact Lie group G . He finds that two conditions are necessary, one essentially being positive linearity and the other being closely related to the Wirthmüller isomorphism.

9.3. The operadic machine for compact Lie groups. We must now distinguish between classical and genuine E_∞ G -operads. A nonequivariant operad \mathcal{C} is said to be an E_∞ operad if $\mathcal{C}_G(n)$ is a universal principal Σ_n -bundle for each n . Thus each $\mathcal{C}_G(j)$ is Σ_j -free and contractible.

Definition 9.6. Let G be a topological group. A classical E_∞ G -operad is a nonequivariant E_∞ operad regarded as a G -trivial G -operad. A G -operad \mathcal{C}_G is a genuine E_∞ G -operad if $\mathcal{C}_G(n)$ is a universal principal (G, Σ_n) -bundle for each n .

Thus the notion of a genuine E_∞ G -operad is precisely the same as our earlier notion of an E_∞ G -operad given in [Definition 5.2](#). We have simply replaced the finite group implicit there by a general topological group here. We emphasize that genuine E_∞ G -operads for compact Lie groups G are useful for other purposes, even though their algebras do not give input to a fully general equivariant infinite

loop space machine. In particular, the linear isometries G -operad plays a central role in the construction of a symmetric monoidal category of G -spectra in which every object is an Ω - G -spectrum [20, 42]. Such a construction is not possible for symmetric or orthogonal G -spectra.

For the classical operadic machine, we start with classical E_∞ G -operads. Here the nonequivariant recognition principle [45] generalizes directly. While [25] focused on finite groups, the basic theory of G -operads and the recognition principle for classical G -spectra were developed for general topological groups in [25, Section 2]. As long as we restrict to those representations V with trivial G -action and to classical G -spectra, everything works exactly the same way as it does nonequivariantly. The approximation theorem, [Theorem 5.5](#), generalizes classically since a check of definitions shows that passage to H -fixed points gives an application of the nonequivariant approximation theorem for each n (that is, for each \mathbb{R}^n), and each closed subgroup H of G .

Now let G again be a compact Lie group. Then the operadic machine, like the Segal machine, generalizes to G provided that we restrict attention to those G -representations and G -spaces whose isotropy groups have finite index in G . Everything in the references [25, 32, 68] works exactly as stated for finite groups except for the approximation theorem, which addresses the homotopical behavior of $\alpha: \mathbb{C}_G X \rightarrow \Omega^V \Sigma^V X$. Even if $G = S^1$, the group completion property of α fails if V is a general non-trivial representation of G , as was first noticed by Segal [73]. His counterexample is presented in [6, Appendix B].

The only further discussion of the compact Lie case of which we are aware is that of Caruso and Waner [12]. They address the approximation theorem in [12, Theorem 1.18]. Although they restrict to finite groups in the *statement* of that result and have written things in a partially stable context, a minor reworking of their proof³⁰ makes clear that the only use of the finiteness of G is to ensure that the G -sets G/H that they encounter are finite. This gives the following conclusion.

Theorem 9.7. *Theorem 5.6 remains true if we start with a compact Lie group G and a genuine E_∞ operad \mathcal{C}_G of G -spaces, provided that we require V to have finite isotropy groups. We do not require the \mathcal{C}_G -space X to have finite isotropy groups.*

9.4. The comparison theorem for compact Lie groups. From here, everything that we have said in this paper applies verbatim to compact Lie groups, provided that we restrict attention to finite index isotropy subgroups. We focused on finite groups only for clarity of exposition. We restate the comparison theorem, [Theorem 6.1](#), in this generality.

Theorem 9.8. *Let G be a compact Lie group and let \mathcal{D}_G be the category of operators over \mathcal{F}_G constructed from a genuine E_∞ operad \mathcal{C}_G of G -spaces. For special \mathcal{D}_G - G -spaces Y , there is a natural zigzag of equivalences of orthogonal G -spectra in $G\mathcal{S}^f$ between $\mathbb{S}_G Y$ and $\mathbb{E}_G Y$.*

Here we define the homotopy groups $\pi_*^H(X)$ of $X \in G\mathcal{S}^f$ for all subgroups H of G exactly as in (6.6), but with the colimit running over those V whose isotropy groups have finite index in G . We are restricting to the finite index subuniverse of the complete universe used in (6.6). Just as in the complete universe case, we define an equivalence in $G\mathcal{S}^f$ to be a π_* -isomorphism. With these equivalences, we then

³⁰We are indebted to Stefan Waner for discussion of this.

have the stable model structure on $G\mathcal{S}^f$ exactly as in [42, Theorem III.4.2]. The discussion in the following epilogue applies verbatim in this more general context.

10. EPILOGUE: MODEL CATEGORICAL INTERPRETATIONS

Our work can be contextualized in terms of model categories,³¹ and we do so now. The essential point is that equivariant infinite loop space theory establishes what in [43] is called a “connective Quillen equivalence” between the model category of grouplike input objects of either the Segalic or operadic sort to the connective part of the stable model category of orthogonal G -spectra. For the Segal machine, this is documented nonequivariantly in [9, 43, 71] and equivariantly in [64, 70]; we shall explain that result with different details here. The result for the operadic machine is documented in [2], although the result was understood much earlier.

In fact, all of the categories in sight have standard and well-known model structures, so we shall just say what those are rather than reproving things from scratch. There are many adjoint pairs of functors in sight. Most are Quillen adjoint pairs and some are (connective) Quillen equivalences. It is reasonable to upgrade the definition of an infinite loop space machine, Definition 0.6, model theoretically to at least include model structures on the source and target categories and to require a characterization principle in the sense of Remark 0.4. That is, one can insist on a connective Quillen equivalence between grouplike input and connective output. Ignoring model structures, it is already apparent that we do have equivalences of homotopical categories in the sense of [18].

On the input data side, our categories have several relevant model structures, and a comparison among the roles they play is illuminating. We explain that in Section 10.2, after recalling some of the relevant structures on underlying ground categories in Section 10.1. On the output side, model structures on the relevant categories of G -spectra are also well understood and will be summarized as we go. The main point that needs investigation is the relationship between these various model structures on the one hand and the infinite loop space machines and their comparison on the other. We discuss that for the (generalized) Segal machine in Section 10.3. Aside from understanding the difference between the classical and genuine dichotomy model theoretically, there is little that is new and we shall be brief. The answer for the operadic machine is easier, as in [2]. We discuss the model categorical relevance of the bar construction in Section 10.4.

Much of what we say in this section applies to general topological groups G and classical G -spectra, but our focus is on finite groups G and genuine G -spectra.

10.1. Model structures on G -spaces and diagram G -spaces. We have been working throughout in topologically enriched G -categories, as we discussed in some detail in Section 1.3. All have standard model structures. The three ground categories in our work are $G\mathcal{U}_*$, $\text{Fun}(\Pi, G\mathcal{U}_*)$ and $\text{Fun}(\Pi_G, G\mathcal{U}_*)$. These are all complete and are all enriched in $G\mathcal{U}_*$. Recall that we write $G\mathcal{T}$ for the full subcategory of $G\mathcal{U}_*$ consisting of nondegenerately based spaces, and we write $\Pi[G\mathcal{T}] = \Pi[\mathcal{T}_G]$ and $\Pi_G[\mathcal{T}_G]$ for the full subcategories of $\text{Fun}(\Pi, G\mathcal{U}_*)$ and $\text{Fun}(\Pi_G, G\mathcal{U}_*)$ satisfying the extra cofibration conditions spelled out in Definition 2.10.

We start with the model structures on $G\mathcal{U}$; they restrict to give model structures on $G\mathcal{U}_*$ [55, Theorem 15.3.6]. The standard q -model structure (q for Quillen)

³¹As the referee very helpfully pointed out.

on $G\mathcal{U}$ has the weak G -equivalences as its weak equivalences and the Serre G -fibrations as its fibrations; these are the G -maps whose H -fixed point maps are weak equivalences or Serre fibrations for all (closed) subgroups H . The cofibrations are the retracts of G -cell complexes, whose attaching maps are defined on G -spheres $G/H \times S^n$. The cofibrations of $G\mathcal{U}_*$ are the based maps whose underlying unbased maps are G -cofibrations, we refer to these as q -equivalences, q -fibrations, and q -cofibrations. A recent discussion of these model structures on G -spaces, with their enrichments, is given in [27, Section 1]. In $G\mathcal{U}_*$, the q -cofibrations satisfy the based HEP (homotopy extension property).

The h -model structure (h for homotopy or Hurewicz) on $G\mathcal{U}$ has the G -homotopy equivalences as its weak equivalences and the Hurewicz G -fibrations (CHP) and Hurewicz G -cofibrations (HEP) as its fibrations and cofibrations.³² We refer to these as h -equivalences, h -fibrations, and h -cofibrations. Since h -fibrations are q -fibrations and h -equivalences are q -equivalences, we have a mixed model structure with m -fibrations = h -fibrations and m -equivalences = q -equivalences; the m -cofibrant objects are the spaces of the homotopy type of CW-complexes [15], [55, §17.4]. As discussed in [56, §4.1] and [55], classical algebraic topology implicitly focused on the mixed model structure on spaces, and the same is true equivariantly. We shall return to the m -model structure in Section 10.4, but we focus on q -model structures until then.

Since $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ and $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ are diagram categories, there is a large body of work that applies to them; see for example [29, 27, 42, 70, 64]. Consider the following diagram of adjunctions

$$G\mathcal{U}_* \begin{array}{c} \xleftarrow{\mathbb{L}} \\ \xrightarrow{\mathbb{R}} \end{array} \text{Fun}(\Pi, \underline{G\mathcal{U}}_*) \begin{array}{c} \xleftarrow{\mathbb{U}} \\ \xrightarrow{\mathbb{P}} \end{array} \text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$$

Since (\mathbb{P}, \mathbb{U}) is an adjoint equivalence, (\mathbb{U}, \mathbb{P}) is also an adjunction, and we write $(\mathbb{L}_G, \mathbb{R}_G)$ for the displayed composite adjunction $(\mathbb{L}\mathbb{U}, \mathbb{P}\mathbb{R})$.

Definition 10.1. We define the level q -model structures on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ and $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ by letting a map $f: X \rightarrow Y$ be a level q -equivalence or level q -fibration if each f_n or each f_α is a q -equivalence of G -spaces. We call these the *classical* model structure on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ and the *genuine* model structure on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$. These are often called projective model structures in the literature.

Remark 10.2. We have given the definition in terms of the q -model structure, but we could instead use the less familiar h or m -model structure, with preference for the last. Everything works the same way, using the foundational details in [4, 3].

These model structures are discussed nonequivariantly in [43, Sections 6 and 17] and in a more general categorical context in [29, Section 4.4]. Equivariant details are given in [70, Sections 4 and 5] and [64, Section 4]. The following central remark uses Theorem 2.38 to describe the adjoint equivalence (\mathbb{P}, \mathbb{U}) model theoretically.

Remark 10.3. We regard the level model structure on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ as its classical model structure. Under the adjoint equivalence (\mathbb{P}, \mathbb{U}) , it transports to a new *classical* model structure on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ whose weak equivalences are those maps that are level G -equivalences *when restricted to G -trivial finite G -sets*. Similarly,

³²Even nonequivariantly, the factorization axioms are subtle. Incorrect arguments in [14, 56, 55] are corrected and generalized by Barthel and Riehl in [4, 3]; the generalization is in a categorical context that includes $G\mathcal{U}$.

we view the level model structure on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ as its genuine model structure. Using [Theorem 2.38](#), we see that, under (\mathbb{P}, \mathbb{U}) , it transports to a new *genuine* model structure on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ whose weak equivalences are the \mathbb{F}_\bullet -level equivalences.

With these definitions, the equivalence (\mathbb{P}, \mathbb{U}) is a Quillen equivalence of model categories with respect to *either* the classical *or* the genuine level model structures on our two categories. The identity functor on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ or on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ is the left and right adjoint of a Quillen adjunction from the classical to the genuine model structure. The latter is a localization of the former.

Remark 10.4. Pragmatically, we are only interested in *special* input. Model theoretically, it therefore makes sense to localize the level model structures at Segal maps to obtain a new model structure on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ in which the fibrant objects are the (classically) special objects and a new model structure on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ in which the fibrant objects are the (genuinely) special objects. These localizations transfer along the equivalence (\mathbb{P}, \mathbb{U}) to give a localization of the classical model structure on $\text{Fun}(\Pi_G, \underline{G\mathcal{U}}_*)$ in which the fibrant objects are the $\mathbb{P}X$, where X is classically special, and a localization of the genuine model structure on $\text{Fun}(\Pi, \underline{G\mathcal{U}}_*)$ in which the fibrant objects are the $\mathbb{U}Y$ such that Y is genuinely special; (\mathbb{P}, \mathbb{U}) gives Quillen equivalences of classical and genuine model structures after localization.

10.2. Quillen equivalences among input model categories. We combine and expand the diagrams [\(4.26\)](#) and [\(4.27\)](#) to discuss the input for our machines model categorically. Assume given an E_∞ operad \mathcal{C}_G . It has associated categories of operators \mathcal{D} and \mathcal{D}_G , the first over \mathcal{F} and under Π and the second over \mathcal{F}_G and under Π_G . Consider the following schematic diagramm where all pairs of arrows represent adjoint functors.

$$(10.5) \quad \begin{array}{ccccc} \text{Fun}(\mathcal{F}, \underline{G\mathcal{U}}_*) & \begin{array}{c} \xleftarrow{\mathbb{P}} \\ \xrightarrow{\mathbb{U}} \end{array} & \text{Fun}(\mathcal{F}_G, \underline{G\mathcal{U}}_*) & \begin{array}{c} \xleftarrow{\mathbb{P}} \\ \xrightarrow{\mathbb{U}} \end{array} & \text{Fun}(\mathcal{W}_G, \underline{G\mathcal{U}}_*) \\ \updownarrow \xi_* \quad \xi^* & & \updownarrow \xi_{G_*} \quad \xi_G^* & & \updownarrow \mathbb{P}_{G\mathcal{F}} \quad \mathbb{U}_{G\mathcal{F}} \\ \text{Fun}(\mathcal{D}, \underline{G\mathcal{U}}_*) & \begin{array}{c} \xleftarrow{\mathbb{P}} \\ \xrightarrow{\mathbb{U}} \end{array} & \text{Fun}(\mathcal{D}_G, \underline{G\mathcal{U}}_*) & \xrightarrow{\mathbb{S}_G} & G\mathcal{S} \\ \updownarrow \mathbb{L} \quad \mathbb{R} & & \updownarrow \mathbb{L}_G \quad \mathbb{R}_G & & \\ \mathcal{C}_G[\underline{G\mathcal{U}}_*] & & & \searrow \mathbb{E}_G & G\mathcal{S} \end{array}$$

Here we write ξ_* and ξ_{G_*} for the *categorical* rather than the derived left adjoints to ξ^* and ξ_G^* .³³ Since $\xi^*\mathbb{U} = \mathbb{U}\xi_G^*$, $\mathbb{P}\xi_* \cong \xi_{G_*}\mathbb{P}$. By definition, $\mathbb{L}_G = \mathbb{L}\mathbb{U}$ and $\mathbb{R}_G = \mathbb{P}\mathbb{R}$. Of course, \mathbb{S}_G and \mathbb{E}_G are the infinite loop space machines compared in [Section 6](#), while the composite $\mathbb{U}_{G\mathcal{F}}\mathbb{P}\mathbb{P}$ is the conceptual Segal machine we started with.

³³We apologize to the reader for the conflict of notation. However, when one interprets the bar construction as a cofibrant approximation, the conflict becomes a reasonably standard one.

We ignore \mathcal{W}_G and $G\mathcal{S}$ in this subsection and consider the six adjoint pairs displayed in the left square and triangle. We focus on finite groups and genuine model structures.

Theorem 10.6. *The categories in the left square and the triangle have q -model structures, and the six adjoint pairs between them are all Quillen equivalences.*

We call all of these *genuine* model structures.

Proof. As noted in Sections 5.2 and 5.4, the five categories on the lefthand side can be described as the categories of algebras over certain monads. As such, they all have right adjoint forgetful functors to their ground categories, which are $G\mathcal{U}_*$, $\text{Fun}(\Pi, G\mathcal{U}_*)$, or $\text{Fun}(\Pi_G, G\mathcal{U}_*)$. The genuine model structures on the ground categories are described in the previous section. We say that a map f in any of our five categories is a weak equivalence or fibration if its underlying map in the ground category is a weak equivalence or fibration. Then [55, Theorem 16.2.5] applies to show that these specifications create the required q -model structures in our five categories. It is then clear that the right adjoints in the left square and triangle preserve fibrations and acyclic fibrations and are thus Quillen right adjoints by [55, Definition 16.2.1]. The discussion in Remark 10.3 applies verbatim to both pairs (\mathbb{P}, \mathbb{U}) , showing that they are Quillen equivalences. That (ξ_*, ξ^*) and (ξ_{G_*}, ξ^{G^*}) are Quillen equivalences follows from Proposition 4.30 and Theorems 4.24 and 4.25, which show that ξ^* and ξ_G^* become equivalences of homotopy categories with inverses the derived versions of ξ_* and ξ_{G_*} . That (\mathbb{L}, \mathbb{R}) and $(\mathbb{L}_G, \mathbb{R}_G)$ are Quillen equivalences follows from Proposition 5.27. \square

Remark 10.7. Using Remark 10.3, we see by the same argument that the previous result is true as stated for classical rather than genuine input to infinite loop space machines for arbitrary topological groups G . In both the classical case and the genuine case, we can localize at Segal maps as in Remark 10.4 and obtain the same conclusions as in Theorem 10.6.

The Segal machine focuses on properties of input data, whereas the operadic machine focuses on structure, as the following remark illuminates.

Remark 10.8. An operad is Σ -free if Σ_j acts freely on $\mathcal{C}(j)$ for all j . A Σ -free operad \mathcal{C}_G of G -spaces is *classically* E_∞ if each $\mathcal{C}_G(j)$ is contractible. This makes sense for any topological group G , and classical nonequivariant operadic infinite loop space theory generalizes effortlessly to give infinite loop space theory with classical (or naive) G -spectra as output. In the body of the paper, we restricted attention to genuine E_∞ operads as defined in Definition 5.2. Of course, such operads are also classically E_∞ , giving forgetful passage from genuine to classical operadic infinite loop space theory.

10.3. \mathcal{F}_G - G -spaces, \mathcal{W}_G - G -spaces, and orthogonal G -spectra. We now turn to the right square of (10.5). We shall be brief since Blumberg has given a good summary [6, Appendix A], which itself is largely an equivariantization of material in [43] and a summary of material in [42]. We first recall some of the model structures on (orthogonal) G -spectra from [42]. Here G is a compact Lie group.

Definition 10.9. The category of orthogonal G -spectra has a level model structure [42, Section III.2]. Using representations as usual to define stable homotopy groups

and stable (weak) equivalences [42, Section III.3], we define the stable model structure on $G\mathcal{S}$ to have weak equivalences the π_* -isomorphisms and cofibrations the q -cofibrations of the level model structure [42, Section III.4]. The fibrant objects are the Ω - G -spectra.

We view $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$ as an intermediary between the input data of the variant Segal machines and the G -spectra output. This is awkward model theoretically since the relevant comparisons are both given by *right adjoint* forgetful functors defined on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$. A drawback of model category theory in general is that its comparisons are designed to deal with Quillen adjunctions, which are intrinsically directional, and the comparison here is the composite of a left adjoint to $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$ and a right adjoint from it to $G\mathcal{S}$. We start with the absolute model structure, which makes sense for any topological group G .

Definition 10.10. In the *absolute* level model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$, a map $Y \rightarrow Z$ is a fibration or weak equivalence if $Y(A) \rightarrow Z(A)$ is a Serre G -fibration or a weak G -equivalence for all $A \in \mathcal{W}_G$.

The following observation about the top right adjunction in (10.5) is immediate and applies to any topological group G .

Proposition 10.11. (\mathbb{P}, \mathbb{U}) is a Quillen adjunction from the genuine model structure on $\text{Fun}(\mathcal{F}_G, \underline{G}\mathcal{U}_*)$ to the absolute level model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$.

Henceforward in this subsection, take G to be a compact Lie group.

Definition 10.12. In the *relative* level model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$, a map $Y \rightarrow Z$ is a fibration or weak equivalence if $Y(S^V) \rightarrow Z(S^V)$ is a Serre G -fibration or a weak G -equivalence for each G -representation V .

By construction, the identity functor is the left adjoint of a Quillen adjunction from the relative to the absolute level model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$. The forgetful functor $\mathbb{U}_{G\mathcal{S}}$ in (10.5) has a left adjoint prolongation functor $\mathbb{P}_{G\mathcal{S}}$, constructed as in [43, Definition 23.1]. To interpret it model theoretically, we use stable model structures on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$. These structures are discussed in [6, A.2, A.4].

Definition 10.13. Define the relative and absolute stable model structures on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$ to have weak equivalences the maps f such that $\mathbb{U}_{G\mathcal{S}}f$ is a π_* -isomorphism. The cofibrations are the maps that are cofibrations in the relative or absolute level model structure. Then a \mathcal{W}_G - G -space Y is fibrant in the absolute stable model structure if and only if the structure map

$$Y(A) \rightarrow \Omega^W Y(\Sigma^W A)$$

is a weak G -equivalence for all $A \in \mathcal{W}_G$ and all G -representations W .

Theorem 10.14. [6, Theorems 1.3, A.13 and A.19], [42, Theorem 4.16]. *Let G be a compact Lie group.*

- (i) *The identity functor is the left adjoint of a Quillen equivalence from the relative to the absolute stable model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$.*
- (ii) *The pair $(\mathbb{P}_{G\mathcal{S}}, \mathbb{U}_{G\mathcal{S}})$ is a Quillen equivalence from the stable model structure on $G\mathcal{S}$ to the relative stable model structure on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$.*

Remark 10.15. Just as we defined the stable model structures on $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$ starting from level model structures, we can define stable model structures on the

four categories in the left square in (10.5). In all of them we say that a map is a stable weak equivalence if it becomes one after passing to $\text{Fun}(\mathcal{F}_G, \underline{G}\mathcal{U}_*)$ and applying \mathbb{P} to land in $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$. Nonequivariantly, this idea goes back to [9] and later [71] and [43, Theorem 18.3]. It is pursued equivariantly in [64, 69, 70]. It is usual to say that grouplike special objects are “very special”. The fibrant objects in these stable model structures are very special. More precisely, the fibrant \mathcal{F} - G -spaces are those that are grouplike and \mathbb{F}_\bullet -special and the fibrant \mathcal{F}_G - G -spaces are those that are grouplike and special. With the language of Remark 0.4, this model theoretic approach is studying a characterization principle rather than a recognition principle: it is blind to group completion.

Say that a \mathcal{W}_G - G -space Y is connective if the G -spectrum $\mathbb{U}_{G,\mathcal{S}}Y$ is connective. The following is an equivariant version of [43, Theorem 0.10], and the proof is the same as there. It remains true if we restrict attention throughout to subgroups of finite index in a compact Lie group.

Theorem 10.16. *Let G be a finite group. Then (\mathbb{P}, \mathbb{U}) is a connective Quillen equivalence from $\text{Fun}(\mathcal{F}_G, \underline{G}\mathcal{U}_*)$ with its stable model structure to $\text{Fun}(\mathcal{W}_G, \underline{G}\mathcal{U}_*)$ with its absolute stable model structure.*

Proof. The unit $\text{Id} \rightarrow \mathbb{U}\mathbb{P}$ of the adjunction is the identity, hence a stable equivalence on all X . The counit $\mathbb{P}\mathbb{U} \rightarrow \text{Id}$ is a stable equivalence when applied to connective stably bifibrant \mathcal{W}_G - G -spaces, just as in [43, Lemma 18.10]. \square

The following is a quibble about the comparison between what we have proven and model categorical language.

Remark 10.17. Recall that a Quillen adjunction $(F, U): \mathcal{C} \rightarrow \mathcal{D}$ in which U creates the weak equivalences in \mathcal{D} is a Quillen equivalence if and only the unit $\eta: Y \rightarrow UFY$ is a weak equivalence for all cofibrant Y [55, Proposition 16.2.3]. In the introduction, we required of a model infinite loop space machine that it induce an equivalence of homotopy categories from grouplike input to connective output, without a requirement about the point-set level transformations that induce the equivalence. The Segal machine \mathbb{S}_G is the composite of the left adjoint in Theorem 10.16 and the right adjoint in Theorem 10.14. It does induce such a connective equivalence of homotopy categories, but the transformations that exhibit the equivalence are not the unit and counit of a single point-set level Quillen adjunction.

As already mentioned, the operadic machine with target category Lewis-May G -spectra admits a cleaner model theoretic interpretation in which the defects of Remarks 10.15 and 10.17 are circumvented [37].

10.4. The bar construction model theoretically. For background, we recall that, classically, many foundational questions were sensibly swept under the rug with the blanket assumption that all given spaces are of the homotopy types of CW complexes. In more modern language, that means that all given spaces are cofibrant in the mixed model structure [15], [55, Section 17.3], and that works just as well for G -spaces for any topological group G .

The classical theorems about transfer of q -model structures along adjunctions (e.g. [55, Theorem 16.2.5]) have analogs for transfer of h -model structures along adjunctions [3, Theorems 6.12-6.14]. From that starting point, we claim that all

of the q -model categories discussed above, including \mathscr{W}_G - G -spaces and G -spectra, can also be given h -model structures and thus also m -model structures. We shall not prolong matters by giving proofs, but the idea should be clear. The main point is to check that the hypotheses of the results of [3] just cited are satisfied by all adjunctions in sight. We leave details to the interested reader.

Therefore, assuming that all of our input G -spaces, including the G -spaces $\mathcal{C}_G(n)$ of operads, have the homotopy types of G -CW complexes, our bar constructions are cofibrant in the relevant mixed model structures, so that our categorical and monadic bar constructions take (levelwise) G -CW homotopy types to G -CW homotopy types. We note here that the categorical bar constructions used in the Segal machine can be viewed as special cases of the monadic bar constructions used in the operadic machine [58, Section 7]. These observations lead to the following folklore theorem. It is labelled as such only because there is as yet no fully worked out proof in the literature, but we believe that arguments in [3, 4] can be adapted to fill the gap.

Folklore Theorem 10.1. *The bar construction approximations that we start from in both the Segal and operadic machines can be interpreted as highly structured cofibrant approximations in m -model structures.*

Remark 10.18. While our focus is on our topological context, it has been understood since the 1970's that bar constructions are “like” cofibrant approximations. One precise formulation of “like” is given in [79] in the context of the homotopical categories of [18].³⁴ There are many papers with results saying implicitly that bar constructions “are” cofibrant approximations in certain contexts, without actually mentioning bar constructions explicitly. See for example [34, Corollary 14.8.8], [23], and [35, Propositions 6.9 and 6.10]. A general categorical discussion of when bar constructions can be interpreted as cofibrant approximations is given by Shulman in [78, Sections 7 and 8].³⁵ In particular, focusing on simplicial model categories, [78, Corollary 8.16] gives an explicit general context in which bar constructions are cofibrant replacements, and that can be reworked topologically.

11. APPENDIX: BEARDING FUNCTORS $\mathscr{D} \rightarrow G\mathscr{U}_*$

As in [57], but working equivariantly, we first generalize Proposition 2.12 from \mathscr{F} to a general G -category of operators \mathscr{D} associated to a G -operad \mathcal{C}_G and then prove the generalization.

Proposition 11.1. *There is a functor $W: \text{Fun}(\mathscr{D}, G\mathscr{U}_*) \rightarrow \text{Fun}(\mathscr{D}, G\mathscr{U}_*)$ together with a natural transformation $\pi: W \rightarrow \text{Id}$ such that, for any functor $X: \mathscr{D} \rightarrow G\mathscr{U}_*$, WX is in $\mathscr{D}[\mathcal{T}_G]$ and $\pi: WX \rightarrow X$ is a levelwise G -homotopy equivalence.*

We break the proof of Proposition 11.1 into steps, following [57, Appendix B].

Definition 11.2 (The definition of $W_n X$ and of $\pi: W_n X \rightarrow X_n$). Let $I = [0, 1]$ have basepoint 1 and the binary operation specified by $(s, t) \mapsto \min\{s, t\}$;³⁶ thus I is a topological abelian monoid with unit 1 and so determines an \mathscr{F} -space $I: \mathscr{F} \rightarrow$

³⁴Bar constructions are not mentioned in [18], but the definition of left deformable functors [18, Section 40] seems designed to accommodate them.

³⁵Those sections are independent of the type theory focus of that paper.

³⁶We are following [57] here, but the product st would work just as well.

\mathcal{F} with n th space I^n . Explicitly, the map $\phi_*: I^m \rightarrow I^n$ associated to a map $\phi: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{F} is specified by

$$(11.3) \quad \phi(s_1, \dots, s_m) = (t_1, \dots, t_n) \text{ where } t_j = \min_{\phi(i)=j} s_i.$$

Here the minimum of the empty set is interpreted as the basepoint 1, so that $t_j = 1$ if $j \notin \text{im}(\phi)$. Note that I is an \mathcal{F} -space; for an injection $\phi: \mathbf{m} \rightarrow \mathbf{n}$, the map $\phi_*: I^m \rightarrow I^n$ is indeed a Σ_ϕ -cofibration. We give I the trivial G -action and regard I as a functor $\mathcal{D} \rightarrow \mathcal{T}_G$ by pullback along $\xi: \mathcal{D} \rightarrow \mathcal{F}$, thus obtaining a \mathcal{D} - G -space.

For a functor $X: \mathcal{D} \rightarrow G\mathcal{U}_*$, we have a product functor $I \times X: \mathcal{D} \rightarrow G\mathcal{U}_*$ with n th based G -space $I^n \times X_n$, and the projection $I \times X \rightarrow X$ is natural with respect to \mathcal{D} . Define $W_n X$ to be the subspace of $I^n \times X_n$ which consists of those points (t, x) , $t = (t_1, \dots, t_n)$, such that $x \in (\iota_t)_*(X_m)$, where $\iota_t: \mathbf{m} \rightarrow \mathbf{n}$ is that ordered injection whose image is the set of j , $1 \leq j \leq n$, such that $t_j = 0$. Note that $\iota_t = \text{id}: \mathbf{n} \rightarrow \mathbf{n}$ if all $t_j = 0$ and that $m = 0$ if no $t_j = 0$. The latter implies that the basepoint $((1, \dots, 1), *)$ of $I^n \times X_n$ is in $W_n X$.

Define a based G -map $\pi: W_n X \rightarrow X_n$ by $\pi(t, x) = x$. Observe that X_n embeds in $W_n X$ as the set of points $((0, \dots, 0), x)$ and is an unbased G -deformation retract of $W_n X$ via the deformation h given by $h((t, x), s) = (t - st, x)$ for $0 \leq s \leq 1$, where $st = (st_1, \dots, st_n)$. Thus π is a levelwise G -homotopy equivalence, although the deformations are not based homotopies.

Quoting [57, Remarks B.2], $W_n X$ should be thought of as obtained from X_n by growing a beard consisting of a thick whisker I^{n-m} attached to each point of $\iota(X_m)$ for each ordered injection $\iota: \mathbf{m} \rightarrow \mathbf{n}$, but with (implicit) identifications of partial whiskers corresponding to compositions of injections. When the underlying Π - G -space of X is $\mathbb{R}Y$ for a based G -space Y , $W_n X$ is $(I \vee Y)^n$, where I is given the basepoint 0 when forming the wedge.

Lemma 11.4. *As n -varies, the $W_n X$ define a subfunctor WX of $I \times X$.*

Proof. The claim is that the action map $\mathcal{D}(\mathbf{m}, \mathbf{n}) \wedge (I^m \times X_m) \rightarrow I^n \times X_n$ restricts to a map $\mathcal{D}(\mathbf{m}, \mathbf{n}) \wedge W_m X \rightarrow W_n X$. Let

$$(\phi; c) = (\phi; c_1, \dots, c_n) \in \mathcal{D}(\mathbf{m}, \mathbf{n}),$$

where $\phi: \mathbf{m} \rightarrow \mathbf{n}$ is a map in \mathcal{F} and c_j is an element of $\mathcal{C}(\phi_j)$, and let (s, x) be an element of $W_m X$. We must show that the element $(\phi; c)(s, x) = (t, (\phi; c)(x))$ of $I^n \times X_n$ is in $W_n X$. Let $\iota_s: \mathbf{p} \rightarrow \mathbf{m}$ be that ordered injection whose image is the set of i , $1 \leq i \leq m$, such that $s_i = 0$ and let $\iota_t: \mathbf{q} \rightarrow \mathbf{n}$ be that ordered injection whose image is the set of j , $1 \leq j \leq n$, such that $t_j = 0$. We are given that $x = \iota_s(x')$ for some $x' \in X_p$, and we must show that $(\phi; c)(x)$ is in $\iota_t(X_q)$. It suffices to show that there exists $(\zeta, d) \in \mathcal{D}(\mathbf{p}, \mathbf{q})$ such that

$$(\phi; c) \circ \iota_s = \iota_t \circ (\zeta; d).$$

Indeed, if such (ζ, d) exists, then

$$(\phi; c)(x) = (\phi; c)(\iota_s(x')) = \iota_t((\zeta; d)(x')),$$

as needed.

Since $t_j = \min_{\phi(i)=j} s_i$, we have that $i \in \text{im}(\iota_s)$ implies that $\phi(i) \in \text{im}(\iota_t)$. There is thus a unique map $\zeta: \mathbf{p} \rightarrow \mathbf{q}$ in \mathcal{F} such the $\iota_t \circ \zeta = \phi \circ \iota_s$. When $\mathcal{D} = \mathcal{F}$, we are done.

For $\mathcal{D} = \mathcal{D}(\mathcal{C}_G)$, the definition of \mathcal{D} gives that

$$(\phi, c) \circ \iota_s = (\phi \circ \iota_s; \bar{c})$$

where $\bar{c} = (\bar{c}_1, \dots, \bar{c}_m)$ for certain \bar{c}_j . Since $\phi \circ \iota_s = \iota_t \circ \zeta$, we have $\bar{c}_j = * \in \mathcal{C}_G(0)$ if $j \notin \text{im}(\iota_t)$. Let d be the q -tuple obtained by deleting from \bar{c} the entries corresponding to those $j \notin \text{im}(\iota_t)$. Using the definition of composition in \mathcal{D} one can check that

$$(\phi \circ \iota_s; \bar{c}) = \iota_t \circ (\zeta; d).$$

□

The following result completes the proof of [Proposition 11.1](#).

Lemma 11.5. *WX is in $\mathcal{D}[\mathcal{T}_G]$.*

Proof. We must show that if $\phi: \mathbf{m} \rightarrow \mathbf{n}$ is an injection in Π and Σ_ϕ is the group of permutations $\sigma: \mathbf{n} \rightarrow \mathbf{n}$ such that $\sigma\phi = \phi$, then $\phi: W_m X \rightarrow W_n X$ is a $(G \times \Sigma_\phi)$ -cofibration. We claim first that

$$\phi(W_m X) = \{(t, x) | t_j = 1 \text{ for all } j \notin \text{im}(\phi)\} \subset W_n X.$$

Since ϕ is an injection and $WX \subset I \times X$, we certainly have $t_j = 1$ if $j \notin \text{im}(\phi)$ for any point $(t, x) \in \phi(W_m X)$. Conversely, consider $(t, x) \in W_n X$ such that $t_j = 1$ for all $j \notin \text{im}(\phi)$. Let $\iota_t: \mathbf{p} \rightarrow \mathbf{n}$ be that ordered injection whose image is the set of j , $1 \leq j \leq n$, such that $t_j = 0$. Our conditions imply that $\text{im}(\iota_t) \subset \text{im}(\phi)$, hence there is a unique injection $\zeta: \mathbf{p} \rightarrow \mathbf{m}$ in Π such that $\iota_t = \phi \circ \zeta$. We have that $x \in \text{im}(\iota_t)$ and thus $x \in \text{im}(\phi)$ and $(t, x) \in \phi(W_m X)$. This proves the claim.

It follows readily from this description that $\phi: W_m X \rightarrow W_n X$ is a $G \times \Sigma_\phi$ -cofibration. To see this, notice that ϕ is induced from $\phi: I^m \rightarrow I^n$ of [\(11.3\)](#), which is a Σ_ϕ -cofibration as mentioned above, and the isomorphism $\phi: W_m X \rightarrow \phi(W_m X)$. □

12. APPENDIX: REALIZATION OF LEVELWISE G -COFIBRATIONS AND G -EQUIVALENCES

We here prove the following results, which generalize [Theorems 1.12](#) and [1.13](#) from finite groups to topological groups G . As before, we assume that the unit of G is a nondegenerate basepoint and we understand subgroups of G to be closed.

Theorem 12.1. *Let $f_*: X_* \rightarrow Y_*$ be a map of Reedy cofibrant simplicial G -spaces such that each f_n is a weak G -equivalence. Then the realization $|f_*|: |X_*| \rightarrow |Y_*|$ is a weak G -equivalence.*

Theorem 12.2. *Let $f_*: X_* \rightarrow Y_*$ be a map of Reedy cofibrant simplicial G -spaces such that each f_n is a G -cofibration. Then the realization $|f_*|: |X_*| \rightarrow |Y_*|$ is a G -cofibration.*

[Theorem 12.1](#) is an immediate consequence of the nonequivariant version and the following result, which is often taken for granted. Jonathan Rubin noticed this gap in the literature and provided the proof that follows.

Lemma 12.3. *Geometric realization of Reedy cofibrant simplicial G -spaces commutes with passage to H -fixed points for subgroups H of G .*

Proof. For a G -space X , X^H is the equalizer exhibited in the diagram

$$X^H \xrightarrow{i} X \xrightleftharpoons[(h)]{\Delta} \prod_{h \in H} X,$$

where (h) denotes multiplication by h on the h th component. When G is not finite, this is an infinite limit, and realization does not generally commute with infinite limits, not even with infinite products. Consider the following diagram, where X_* is a simplicial G -space, K is a space, and $f: K \rightarrow |X_*|$ is a map such that $\Delta \circ f = (h) \circ f$.

$$\begin{array}{ccccc} |X_*^H| & \xrightarrow{|i_*|} & |X_*| & \xrightleftharpoons[|(h)_*|]{|\Delta_*|} & |\prod_{h \in H} X_*| \\ & \swarrow \tilde{f} & \uparrow f & \searrow \Delta & \downarrow \pi \\ & & K & \xrightarrow{(h)} & \prod_{h \in H} |X_*| \end{array}$$

The top row is an equalizer and π is the natural map. The maps to products have components given by multiplication by h , and the two evident triangles clearly commute. Moreover, $|\Delta_*| \circ f = |(h)_*| \circ f$ by inspection of the unique representation of points $f(k) \in |X_*|$ in nondegenerate form [45, Lemma 11.3]. Since the top row is an equalizer, there is a unique map $\tilde{f}: K \rightarrow |X_*^H|$ such that $|i_*| \circ \tilde{f} = f$. Therefore $|i_*|$ is the equalizer of Δ and (h) , giving the conclusion. \square

To prove [Theorem 12.2](#), we will need the following two standard results about G -cofibrations. Recall that we are working in the category of compactly generated weak Hausdorff spaces, so that all cofibrations are closed inclusions. We use the convention of identifying the domain of a cofibration with its image.

Proposition 12.4. *Consider the following diagram in G -spaces.*

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & C & & \\ \downarrow i & \searrow \alpha & \downarrow & \searrow \gamma & \\ & A' & \xrightarrow{\quad} & C' & \\ & \downarrow i' & \downarrow & \downarrow & \\ B & \xrightarrow{\quad} & D & & \\ \downarrow \beta & & \downarrow & & \\ & B' & \xrightarrow{\quad} & D' & \end{array}$$

Assume that the front and back faces are pushouts, α , β , γ , i and i' are G -cofibrations, and $A' \cap B = A$ (as subsets of B'). Then the map $D \rightarrow D'$ is also a G -cofibration.

Proof. The nonequivariant result is [39, Proposition 2.5] and, as pointed out there, the equivariant proof goes through the same way. \square

The following result is given in [8, Proposition A.4.9] and [39, Lemma 3.2.(a)].

Proposition 12.5. *Let $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$ and $B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \dots$ be diagrams of G -cofibrations and let $f_i: A_i \rightarrow B_i$ be a map of diagrams such that*

each f_i is a G -cofibration. Assume moreover that for every $i \geq 1$, $A_{i-1} = A_i \cap B_{i-1}$. Then the induced map $\operatorname{colim}_i A_i \rightarrow \operatorname{colim}_i B_i$ is a G -cofibration.

The following observation is the key to using these results to prove [Theorem 1.13](#).

Lemma 12.6. *Let $f_*: X_* \rightarrow Y_*$ be a map of simplicial G -spaces that is levelwise injective. Then for all $n \geq 1$ and all $0 \leq i \leq n-1$,*

$$s_i(Y_{n-1}) \cap f_n(X_n) = f_n s_i(X_{n-1})$$

and

$$L_n Y \cap f_n(X_n) = f_n(L_n X).$$

Proof. For the first equality, one of the inclusions is obvious. For the other inclusion, take $y = s_i(y') = f_n(x)$. Then $y' = d_i s_i(y') = d_i f_n(x) = f_{n-1} d_i(x)$. Thus, $y = s_i f_{n-1} d_i(x) = f_n s_i d_i(x) \in f_n s_i(X_{n-1})$, as wanted. The second statement is obtained from the first by taking the union over all i . \square

Proposition 12.7. *Let X_* and Y_* be Reedy cofibrant simplicial G -spaces, and let $f_*: X_* \rightarrow Y_*$ be a level G -cofibration. Then for all $n \geq 1$, $L_n X \rightarrow L_n Y$ is a G -cofibration.*

Proof. We proceed by induction on n . Note that when $n = 1$, $L_1 X = s_0 X_0$ which is homeomorphic to X_0 , so there is nothing to prove. For $n > 1$, let $k \leq n-1$. Just as nonequivariantly, we have a pushout square

$$(12.8) \quad \begin{array}{ccc} s_k(\bigcup_{i=0}^{k-1} s_i(X_{n-2})) & \longrightarrow & \bigcup_{i=0}^{k-1} s_i(X_{n-1}) \\ \downarrow & & \downarrow \\ s_k(X_{n-1}) & \longrightarrow & \bigcup_{i=0}^k s_i(X_{n-1}). \end{array}$$

We show by induction on n and k that the right vertical map is a G -cofibration. By the inductive hypothesis for $n-1$ and the fact that X_* is Reedy cofibrant, we have that the composite

$$\bigcup_{i=0}^{k-1} s_i(X_{n-2}) \longrightarrow \bigcup_{i=0}^k s_i(X_{n-2}) \longrightarrow \dots \longrightarrow \bigcup_{i=0}^{n-2} s_i(X_{n-2}) = L_{n-1} X \longrightarrow X_{n-1}$$

is a G -cofibration. Since s_k is a G -homeomorphism onto its image, the left hand map in the square is a G -cofibration. Therefore, the righthand map in [\(12.8\)](#) is also a G -cofibration by cobase change.

We apply [Proposition 12.4](#) to the pushout in [\(12.8\)](#) to deduce that the maps

$$\bigcup_{i=0}^k s_i(X_{n-1}) \longrightarrow \bigcup_{i=0}^k s_i(Y_{n-1})$$

are G -cofibrations. In particular, by induction on n and k , $L_n X \rightarrow L_n Y$ is a G -cofibration. The required intersection condition follows from [Lemma 12.6](#). \square

Proof of [Theorem 12.2](#). Recall from [Section 8.1](#) that $|X_*|$ is the colimit of its filtration pieces $F_p |X|$, and that these can be built by the iterated pushout squares [\(8.2\)](#). Note that all of the vertical maps in diagram [\(8.2\)](#) are G -cofibrations. There is a similar diagram for $|Y_*|$ and a map of diagrams induced by f_* . The maps between the three corners of the upper squares are G -cofibrations. By [Proposition 12.4](#) and [Lemma 12.6](#), the map between the pushouts of the upper squares is a G -cofibration.

We assume by induction that the map $F_{p-1}|X| \rightarrow F_{p-1}|Y|$ is a G -cofibration, so again we have that all three maps between the corners of the lower pushout square for X and the one for Y are G -cofibrations. Lemma 12.6 implies again that the intersection condition necessary to apply Proposition 12.4 holds, and we can deduce by induction that the map $F_p|X| \rightarrow F_p|Y|$ is a G -cofibration.

Lastly, we check the intersection condition of Proposition 12.5, namely that $f(F_p|X|) \cap F_{p-1}|Y| = f(F_{p-1}|X|)$. One inclusion is obvious. To see the other inclusion, take an element $(u, f_p x)$ in $f(F_p|X|) \setminus f(F_{p-1}|X|) = (\Delta_p \setminus \partial\Delta_p) \times f_p(X_p \setminus L_p X)$. Since f_p is injective, we can again use Lemma 12.6 to see that $(u, f_p x) \in F_p Y \setminus F_{p-1} Y$. Thus an element in the intersection must be in $F_{p-1}|X|$. By Proposition 12.5, we conclude that the map

$$|f_*|: |X_*| \rightarrow |Y_*|$$

is a G -cofibration. □

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