THE HOMOTOPY THEORY OF EQUIVARIANT POSETS

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Abstract. Let G be a discrete group. We prove that the category of G-posets admits a model structure that is Quillen equivalent to the standard model structure on G-spaces. As is already true nonequivariantly, the three classes of maps defining the model structure are not well understood calculationally. To illustrate, we exhibit some examples of cofibrant and fibrant posets and an example of a non-cofibrant finite poset.

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1. Introduction

In [20], Thomason proved that categories model the homotopy theory of topological spaces by proving that the category **Cat** of (small) categories has a model structure that is Quillen equivalent to the standard model structure on the category **Top** of topological spaces. In [16], Raptis proved that the category of posets also models the homotopy theory of topological spaces by showing that the category **Pos** of posets has a model structure that is Quillen equivalent to the Thomason model structure on **Cat**. It is natural to expect this to hold since Thomason proved in [20, Proposition 5.7] that cofibrant categories in his model structure are posets. The first and third authors rediscovered this, observing that a geodesic proof, if not the statement, of that result is already contained in Thomason's paper. This implies that all of the algebraic topology of spaces can in principle be worked out in the category of posets. It can also be viewed as a bridge between the combinatorics of partial orders and algebraic topology.

In this paper we prove an analogous result for the category of G-spaces for a discrete group G. For a category $\mathcal{C}$, let $G\mathcal{C}$ denote the category of objects with a (left) action of G and maps that preserve the action. In [3], Bohmann, Mazur,

The second author was supported by SNSF grant 158932.
The third author was supported by an NSF MSRFP grant.

Osorno, Ozornova, Ponto, and Yarnall proved in precise analogy to Thomason's result that $G\mathbf{Cat}$ models the homotopy theory of G-spaces. Here we prove the pushout of the results of Raptis and Bohmann, et al: the category $G\mathbf{Pos}$ of G-posets admits a model structure that is Quillen equivalent to the model structure on the category $G\mathbf{Cat}$ of G-categories and therefore also Quillen equivalent to the model structures on $G\mathbf{sSet}$ and $G\mathbf{Top}$. Just as the model structure on $\mathbf{Pos}$ is implicit in Thomason's paper [20], we shall see that the model structure on $G\mathbf{Pos}$ is implicit in the six author paper [3].

While the background makes this an expected result, it is perhaps surprising, at least psychologically. There is relatively little general study of equivariant posets in either the combinatorial or topological literature, especially not from a homotopy theoretic perspective. One thinks of group actions as permutations, as exemplified by the symmetric groups, and it does not come naturally to think of a general theory of groups acting by order-preserving maps of posets. However, our theorem says that group actions on posets abound: every G-space is weakly equivalent to the classifying G-space of a G-poset, where a map f of G-spaces is a weak equivalence if its fixed point maps f^H are weak equivalences for all subgroups H of G. The result can be viewed as a formal bridge between equivariant combinatorics and equivariant algebraic topology.

The combinatorial literature seems to start with Stanley's paper [17], which restricts to finite posets and focuses on the connection with representation theory. A paper of Babson and Kozlov [1] about G-posets X focuses on problems arising from the fact that the orbit category X/G is generally not a poset. There is considerable group theory literature about posets of subgroups of G with G acting by conjugation, starting from Quillen's paper [15]. That led Thévenaz and Webb to an equivariant generalization of Quillen's Theorem A applicable to G-posets [19]. In turn, that led to Welker's paper [21], which considers the order G-complex associated to a G-poset, again with group theoretic applications in mind.

Let $\mathcal{O}_G$ denote the orbit category of G. Its objects are the G-sets G/H and its morphisms are the G-maps. Just as for G-spaces, G-simplicial sets (that is, simplicial G-sets), and G-categories, it is natural to start with the levelwise (or projective) model structure on the category $\mathcal{O}_G$-$\mathbf{Pos}$ of contravariant functors $\mathcal{O}_G \longrightarrow \mathbf{Pos}$. As a functor category, $\mathcal{O}_G$-$\mathbf{Pos}$ inherits a model structure from $\mathbf{Pos}$. Its fibrations and weak equivalences are defined levelwise. It is standard that this gives a compactly generated model structure (e.g. [10, 11.6.1]).[1]

Define the fixed point diagram functors

$$\Phi\colon G\mathbf{Pos} \longrightarrow \mathcal{O}_G\text{-}\mathbf{Pos} \quad \text{and} \quad \Phi\colon G\mathbf{Cat} \longrightarrow \mathcal{O}_G\text{-}\mathbf{Cat}$$

by

$$\Phi(X)(G/H) = X^H.$$

These functors Φ have left adjoints, denoted Λ; in both cases, Λ sends a contravariant functor Y defined on $\mathcal{O}_G$ to $Y(G/e)$.

[1] Compactly generated is a variant of cofibrantly generated that applies when only countable colimits are needed in the small object argument, that is, when transfinite colimits are unnecessary and irrelevant, as they are in all of the model structures we shall consider. This variant is discussed in detail in [13, §15.2]. It seems reasonable to eliminate transfinite verbiage whenever possible, and that would shorten and simplify some of the work in the sources we shall cite.

We prove that $G\mathbf{Pos}$ inherits a model structure from $\mathcal{O}_G$-$\mathbf{Pos}$. The analogue for $G\mathbf{Cat}$ is [3, Theorem A]. After recalling details of the model structures already cited, we shall prove the following theorem.

Theorem 1.1. *The functor Φ creates a compactly generated proper model structure on $G\mathbf{Pos}$, so that a map of G-posets is a weak equivalence or fibration if it is so after applying Φ. The adjunction (Λ, Φ) is a Quillen equivalence between $G\mathbf{Pos}$ and $\mathcal{O}_G$-$\mathbf{Pos}$.*

Replacing $\mathbf{Pos}$ with $\mathbf{Cat}$ in Theorem 1.1 gives the statement of [3, Theorem A]. The strategy of proof in [3] is to verify general conditions on a model category $\mathcal{C}$ that ensure that $G\mathcal{C}$ inherits a model structure from $\mathcal{O}_G$-$\mathcal{C}$.[2] The cited general conditions are taken from a paper of the second author [18]. Our proof of Theorem 1.1 will proceed in the same way. The following result is a formal consequence of Theorem 1.1 and its analogue for $\mathbf{Cat}$.

Theorem 1.2. *The adjunction (P, U) between $G\mathbf{Cat}$ and $G\mathbf{Pos}$ is a Quillen equivalence. Therefore, $G\mathbf{Pos}$ is Quillen equivalent to $G\mathbf{sSet}$ and $G\mathbf{Top}$.*

The following diagram displays the relevant equivariant Quillen equivalences.

$$\begin{array}{ccccccc}
G\mathbf{Top} & \underset{S_*}{\overset{|-|}{\leftrightarrows}} & G\mathbf{sSet} & \underset{\mathrm{Ex}^2 N}{\overset{\Pi\,\mathrm{Sd}^2}{\rightleftarrows}} & G\mathbf{Cat} & \underset{U}{\overset{P}{\rightleftarrows}} & G\mathbf{Pos} \\
\Lambda \uparrow\downarrow \Phi & & \Lambda \uparrow\downarrow \Phi & & \Lambda \uparrow\downarrow \Phi & & \Lambda \uparrow\downarrow \Phi \\
\mathcal{O}_G\text{-}\mathbf{Top} & \underset{S_*}{\overset{|-|}{\leftrightarrows}} & \mathcal{O}_G\text{-}\mathbf{sSet} & \underset{\mathrm{Ex}^2 N}{\overset{\Pi\,\mathrm{Sd}^2}{\rightleftarrows}} & \mathcal{O}_G\text{-}\mathbf{Cat} & \underset{U}{\overset{P}{\rightleftarrows}} & \mathcal{O}_G\text{-}\mathbf{Pos}
\end{array}$$

The definitions of Π, Sd, Ex, and N are recalled in the next section.

All of the vertical adjunctions and the adjunctions on the bottom row are Quillen equivalences, hence so are all of the adjunctions on the top row. Applied to the righthand square, this gives the proof of Theorem 1.2. Applied to the middle square, this gives [3, Theorem B], which is the equivariant version of Thomason's comparison between $\mathbf{sSet}$ and $\mathbf{Cat}$.

Remark 1.3. Both equivariantly and nonequivariantly, replacing $\mathbf{Cat}$ by $\mathbf{Pos}$ ties in the Thomason model structure to more classical algebraic topology. The composite $N \circ U \colon \mathbf{Pos} \longrightarrow \mathbf{sSet}$ coincides with the composite of the functor that sends a poset to its order complex and the canonical functor from ordered simplicial complexes to simplicial sets, and the same is true equivariantly. It also ties in the Thomason model structure to finite T_0-spaces and, more generally T_0-Alexandroff spaces, or A-spaces, since the categories of posets and A-spaces are isomorphic.

An interesting and unfortunate feature of all of the model structures discussed in this paper is that the classes of weak equivalences, cofibrations, and fibrations are defined formally, using non-constructive arguments. In no case do we have a combinatorially accessible description of any of these classes of maps. Even in the

[2] There are two slightly different ways to equip $G\mathcal{C}$ with a model structure, either transferring the model structure from $\mathcal{O}_G$-$\mathcal{C}$, as we shall do, or from copies of $\mathcal{C}$ via all of the fixed point functors, as in [3, 18].

case when G is trivial very little is known about the structure. In [6, Theorem 2.2.11], Cisinski gives a characterization of the subcategory of weak equivalences in **Cat** through a global characterization, but that does not allow us to determine whether or not a particular morphism is a weak equivalence.

The state of the art for fibrant and cofibrant objects is similarly sparse. The problem of determining the cofibrant posets has recently been studied by Bruckner and Pegel [4], who show in particular that every poset with at most five elements is cofibrant. In §6, we prove that all finite posets of dimension one are cofibrant and give an example of a six element poset that is *not* cofibrant.[3]

The problem of determining the fibrant categories has recently been studied by Meier and Ozornova [14]. In §7, we use work of Droz and the third author [8] to obtain a more concrete understanding of the posets that the main theorem of [14] shows to be fibrant.

Before turning to the equivariant generalizations, we review and reprove the nonequivariant theorems, giving some new details that streamline and clarify the key arguments.

2. Background

We recall as much as we need about the definitions of the nonequivariant versions of the functors in the diagram above and describe the relevant nonequivariant model structures. Of course, the nerve $N\mathcal{C}$ of a category $\mathcal{C}$ is the simplicial set with

$$(N\mathcal{C})_n = \big\{x_0 \longrightarrow \cdots \longrightarrow x_n \in \mathcal{C}\big\}.$$

Define $(\operatorname{Sd}\Delta)(n)$ to be the nerve of the poset of nonempty subsets of $\{0, 1, \cdots, n\}$. Then $\operatorname{Sd}\Delta$ is a covariant functor $\Delta \longrightarrow \mathbf{sSet}$. Let $K\colon \Delta^{op} \longrightarrow \mathbf{Set}$ be a simplicial set. The subdivision $\operatorname{Sd} K$ is the simplicial set defined conceptually as the tensor product of functors (given by the evident left Kan extension)

$$\operatorname{Sd} K = K \otimes_{\Delta} \operatorname{Sd}\Delta.$$

The functor Ex is the right adjoint of Sd; we will not need a description of it.

The fundamental category[4] ΠK has object set K_0 and morphism set freely generated by K_1, where $x \in K_1$ is viewed as a morphism $d_1 x \longrightarrow d_0 x$, subject to the relations

$$d_1 y = (d_0 y) \circ (d_2 y) \quad \text{for each } y \in K_2 \quad \text{and} \quad s_0 x = \mathrm{id}_x \quad \text{for each } x \in K_0.$$

The functor U is the full and faithful functor that sends a poset X to X regarded as the category with objects the elements of X and a morphism $x \longrightarrow y$ whenever $x \leq y$. The image of U consists of skeletal categories with at most one morphism $x \longrightarrow y$ for each pair of objects (x, y). The functor P sends a category $\mathcal{C}$ to the poset $P\mathcal{C}$ with points the equivalence classes $[c]$ of objects of $\mathcal{C}$, where $c \sim d$ if there are morphisms $c \longrightarrow d$ and $d \longrightarrow c$ in $\mathcal{C}$. The partial order $\leq$ is defined by $[c] \leq [d]$ if there is a morphism $c \longrightarrow d$ in $\mathcal{C}$, a condition independent of the choice of representatives in the equivalence classes. Note crucially that $P \circ U$ is the identity functor. We often drop the notation U, regarding posets as categories.

We recall the specification of the model structures that we are starting from.

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[3]Amusingly, when we found this example we did not know that it is the smallest possible one.

[4]Following [20], the functor Π is generally denoted c, or sometimes *cat*, in the literature.

Definition 2.1. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between (small) categories is a fibration or weak equivalence if $\mathrm{Ex}^2 NF$ is a fibration or weak equivalence. An order preserving function $f\colon X \longrightarrow Y$ between posets is a fibration or weak equivalence if Uf is a fibration or weak equivalence; that is, f is a fibration or weak equivalence if it is so when considered as a functor.

As noted by Thomason [20, Proposition 2.4], F is a weak equivalence if and only if NF is a weak equivalence.

Notation 2.2. Let $\mathcal{I}$ denote the set of generating cofibrations $\partial\Delta[n] \longrightarrow \Delta[n]$ and let $\mathcal{J}$ denote the set of generating acyclic cofibrations $\Lambda^k[n] \longrightarrow \Delta[n]$ for the standard model structure on **sSet**.

Theorem 2.3 (Thomason)**.** *With these fibrations and weak equivalences,* **Cat** *is a compactly generated proper model category whose sets of generating cofibrations and generating acyclic cofibrations are* $\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ *and* $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$*. Via the adjunction* $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2 N)$*, this model structure is Quillen equivalent to the standard model structure on* **sSet**.

Remark 2.4. In contrast to more recent papers, which use but do not always need transfinite colimits, Thomason's paper preceded the formal introduction of cofibrantly generated model categories, and he neither used nor needed such colimits; our statement is a reformulation of what he actually proved.

Theorem 2.5 (Raptis)**.** *With these fibrations and weak equivalences,* **Pos** *is a compactly generated proper model category whose sets of generating cofibrations and generating acyclic cofibrations are* $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ *and* $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$*. Via the adjunction* (P,U)*, this model structure is Quillen equivalent to the Thomason model structure on* **Cat**.

3. The proofs of Theorems 2.3 and 2.5

The proofs of the model axioms in [20, 16] can be streamlined by use of a slight variant of Kan's transport theorem [10, Theorem 11.3.2]. It is proven in [13, 16.2.5].

Theorem 3.1 (Kan)**.** *Let* $\mathcal{C}$ *be a compactly generated model category with generating cofibrations* $\mathcal{I}$ *and generating acyclic cofibrations* $\mathcal{J}$*. Let* $\mathcal{D}$ *be a bicomplete category, and let* $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ *be a pair of adjoint functors. Assume that*

(i) *all objects in the sets* $F\mathcal{I}$ *and* $F\mathcal{J}$ *are compact and*
(ii) *the functor* U *takes relative* $F\mathcal{J}$*-cell complexes to weak equivalences.*

Then there is a compactly generated model structure on $\mathcal{D}$ *such that* $F\mathcal{I}$ *is the set of generating cofibrations,* $F\mathcal{J}$ *is the set of generating acyclic cofibrations, and the weak equivalences and fibrations are the morphisms* f *such that* Uf *is a weak equivalence or fibration. Moreover,* $(F \dashv U)$ *is a Quillen pair.*

Remark 3.2. It is clear that if $\mathcal{C}$ is right proper then so is $\mathcal{D}$. Since the standard model structure on **sSet** is right proper, so are the model structures on **Cat** and **Pos** described below. It is less clear that they are left proper, as we shall discuss.

Compactly generated makes sense when the generating sets are compact in the sense of [13, 15.1.6], as we require in condition (i). In Theorem 2.5, the domain posets of all maps in $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ and $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$ are finite since they are obtained from simplicial sets with only finitely many 0-simplices. Therefore they are compact relative to all of **Cat** and in particular are compact relative to $P\Pi\,\mathrm{Sd}^2\,\mathcal{I}$ and $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$. This shows that (i) holds, and we need only prove (ii) to complete the proof of the model axioms in Theorem 2.5.

Since we are working with compact generation, a relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex $i\colon A \longrightarrow X = \operatorname{colim} X_n$ is the colimit of a sequence of maps of posets $X_n \longrightarrow X_{n+1}$, where $X_0 = A$ and X_{n+1} is a pushout

$$\begin{array}{ccc} K_n & \xrightarrow{f} & X_n \\ {\scriptstyle j}\downarrow & & \downarrow \\ L_n & \longrightarrow & X_{n+1} \end{array} \tag{3.3}$$

in **Pos** in which j is a coproduct of maps in $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$. We must prove that such a map i, or rather Ui, is a weak equivalence in **Cat**. The only subtlety in the proof of Theorem 2.5 is that pushouts in **Cat** between maps in **Pos** are generally not posets. Rather, pushouts in **Pos** are constructed by taking pushouts in **Cat** and then applying the left adjoint P. However, results already in [20] show that we do not encounter that problem when constructing relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complexes, as we now explain.

To deal with pushouts when proving Theorem 2.3, Thomason introduced the notion of a Dwyer map.

Definition 3.4. Let $\mathcal{S}$ be a subcategory of a category $\mathcal{C}$. Then $\mathcal{S}$ is called a *sieve in* $\mathcal{C}$ if for every morphism $f\colon c \longrightarrow s$ in $\mathcal{C}$ with $s \in \mathcal{S}$, c and f are in $\mathcal{S}$. Dually, $\mathcal{S}$ is a *cosieve* if for every morphism $f\colon s \longrightarrow c$ in $\mathcal{C}$ with $s \in \mathcal{S}$, c and f are in $\mathcal{S}$. In either case, $\mathcal{S}$ must be a full subcategory of $\mathcal{C}$. Observe that if a sieve factors as a composite of inclusions $\mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{C}$, then $\mathcal{S} \longrightarrow \mathcal{T}$ is again a sieve.

Definition 3.5. A functor $k\colon \mathcal{S} \longrightarrow \mathcal{C}$ in **Cat** or in **Pos** is a Dwyer map if k is the inclusion of a sieve and k factors as a composite

$$\mathcal{S} \xrightarrow{i} \mathcal{T} \xrightarrow{j} \mathcal{C},$$

where j is the inclusion of a cosieve and i is an inclusion with a right adjoint $r\colon \mathcal{T} \longrightarrow \mathcal{S}$ such that the unit $\mathrm{id} \longrightarrow r \circ i$ of the adjunction is the identity.

The following sequence of results shows that Theorem 2.5 is directly implied by details in Thomason's paper [20] that he used to prove Theorem 2.3. Except that we add in the trivial statement about coproducts, the first is [20, Lemma 5.6].

Lemma 3.6. *The following statements about posets hold.*

- *(i) For any simplicial set K, $\Pi\,\mathrm{Sd}^2\,K$ is a poset.*
- *(ii) Any subcategory of a poset is a poset.*
- *(iii) Any coproduct of posets in* **Cat** *is a poset.*
- *(iv) If $j\colon K \longrightarrow L$ is a Dwyer map between posets and $f\colon K \longrightarrow X$ is a map of posets, then the pushout Y in* **Cat** *of j and f is a poset.*
- *(v) The (directed) colimit in* **Cat** *of any sequence of maps of posets is a poset.*

The second is [20, Proposition 4.2].

Lemma 3.7. *Let $K \subset L$ be an inclusion of simplicial sets that arises from an inclusion of ordered simplicial complexes. Then the induced map $\Pi\,\mathrm{Sd}^2 K \longrightarrow \Pi\,\mathrm{Sd}^2 L$ is a Dwyer map in* **Cat** *and therefore, by Lemma 3.6(i), in* **Pos**.

For completeness, we state an analogue to Lemma 3.6 about Dwyer maps in **Cat**. It combines part of [20, Proposition 4.3] with the correct parts of [20, Lemma 5.3]. We again add in a trivial statement about coproducts.

Lemma 3.8. *The following statements about Dwyer maps in* **Cat** *hold.*

(i) Any composite of Dwyer maps is a Dwyer map.
(ii) Any coproduct of Dwyer maps is a Dwyer map.
(iii) If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer map and $f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a functor, then the pushout $k\colon \mathcal{C} \longrightarrow \mathcal{D}$ of j along f is a Dwyer map.
(iv) For Dwyer maps $\mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$, $n \geq 0$, the induced map $\mathcal{C}_0 \longrightarrow \operatorname{colim} \mathcal{C}_n$ is a Dwyer map.

Therefore the same statements hold for Dwyer maps in **Pos**.

Corollary 3.9. *If A is a poset and $i\colon A \longrightarrow X$ is a relative $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in* **Cat**, *then X is a poset and i is both a Dwyer map and a relative $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in* **Pos**. *The same statement holds for relative $\Pi\,\mathrm{Sd}^2\,\mathcal{I}$-complexes.*

Remark 3.10. Once the model structures on **Pos** and **Cat** are in place, the results above imply that a map f between posets is a cofibration in **Pos** if and only if f is a cofibration in **Cat**.

The real force of the introduction of Dwyer maps comes from the following result. It combines Thomason's [20, Proposition 4.3 and Corollary 4.4].

Proposition 3.11. *If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer map in* **Cat**, *$f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a functor, and $\mathcal{D}$ is their pushout, then the canonical map*

$$N\mathcal{L} \cup_{N\mathcal{K}} N\mathcal{C} \longrightarrow N(\mathcal{L} \cup_{\mathcal{K}} \mathcal{C}) = N\mathcal{D}$$

is a weak equivalence. The same statement holds in **Pos**. *Therefore, if f is a weak equivalence, then so is the pushout $g\colon \mathcal{L} \longrightarrow \mathcal{D}$ of f along j.*

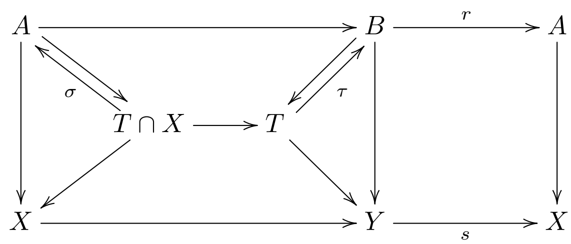
The last statement is inherited from the corresponding statement in **sSet**.

Remark 3.12. The incorrect part of [20, Lemma 5.3] states that a retract of a Dwyer map is a Dwyer map. As noticed by Cisinski [5], that is not true. He gave an example to show that a retract of a cofibration in **Cat** need not be a Dwyer map, which invalidates the proof that **Cat** is left proper given in [20, Corollary 5.5]. He introduced the slightly more general notion of a pseudo Dwyer map to get around this. He proved that a retract of a pseudo Dwyer map is a pseudo Dwyer map, so that any cofibration in **Cat** is a pseudo Dwyer map. He then used that to give a correct proof that **Cat** is left proper, and he observed that our Lemmas 3.7 and 3.8 remain true with Dwyer maps replaced by pseudo Dwyer maps.

The problem discussed in the remark does not arise when dealing with **Pos**, where Dwyer maps and pseudo Dwyer maps coincide, as follows directly from the definition of the latter. Since we are omitting that definition, we give a simple direct proof of the following result. Once the model structure is in place, it gives that cofibrations in **Pos** are Dwyer maps. This highlights the technical convenience of posets, as compared with general categories.

Lemma 3.13. *A retract of a Dwyer map in* **Pos** *is a Dwyer map. Therefore retracts in* **Pos** *of relative* $\Pi\,\mathrm{Sd}^2\,\mathfrak{I}$*-complexes are Dwyer maps.*

Proof. Consider the following diagram of posets, which commutes with σ and τ omitted. All unlabeled arrows are inclusions.



We assume that r restricts to the identity on A and s restricts to the identity on X. We also assume that $B \longrightarrow Y$ is a sieve, $T \longrightarrow Y$ is a cosieve, and τ is right adjoint to the inclusion $B \longrightarrow T$ with unit the identity, so that τ restricts to the identity on B. We define σ to be the restriction of $r \circ \tau$ to $T \cap X$. The following observations prove that $A \longrightarrow X$ is a Dwyer map.
(i) The restriction $T \cap X \longrightarrow X$ of the cosieve $T \longrightarrow Y$ is again a cosieve.
Proof. If $w \in T \cap X$ and $w \leq x$ in X, then $x \in T$, hence $x \in T \cap X$.
(ii) The restriction $A \longrightarrow X$ of the sieve $B \longrightarrow Y$ is again a sieve.
Proof. If $a \in A$, $x \in X$, and $x \leq a$, then $x \in B$ since $B \longrightarrow Y$ is a sieve, and then $x = s(x) = r(x) \leq r(a) = a$ in A.
(iii) σ is right adjoint to the inclusion $A \longrightarrow T \cap X$, with unit the identity map.
Proof. σ restricts to the identity on A since if $a \in A$, then

$$\sigma(a) = (r \circ \tau)(a) = r(a) = a.$$

For the adjunction, we must show that if $a \in A$ and $x \in T \cap X$, then $a \leq x$ if and only if $a \leq \sigma(x)$. If $a \leq x$, then $a = \sigma(a) \leq \sigma(x)$. Suppose $a \leq \sigma(x)$ and note that $\sigma(x) = (r \circ \tau)(x) = (s \circ \tau)(x)$. Since τ is right adjoint to $B \longrightarrow T$, the counit of the adjunction gives that $\tau(y) \leq y$ for any $y \in T$. Thus $(s \circ \tau)(x) \leq s(x) = x$. $\square$

Proof of Theorems 2.3 and 2.5. The heart of Thomason's proof of Theorem 2.3 is the verification of condition (ii) of Theorem 3.1. Since coproducts and colimits of weak equivalences are weak equivalences, this reduces to showing that the pushouts in the construction of relative $\mathcal{J}$-complexes are weak equivalences. But that is immediate from Proposition 3.11. Since a relative $P\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in **Pos** is a special case of a relative $\Pi\,\mathrm{Sd}^2\,\mathcal{J}$-complex in **Cat**, condition (ii) of Theorem 3.1 holds in **Pos** since it is a special case of the condition in **Cat**. This proves that **Cat** and **Pos** are compactly generated model categories. In view of Lemma 3.13, Proposition 3.11 also implies that **Pos** is left proper and therefore proper. As pointed out in Remark 3.12, Cisinski [5] proves that **Cat** is left proper and therefore proper.

It remains to show that the adjunctions $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2\,N)$ and (P, U) are Quillen equivalences. To show that $(\Pi\,\mathrm{Sd}^2, \mathrm{Ex}^2\,N)$ is a Quillen equivalence, it suffices to show that the composite $\mathrm{Ex}^2\,N$ induces an equivalence between the homotopy categories of **Cat** and **sSet**. Quillen [11, Ch. VI, Corollaire 3.3.1] proved that the nerve N induces an equivalence. Kan [9, Ch. III, Theorem 4.6] proved that Ex and therefore Ex^2 induces an equivalence by showing that there is a natural weak equivalence $K \longrightarrow \mathrm{Ex}\,K$ for simplicial sets K.

To show that (P, U) is a Quillen equivalence, it suffices to show that for all cofibrant categories $\mathcal{C} \in \mathbf{Cat}$ and all fibrant posets $X \in \mathbf{Pos}$, a functor $f\colon \mathcal{C} \longrightarrow UX$ is a weak equivalence if and only if its adjunct $\widetilde{f}\colon P\mathcal{C} \longrightarrow X$ is a weak equivalence. Since $\mathcal{C}$ is cofibrant, it is a poset, hence $\mathcal{C} = UY$ for a poset Y. But then $U\widetilde{f} = f$ and the conclusion holds by the definition of weak equivalences in **Pos**. □

Remark 3.14. The fact that $\Pi \operatorname{Sd}^2 K$ is a poset for any simplicial set K is closely related to the less well-known fact that $\operatorname{Sd}^2 \mathcal{C}$ is a poset for any category $\mathcal{C}$. However, the subdivision functor on **Cat** plays no role in Thomason's work or ours. The relation between these subdivision functors is studied in [7] and [12].

4. Equivariant Dwyer maps and cofibrations

To mimic the arguments just given equivariantly, we introduce equivariant Dwyer maps and relate them to cofibrations in **Pos**.

Definition 4.1. A functor $k\colon \mathcal{S} \longrightarrow \mathcal{C}$ in $G\mathbf{Cat}$ or in $G\mathbf{Pos}$ is a Dwyer G-map if k is the inclusion of a sieve and k factors in $G\mathbf{Cat}$ as a composite

$$\mathcal{S} \xrightarrow{\ i\ } \mathcal{T} \xrightarrow{\ j\ } \mathcal{C},$$

where j is the inclusion of a cosieve and i is an inclusion with a right adjoint $r\colon \mathcal{T} \longrightarrow \mathcal{S}$ in $G\mathbf{Cat}$ such that the unit $\mathrm{id} \longrightarrow r \circ i$ of the adjunction is the identity.[5]

The following two lemmas are immediate from the definition.

Lemma 4.2. *If k is a Dwyer G-map, then k^H is a Dwyer map for any subgroup H of G.*

Regard the G-set G/H as a discrete G-category (identity morphisms only).

Lemma 4.3. *If $j\colon K \subset L$ is a Dwyer map and H is a subgroup of G, then $\mathrm{id} \times j\colon G/H \times K \longrightarrow G/H \times L$ is a Dwyer G-map.*

We have the equivariant analogues of Lemma 3.8 and Corollary 3.9, with the same proofs.

Lemma 4.4. *The following statements about Dwyer G-maps in $G\mathbf{Cat}$ hold.*

(i) Any composite of Dwyer G-maps is a Dwyer G-map.
(ii) Any coproduct of Dwyer G-maps is a Dwyer G-map.
(iii) If $j\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a Dwyer G-map and $f\colon \mathcal{K} \longrightarrow \mathcal{C}$ is a G-map, then the pushout $k\colon \mathcal{C} \longrightarrow \mathcal{D}$ of j along f is a Dwyer G-map.
(iv) For Dwyer G-maps $\mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$, $n \geq 0$, the induced map $\mathcal{C}_0 \longrightarrow \operatorname{colim} \mathcal{C}_n$ is a Dwyer G-map.

Therefore the same statements hold for Dwyer G-maps in $G\mathbf{Pos}$.

Let $G\Pi \operatorname{Sd}^2 \mathcal{I}$ and $G\Pi \operatorname{Sd}^2 \mathcal{J}$ denote the sets of all G-maps that are of the form $\mathrm{id} \times j\colon G/H \times K \longrightarrow G/H \times L$, where j is in $\Pi \operatorname{Sd}^2 \mathcal{I}$ or $\Pi \operatorname{Sd}^2 \mathcal{J}$. These are the generating cofibrations and generating acyclic cofibrations in $G\mathbf{Cat}$.

Corollary 4.5. *If A is a G-poset and $i\colon A \longrightarrow X$ is a relative $G\Pi \operatorname{Sd}^2 \mathcal{J}$-complex in $G\mathbf{Cat}$, then X is a G-poset and i is both a Dwyer G-map and a relative $G\Pi \operatorname{Sd}^2 \mathcal{J}$-complex in $G\mathbf{Pos}$. The same statement holds for relative $G\Pi \operatorname{Sd}^2 \mathcal{I}$-complexes.*

---

[5] Since the unit is the identity, the pair (i, r) is automatically an adjunction in the 2-category of G-objects in **Cat**, equivariant functors, and equivariant natural transformations.

We also have the equivariant analogue of Lemma 3.13.

Lemma 4.6. *A retract of a Dwyer G-map in G**Pos** is a Dwyer G-map. Therefore all cofibrations in G**Pos** are Dwyer G-maps.*

We require a description of pushouts inside G**Pos**. The following is a simplification of [3, Lemma 2.5].

Lemma 4.7. *Let $j\colon K \longrightarrow L$ be a sieve of G-posets and $f\colon K \longrightarrow X$ be a map of G-posets. Consider the set $Y = (L \setminus K) \amalg X$ with the order relation given by restriction on $L \setminus K$ and on X, with the additional relation that for $x \in X$ and $y \in L \setminus K$, $x \leq y$ if there exists $w \in K$ such that $x \leq f(w)$ and $j(w) \leq y$. Then Y is a G-poset and the following diagram is a pushout in G**Pos**, where k is the inclusion of the summand X and g is the sum of f on K and the identity on $L \setminus K$.*

(4.8)
$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ L & \xrightarrow[g]{} & Y \end{array}$$

Moreover, if j is a Dwyer map with factorization $K \xrightarrow{\iota} S \xrightarrow{\nu} L$ and retraction $r\colon S \longrightarrow K$, then for $x \in X$ and $y \in L \setminus K$, $x \leq y$ if and only if $y = \nu(z)$ for some $z \in S$ such that $x \leq (f \circ r)(z)$.

Proof. First, note that Y is well-defined, since $L \setminus K$ is a G-subposet of L. Indeed, if $y \in L \setminus K$ and $gy \in K$ then $y = g^{-1}gy \in K$, a contradiction. The relation $\leq$ on Y is reflexive and anti-symmetric since L and X are posets. Transitivity requires a straightforward verification in the two non-trivial cases when $x \leq y$ and $y \leq z$ with either $x, y \in X$ and $z \in L \setminus K$ or $x \in X$ and $y, z \in L$. Thus Y is a poset.

Clearly the map k is order-preserving. Using that j is a sieve, we see that g is order-preserving by the definition of the order on Y. The square (4.8) is clearly a pushout of sets. Thus to show that it is a pushout of posets it suffices to show that for any commutative square

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \ell} \\ L & \xrightarrow[h]{} & Z \end{array}$$

of posets, the induced map $Y \longrightarrow Z$ is order-preserving. The only case that is non-trivial to check is when $x \leq y$ with $x \in X$ and $y \in L \setminus K$. We must show that $\ell(x) \leq h(y)$. By assumption, there is an element $w \in K$ such that $x \leq f(w)$ and $j(w) \leq y$. It follows that

$$\ell(x) \leq (\ell \circ f)(w) = (h \circ j)(w) \leq h(y),$$

as desired.

For the last statement of the lemma, if $y = \nu(z)$ where $z \in S$ and $x \leq (f \circ r)(z)$, let $w = r(z)$. Then $x \leq f(w)$ and $j(w) = (\nu \circ \iota \circ r)(z) \leq \nu(z) = y$ by the counit of the adjunction (ι, r). Conversely, let $j(w) \leq y$ and $x \leq f(w)$. Since ν is a cosieve, $j(w) = (\nu \circ \iota)(w) \leq y$ implies $y = \nu(z)$ for some $z \in S$ with $\iota(w) \leq z$, and then $w = (r \circ \iota)(w) \leq r(z)$ so that $x \leq f(w)$ implies $x \leq (f \circ r \circ \iota)(w) \leq (f \circ r)(z)$. $\square$

Using this description we can show that pushouts along Dwyer G-maps are preserved when taking H-fixed points for any subgroup H of G. The statement about fixed points is a modification of [3, Proposition 2.4].

Lemma 4.9. *Let $j\colon K \longrightarrow L$ be a Dwyer G-map of G-posets, such as a retract of a relative $G\amalg \operatorname{Sd}^2 \mathcal{I}$-cell complex, and let $f\colon K \longrightarrow X$ be any map of G-posets. Form the pushout diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow \\ L & \longrightarrow & Y \end{array}$$

*in G**Cat**. Then Y is a G-poset and the diagram remains a pushout after taking H-fixed points for any subgroup H of G.*

Proof. Ignoring the G-action, the left vertical arrow is a Dwyer map of posets. Therefore Y is a poset by Lemma 3.6(iv) and is thus a G-poset. Fix a subgroup H of G; by Lemma 4.2 j^H is a Dwyer map, and thus the description from Lemma 4.7 can be used for $X^H \cup_{K^H} L^H$. □

Example 4.10. Let G be the cyclic group of order two. Let L be the three object G-poset depicted by $0 \longrightarrow 2 \longleftarrow 1$ equipped with the action that interchanges 0 and 1, but fixes 2. Let K be the G-subposet that consists of the elements 0 and 1. Then the inclusion $K \longrightarrow L$ is a sieve but *not* a Dwyer G-map. If $X = *$ is the terminal G-poset and $K \longrightarrow X$ is the unique map, then the pushout $L \cup_K X$ in G**Pos** is the G-poset depicted by $* \longrightarrow 2$, with trivial G-action. Thus its G-fixed point poset is also $* \longrightarrow 2$. However, the pushout $L^G \amalg_{K^G} X^G$ is the discrete poset with two elements $*$ and 2.

5. The proof of Theorem 1.1

For our equivariant model structures, we start with the following general result, which puts together results of the second author [18, Proposition 2.6, Theorem 2.10] with augmentations of those results due to Bohmann, et al [3, Propositions 1.4, 1.5, and 1.6], all reformulated in our simpler compactly generated setting. Recall that $\mathcal{O}_G$ denotes the orbit category of G.

Definition 5.1. For a category $\mathcal{C}$, let $G\mathcal{C}$ denote the category of G-objects in $\mathcal{C}$ and let $\mathcal{O}_G$-$\mathcal{C}$ denote the category of contravariant functors $\mathcal{O}_G \longrightarrow \mathcal{C}$. Assuming that $\mathcal{C}$ has coproducts, define a functor

$$\otimes\colon G\mathbf{Set} \times \mathcal{C} \longrightarrow G\mathcal{C}$$

by $S \otimes X = \amalg_S X$, the coproduct of copies of X indexed by elements of S, with G-action induced from the action of G on S by permutation of the copies of X.

We have an adjunction (Λ, Φ) between $G\mathcal{C}$ and $\mathcal{O}_G$-$\mathcal{C}$. The left adjoint Λ sends a functor $\mathcal{O}_G \longrightarrow \mathcal{C}$ to its value on G/e and the right adjoint Φ sends a G-object to its fixed point functor.

Theorem 5.2. *Let $\mathcal{C}$ be a compactly generated model category. Assume that for each subgroup H of G, the H-fixed point functor $(-)^H\colon G\mathcal{C} \longrightarrow \mathcal{C}$ satisfies the following properties.*

(i) It preserves colimits of sequences of maps $i_n\colon X_n \longrightarrow X_{n+1}$ *in* $G\mathcal{C}$*, where each* i_n *is a cofibration in* $\mathcal{C}$*.*
(ii) It preserves coproducts.
(iii) It preserves pushouts of diagrams in which one leg is given by a coproduct of maps of the form

$$\mathrm{id} \otimes j\colon G/J \otimes X \longrightarrow G/J \otimes Y,$$

where j *is a generating cofibration (or generating acyclic cofibration) of* $\mathcal{C}$ *and* J *is a subgroup of* G*.*[6]
(iv) For any object X *of* $\mathcal{C}$*, the natural map*

$$(G/J)^H \otimes X \longrightarrow (G/J \otimes X)^H$$

is an isomorphism in $\mathcal{C}$*.*

Then $G\mathcal{C}$ *admits a compactly generated model structure, where a map* f *in* $G\mathcal{C}$ *is a fibration or weak equivalence if each fixed point map* f^H *is a fibration or weak equivalence, so that* $\Phi(f)$ *is a fibration or weak equivalence in* $\mathcal{O}_G$*-*$\mathcal{C}$*. The generating (acyclic) cofibrations are the* G*-maps* $\mathrm{id} \otimes j\colon G/J \otimes K \longrightarrow G/J \otimes L$*, where the maps* $j\colon K \longrightarrow L$ *are the generating (acyclic) cofibrations of* $\mathcal{C}$*. Moreover,* (Λ, Φ) *is then a Quillen equivalence between* $G\mathcal{C}$ *and* $\mathcal{O}_G$*-*$\mathcal{C}$*. Further, if* $\mathcal{C}$ *is left or right proper, then so is* $G\mathcal{C}$*.*

By [3, 1.3], the model structure is functorial with respect to Quillen pairs.

Theorem 5.3. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be compactly generated model categories satisfying the assumptions of Theorem 5.2 and let* (L, R) *be a Quillen pair between them. Then there is an induced Quillen pair between* $G\mathcal{C}$ *and* $G\mathcal{D}$*, and it is a Quillen equivalence if* (L, R) *is a Quillen equivalence.*

Proof of Theorem 1.1. We need only verify conditions (i)-(iv) of Theorem 5.2 when $\mathcal{C} = \mathbf{Pos}$. Cofibrations in **Pos** are inclusions and if $x \in X = \operatorname{colim} X_n$, then $x \in X^H$ if and only if $x \in X_n^H$ for a large enough n; thus condition (i) holds. Condition (ii) holds by the definition of coproducts in **Cat**. Since the action of G on $G/J \otimes X$ comes from the action of G on G/J, condition (iv) holds as well.

It remains to check condition (iii). By Lemma 3.7, the generating (acyclic) cofibrations in **Pos** are Dwyer maps. Consider a pushout diagram in $G\mathbf{Cat}$

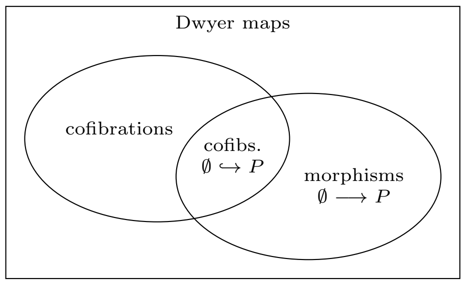
$$\begin{array}{ccc} \coprod_{i\in I} G/J_i \otimes K_i & \xrightarrow{\ \amalg f_i\ } & X \\ {\scriptstyle \amalg \mathrm{id} \otimes j_i}\downarrow & & \downarrow \\ \coprod_{i\in I} G/J_i \otimes L_i & \longrightarrow & Y \end{array}$$

where each $j_i : K_i \longrightarrow L_i$ is a Dwyer map and $f_i : G/J_i \otimes K_i \longrightarrow X$ is a map of G-posets. Condition (iii) holds if, for any such diagram, Y is a G-poset (hence Y^H is also a poset) and the diagram remains a pushout after passage to H-fixed points. This is a special case of Lemma 4.9. □

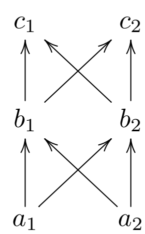
[6] We don't need to assume the condition for acyclic cofibrations, but we do so for convenience.

6. Cofibrant posets

Since every cofibrant object in **Cat** is a poset and, by Remark 3.10, a poset is cofibrant in **Pos** if and only if it is cofibrant in **Cat**, it follows that **Pos** and **Cat** have the same cofibrant objects. We have an explicit cofibrant replacement functor for **Pos**, namely double subdivision. While this does give a large class of cofibrant objects, it does not help to determine whether or not a given poset is cofibrant. By Lemma 3.13, any cofibration in **Pos** is a Dwyer map and it follows immediately from the definition of Dwyer maps that the map $\emptyset \longrightarrow P$ is a Dwyer map for any poset P. Our understanding is summarized in the following picture:



It is not difficult to show that most of the sections in this Venn diagram are nonempty; the only difficulty is to show that there exist morphisms $\emptyset \longrightarrow P$ which are not cofibrations. As the referee pointed out to us, it is not hard to find infinite posets that are not cofibrant, such as the natural numbers with its reverse ordering. However, as far as we know ours is the first example of a finite poset that is not cofibrant. Specifically, in Proposition 6.2 we show that the following model of the 2-sphere, which is a finite poset A whose classifying space is homeomorphic to S^2, is not cofibrant in **Pos**.



This example of a finite, non-cofibrant poset is minimal in dimension and in cardinality. We prove in Proposition 6.5 that every one-dimensional finite poset is cofibrant, and Bruckner and Pegel [4] have shown that every poset with at most five elements is cofibrant.

We first give a tool for showing that posets are not cofibrant.

Lemma 6.1. *Let A be a nonempty finite poset. Suppose that A satisfies the following condition: for any pushout square*

$$\begin{array}{ccc} \Pi\,\mathrm{Sd}^2\,\partial\Delta[n] & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \Pi\,\mathrm{Sd}^2\,\Delta[n] & \longrightarrow & Y \end{array}$$

if A is a retract of Y then it is also a retract of X. Then A is not cofibrant in **Pos**.

Proof. Assume that A is cofibrant. We prove that A must be empty, a contradiction. Since **Pos** is compactly generated and A is cofibrant, A is a retract of a sequential colimit $\operatorname{colim}_n X_n$, where $X_0 = \emptyset$ and $X_i \longrightarrow X_{i+1}$ is a pushout of a coproduct of generating cofibrations for $i \geq 0$. Since A is finite, the inclusion $A \longrightarrow \operatorname{colim}_n X_n$ factors through some X_n, and then A is a retract of X_n. Assume $n > 0$. Since A is finite, the inclusion $A \longrightarrow X_n$ factors through a pushout Y_n obtained by attaching only finitely many generating cofibrations to X_{n-1}, and then A is a retract of Y_n. We can now use the assumed condition on A to induct downwards one generating cofibration at a time; our condition ensures that A is a retract of X_{n-1}. Iterating, we deduce that A is a retract of $X_0 = \emptyset$ and thus $A = \emptyset$. $\square$

We will also need the following explicit description of the generating cofibrations

$$\Pi \operatorname{Sd}^2 \partial\Delta[n] \longrightarrow \Pi \operatorname{Sd}^2 \Delta[n].$$

An element of the poset $\Pi \operatorname{Sd}^2 \Delta[n]$ is a sequence of strict inclusions

$$S_0 \subset \ldots \subset S_k$$

of nonempty subsets of $\mathbf{n} = \{0, \ldots, n\}$. We can identify such a sequence with the totally ordered set $\{S_0, \ldots, S_k\}$. With this identification the order relation on $\Pi \operatorname{Sd}^2 \Delta[n]$ is given by subset inclusion. The poset $\Pi \operatorname{Sd}^2 \partial\Delta[n]$ is the subposet of $\Pi \operatorname{Sd}^2 \Delta[n]$ given by the sequences $S_0 \subset \ldots \subset S_k$ with $S_k \neq \mathbf{n}$.

We are now ready to show that our model A of the 2-sphere is not cofibrant.

Proposition 6.2. *The finite poset A is not cofibrant in* **Pos**.

Proof. We will show that A satisfies the condition in Lemma 6.1; since A is nonempty, this implies that A is not cofibrant.

Let Y be the pushout of a diagram of the form

$$\Pi \operatorname{Sd}^2 \Delta[n] \longleftarrow \Pi \operatorname{Sd}^2 \partial\Delta[n] \longrightarrow X,$$

where X is any poset. We use the explicit description of the pushout from Lemma 4.7. Suppose that A is a retract of Y, so that id_A admits a factorization $A \xrightarrow{i} Y \xrightarrow{r} A$.

Consider the map $(\Pi \operatorname{Sd}^2 \Delta[n]) \setminus \{\mathbf{n}\} \longrightarrow \Pi \operatorname{Sd}^2 \partial\Delta[n]$ defined by

$$S_0 \subset \ldots \subset S_k \quad \longmapsto \begin{cases} S_0 \subset \ldots \subset S_{k-1} & \text{if } S_k = \mathbf{n}, \\ S_0 \subset \ldots \subset S_k & \text{otherwise.} \end{cases}$$

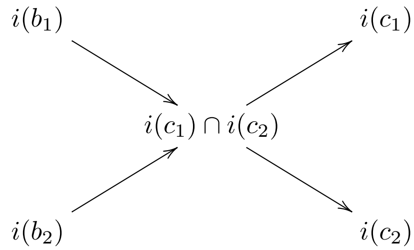
This induces a map $p\colon Y \setminus \{\mathbf{n}\} \longrightarrow X$. We show that $\mathbf{n} \notin i(A)$, and that the composite

$$A \xrightarrow{i} Y \setminus \{\mathbf{n}\} \xrightarrow{p} X \longrightarrow Y \xrightarrow{r} A \tag{6.3}$$

is the identity on A. From this we can conclude that A is a retract of X.

Since $\mathbf{n} \in Y$ is not a codomain of a non-identity arrow, the only elements of A that i could send to $\mathbf{n}$ are a_1 and a_2. We show more generally, that $i(a_1), i(a_2) \in X$. If $i(a_1) \in Y \setminus X$ or $i(a_2) \in Y \setminus X$, then $i(b_1), i(b_2), i(c_1), i(c_2) \in Y \setminus X$. Considering $i(c_1)$ and $i(c_2)$ as totally ordered sets of nonempty subsets of $\mathbf{n}$, the intersection

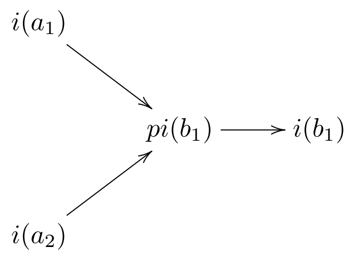
$i(c_1) \cap i(c_2)$ is an element of $Y \setminus X$ and we have a diagram



in Y. Applying the retraction $r\colon Y \longrightarrow A$ to this diagram yields an arrow between b_1 and b_2 or an arrow between c_1 and c_2. Both cases are impossible. We have shown that $i(a_1), i(a_2) \in X$ and thus that $\mathbf{n} \notin i(A)$.

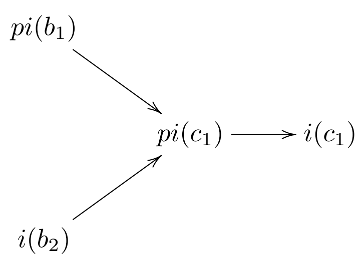
We can also show by the same argument as above that $i(b_1)$ and $i(b_2)$ cannot both belong to $Y \setminus X$.

It remains to show that the composite (6.3) is the identity. Recall that $i(a_1)$, $i(a_2)$ and at least one of $i(b_1)$, $i(b_2)$ belong to X. By symmetry we can assume that $i(b_2) \in X$. We need to show that $rpi(b_1) = b_1$, $rpi(c_1) = c_1$ and $rpi(c_2) = c_2$. Implicitly, we will use that any arrow in Y from an element in X to an element z in $Y \setminus X$ factors through $p(z)$. Since $i(a_1) \leq i(b_1)$ and $i(a_2) \leq i(b_1)$, we have a diagram



in Y. By applying r to this diagram, we deduce that $rpi(b_1) = b_1$ since there is no arrow between a_1 and a_2.

Applying r to the diagram



in Y, we deduce that $rpi(c_1) = c_1$. By symmetry, we also have $rpi(c_2) = c_2$. We have shown that A is a retract of X. □

Corollary 6.4. *Not all finite posets in Thomason's model structure on* **Cat** *are cofibrant.*

The above proof used many special properties of A and thus cannot be used in general to determine which objects are cofibrant. However, there is one class of posets that we can prove are cofibrant: the one-dimensional finite ones. We say that a poset P is (at most) one-dimensional if in any pair of composable morphisms at least one is an identity morphism.

Proposition 6.5. *Every one-dimensional finite poset X is cofibrant.*

Proof. We proceed by induction on the number m of elements of X. If $m = 0$, then $X = \emptyset$ and is thus cofibrant. Now suppose that $m \geq 1$. If X has no non-identity morphisms (is zero-dimensional), then X can be built up by attaching singleton sets $\Pi \operatorname{Sd}^2 \Delta[0]$ to $\emptyset$ and is thus cofibrant.

Otherwise, let a be the domain of a non-identity morphism. Set $A = X \setminus \{a\}$. By the induction hypothesis A is cofibrant. Let $Y = \{y_0, \dots, y_n\}$ be the set of elements $y \in X$ such that there exists a non-identity morphism $a \longrightarrow y$ in X.

Let CY denote the cone on Y obtained by adding a least element $*$ to Y. Note that $Y \longrightarrow CY$ is an inclusion of a cosieve. Thus $X \cong A \cup_Y CY$ by a dual version of Lemma 4.7.

We distinguish the two cases $n = 0$ and $n > 0$. If $n = 0$, we glue $\Pi \operatorname{Sd}^2 \Delta[1]$ to A along a cofibration in such a way that X is a retract of the resulting pushout, and therefore cofibrant. The inclusion of the vertex 0 into $\Delta[1]$ is a cofibration. Applying $\Pi \operatorname{Sd}^2$ to this cofibration yields the inclusion of the poset $\{\{0\}\}$ into $\Pi \operatorname{Sd}^2 \Delta[1]$. Identifying the element $\{0\}$ with y_0, we show that X is a retract of the pushout $A \cup_Y \Pi \operatorname{Sd}^2 \Delta[1]$. Let $X \longrightarrow A \cup_Y \Pi \operatorname{Sd}^2 \Delta[1]$ be the map

$$x \mapsto \begin{cases} \{0\} \subset \mathbf{1} & \text{if } x = y_0 \\ \mathbf{1} & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

The map $\Pi \operatorname{Sd}^2 \Delta[1] \longrightarrow CY$,

$$S_0 \subset \ldots \subset S_k \quad \mapsto \begin{cases} y_0 & \text{if } S_0 = \{0\} \\ * & \text{otherwise} \end{cases}$$

induces a retraction $A \cup_Y \Pi \operatorname{Sd}^2 \Delta[1] \longrightarrow X$ of the map $X \longrightarrow A \cup_Y \Pi \operatorname{Sd}^2 \Delta[1]$ above. Thus X is cofibrant if $n = 0$ and we now assume that $n > 0$.

Similarly to the case $n = 0$, we glue $\Pi \operatorname{Sd}^2 \Delta[n]$ to A along a cofibration in such a way that X is a retract of the resulting pushout, and therefore cofibrant.

The inclusion of the set of vertices of $\Delta[n]$ into $\Delta[n]$ is a cofibration. Applying $\Pi \operatorname{Sd}^2$ to this cofibration yields the inclusion of the discrete poset $\{\{i\} \,|\, 0 \leq i \leq n\}$ into $\Pi \operatorname{Sd}^2 \Delta[n]$. Identifying the element $\{i\}$ with y_i, let Z denote the pushout $A \cup_Y (\Pi \operatorname{Sd}^2 \Delta[n])$. We claim that X is a retract of Z. Indeed, let $j \colon X \longrightarrow Z$ be the map

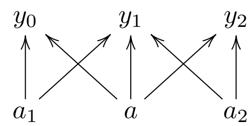
$$x \mapsto \begin{cases} \{i\} \subset \mathbf{n} & \text{if } x = y_i \\ \mathbf{n} & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

The map $\Pi \operatorname{Sd}^2 \Delta[n] \longrightarrow CY$,

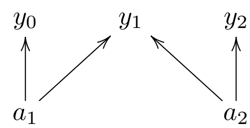
$$S_0 \subset \ldots \subset S_k \quad \mapsto \begin{cases} y_i & \text{if } S_0 = \{i\} \\ * & \text{otherwise} \end{cases}$$

induces a map $r \colon Z \longrightarrow X$ such that $rj = \mathrm{id}_X$ as desired. $\square$

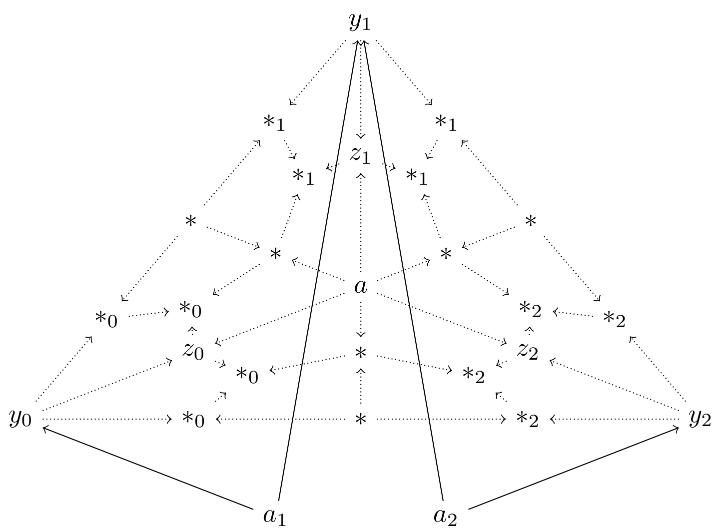
We illustrate the induction step of this proof using the following poset X:



After removing a we obtain the following poset A, which by induction hypothesis is cofibrant.



The poset Z in the proof above can be pictured as follows.



Here each vertex is a distinct object of Z (although we have not given the objects distinct names), and the edges give all of the non-identity morphisms of Z. The inclusion $j\colon X \longrightarrow Z$ maps a_i to a_i, y_k to z_k and a to a. The retraction r is defined by

$$r(z_k) = r(y_k) = r(*_k) = y_k \qquad r(a_i) = a_i \qquad r(a) = r(*) = a.$$

The essential point is that, even in such simple cases as in this section, proving that a poset is or is not cofibrant is a non-trivial exercise.

7. Fibrant posets

In this section we give a class of examples of fibrant posets. Before we begin we give several easy lemmas needed in the proofs. First, we show that when proving a category is fibrant it suffices to consider its connected components. Here, a category is connected if any two objects are connected by a finite zigzag of morphisms. A component of a category is a maximal connected full subcategory, and any category is the disjoint union of its components.

Lemma 7.1. *Let $\mathcal{C} \in$ **Cat** or **Pos**. Then $\mathcal{C}$ is fibrant if and only if all of its components are so.*

Proof. The image of a connected category under a functor lies in a single component. Since each $\Pi \operatorname{Sd}^2 \Lambda^k[n]$ is connected, any functor $\Pi \operatorname{Sd}^2 \Lambda^k[n] \longrightarrow \mathcal{C}$ lands in a single component and similarly for $\Pi \operatorname{Sd}^2 \Delta[n]$. A category $\mathcal{C}$ is fibrant if and only if for every functor $f : \Pi \operatorname{Sd}^2 \Lambda^k[n] \longrightarrow \mathcal{C}$, there exists a functor $h\colon \Pi \operatorname{Sd}^2 \Delta[n] \longrightarrow \mathcal{C}$ such that the diagram

$$\begin{array}{ccc} \Pi \operatorname{Sd}^2 \Lambda^k[n] & \xrightarrow{\ f\ } & \mathcal{C} \\ \downarrow & \nearrow_{h} & \\ \Pi \operatorname{Sd}^2 \Delta[n] & & \end{array}$$

commutes, and this holds if and only if it holds with $\mathcal{C}$ replaced by each of its components. □

Second, we record the following result relating pullbacks and pushouts to binary products $\times$ and binary coproducts $\cup$ inside a poset P. Its proof is an exercise using that there is at most one morphism between any two objects of P.

Lemma 7.2. *If the pullback of a given pair of maps $x \longrightarrow a \longleftarrow y$ exists, it is the product $x \times y$, and if the product $x \times y$ exists, it is the pullback of any pair of maps $x \longrightarrow a \longleftarrow y$. Dually, if the pushout of a given pair of maps $x \longleftarrow a \longrightarrow y$ exists, it is the coproduct $x \cup y$, and if the coproduct $x \cup y$ exists, it is the pushout of any pair of maps $x \longleftarrow a \longrightarrow y$.*

The following addendum implies that a poset with binary products or coproducts is contractible, meaning that its classifying space is contractible.

Lemma 7.3. *If P is a poset containing an object c such that either $c \times x$ exists for any $x \in P$ or $c \cup x$ exists for any $x \in P$, then P is contractible.*

Proof. We prove the lemma in the first case; the second case follows by duality. Let P/c be the poset of all elements x over c; this means that $x \leq c$, or, thinking of P and P/c as categories, that there is a morphism $x \longrightarrow c$; it is contractible since it has the terminal object $c \longrightarrow c$. Since P is a poset, there is at most one morphism $x \longrightarrow c$ for any object x and the functor $P \longrightarrow P/c$ that sends an object y to $c \times y \longrightarrow c$ is right adjoint to the forgetful functor that sends $x \longrightarrow c$ to x. Therefore the classifying space of P is homotopy equivalent to that of P/c. □

In [14], Meier and Ozornova construct examples of fibrant categories. They start from the notion of a partial model category, which is a weakening of the notion of a model category. Recall that a homotopical category $(\mathcal{C}, \mathcal{W})$ is a category $\mathcal{C}$ together with a subcategory $\mathcal{W}$, whose maps we call weak equivalences, such that every object of $\mathcal{C}$ is in $\mathcal{W}$ and $\mathcal{W}$ satisfies the 2 out of 6 property: if morphisms $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so are f, g, h, and $h \circ g \circ f$.

Definition 7.4 ([2, §1.1])**.** A *partial model category* is a homotopical category $(\mathcal{C}, \mathcal{W})$ such that $\mathcal{W}$ contains subcategories $\mathcal{U}$ and $\mathcal{V}$ that satisfy the following properties.

(i) $\mathcal{U}$ is closed under pushouts along morphisms in $\mathcal{C}$ and $\mathcal{V}$ is closed under pullbacks along morphisms in $\mathcal{C}$.
(ii) The morphisms of $\mathcal{W}$ admit a functorial factorization into a morphism in $\mathcal{U}$ followed by a morphism in $\mathcal{V}$.

In (i), it is implicitly required that the cited pushouts and pullbacks exist in $\mathcal{C}$. For example, if $\mathcal{C}$ has a model structure with weak equivalences $\mathcal{W}$ then it has a partial model structure, with $\mathcal{U}$ being the subcategory of acyclic cofibrations and $\mathcal{V}$ being the subcategory of acyclic fibrations.

Theorem 7.5 ([14, Main Theorem])**.** *If $(\mathcal{C}, \mathcal{W})$ is a homotopical category that admits a partial model structure, then $\mathcal{W}$ is fibrant in the Thomason model structure on* **Cat***.*

In the present context, it is very natural to consider those partial model structures such that $\mathcal{C}$ is a poset. In [8], Droz and Zakharevich classified all of the model structures on posets.

Theorem 7.6 ([8, Theorem B])**.** *Let P be a poset which contains all finite products and coproducts, and let $\mathcal{W}$ be a subcategory that contains all objects of P. Then P has a model structure with $\mathcal{W}$ as its subcategory of weak equivalences if and only if the following two properties hold.*

(i) If a composite gf of morphisms in P is in $\mathcal{W}$, then both f and g are in $\mathcal{W}$.

(ii) There is a functor $\chi\colon P \longrightarrow P$ that takes all maps in $\mathcal{W}$ to identity maps and has the property that for every object $x \in P$, the four canonical maps of the diagram

$$\begin{array}{ccc} \chi(x) \times x & \longrightarrow & \chi(x) \\ \downarrow & & \downarrow \\ x & \longrightarrow & \chi(x) \cup x \end{array}$$

in P are weak equivalences.

These two results have the following consequence.

Proposition 7.7. *Let P be a poset satisfying the following conditions:*

(i) P contains an object c such that $c \times x$ and $c \cup x$ exist in P for any other object $x \in P$.

(ii) For any two objects $a, b \in P$, either $a \times b$ exists or there does not exist an $x \in P$ such that $x \leq a$ and $x \leq b$. Dually, either $a \cup b$ exists or there does not exist an $x \in P$ such that $x \geq a$ and $x \geq b$.

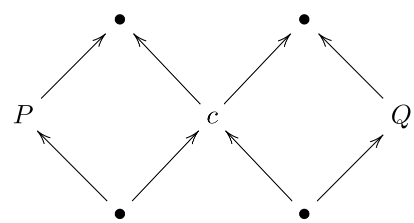
Then P is a component of the weak equivalences in a model category and is therefore fibrant in **Pos***. Moreover, P is contractible.*

Proof. Consider the poset $\widetilde{P}$ whose objects are those of P and two further objects, $\emptyset$ and $*$. The morphisms are those of P and those dictated by requiring $\emptyset$ to be an initial object and $*$ to be a terminal object (so there is no morphism $* \longrightarrow \emptyset$). Condition (ii) ensures that $\widetilde{P}$ has all finite products and coproducts. Indeed, if $a, b \in P$ and $a \times b$ does not exist in P, then $a \times b = \emptyset$ in $\widetilde{P}$ and dually for coproducts. For all $x \in \widetilde{P}$, $x \times * = x$, $x \times \emptyset = \emptyset$, $x \cup \emptyset = x$, and $x \cup * = *$.

Let $\mathcal{W}$ be the union of P and the discrete subcategory $\{\emptyset, *\}$ of P. Although $\widetilde{P}$ is connected, P is one of the three components of $\mathcal{W}$, the other two being the discrete components $\{\emptyset\}$ and $\{*\}$ (which are clearly fibrant). Theorem 7.6 implies that $\widetilde{P}$ has a model structure with $\mathcal{W}$ as its subcategory of weak equivalences. Indeed, condition (i) is clear and, for condition (ii), we define $\chi\colon \widetilde{P} \longrightarrow \widetilde{P}$ by mapping all of P to c (and its identity morphism), mapping $\emptyset$ to $\emptyset$, and mapping $*$ to $*$. Therefore

$\mathcal{W}$ is fibrant by Theorem 7.5, hence P is fibrant by Lemma 7.1; P is contractible by Lemma 7.3. □

For example, if P and Q are any posets satisfying condition (ii) of Proposition 7.7 then the following poset is fibrant:



Finally, we prove a partial converse to Proposition 7.7 which shows that in many cases the connected fibrant posets constructed by Theorem 7.5 are contractible.

Definition 7.8. A map $f\colon a \longrightarrow b$ in a poset P is *maximal* if there do not exist any non-identity morphisms $z \longrightarrow a$ or $b \longrightarrow z$.

For example, the composition of a sequence of maximal length in P is maximal.

Proposition 7.9. *Let $(\mathcal{W}, \mathcal{U}, \mathcal{V})$ be a partial model structure on a poset P and let Q be a connected component of $\mathcal{W}$ that contains a maximal map. Then Q contains an object c such that $c \times x$ and $c \cup x$ exist in Q for any other object $x \in Q$. Therefore Q is contractible.*

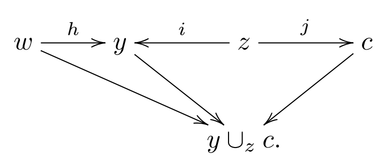
Proof. Let $f\colon a \longrightarrow b$ be a maximal map in Q and factor it as a map $a \longrightarrow c$ in $\mathcal{U}$ followed by a map $c \longrightarrow b$ in $\mathcal{V}$, using the functorial factorization. Since Q is a connected component of $\mathcal{W}$, c is in Q.

First, we claim that any morphism $g\colon z \longrightarrow c$ in Q is in $\mathcal{U}$. Factor g as a morphism $z \longrightarrow w$ in $\mathcal{U}$ followed by a morphism $w \longrightarrow c$ in $\mathcal{V}$. Since $\mathcal{V}$ is closed under pullbacks, $a \times_c w \longrightarrow a$ exists and is in $\mathcal{V}$. However, since f is maximal in $\mathcal{W}$, we must have $a \times_c w = a$, so there exists a morphism $a \longrightarrow w$. By Lemma 7.2, the pushout $c \cup_a w$ of $a \longrightarrow c$ along $a \longrightarrow w$ is $w \cup c$, and $w \cup c = c$ since P is a poset and there is a map $w \longrightarrow c$. But then $w \longrightarrow c$ is the pushout of a morphism in $\mathcal{U}$, so it is also in $\mathcal{U}$. Thus g is the composite of two morphisms in $\mathcal{U}$, so it is also in $\mathcal{U}$, as claimed. Dually, any morphism $c \longrightarrow z$ in Q is in $\mathcal{V}$.

Now let x be any object in Q. Since Q is connected, there is a finite zigzag of morphisms of Q connecting x to c. If the zigzag ends with

$$w \xrightarrow{\ h\ } y \xleftarrow{\ i\ } z \xrightarrow{\ j\ } c,$$

then j is in $\mathcal{U}$, so $y \cup_z c$ exists and we can shorten the zigzag via the diagram



The dual argument applies to shorten the zigzag if it ends with

$$w \xleftarrow{\ h\ } y \xrightarrow{\ i\ } z \xleftarrow{\ j\ } c.$$

Inductively, we can shorten any zigzag to one of either of the forms

$$x \longleftarrow z \longrightarrow c \qquad \text{or} \qquad x \longrightarrow z \longleftarrow c.$$

We show that $c \cup x$ and $c \times x$ exist in the first case; the same is true in the second case by symmetry. Since $z \longrightarrow c$ is in $\mathcal{U}$, $c \cup_z x$ exists, and it is $c \cup x$ by Lemma 7.2. Since $c \longrightarrow c \cup x$ is in $\mathcal{V}$, $c \times_{c \cup x} x$ also exists, and it is $c \times x$ by Lemma 7.2 again.

Thus Q contains an object c such that $c \times x$ and $c \cup x$ exists for any object $x \in Q$, as claimed, and it follows from Lemma 7.3 that Q is contractible. □

Acknowledgement

We thank Viktoriya Ozornova for pointing out some misleading typos and a mistake in our original proof of Proposition 6.5 and for alerting us to the work of Bruckner and Pegel. We are grateful to the referee for detailed comments and, in particular, for pointing out that infinite non-cofibrant posets are known.

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