EQUIVARIANT ORIENTATIONS AND THOM ISOMORPHISMS

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Contents

Let G be a compact Lie group and let E_G^* be an $RO(G)$-graded cohomology theory on G-spaces. We shall explain a sensible way to think about orientations and the Thom isomorphism theorem in the theory E_G^*, offering an alternative to the approach given by Costenoble and Waner in [7]. Both approaches generalize the restricted theory given by Lewis and myself in [15], and both grew out of joint work of Costenoble, Waner, and myself [5].

In the study of nonequivariant bundles and their orientations, an innocuous first step is to assume that the base space is (path) connected. The analogous equivariant assumption is that the G-set of components of the base space is a single orbit, but this assumption doesn't get us very far. There is an entirely satisfactory theory of equivariant Thom isomorphisms and Poincaré duality under the much more stringent hypothesis that the base space X be G-connected, in the sense that each X^H is non-empty and (path) connected. This is developed in detail in [15, III§6 and X§5]. The basic problem, then, is to generalize that theory to a less restricted class of base spaces. The obvious approach is to parametrize changes of fiber representation on the fundamental groupoid. However, doing this directly leads to a fairly complicated, and hard to compute, generalization of equivariant cohomology [5, 6, 7, 8]. We seek a variant approach that allows us to work within the framework of $RO(G)$-graded cohomology theories, so that we can apply rather than generalize the pre-existing theory of [15].

Our essential idea is that, to obtain a satisfactory general theory, it seems reasonable to give up the idea that the Thom isomorphism must be a single isomorphism. Rather, we shall define it to be an appropriate family of isomorphisms. More precisely, we shall define an E_G^*-orientation of a G-vector bundle, or more generally of a spherical G-fibration, to be a suitably coherent family of cohomology classes. Each class in the family will determine a Thom isomorphism, and these isomorphisms will be nicely related. If the base space is G-connected, then all of these Thom

classes and Thom isomorphisms will be determined by one member of the family. In general, all members of the family will be determined by choosing members of the family indexed on components of fixed point spaces X^H which do not contain any K-fixed points for K larger than H.

We begin in Sections 1–3 by defining the fundamental groupoid $\pi(X)$, discussing functors defined on it, and constructing the family of "H-connected covers" of a G-space X. This construction can be expected to have other uses in equivariant algebraic topology. Many arguments in algebraic topology begin with the statement "We may assume without loss of generality that X is connected". Our H-connected covers give a tool that often allows us to give the same start to equivariant arguments. However, the reader is warned that there are some prices to be paid, beyond the intrinsic complexity. Probably the most significant is that the H-connected cover of a finite G-CW complex will in general be infinite dimensional. For this reason, I have not yet succeeded in obtaining a satisfactory treatment of Poincaré duality that starts from the Thom isomorphism theorem given here.

We define the notion of an orientable spherical G-fibration in Section 4. This does not depend on our H-connected covers, but we use these in our definition of an E_G^*-orientation of a spherical G-fibration in Section 5. Our Thom isomorphism theorem in E_G^*-cohomology follows directly from the definition and the work in [15]. We specialize to ordinary $RO(G)$-graded cohomology with Burnside ring coefficients in Section 6. Nonequivariantly, orientability as defined topologically is equivalent to cohomological orientability with integer coefficients. We prove that perhaps the most natural topological notion of equivariant orientability is equivalent to cohomological orientability with Burnside ring coefficients. We briefly mention other examples in Section 7.

The theory here was suggested by joint work with Steve Costenoble and Stefan Waner [5], and later Igor Kriz, that began in the late 1980's and is still largely unpublished. In that work, we take the more direct approach, confronting head-on the problem of parametrizing the change of fiber representations of a G-vector bundle by the equivariant fundamental groupoid of the base space. Rather than giving up the idea that an orientation is a single cohomology class, we construct more complicated cohomology theories in which a single class can encode all the complexity. That approach has been exploited in a series of papers on this and related topics by Costenoble and Waner [6, 7, 8, 9, 10]. A fully coherent theory of orientation will require a comparison of that approach to the one given here. Although Costenoble, Kriz, Waner, and I sketched out such a comparison some years ago, the details have yet to be worked out. It is a pleasure to thank Costenoble, Waner, and Kriz for numerous discussions of this material.

This paper is a very small token of thanks to Mel Rothenberg, my colleague and friend for the last 32 years. I wish I had a better paper to offer, since this one should have a sign on it saying "speculative, may not be useful", but it is in one of the areas that Mel has pioneered (e.g. [18]) and that I have in part learned from him. It has been a privilege to work with him all these years to help make Chicago a thriving center of topology.

1. The fundamental groupoid and categorical definitions

We here recall our preferred definition of $\pi(X)$ and give some categorical language that will help us define structures in terms of it. An equivalent definition appears in [11, 10.7], and a definition in terms of Moore loops is given in [17, App].

We assume that G is a compact Lie group, and we only consider closed subgroups. Let $\mathscr{O}_G$ denote the topological category of orbit spaces G/H and G-maps between them, where H runs through the (closed) subgroups of G. Let $h\mathscr{O}_G$ be the homotopy category of $\mathscr{O}_G$. Of course, $\mathscr{O}_G = h\mathscr{O}_G$ if G is finite. The following observation describes the structure of $h\mathscr{O}_G$ for general compact Lie groups G. Recall that if $\alpha : G/H \longrightarrow G/K$ is a G-map with $\alpha(eH) = gK$, then $g^{-1}Hg \subset K$.

Lemma 1.1. *Let $j : \alpha \longrightarrow \beta$ be a G-homotopy between G-maps $G/H \longrightarrow G/K$. Then j factors as the composite of α and a homotopy $c : G/H \times I \longrightarrow G/H$ such that $c(eH, t) = c_t H$, where $c_0 = e$ and the c_t specify a path in the identity component of the centralizer $C_G H$ of H in G.*

Proof. Let $j(eH, t) = g_t K$. Since we can lift this path in G/K to a path in G starting at g_0, we may assume that the g_t specify a path in G. Now $g_t^{-1}Hg_t \subset K$ for all t, so we can define $d : H \times I \longrightarrow K$ by $d(h, t) = g_t^{-1}hg_t$. Since the adjoint $\tilde{d} : I \longrightarrow \mathrm{Map}(H, K)$ is a path through homomorphisms, the Montgomery-Zippen Theorem [4, 38.1] implies that there are elements $k_t \in K$ such that $k_0 = e$ and $d(h, t) = k_t^{-1}g_0^{-1}hg_0k_t$. Define $\zeta : K \longrightarrow \mathrm{Hom}(g_0^{-1}Hg_0, K)$ by $\zeta(k)(h') = k^{-1}h'k$. The image of ζ may be identified with K/L, where L is the subgroup of elements k such that $\zeta(k) = \zeta(e)$ is the inclusion of $g_0^{-1}Hg_0$ in K. It follows that ζ is a bundle over its image. We may regard $\tilde{d}$ as a path in $\mathrm{Hom}(g_0^{-1}Hg_0, K)$, and we can lift it to a path $k : I \longrightarrow K$ with $k(0) = e$. Thus we may assume that the k_t specify a path in K. Now define $c_t = g_t k_t^{-1}g_0^{-1}$. Then $j(eH, t) = c_t g_0 K$, $c_0 = e$, and the c_t are in $C_G H$, as desired. $\square$

Definition 1.2. Let X be a G-space. Define the fundamental groupoid $\pi(X)$ as follows. Its objects are the pairs $(G/H, x)$, where $H \subset G$ and $x \in X^H$; we think of this pair as the G-map $x : G/H \longrightarrow X$ that sends eH to x. The morphisms $(G/H, x) \longrightarrow (G/K, y)$ are the equivalence classes $[\alpha, \omega]$ of pairs (α, ω), where $\alpha : G/H \longrightarrow G/K$ is a G-map and $\omega : G/H \times I \longrightarrow X$ is a G-homotopy from x to $y \circ \alpha$. Here two such pairs (α, ω) and (α', ω') are equivalent if there are G-homotopies $j : \alpha \simeq \alpha'$ and $k : \omega \simeq \omega'$ such that

$$k(a, 0, t) = x(a) \quad \text{and} \quad k(a, 1, t) = y \circ j(a, t)$$

for $a \in G/H$ and $t \in I$. If G is finite, then $\alpha = \alpha'$ and j is constant. Composition is evident. Define a functor $\varepsilon = \varepsilon_X : \pi(X) \longrightarrow h\mathscr{O}_G$ by sending $(G/H, x)$ to G/H and sending $[\alpha, \omega]$ to the homotopy class $[\alpha]$ of α. A G-map $f : X \longrightarrow Y$ induces a functor $f_* : \pi(X) \longrightarrow \pi(Y)$ such that $\varepsilon_Y \circ f_* = \varepsilon_X$. A G-homotopy $h : f \simeq f'$ induces a natural isomorphism $h_\# : f_* \longrightarrow f'_*$.

To be precise about orientability and orientations, we need some abstract definitions and constructions, which are taken from joint work with Costenoble and Waner [5]. The first encodes the formal structure of the fundamental groupoid.

Definition 1.3. A *groupoid over a small category $\mathscr{B}$* is a small category $\mathscr{C}$ together with a functor $\varepsilon : \mathscr{C} \longrightarrow \mathscr{B}$ that satisfies the following properties. For an object b

of $\mathscr{B}$, the fiber $\mathscr{C}_b$ is the subcategory of $\mathscr{C}$ consisting of the objects and morphisms that ε maps to b and id_b.

(i) For each object b of $\mathscr{B}$, $\mathscr{C}_b$ is either empty or a groupoid (in the sense that each of its morphisms is an isomorphism).

(ii) (Source lifting) For each object $y \in \mathscr{C}$ and each morphism $\beta : a \longrightarrow \varepsilon(y)$ in $\mathscr{B}$, there is an object $x \in \mathscr{C}$ such that $\varepsilon(x) = a$ and a morphism $\gamma : x \longrightarrow y$ in $\mathscr{C}$ such that $\varepsilon(\gamma) = \beta$.

(iii) (Divisibility) For each pair of morphisms $\gamma : x \longrightarrow y$ and $\gamma' : x' \longrightarrow y$ in $\mathscr{C}$ and each morphism $\beta : \varepsilon(x) \longrightarrow \varepsilon(x')$ in $\mathscr{B}$ such that $\varepsilon(\gamma) = \varepsilon(\gamma') \circ \beta$, there is a morphism $\delta : x \longrightarrow x'$ in $\mathscr{C}$ such that $\varepsilon(\delta) = \beta$ and $\gamma' \circ \delta = \gamma$.

Remark 1.4. We say that $\mathscr{C}$ has *unique divisibility* if the morphism δ asserted to exist in (iii) is unique. This holds for fundamental groupoids when G is finite, but not when G is a general compact Lie group. When it holds, $\mathscr{C}$ is exactly a "catégorie fibrée en groupoides" over $\mathscr{B}$ as defined by Grothendieck [13, p. 166].

Definitions 1.5. Let $\mathscr{C}$ be a groupoid over $\mathscr{B}$.

(i) $\mathscr{C}$ is *skeletal over* $\mathscr{B}$ if each fiber $\mathscr{C}_b$ is skeletal (has a single object in each isomorphism class of objects).

(ii) $\mathscr{C}$ is *faithful over* $\mathscr{B}$ if ε is faithful (injective on hom sets).

(iii) $\mathscr{C}$ is *discrete over* $\mathscr{B}$ if each $\mathscr{C}_b$ is discrete (has only identity morphisms).

Lemma 1.6. *$\mathscr{C}$ is discrete over $\mathscr{B}$ if and only if it is skeletal and faithful over $\mathscr{B}$.*

Proof. Clearly, if $\mathscr{C}$ is skeletal and faithful, then it is discrete. If $\mathscr{C}$ is discrete, then it is clearly skeletal, and it is faithful by Remark 1.7(ii) below. $\square$

Remarks 1.7. Let $\mathscr{C}$ be a groupoid over $\mathscr{B}$.

(i) If $\mathscr{C}$ is skeletal over $\mathscr{B}$, then divisibility implies that the object x asserted to exist in the source lifting property is unique. If $\mathscr{C}$ is skeletal and faithful over $\mathscr{B}$, then the morphism asserted to exist in the source lifting property is also unique. If $\mathscr{C}$ is faithful over $\mathscr{B}$, then it is uniquely divisible.

(ii) By divisibility, any two morphisms $x \longrightarrow y$ of $\mathscr{C}$ over the same morphism of $\mathscr{B}$ differ by precomposition with an automorphism of x over the identity morphism of $\varepsilon(x)$. Thus $\mathscr{C}$ is faithful over $\mathscr{B}$ if and only if the only automorphisms in each $\mathscr{C}_b$ are identity maps.

(iii) If $\gamma : x \longrightarrow y$ is a morphism of $\mathscr{C}$ such that $\varepsilon(\gamma)$ is an isomorphism, then γ is an isomorphism, as we see by application of divisibility to the equality $\varepsilon(\gamma)\varepsilon(\gamma)^{-1} = \mathrm{id}$. If every endomorphism of every object of $\mathscr{B}$ is an isomorphism, as holds in $\mathscr{O}_G$, then every endomorphism of every object of $\mathscr{C}$ is an isomorphism.

Construction 1.8. Let $\mathscr{C}$ be a groupoid over $\mathscr{B}$. We construct the discrete groupoid over $\mathscr{B}$ associated to $\mathscr{C}$.

(i) Construct a faithful groupoid $\mathscr{C}/\varepsilon$ with the same objects as $\mathscr{C}$ by setting

$$\mathscr{C}/\varepsilon(x, y) = \mathrm{Im}(\varepsilon : \mathscr{C}(x, y) \longrightarrow \mathscr{B}(\varepsilon(x), \varepsilon(y)).$$

The quotient functor $\mathscr{C} \longrightarrow \mathscr{C}/\varepsilon$ is the universal map from $\mathscr{C}$ into a faithful groupoid over $\mathscr{B}$.

(ii) Construct a skeletal subgroupoid $\mathscr{C}'$ of $\mathscr{C}$ by choosing one object in each isomorphism class of objects of $\mathscr{C}_b$ for each object b of $\mathscr{B}$ and letting $\mathscr{C}'$ be

the resulting full subcategory of $\mathscr{C}$. The inclusion $\mathscr{C}' \longrightarrow \mathscr{C}$ is an adjoint equivalence of categories over $\mathscr{B}$; its left inverse is a retraction ρ obtained from any choice of isomorphisms from each object of each $\mathscr{C}_b$ to an object of $\mathscr{C}_b'$. We call $\mathscr{C}'$ a skeleton of $\mathscr{C}$.

(v) By Remark 1.7(iii), passage from $\mathscr{C}$ to $\mathscr{C}/\varepsilon$ creates no new isomorphisms, so that we can make the same choices of objects for $\mathscr{C}$ and for $\mathscr{C}/\varepsilon$ when forming skeleta. Then $\mathscr{C}'/\varepsilon = (\mathscr{C}/\varepsilon)'$. We call this category the discrete groupoid over $\mathscr{B}$ associated to $\mathscr{C}$.

Lemma 1.9 (Joyal). *A discrete groupoid $\mathscr{C}$ over $\mathscr{B}$ determines and is determined by an associated contravariant functor $\Gamma : \mathscr{B} \longrightarrow Sets$.*

Proof. Given $\mathscr{C}$, define Γ as follows. For an object b of B, $\Gamma(b)$ is the set of objects of $\mathscr{C}_b$. For a morphism $\beta : a \longrightarrow b$ of $\mathscr{B}$ and an object y of $\mathscr{C}_b$, $\Gamma(\beta)(y)$ is the unique object of $\mathscr{C}_a$ that is the source of a map covering β. Given Γ, define $\mathscr{C}$ as follows. Its objects are pairs (b, y), where b is an object of $\mathscr{B}$ and $y \in \Gamma(b)$. A morphism $(a, x) \longrightarrow (b, y)$ is a morphism $\beta : a \longrightarrow b$ of $\mathscr{B}$ such that $\Gamma(\beta)(y) = x$. Composition, and the functor $\varepsilon : \mathscr{C} \longrightarrow \mathscr{B}$, are evident. □

Now return to the fundamental groupoid.

Notations 1.10. Let $\pi_0(X)$ denote the discrete groupoid over $h\mathscr{O}_G$ associated to the fundamental groupoid $\pi(X)$. The quotient functor $\pi(X) \longrightarrow \pi(X)/\varepsilon$ identifies $[\alpha, \omega]$ and $[\alpha', \omega']$ whenever $[\alpha] = [\alpha']$, so that functors defined on $\pi(X)$ factor through $\pi(X)/\varepsilon$ if their values on morphisms are independent of paths. The category $\pi_0(X)$ is obtained from $\pi(X)/\varepsilon$ by choosing one point in each component of each fixed point space. The notation $\pi_0(X)$ is justified since the associated contravariant functor $h\mathscr{O}_G \longrightarrow Sets$ can be identified with the evident functor that sends an orbit G/H to the set of components $\pi_0(X^H)$.

2. FUNCTORS TO THE CATEGORY OF G-SPACES OVER ORBITS

In the next section, we show how to construct a system of interrelated "H-connected covers" associated to a given G-space X. The interrelationships will be encoded in terms of functors defined on $\pi(X)$. We describe the target category and the abstract nature of the functors we will be concerned with in this section.

Let $\mathscr{U}$ be the category of compactly generated, weak Hausdorff spaces and let $G\mathscr{U}$ be the category of G-spaces.

Definition 2.1. Define $G\mathscr{U}/\mathscr{O}_G$ to be the category of *G-spaces over G-orbits*. The objects of this category are G-maps $\chi : X \longrightarrow G/H$ and the morphisms are commutative diagrams of G-maps

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \psi} \\
G/H & \xrightarrow[\ \alpha\ ]{} & G/K.
\end{array}
$$

There is an evident notion of a homotopy between such morphisms and a resulting homotopy category $hG\mathscr{U}/\mathscr{O}_G$. Let $\varepsilon : hG\mathscr{U}/\mathscr{O}_G \longrightarrow h\mathscr{O}_G$ be the evident augmentation functor; it forgets the G-spaces and remembers the G-orbits.

We think of a G-map $\chi : X \longrightarrow G/H$ as having the total space X and base space G/H, although we do not require χ to be a fibration. Let $V = \chi^{-1}(eH) \subset X$. Then V is an H-space, and the action of G on X induces a G-map $\xi : G \times_H V \longrightarrow X$ that is easily verified to be a bijection. To avoid point-set pathology, we agree to restrict attention to completely regular (e.g., normal) G-spaces. For such X, ξ is a homeomorphism of G-spaces. The following remark gives a more concrete, but less canonical, description of the category $hG\mathscr{U}/\mathscr{O}_G$.

Remark 2.2. For a commutative diagram of G-maps

$$
\begin{array}{ccc}
G \times_H V & \xrightarrow{\ f\ } & G \times_K W \\
\downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \psi} \\
G/H & \xrightarrow{\ \alpha\ } & G/K
\end{array}
$$

with $\alpha(eH) = gK$, define $\tilde{f} : V \longrightarrow W$ by $\tilde{f}(v) = g^{-1}f(v)$. Then $\tilde{f}$ is an H-map, where H acts on W by $hw = g^{-1}hgw$, and

$$(2.3) \qquad\qquad\qquad f(j,v) = (jg, \tilde{f}(v))$$

for $j \in G$. We call f *the G-map associated to the pair* $(\tilde{f}, g)$. Suppose that maps f_0 and f_1 over homotopic G-maps α_0 and α_1 are associated to pairs $(\tilde{f}_0, g_0)$ and $(\tilde{f}_1, g_1)$. Then (f_0, α_0) and (f_1, α_1) are homotopic if and only if there is a path $\tilde{f}_t$ connecting $\tilde{f}_0$ to $\tilde{f}_1$ in the space of H-maps $V \longrightarrow W$. The point is that, by Lemma 1.1, the homotopy $\alpha_0 \simeq \alpha_1$ can be written in the form $\alpha_t(eH) = c_t g_0 K$, where c_t is a path in $C_G H$ starting at e, and the conjugate H-action on W is then the same throughout the homotopy.

Definition 2.4. Let $\mathscr{C}$ be a groupoid over $h\mathscr{O}_G$. A $\mathscr{C}$-*space* is a functor $Y : \mathscr{C} \longrightarrow hG\mathscr{U}/\mathscr{O}_G$ over $h\mathscr{O}_G$. Given a map $A : \mathscr{C} \longrightarrow \mathscr{C}'$ of groupoids over $h\mathscr{O}_G$, a map $\phi : Y \longrightarrow Y'$ from a $\mathscr{C}$-space Y to a $\mathscr{C}'$-space Y' is a natural transformation $\phi : Y \longrightarrow Y' \circ A$ over $h\mathscr{O}_G$.

There is a less conceptual but perhaps more easily understood version of this definition in terms of our concrete description of $G\mathscr{U}/\mathscr{O}_G$. We shall work throughout in terms of this alternative version.

Lemma 2.5. *A $\mathscr{C}$-space Y determines and is determined by the following data.*
  (i) *An H-space $Z(x)$ for each object x in the fiber $\mathscr{C}_{G/H}$ of $\mathscr{C}$.*
  (ii) *A homotopy class of H-maps $Z(\gamma, g) : Z(x) \longrightarrow Z(y)$ for each morphism $\gamma : x \longrightarrow y$ of $\mathscr{C}$ and element g of G such that $\varepsilon(\gamma)(eH) = gK$, where $\varepsilon(\gamma) : G/H \longrightarrow G/K$; here H acts on $Z(y)$ by $ha = (g^{-1}hg)a$ for $a \in Z(y)$.*
These data must satisfy the following properties:
  (iii) *In (ii), $Z(\gamma, gk) \simeq k^{-1}Z(\gamma, g)$ for $k \in K$.*
  (iv) *$Z(\mathrm{id}, e) \simeq \mathrm{id}$ and $Z(\gamma', g') \circ Z(\gamma, g) \simeq Z(\gamma' \circ \gamma, gg')$ when $\gamma' \circ \gamma$ is defined.*
Given $A : \mathscr{C} \longrightarrow \mathscr{C}'$, a map $\phi : Y \longrightarrow Y'$ from a $\mathscr{C}$-space Y to a $\mathscr{C}'$-space Y' determines and is determined by H-maps $\zeta = \zeta_x : Z(x) \longrightarrow Z'(Ax)$ for objects $x \in \mathscr{C}_{G/H}$ such that $Z'(A\gamma, g) \circ \zeta_x \simeq \zeta_y \circ Z(\gamma, g)$ for pairs (γ, g) as in (ii).

Proof. Given the specified data, set $Y(x) = G \times_H Z(x)$ and let $Y(\gamma)$ be the homotopy class of the G-map associated to the pair $(Z(\gamma, g), g)$; property (iii) ensures that $Y(\gamma)$ is independent of the choice of g. Conversely, given Y, let $Z(x) \subset Y(x)$

be the H-space over the orbit eH and let $Z(\gamma, g) : Z(x) \longrightarrow Z(y)$ be the composite of $Y(\gamma)$ and multiplication by g^{-1}. Similarly, given ζ, let $\phi_x : Y(x) \longrightarrow Y'A(x)$ be the G-map associated to the pair (ζ_x, e) and, conversely, given ϕ, let ζ_x be the restriction $Z(x) \longrightarrow Z'(Ax)$ of ϕ_x. $\square$

3. Coherent families of connected covers

A standard tool in equivariant algebraic topology is to study G-spaces by means of their diagrams of fixed point spaces. On the diagram level, it is quite trivial to give a notion of an H-connected cover. The following definition encodes that trivial starting point of our work.

Definition 3.1. An $\mathscr{O}_G$-space is a continuous contravariant functor $\mathscr{O}_G \longrightarrow \mathscr{U}$, and a map of $\mathscr{O}_G$-spaces is a natural transformation. Let $\mathscr{O}_G \mathscr{U}$ denote the category of $\mathscr{O}_G$-spaces. For a G-space X, define the fixed point $\mathscr{O}_G$-space ΦX by $(\Phi X)(G/H) = X^H$. For a fixed point $x \in X^H$, let (X^H, x) denote the component of x in X^H. Define the H-connected cover of ΦX at x to be the sub $\mathscr{O}_H$-space $\Phi(X, x)$ of the $\mathscr{O}_H$-space X such that $\Phi(X, x)(H/J) = (X^J, x)$ for $J \subset H$.

We can lift this essentially nonequivariant structure to the equivariant level by means of a construction due to Elmendorf [12]; see also [16, V§3 and VI§6]. We shall gradually make sense of and prove the following result in this section.

Theorem 3.2. *Let X be a G-space. There is a $\pi(X)$-space $\bar{X}$ such that, for $x : G/H \longrightarrow X$, $\bar{X}(x)$ is the H-connected cover of X at x. There is a natural map of $\pi(X)$-spaces $\bar{\varepsilon} : \bar{X} \longrightarrow X$, where X is regarded as a constant $\pi(X)$-space. For $J \subset H$, $\bar{\varepsilon}_x^J : \bar{X}(x)^J \longrightarrow X^J$ is the composite of a canonical weak equivalence $\bar{X}(x)^J \longrightarrow (X^J, x)$ and the inclusion $(X^J, x) \longrightarrow X^J$.*

Here we are thinking of $\pi(X)$-spaces in terms of the data specified in Lemma 2.5. We recall the main properties of Elmendorf's construction.

Theorem 3.3. *There is a functor $\Psi : \mathscr{O}_G \mathscr{U} \longrightarrow G\mathscr{U}$ and a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{Id}$ such that, for an $\mathscr{O}_G$-space T, each $\varepsilon(G/H) : (\Psi T)^H \longrightarrow T(G/H)$ is a homotopy equivalence. If X is a G-CW complex, then*

$$[X, \Psi T]_G \cong [\Phi X, T]_{\mathscr{O}_G}.$$

For an $\mathscr{O}_G$-space T, evaluation of ε at G/e gives a G-map

$$\varepsilon(G/e) : \Psi(T)^G \longrightarrow T(G/e).$$

When $T = \Phi X$, so that $T(G/e) = X$,

$$\varepsilon(G/e)^J = \varepsilon(G/J) : (\Psi\Phi X)^J \longrightarrow X^J.$$

Thus $\varepsilon(G/e) : \Psi\Phi X \longrightarrow X$ is a weak G-equivalence for any G-space X, and $\varepsilon(G/e)$ is a G-homotopy equivalence if X is a G-CW complex.

Definition 3.4. The H-connected cover of X at $x \in X^H$ is the H-space

$$\bar{X}(x) = \Psi\Phi(X, x).$$

Thus we have homotopy equivalences $\varepsilon(H/J) : \bar{X}(x)^J \longrightarrow (X^J, x)$ for $J \subset H$. Applying Ψ to the inclusion of $\mathscr{O}_H$-spaces $\Phi(X, x) \subset \Phi X$ and composing with $\varepsilon(G/e) : \Psi\Phi X \longrightarrow X$, we obtain an H-map $\bar{\varepsilon}_x : \bar{X}(x) \longrightarrow X$ such that $\bar{\varepsilon}_x^J$ is the composite of the homotopy equivalence $\varepsilon(H/J)$ and the inclusion $(X^J, x) \longrightarrow X^J$.

It remains to discuss the functoriality and naturality of this construction, which is the crux of the matter. We recall that the functor $\Psi : \mathscr{O}_G\mathscr{U} \longrightarrow G\mathscr{U}$ is given by a categorical two-sided bar construction:

$$\Psi T = B(T, \mathscr{O}_G, \bar{\mathscr{O}}_G).$$

Here $\bar{\mathscr{O}}_G : \mathscr{O}_G \longrightarrow \mathscr{U}$ is the covariant functor that sends the object G/H of $\mathscr{O}_G$ to the space G/H. The construction is suitably functorial in all three of its variables. An alternative description of ΨT may make the functoriality clearer. Define a small topological G-category $\mathscr{C}(T, G)$ as follows. The object G-space of $\mathscr{C}(T, G)$ is the disjoint union of the G-spaces $T(G/H) \times G/H$, where G acts on the orbit factors. A morphism $\alpha : (t, c) \longrightarrow (t', c')$ is a G-map $\alpha : G/H \longrightarrow G/H'$ such that $\alpha_*(c) = c'$ and $\alpha^*(t') = t$, where the subscript and superscript $*$'s indicate the evaluation of covariant and contravariant functors. There is an evident topology and G-action on the set of morphisms such that the source, target, identity, and composition functions are continuous G-maps. Up to canonical homeomorphism of G-spaces,

$$\Psi T = B\mathscr{C}(T, G).$$

For a homomorphism $\mu : G \longrightarrow G'$, a G-functor $\mathscr{C}(T, G) \longrightarrow \mathscr{C}(T', G')$ induces a G-map $\Psi T \longrightarrow \Psi T'$, where G acts on the targets by pullback along μ; similarly, a G-natural transformation induces a G-homotopy.

Let $[\alpha, \omega] : x \longrightarrow y$ be a morphism in $\pi(X)$. Let $\alpha : G/H \longrightarrow G/K$ be given by $\alpha(eH) = gK$ and let $c(g) : H \longrightarrow K$ be the conjugacy injection that sends h to $g^{-1}hg$. By Lemma 1.1, if we change α in its homotopy class, then we replace g by cg for some c in the identity component of $C_G H$. Therefore, although $c(g)$ depends on the choice of g in its coset, it does not depend on the choice of α in its homotopy class. The homomorphism $c(g)$ determines a functor $\mathscr{O}_H \longrightarrow \mathscr{O}_K$ that sends H/J to $K/g^{-1}Jg$, and we also have the H-map $H/J \longrightarrow K/g^{-1}Jg$ that sends hJ to $(g^{-1}hg)(g^{-1}Jg)$. Using the functoriality of the two-sided bar construction, there results an H-map

$$\bar{X}([\alpha, \omega], g) : \bar{X}(x) \longrightarrow \bar{X}(y).$$

The properties specified in (iii) and (iv) of Lemma 2.5 are satified. Here (iii) is not obvious since a homotopy is required, but it is easy to check that the two maps specified there are obtained by passage to classifying spaces of categories from naturally equivalent functors and are therefore homotopic.

Intuitively, this transport along paths ensures that our H-connected covers are related by the evident commutative diagrams to inclusions of components of fixed point spaces. The naturality of the construction with respect to G-maps $X \longrightarrow Y$ is checked similarly.

4. Orientability of spherical G-fibrations

Nonequivariantly, there is only one sensible definition of an orientation of a vector bundle, but this is a calculational fact that does not extend to the equivariant setting. The point is that

$$\mathbb{Z}/2 \cong \pi_0(O(n)) \cong \pi_0(PL(n)) \cong \pi_o(Top(n)) \cong \pi_0(F(n))$$

for all $n \geq 1$, including $n = \infty$. Nothing like this holds equivariantly. There are (at least) eight different reasonable orientation theories for G-vector bundles, corresponding to the linear, piecewise linear, topological, and homotopical categories

and their stable variants. Similarly, there are six orientation theories for PL G-bundles, four for topological G-bundles, and two for spherical G-fibrations. We shall focus on the stable spherical G-fibration case, but the modifications for the other cases are easily imagined. A general framework is given in [5]. We begin in this section with the simpler notion of *orientability*. Even this depends on the type of G-bundle or G-fibration we consider. By a G-fibration, we understand a map that satisfies the G-covering homotopy property (G-CHP).

Definitions 4.1. (i) Let $\mathscr{O}_G \backslash G\mathscr{U}/\mathscr{O}_G$ be the category of G-spaces $\xi : X \longrightarrow G/H$ with sections $\sigma : G/H \longrightarrow X$ and section-preserving maps of G-spaces over orbits. For an H-representation V, let S^V be the one-point compactification of V. We have a G-fibration $G \times_H S^V \longrightarrow G/H$ with section given by the points at infinity. Define the category $\mathscr{F}_n$ of n-sphere G-fibrations to be the full subcategory of $\mathscr{O}_G \backslash G\mathscr{U}/\mathscr{O}_G$ whose objects are the G-fibrations with section that are fiber G-homotopy equivalent to some $G \times_H S^V$.
(ii) A homotopy between maps in $\mathscr{F}_n$ is a section-preserving homotopy; compare Remark 2.2. The homotopy category $h\mathscr{F}_n$ is a groupoid over the category $h\mathscr{O}_G$.
(iii) Define the stable homotopy category $sh\mathscr{F}_n$ of n-sphere G-fibrations over orbits to have the same objects as $h\mathscr{F}_n$ and stable homotopy classes of maps. Then $sh\mathscr{F}_n$ is also a groupoid over $h\mathscr{O}_G$, and we have a canonical map $i : h\mathscr{F}_n \longrightarrow sh\mathscr{F}_n$ of groupoids over $h\mathscr{O}_G$.

To control the colimits implicit in (iii), let U be the direct sum of countably many copies of each irreducible orthogonal representation of G; since any representation of $H \subset G$ extends to a representation of G on a possibly larger vector space, U is also the sum of countably many copies of each irreducible representation of H. For a G-representation W, $G \times_H S^W \cong G/H \times S^W$ over G/H, and we have the fiberwise smash products $X \wedge W$ of spherical G-fibrations of dimension n and such trivial G-fibrations. The set of stable maps $X \longrightarrow Y$ is the colimit over $W \subset U$ of the set of maps of spherical G-fibrations $X \wedge W \longrightarrow Y \wedge W$.

Restricting objects and morphisms appropriately, we obtain analogous definitions for vector bundles (or better, their fiberwise one-point compactifications), piecewise linear bundles, and topological bundles. We need a lemma before we can explain what it means for a spherical G-fibration $p : E \longrightarrow B$ to be orientable.

Lemma 4.2. *An n-sphere G-fibration $p : E \longrightarrow B$ determines a functor $p^* : \pi(B) \longrightarrow h\mathscr{F}_n$ over $h\mathscr{O}_G$. A map*

$$
\begin{array}{ccc}
D & \xrightarrow{\ f\ } & E \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{\ d\ } & B
\end{array}
$$

of n-sphere G-fibrations determines a natural isomorphism $f^* : q* \longrightarrow p^* \circ f_*$ of functors $\pi(A) \longrightarrow h\mathscr{F}_n$ over $h\mathscr{O}_G$. If

$$
\begin{array}{ccc}
D \times I & \xrightarrow{\ h\ } & E \\
\downarrow{\scriptstyle q \times \mathrm{id}} & & \downarrow{\scriptstyle p} \\
A \times I & \xrightarrow{\ j\ } & B
\end{array}
$$

is a homotopy between maps of spherical G-fibrations (f_0, d_0) and (f_1, d_1), then
$f_1^* = p^* j_\# \circ f_0^*$.

Proof. This is an exercise in pulling back spherical G-fibrations along G-maps $x :
G/H \longrightarrow B$ and using the G-covering homotopy property.   □

Definition 4.3. A spherical G-fibration $p : E \longrightarrow B$ is *orientable (in the stable
sense)* if the composite of $p^* : \pi(B) \longrightarrow h\mathscr{F}_n$ and $i : h\mathscr{F}_n \longrightarrow sh\mathscr{F}_n$ has the
property that $ip^*[\alpha, \omega] = ip^*[\alpha', \omega']$ for every pair of morphisms $[\alpha, \omega]$ and $[\alpha', \omega']$
such that $[\alpha] = [\alpha']$. Intuitively, over a given $[\alpha]$, the stable homotopy class of the
map of fibers over orbits induced by a path between orbits is independent of the
choice of path.

In the language of Construction 1.8, orientability requires ip^* to factor through
the universal faithful groupoid associated to $\pi(B)$. We must distinguish between
orientability in the stable sense and orientability in the unstable sense since it is
possible to have $ip^*[\alpha, \omega] = ip^*[\alpha', \omega']$ but $p^*[\alpha, \omega] \neq p^*[\alpha', \omega']$.

The following lemma is easily verified no matter how we define orientability.

Lemma 4.4. *Each $G \times_H S^V$ is an orientable spherical G-fibration.*

Nonequivariantly, when defining orientations of bundles, we implicitly compare
fibers to $\mathbb{R}^n$ with its standard two orientations. This amounts to choosing a skeleton
of the category of n-dimensional vector spaces. Equivariantly, we must orient the
$G \times_H S^V$ and use their orientations as references, and we must start by fixing a
skeleton of $sh\mathscr{F}_n$. We have already discussed how to do this in Construction 1.8.

Definition 4.5. Let $\mathrm{Sph}\mathscr{F}_n$ denote the discrete groupoid over $h\mathscr{O}_G$ associated to
$sh\mathscr{F}_n$ and let $\rho : sh\mathscr{F}_n \longrightarrow \mathrm{Sph}\mathscr{F}_n$ be the canonical equivalence of categories.
Explicitly, for each $H \subset G$, choose one homomorphism $f : H \longrightarrow O(n)$ in each
conjugacy class and let $V_f = \mathbb{R}^n$ with H acting through f. Choose one S^V in each
stable homotopy class of such H-linear n-spheres. The objects of $\mathrm{Sph}\mathscr{F}_n$ are the
resulting n-sphere G-fibrations $G \times_H S^V$. For each n-sphere G-fibration X over
G/H, we have an isomorphism $\lambda : X \longrightarrow G \times_H S^V$ in $sh\mathscr{F}_n$, and these chosen
isomorphisms determine ρ.

Definition 4.6. Let $p : E \longrightarrow B$ be an n-sphere G-fibration. Define $p^\#$ to be the
composite of $p^* : \pi(B) \longrightarrow sh\mathscr{F}_n$ and $\rho : sh\mathscr{F}_n \longrightarrow \mathrm{Sph}\mathscr{F}_n$. We continue to write
$p^\#$ for its restriction to a skeleton $\mathrm{sk}\pi(B)$ of $\pi(B)$.

Now recall Notations 1.10. The following immediate observation gives a con-
ceptual characterization of orientability in terms of the relationship between the
fundamental groupoid and the component groupoid of B.

Lemma 4.7. *The n-sphere G-fibration $p : E \longrightarrow B$ is orientable if and only if
$p^\# : sk\pi(B) \longrightarrow Sph\mathscr{F}_n$ factors through the associated discrete groupoid $\pi_0(B)$.*

5. Coherent families of Thom classes and Thom isomorphisms

Let E be a commutative ring G-spectrum, in the classical homotopical sense:
we have a unit map $S \longrightarrow E$ and a product $E \wedge E \longrightarrow E$ satisfying the usual
unity, associativity, and commutativity diagrams in the stable homotopy category
of G-spectra of [15]; see also [16, XIII§5]. We are interested in the $RO(G)$-graded
cohomology theory E_G^* represented by E. Evaluated on G-spaces, we understand

the unreduced theory, writing $\tilde{E}^*_G$ for the reduced theory on based G-spaces. We shall make use of the precise treatment of $RO(G)$-grading given in [16, XIII§§1,2]. We may regard E as an H-ring spectrum for any $H \subset G$, and we write E^*_H for the theory on H-spaces represented by E. For an H-space Y and G-representation ρ,

$$E^\rho_G(G \times_H Y) \cong E^\rho_H(Y).$$

We begin with a generalization of the notion of a cohomology class of a G-space.

Definition 5.1. Let $\mathscr{C}$ be a groupoid over $h\mathscr{O}_G$, let $q : \mathscr{C} \longrightarrow \mathrm{Sph}\mathscr{F}_n$ be a map of groupoids over $h\mathscr{O}_G$, and let $Y : \mathscr{C} \longrightarrow G\mathscr{U}/h\mathscr{O}_G$ be a $\mathscr{C}$-space. For an object $x \in \mathscr{C}_{G/H}$, write $q(x) = G \times_H S^{V(x)}$, and describe Y as in Lemma 2.5 in terms of a system of H-spaces $Z(x)$. *An E^*_G-cohomology class ν indexed on q of the $\mathscr{C}$-space Y* consists of an element $\nu(x) \in E^{V(x)}_H(Z(x))$ for each object $x \in \mathscr{C}_{G/H}$. The $\nu(x)$ are required to be *compatible under restriction* in the sense that

$$Z(\gamma, g)^*(\nu(y)) = \nu(x),$$

where $\gamma : x \to y$ is a morphism of $\mathscr{C}$ and g is an element of G such that $\varepsilon(\gamma)(eH) = gK$; compare Lemma 2.5(ii). Here

$$Z(\gamma, g)^* : E^{V(y)}_K(Z(y)) \longrightarrow E^{V(x)}_H(Z(x))$$

is the composite of restriction

$$E^{V(y)}_K(Z(y)) \longrightarrow E^{V(y)}_H(Z(y))$$

along $c(g) : H \longrightarrow K$ and the map

$$E^{V(y)}_H(Z(y)) \longrightarrow E^{V(x)}_H(Z(x))$$

induced by the H-map $Z(\gamma, g) : Z(x) \longrightarrow Z(y)$ and the inverse of the stable H-equivalence $\tilde{f} : S^{V(x)} \longrightarrow S^{V(y)}$ such that $(\tilde{f}, g)$ determines the stable G-equivalence $q(\gamma) : G \times_H S^{V(x)} \longrightarrow G \times_K S^{V(y)}$; compare Remark 2.2.

The simultaneous functoriality in the grading and the space that we have used is explained in [16, XIII§§1,2]. Essentially, this is just an exercise in the use of the suspension isomorphism in $RO(G)$-graded cohomology.

We shall apply this definition with $\mathscr{C} = \pi(B)$, taking q to be the functor $p^\# : \pi(B) \longrightarrow \mathrm{Sph}\mathscr{F}_n$ associated to an n-sphere G-fibration p. The relevant $\pi(B)$-space is the Thom $\pi(B)$-space $\bar{T}(p)$ given by the following definition.

Definition 5.2. Let $p : E \longrightarrow B$ be an n-sphere G-fibration. Define the (based) *Thom G-space Tp* to be the quotient space $E/\sigma B$. For example, if we start with a G-vector bundle ξ, then its Thom space is obtained from the fiberwise one-point compactification of ξ by identifying all of the points at infinity. We have the map of $\pi(B)$-spaces $\bar{\varepsilon} : \bar{B} \longrightarrow B$ of Theorem 3.2. Define the *Thom $\pi(B)$-space $\bar{T}(p)$* by letting $\bar{T}(p)(x) = T(\bar{p}(x))$, $x \in B^H$, be the Thom H-space of the pullback $\bar{p}(x)$ of p along $\bar{\varepsilon} : \bar{B}(x) \longrightarrow B$.

The point of the definition is that the H-space $\bar{B}(x)$ is H-connected and, as we now recall, orientation theory for n-sphere G-fibrations over G-connected base spaces is well understood.

We first define orientations of spherical G-fibrations over orbits, then define orientations of spherical G-fibrations over G-connected base spaces, and finally give our new definition of orientations of general spherical G-fibrations.

Definition 5.3. The Thom G-space of $\xi : G \times_H S^V \longrightarrow G/H$ is $G_+ \wedge_H S^V$, and

$$\tilde{E}_G^V(G_+ \wedge_H S^V) \cong \tilde{E}_H^V(S^V) \cong E_H^0(pt).$$

An E_G^*-*orientation*, or *Thom class*, μ of ξ is an element $\mu \in \tilde{E}_G^V(G_+ \wedge_H S^V)$ that maps under this isomorphism to a unit of the ring $E_H^0(pt)$.

Definition 5.4. Let $p : E \longrightarrow B$ be an n-sphere G-fibration over a G-connected base space B. For any $x \in B^G$, $p^{-1}(x)$ is a based G-space of the homotopy type of S^V for some n-dimensional representation V of G, and V is independent of the choice of x. Moreover, for all $x : G/H \longrightarrow B$, the pullback of p along x is fiber G-homotopy equivalent to $G \times_H S^V$. An E_G^*-*orientation*, or *Thom class*, μ of p is an element $\mu \in \tilde{E}_G^V(Tp)$ that pulls back to an orientation along each orbit inclusion x.

Definition 5.5. Let $p : E \longrightarrow B$ be an n-sphere G-fibration. An E_G^*-*orientation*, or *Thom class*, of p is an E_G^*-cohomology class μ indexed on $p^\# : \pi(B) \longrightarrow \mathrm{Sph}\mathscr{F}_n$ of the Thom $\pi(B)$-space $\bar{T}(p)$ such that, for each $x \in B^H$, $\mu(x) \in E_H^{V(x)}(\bar{T}(x))$ is an orientation of the pullback $\bar{p}(x)$ of p along $\varepsilon : \bar{B}(x) \longrightarrow B$. We say that p is E_G^*-*orientable* if it has an E_G^*-orientation.

Here, for $x \in B^H$, $V(x)$ is the fiber H-representation at x, so that $S^{V(x)}$ is stably G-homotopy equivalent to $p^{-1}(x)$, the equivalence being fixed by the specification of $p^\#$. Observe that the equivalence fixes a stable H-map

(5.6) $$\qquad\qquad i(x) : S^V \simeq p^{-1}(x) \longrightarrow \bar{T}(p)(x).$$

The following observation should help clarify the force of the compatibility condition required of our orientations on H-connected covers.

Lemma 5.7. *Let p be an n-sphere G-fibration. The following diagram commutes for a morphism $[\alpha, \omega] : x \longrightarrow y$ in $\pi(B)$ with $\alpha(eH) = gK$:*

$$
\begin{array}{ccccc}
E_K^{V(y)}(\bar{T}(p)(y)) & \xrightarrow{\ i(y)^*\ } & \tilde{E}_K^{V(y)}(S^{V(y)}) & \cong & E_K^0(pt) \\
{\scriptstyle \bar{T}(p)([\alpha,\omega],g)^*}\Big\downarrow & & \Big\downarrow{\scriptstyle p^\#([\alpha,\omega],g)^*} & & \Big\downarrow{\scriptstyle c(g)} \\
E_H^{V(x)}(\bar{T}(p)(x)) & \xrightarrow[\ i(x)^*\ ]{} & \tilde{E}_H^{V(x)}(S^{V(x)}) & \cong & E_H^0(pt).
\end{array}
$$

Proof. The map $p^\#([\alpha, \omega], g)^*$ is defined exactly as was $\bar{T}(p)([\alpha, \omega], g)^*$ in Definition 5.1, and the left square is a naturality diagram. The right square commutes by a direct unravelling of definitions. $\square$

Remark 5.8. If the horizontal arrows are isomophisms, then the left vertical arrow is determined by the right vertical arrow and the compatibility reduces to a question of compatible units in the rings comprising the Mackey functor $\mathbf{E}^0$ with $\mathbf{E}^0(G/H) = E_H^0(pt)$. As we shall see in the next section, this is exactly what happens when p is orientable and we specialize to ordinary cohomology with Burnside ring coefficients.

Compatible Thom isomorphisms follow immediately from [15, X§5], where a generalization of the following theorem is proven.

Theorem 5.9. *Let* $p : E \longrightarrow B$ *be an* n-*sphere* G-*fibration over a* G-*connected base space* B *and let* $\mu \in \tilde{E}_G^V(T(p))$ *be a Thom class. Then cupping with* μ *defines a Thom isomorphism*

$$\theta = \theta(p) : E^\rho(B) \longrightarrow \tilde{E}_G^{\rho+V}(T(p))$$

for all $\rho \in RO(G)$.

Again, we refer to [16, XIII§§1,2] for precision about the grading.

Theorem 5.10. *Let* $p : E \longrightarrow B$ *be an* n-*sphere* G-*fibration and let* $\{\mu(x)\}$ *be a Thom class of* p. *Then the* $\mu(x)$, $x \in B^H$, *give rise to Thom isomorphisms*

$$\theta(\bar{p}(x)) : E_H^*(\bar{B}(x)) \longrightarrow E_H^{*+V(x)}(\bar{T}(p)(x)),$$

where the H-*space* $S^{V(x)}$ *is stably equivalent to* $p^{-1}(x)$. *Moreover, the following diagrams are commutative for* $[\alpha, \omega] : x \longrightarrow y$, *where* $\alpha(eH) = gK$ *and* $\rho \in RO(K)$:

$$
\begin{array}{ccc}
E_K^\rho(\bar{B}(y)) & \xrightarrow{\theta(\bar{p}(y))} & \tilde{E}_K^{\rho+V(y)}(\bar{T}(p)(y)) \\
{\scriptstyle\bar{B}([\alpha,\omega],g)^*}\Big\downarrow & & \Big\downarrow{\scriptstyle\bar{T}(p)([\alpha,\omega],g)^*} \\
E_H^\rho(\bar{B}(x)) & \xrightarrow[\theta(\bar{p}(x))]{} & \tilde{E}_H^{\rho+V(x)}(\bar{T}(p)(x)).
\end{array}
$$

Here the vertical arrows are as specified in Definition 5.1.

6. Orientations in ordinary equivariant cohomology

We have formalized the intuitive geometrical notion of orientability in Definition 4.3 and have expressed this notion categorically in Lemma 4.7. It is natural to hope that this notion coincides with the notion of orientability with respect to a suitable cohomology theory.

Nonequivariantly, the relevant theory is integral cohomology. The real reason this works is that orientability is a stable notion and $\mathbb{Z}$ coincides with the zeroth stable homotopy group of spheres. Equivariantly, the analogue of $\mathbb{Z}$ is the Burnside ring $A(G)$, which is the zeroth equivariant stable homotopy group of spheres. As was first explained by Bredon [3], ordinary equivariant cohomology theories are indexed on coefficient systems, namely contravariant functors $M : h\mathcal{O}_G \longrightarrow$ Ab, where Ab denotes the category of abelian groups. We have the Burnside ring coefficient system $\mathbf{A}$ such that $\mathbf{A}(G/H) = A(H)$. As was proven in [14], the ordinary cohomology theory indexed on M extends to an $RO(G)$-graded theory if and only if the coefficient system M extends to a Mackey functor. See [15, V§9] or [16, IX§4] for a discussion of Mackey functors in the context of compact Lie groups. The Burnside ring coefficient system does so extend, hence we have the ordinary $RO(G)$-graded cohomology theory $H_G^*(-; \mathbf{A})$. It is represented by an Eilenberg-Mac Lane G-spectrum $H\mathbf{A}$ [16, XIII§4], and $H\mathbf{A}$ is a commutative ring G-spectrum.

We abbreviate $H\mathbf{A}_G^*$-orientability to $\mathbf{A}$-orientability and $H_G^*(X; \mathbf{A})$ to $H_G^*(X)$. We proceed to relate orientability to $\mathbf{A}$-orientability, beginning with the case of G-fibrations over G-connected base spaces.

Theorem 6.1. *Let* $p : E \longrightarrow B$ *be an* n-*sphere* G-*fibration, where* B *is* G-*connected. Let* $x \in B^G$, *let* V *be the fiber* G-*representation at* x, *and consider the map* $i : S^V \simeq p^{-1}(x) \subset T(p)$. *The following statements are equivalent.*

(i) *p is orientable.*

(ii) *p is **A**-orientable.*

(iii) *$i^* : \tilde{H}_G^V(T(p)) \longrightarrow \tilde{H}_G^V(S^V) \cong A(G)$ is an isomorphism.*

Proof. By G-CW approximation, we may assume without loss of generality that B is a G-CW complex with a single G-fixed base vertex x. Let B^q be the q-skeleton of B, let $E^q = p^{-1}(B^q)$, and let p^q be the restriction of p to E^q. Let $C^q = T(p^q)/T(p^{q-1})$. Observe that $S^V \simeq T(p^0)$. If $c : G/H_c \times D^q \longrightarrow B^q$ is the characteristic map of a q-cell of B, then the pullback of p along c is trivial and is thus equivalent to $G/H_c \times D^q \times S^V$. Moreover, the equivalence is determined by a choice of path connecting x to $c(e, 0)$. These equivalences determine an equivalence between the wedge over all q-cells c of the G-spaces $(G/H_c)_+ \wedge S^q \wedge S^V$ and the quotient G-space C^q.

Consider cohomology in degrees $V + i$, where i is an integer. We have

$$\tilde{H}_G^{V+i}((G/H)_+ \wedge S^q \wedge S^V) \cong H_H^{i-q}(pt).$$

This is zero unless $i \geq q$ and it is $A(H)$ when $i = q$, by the dimension axiom. We conclude by long exact sequences and $\lim^1$ exact sequences that

$$\tilde{H}^{V-i}(T(p^q)) = \tilde{H}^{V-i}(T(p)) = 0$$

for $i \geq 1$ and there is an exact sequence

$$0 \longrightarrow H_G^V(Tp) \overset{i^*}{\longrightarrow} H_G^V(S^V) \overset{\delta}{\longrightarrow} H_G^{V+1}(C^1).$$

Here δ may be viewed as a map $A(G) \longrightarrow \prod A(H_c)$ of $A(G)$-modules, where the product runs over the 1-cells c. A 1-cell c is specified by a loop at x in B^{H_c}. The component of δ in $A(H_c)$ can be interpreted geometrically as the difference between the identity map of S^V and the stable H-equivalence of S^V obtained by the action of this loop on S^V. The three statements of the theorem are each equivalent to the assertion that $\delta = 0$. $\square$

Observe the relevance of our definition of orientability *in the stable sense*. The conclusion would fail if we defined orientability in the unstable sense.

Before generalizing this result, we recall a standard fact about conjugation homomorphisms between Burnside rings. Let $\alpha : G/H \longrightarrow G/K$ be given by $\alpha(eH) = gK$ and consider $c(g) : H \longrightarrow K$. Since $c(k)^* : A(K) \longrightarrow A(K)$ is the identity for $k \in K$, by inspection of the the standard inclusion of $A(K)$ into a product of copies of $\mathbb{Z}$ (e.g. [15, V§2]), we see that $c(g)^* : A(K) \longrightarrow A(H)$ is independent of the choice of g in its coset gK. It is also independent of the choice of α in its homotopy class, by Lemma 1.1. We write $c(g)^* = c(\alpha)^*$.

Theorem 6.2. *Let p be an n-sphere G-fibration. The following statements are equivalent.*

(i) *p is orientable.*

(ii) *Each $\bar{p}(x)$ is orientable.*

(iii) *Each $\bar{p}(x)$ is **A**-orientable.*

(iv) *p is **A**-orientable.*

Moreover, an $H\mathbf{A}$-orientation μ of p is specified by a collection of units $\nu(x) \in A(H)$ for points $x \in B^H$ of the discrete groupoid $\pi_0(B)$ that satisfy the compatibility condition $c(\alpha)^*(\nu(y)) = \nu(x)$ for a map $\gamma : x \longrightarrow y$ of $\pi_0(B)$ with $\varepsilon(\gamma) = \alpha$. Equivalently, μ is specified by an automorphism of the functor $p^\# : sk\pi(B) \longrightarrow Sph\mathscr{F}_n$ over $h\mathscr{O}_G$.

Proof. Since the notion of orientability of p depends only on the behavior of the pullbacks of p along paths and paths lie in connected components, the equivalence of (i) and (ii) is immediate from the properties of H-connected covers given in Theorem 3.2. The equivalence of (ii) and (iii) is part of the previous theorem, and it is trivial that (iv) implies (iii), by consideration of pullbacks. Thus assume (iii) and consider the diagram of Lemma 5.7 with $E = H\mathbf{A}$. Its horizontal arrows are isomorphisms by the previous theorem, and Remark 5.8 applies to give the specified description of an $\mathbf{A}$-orientation in terms of units of Burnside rings. In particular, we may take $\nu(x)$ to be the identity element for all x, and this shows that p is $\mathbf{A}$-orientable. Finally, the group of automorphisms of an object $G \times_H S^V$ of $\mathrm{Sph}\mathscr{F}_n$ is canonically isomorphic to the group of stable H-equivalences of S^V and thus to a copy of the group of units of the Burnside ring $A(H)$. For our functor $p^{\#} : \mathrm{sk}\pi(B) \longrightarrow \mathrm{Sph}\mathscr{F}_n$, the compatibility condition on units required of an $\mathbf{A}$-orientation can be interpreted as the naturality condition required of an automorphism of functors. $\square$

Let $\mathbf{A}\text{-}Or(p)$ denote the set of $\mathbf{A}$-orientations of an orientable n-sphere G-fibration p.

Corollary 6.3. *By multiplication of units or, equivalently, by composition of automorphisms of the functor $p^{\#} : sk\pi(B) \longrightarrow Sph\mathscr{F}_n$ over $h\mathscr{O}_G$, $\mathbf{A}\text{-}Or(p)$ acquires a structure of commutative group.*

Nonequivariantly, there are both topological and cohomological notions of an orientation, and these notions coincide. Equivariantly, we have explained a cohomological notion of an orientation. There is also a topological notion, defined in [5]. However, these two notions do not coincide. To explain this, we sketch the definition given in [5]. Working in the category of groupoids over $h\mathscr{O}_G$, consider maps into $\mathrm{Sph}\mathscr{F}_n$. In [5], we construct and characterize a particular map $\rho : \mathscr{C}_n \longrightarrow \mathrm{Sph}\mathscr{F}_n$ such that $\mathscr{C}_n$ is faithful over $h\mathscr{O}_G$ and any map from a faithful groupoid over $h\mathscr{O}_G$ into $\mathrm{Sph}\mathscr{F}_n$ factors up to isomorphism through at least one map into $\mathscr{C}_n$; this is a weak universal property of ρ, which, intuitively, is a kind of universal orientation.

Fix an orientable n-sphere G-fibration $p : E \longrightarrow B$ for the rest of the section. The functor $p^{\#} : sk\pi(B) \longrightarrow \mathrm{Sph}\mathscr{F}_n$ factors through the discrete groupoid $\pi_0(B)$, and we now agree to write $p^{\#}$ for the resulting functor defined on $\pi_0(B)$. The topological notion of an orientation is a pair (ζ, η) consisting of a functor $\zeta : \pi_0(B) \longrightarrow \mathscr{C}_n$ over $h\mathscr{O}_G$ together with a natural isomorphism $\eta : p^{\#} \longrightarrow \rho \circ \zeta$. Let $Or(p)$ denote the set of such orientations of p. Precomposing with automorphisms of $p^{\#}$ for fixed ζ, we obtain a free right action of $\mathbf{A}\text{-}Or(p)$ on $Or(p)$. Call the orbit set $Or(p)/\mathbf{A}$; it can be identified with the set of those functors $\zeta : \pi_0(B) \longrightarrow \mathscr{C}_n$ that can be part of an orientation (ζ, η). Let $F(p)$ be the set of all functors $\zeta : \pi_0(B) \longrightarrow \mathscr{C}_n$ over $h\mathscr{O}_G$ such that $p^{\#}$ and $\rho \circ \zeta$ agree on objects. In general, not all such functors are components of orientations, and we have an inclusion $\alpha : Or(p)/\mathbf{A} \longrightarrow F(p)$.

Let $\mathbf{A}^{\times} : h\mathscr{O}_G \longrightarrow \mathrm{Ab}$ be the contravariant functor that sends G/H to the group of units of $A(H)$ and continue to write $\mathbf{A}^{\times}$ for its composite with $\varepsilon : \mathscr{C} \longrightarrow h\mathscr{O}_G$ for any groupoid $\mathscr{C}$ over $h\mathscr{O}_G$. By analyzing the obstruction to the construction of η such that (ζ, η) is an orientation, one arrives at the following proposition. We omit the proof, as it is not very illuminating. The essential ingredients are the cited weak universal property of ρ and the fact that $\mathrm{Sph}\mathscr{F}_n$ is a uniquely divisible groupoid over $h\mathscr{O}_G$.

Proposition 6.4. *There is an exact sequence of pointed sets*

$$* \longrightarrow Or(p)/\mathbf{A} \xrightarrow{\alpha} F(p) \xrightarrow{\beta} H^1(\pi_0(B), \mathbf{A}^\times) \longrightarrow *.$$

Thus $H^1(\pi_0(B), \mathbf{A}^\times)$ measures the difference between topological and cohomological orientations: if β is a bijection, the notions are equivalent.

7. CONCLUDING REMARKS

Whenever one has cohomological orientations of a class of G-vector bundles that are sufficiently natural in G, one will have cohomological orientations in the sense that we have defined. Since this paper was written around the deadline for submissions to this volume, I have not had time to check details of the following two examples, but they are surely correct. Here it makes sense to use the variant of the theory appropriate to unstable G-vector bundles rather than to stable G-fibrations. Clearly orientations in the former sense give rise to orientations in the latter sense.

The methods of [2] should give the following result; compare [1].

Example 7.1. Complex G-vector bundles admit canonical KU_G^*-orientations. Real G-vector bundles with Spin structures and dimension divisible by eight admit canonical KO_G^*-orientations.

Tautological orientations should give the following result.

Example 7.2. Complex G-vector bundles admit canonical MU_G^*-orientations. Real G-vector bundles admit canonical MO_G^*-orientations.

At the most structured extreme, as in the nonequivariant case, we have the following observation.

Example 7.3. A spherical G-fibration is π_G^*-orientable if and only if its pullbacks to H-connected covers are stably fiber homotopy trivial with suitably compatible trivializations.

To obtain a Poincaré duality theorem along the present lines, one would have to prove an Atiyah duality theorem for the H-connected covers of smooth compact G-manifolds M. That is, if M embeds in V with normal bundle ν, one might hope that the H-spaces $\bar{T}(\nu)(x)$ and $\bar{M}(x)$ are V-dual for $x \in M^H$. Although $\bar{M}(x)$ is infinite dimensional, one has complete homotopical control on its fixed point spaces, which are homotopy equivalent to smooth manifolds. I have not explored this question.

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