MULTICATEGORIES ASSOCIATED TO OPERADS

Abstract. We here show how to construct multicategories from operads and from categories of operators in a general 2-categorical setting. This gives the starting point of a new approach to equivariant multiplicative infinite loop space theory and is likely to have other applications.

Contents

Introduction and statement of results

The starting point of this paper is the observation that any operad $\mathscr{O}$ in any symmetric monoidal category has an intrinsic pairing. It is not true in general that the pairing leads to a multicategory of $\mathscr{O}$-algebras. However, that is true for operads of categories that have extra structure prescribed by certain pseudofunctors. The need

1991 *Mathematics Subject Classification.* Primary 55P42, 55P43, 55P91; Secondary 18A25, 18E30, 55P48, 55U35.

Received by the editor May 8, 2000.

to use pseudofunctors forces us to work 2-categorically. The multicategories needed in multiplicative infinite loop space theory are specializations of those constructed here. For that theory, we must strictify the 2-categorical structures in this paper, starting from the strictification of additive structure given in [9, 5]. Since we have two very different implementations of that strategy, one quite general but also quite categorically intensive [8, 7] and the other intrinsically equivariant but considerably simpler categorically [6], we have chosen to separate out here the preliminaries that are common to both implementations.

We fix a bicomplete cartesian closed category $\mathscr{V}$ and work in the 2-category $\mathbf{Cat}(\mathscr{V})$ of categories internal to $\mathscr{V}$ throughout this paper. We abbreviate notation.

Notation 0.1. Throughout this paper, let $\mathscr{K}$ denote the 2-category $\mathbf{Cat}(\mathscr{V})$.

The motivating example is $\mathscr{V} = G\mathscr{U}$, the category of G-spaces for a finite group G, but equivariance and topology play no role whatsoever in the category theory here. We have many other examples in mind. Quite generally, we can take $\mathscr{V}$ to be the cartesian monoidal category of cocommutative comonoids in *any* bicomplete closed symmetric monoidal category [10]. This gives many algebraic examples as well as many topological ones.

We show in Section 2 that any operad $\mathscr{O}$ in any symmetric monoidal category $\mathscr{W}$ has a certain intrinsic pairing $\wedge$. Taking $\mathscr{W} = \mathscr{K}$, we say that $\mathscr{O}$ is "pseudo-commutative" if the pairing has a certain symmetry property; the word comes from work of Hyland and Power [11], which plays a motivating role. As we observed in [9], many operads in $\mathscr{K}$, including those most relevant to us, are "chaotic". All chaotic operads are pseudo-commutative.

In [9, Section ection 2], we defined the 2-category $\mathscr{O}$-$\mathbf{PsAlg}$ of $\mathscr{O}$-pseudoalgebras and $\mathscr{O}$-pseudomorphisms, together with its sub 2-category $\mathbf{Mult_{st}}(\mathscr{O})$ of (strict) $\mathscr{O}$-algebras and (strict) morphisms. We think of $\mathscr{O}$-$\mathbf{PsAlg}$ as providing underlying additive structure. We give preliminaries about multicategories in Section 1; multicategories are understood to be symmetric throughout this paper and multifunctors are understood to be symmetric multifunctors unless otherwise stated. We introduce the notion of a double multicategory in Section 1.2. With this definitional framework, we prove the following result in Section 3.1.

Theorem 0.2. *For any pseudo-commutative operad $\mathscr{O}$ in $\mathbf{Cat}(\mathscr{V})$, there is a double multicategory $\mathbf{Mult}(\mathscr{O})$ of $\mathscr{O}$-pseudoalgebras and multilinear k-functors, and there is a submulticategory $\mathbf{Mult_{st}}(\mathscr{O})$ of (strict) $\mathscr{O}$-algebras and multilinear k-functors.*

We emphasize that although the 1-cells of the underlying 2-category $\mathscr{O}$-$\mathbf{AlgSt}$ of $\mathbf{Mult_{st}}(\mathscr{O})$ are given by functors, the k-morphisms for $k \geq 2$ are only pseudofunctors. They depend on the pseudofunctors that give the pseudo-commutativity of $\wedge$.

As usual in categorical work such as the proof of Theorem 0.2, rigor requires a slew of coherence diagrams. The reader should not let them be a distraction since we are interested primarily in structures starting from chaotic operads, and any well formulated coherence diagram derived only from the operad structure will commute automatically, by [9, Lemma 1.16].

Categories of operators were introduced in [20] and play a key role in infinite loop space theory. They are discussed in the present categorical framework in [5]. There we described the 2-category $\mathscr{D}$-$\mathbf{PsAlg}$ of $\mathscr{D}$-pseudoalgebras and $\mathscr{D}$-pseudomorphisms together with its sub 2-category $\mathscr{D}$-$\mathbf{AlgSt}$ of (strict) $\mathscr{D}$-algebras and (strict) $\mathscr{D}$-morphisms, where $\mathscr{D}$ is a category of operators in $\mathscr{K}$. We define pseudo-commutative categories of operators in Section 4.1 and prove the following companion to Theorem 0.2 in Section 5.1.

Theorem 0.3. *For any pseudo-commutative category of operators $\mathscr{D}$ over $\mathscr{F}$ in $\mathbf{Cat}(\mathscr{V})$, there is a double multicategory $\mathbf{Mult}(\mathscr{D})$ of $\mathscr{D}$-pseudoalgebras and multilinear k-functors and there is a sub double multicategory $\mathbf{Mult_{st}}(\mathscr{D})$ of $\mathscr{D}$-algebras and multilinear k-functors.*

Here again, we emphasize that although the 1-cells of the underlying 2-category $\mathscr{D}$-**AlgSt** of $\mathbf{Mult_{st}}(\mathscr{D})$ are 2-functors, the k-morphisms for $k \geq 2$ are only pseudo-functors since they depend on the product $\wedge$ that defines the pseudo-commutative pairing on $\mathscr{D}$.

In [5], we constructed a category of operators $\mathscr{D}(\mathscr{O})$ from an operad $\mathscr{O}$ in $\mathscr{K}$. We are only interested in categories of operators of this form. With $\mathscr{D} = \mathscr{D}(\mathscr{O})$, we also constructed an adjoint pair of 2-functors $(\mathbb{L}, \mathbb{R})$ between $\mathscr{O}$-**PsAlg** and $\mathscr{D}$-**PsAlg** that restricts to an adjoint pair of 2-functors between $\mathscr{O}$-**AlgSt** and $\mathscr{D}$-**AlgSt**. We prove the two statements of the following result in Section 4.2 and Section 5.2. We shall not belabor categorical language, but the adjunction $(\mathbb{L}, \mathbb{R})$ extends to the multicategorical context.

Theorem 0.4. *Let $\mathscr{O}$ be a pseudo-commutative operad in $\mathbf{Cat}(\mathscr{V})$ and let $\mathscr{D} = \mathscr{D}(\mathscr{O})$. Then $\mathscr{D}$ is a pseudo-commutative category of operators and the 2-functor*

$$\mathbb{R}\colon \mathscr{O}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}$$

extends to a multifunctor $\mathbf{Mult}(\mathscr{O}) \longrightarrow \mathbf{Mult}(\mathscr{D})$ that restricts to a multifunctor $\mathbf{Mult_{st}}(\mathscr{O}) \longrightarrow \mathbf{Mult_{st}}(\mathscr{D})$.

We are especially interested in equivariant situations. Let G be a finite group. We can do everything that we have discussed starting in the category $G\mathscr{V}$ of G-objects in $\mathscr{V}$ and G-maps since it satisfies our original hypotheses on $\mathscr{V}$. Our ground 2-category then becomes $Cat(G\mathscr{V})$, which we denote by $\mathscr{K}_G$. As discussed for G-spaces in [19, Section 4], there is then a variant of everything above in which we replace $\mathscr{F}$ by the category $\mathscr{F}_G$ of finite G-sets to define categories of operators $\mathscr{D}_G$ over $\mathscr{F}_G$. As we explain in Section 6.1, there is a notion of a pseudo-commutative category of operators over $\mathscr{F}_G$ such that a pseudo-commutative category of operators $\mathscr{D}$ over $\mathscr{F}$ has an associated pseudo-commutative category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$. We restrict attention to examples of this form. With this restriction, we have the following analog of Theorem 0.3 and a comparison between the two, as we prove in Section 6.2 and Section 6.3

Theorem 0.5. *For any pseudo-commutative category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$ in $\mathbf{Cat}(g\mathscr{V})$, there is a double multicategory $\mathbf{Mult}(\mathscr{D}_\mathbf{G})$ of $\mathscr{D}_G$-pseudoalgebras and multilinear k-functors and there is a sub double multicategory $\mathbf{Mult_{st}}(\mathscr{D}_\mathbf{G})$ of $\mathscr{D}_G$-algebras and multilinear k-functors.*

Theorem 0.6. *There is an adjoint equivalence $(\mathbb{P}, \mathbb{U})$ between the categories $\mathscr{D}$-**PsAlg** and $\mathscr{D}_G$-**PsAlg**, and $\mathbb{P}$ extends to a multifunctor*

$$\mathbb{P}\colon \mathbf{Mult}(\mathscr{D}) \longrightarrow \mathbf{Mult}(\mathscr{D}_\mathbf{G}).$$

Finally, going back to an operad $\mathscr{O}$ in $\mathscr{K}$, in Section 7.1 we illustrate definitions by proving the following theorem about the free $\mathscr{O}$-algebra functor $\mathbb{O}$.

Theorem 0.7. *For a chaotic operad $\mathscr{O}$ in $\mathscr{K}$, the functor $\mathbb{O}$ from $\mathscr{V}$-categories to $\mathscr{O}$-algebras extends to a multifunctor*

$$\mathbb{O}\colon \mathbf{Mult}(\mathscr{K}) \longrightarrow \mathbf{Mult}_{\mathbf{st}}(\mathscr{O}).$$

As we explain in Section 7.2, this multifunctor is *not* symmetric.

1. Multicategories and double multicategories

1.1. **Multicategories.** We shall not repeat the complete definition of a multicategory given in such sources as [3, 14, 21]. A multicategory $\mathscr{M}$ enriched in a cartesian monoidal category $\mathscr{V}$ has a class $\mathbf{Ob}(\mathscr{M})$ of objects and for each sequence $\underline{a} = \{a_1, \cdots, a_k\}$ of objects and each object b an object in $\mathscr{V}$ of k-morphisms, often called k-linear maps even when there is no linear structure in sight,

$$\mathscr{M}_k(\underline{a}; b) = \mathscr{M}_k(a_1, \cdots, a_k; b).$$

Throughout, we understand multicategories to be symmetric, so that the symmetric group Σ_k acts from the right via $\mathscr{V}$-maps

$$\sigma\colon \mathscr{M}_k(a_1, \cdots, a_k; b) \longrightarrow \mathscr{M}_k(a_{\sigma(1)}, \cdots, a_{\sigma(k)}; b).$$

For each object a there is a unit 1-morphism $a \longrightarrow a$, given formally by a $\mathscr{V}$-map $\mathrm{id}_a\colon * \longrightarrow \mathscr{M}_1(a; a)$, and there are composition $\mathscr{V}$-maps

$$\gamma\colon \mathscr{M}_k(\underline{b}; c) \times \mathscr{M}_{j_1}(\underline{a}_1; b_1) \times \cdots \times \mathscr{M}_{j_k}(\underline{a}_k; b_k) \longrightarrow \mathscr{M}_j(\underline{a_1, \cdots, a_k}; c), \tag{1.1}$$

where $j = j_1 + \cdots + j_k$ and

$$\underline{a_1, \cdots, a_k} = \{a_{1,1}, \cdots, a_{1,j_1}, \cdots, a_{k,1}, \cdots, a_{k,j_k}\}.$$

The γ are subject to direct generalizations of the associativity, unit, and equivariance properties required of an operad in [15, pp 1-2]. These properties are spelled out diagrammatically in [3, 2.1] and, with exceptional care, in [21, Chapter 11].[1]

A multicategory with one object is the same thing as an operad in $\mathscr{V}$, and multicategories are often called colored operads, with objects thought of as colors. The objects and 1-morphisms of a multicategory $\mathscr{M}$ specify its underlying category enriched in $\mathscr{V}$, which is often also denoted $\mathscr{M}$ by abuse of notation. In our examples, $\mathscr{V}$ is usually replaced by $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$, so that we have an underlying $\mathscr{V}$-2-category as discussed in [9, Section ection 1.1].

There is a canonical multicategory $\mathbf{Mult}(\mathscr{C})$ associated to a symmetric monoidal $\mathscr{V}$-category $(\mathscr{C}, \otimes)$. Its objects are those of $\mathscr{C}$, and

$$\mathbf{Mult}_k(\mathscr{C})(a_1, \cdots, a_k; b) = \mathscr{C}(a_1 \otimes \cdots \otimes a_k, b).$$

It has the evident symmetric group actions and units. In schematic elementwise notation, using the notations of (1.1), the composite of a k-morphism $F\colon \underline{b} \longrightarrow c$ with $(e_1, \cdots, e_k)$, where $E_r\colon \underline{a}_r \longrightarrow b_r$ is a j_r-morphism for $1 \leq r \leq k$, is the composite

$$\otimes_{1\leq r\leq k} \otimes_{1\leq s\leq j_r} a_{r,s} \xrightarrow{\otimes_r E_r} \otimes_{1\leq r\leq k} b_r \xrightarrow{F} c \tag{1.2}$$

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[1] The colored operads in [21] are symmetric multicategories with a set of objects, called colors, but the generalization to a class of objects is evident.

More formally, multicomposition is given by the composite

$$\begin{tikzcd} \mathscr{C}(\underline{b};c)\otimes\mathscr{C}(\underline{a}_1;b_1)\otimes\cdots\otimes\mathscr{C}(\underline{a}_k;b_k) \arrow[d, "\mathrm{id}\otimes(\otimes)^k"] \\ \mathscr{C}(\underline{b};c)\otimes\mathscr{C}(\underline{a}_1\otimes\cdots\otimes\underline{a}_k;\underline{b}) \arrow[d, "\circ"] \\ \mathscr{C}(\underline{a}_1\otimes\cdots\otimes\underline{a}_k;c) \end{tikzcd} \tag{1.3}$$

Remark 1.4. If $\mathscr{C}$ is closed, we can interpret $\mathscr{C}(-,-)$ as the internal hom and so enrich $\mathbf{Mult}(\mathscr{C})$ in $\mathscr{C}$.

A (strict) morphism $F\colon \mathscr{M}\longrightarrow\mathscr{N}$ of multicategories, called a multifunctor, is a function $F\colon \mathbf{Ob}(\mathscr{M})\longrightarrow\mathbf{Ob}(\mathscr{N})$ together with Σ_k-equivariant maps in $\mathscr{V}$

$$F\colon \mathscr{M}_k(a_1,\cdots,a_k;b)\longrightarrow\mathscr{N}_k(Fa_1,\cdots,Fa_k;Fb)$$

for all objects a_i and b such that $F(\mathrm{id}_a)=\mathrm{id}_{F(a)}$ and F preserves composition.

1.2. **Double multicategories.** We have described (symmetric) multicategories enriched in a cartesian monoidal category $\mathscr{V}$, focusing on the objects $\mathscr{M}_k(\underline{a};b)$ in $\mathscr{V}$ of k-morphisms for k-tuples of objects $\underline{a}$ and objects b. However, there is further relevant enrichment in all of our examples. All of our multicategories are enriched in **Cat**. They have a class of objects and an (ordinary) category of k-morphisms $\mathscr{M}(\underline{a};b)$ whose objects are the k-morphisms $\underline{a}\longrightarrow b$. That is, we fix $\underline{a}$ and b and consider the underlying set of $\mathscr{M}(\underline{a};b)$. Such enrichment in **Cat** is familiar in the literature, but it is only part of the full structure that is present in all examples we know, the point being that it is unnatural to only consider fixed $\underline{a}$ and b.

It is standard category theory that a 2-category $\mathscr{K}$, or category enriched in **Cat**, has an associated double category, or category internal to **Cat**. The 0-cells and 1-cells of $\mathscr{K}$ give the category of objects, alias the horizontal category. The category of morphisms, alias the vertical category, has objects the 1-cells. For 1-cells $e\colon a\longrightarrow b$ and $f\colon c\longrightarrow d$, a morphism $e\longrightarrow f$ is a pair of 1-cells (h,j) and a 2-cell ω as displayed in the diagram

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\omega"] & b \arrow[d, "j"] \\ c \arrow[r, "f"'] & d. \end{tikzcd}$$

There seems to be no definition of a double multicategory, or a multicategory internal to **Cat**, in the literature,[2] and this is not the place for a formal categorical treatment. We shall just give the idea. The important point is that the double category associated to a 2-category is implicit in its structure, whereas the analogous information for multicategories is generally not, so should be added to the definition.

It is implicit in the multicategory $\mathbf{Mult}(\mathscr{K})$ associated to $\mathscr{K}=\mathbf{Cat}(\mathscr{V})$, the category of categories internal to $\mathscr{V}$. Its objects are the $\mathscr{V}$-categories and its k-morphisms are the $\mathscr{V}$-functors $E\colon \mathscr{X}_1\times\cdots\times\mathscr{X}_k\longrightarrow\mathscr{Y}$. It has an associated

[2] The notion we have in mind seems quite different from the notions of fc-multicategories of [14] and of cyclic double multicategories of [1].

double multicategory structure whose horizontal morphisms are the k-morphisms (for all k). For k-morphisms $E\colon \underline{\mathscr{X}} \longrightarrow \mathscr{Y}$ and $F\colon \underline{\mathscr{X}}' \longrightarrow \mathscr{Y}'$, a vertical morphism $E \longrightarrow F$ is a k-tuple of 1-morphisms $H_i\colon \mathscr{X}_i \longrightarrow \mathscr{X}_i'$, a 1-morphism $J\colon \mathscr{Y} \longrightarrow \mathscr{Y}'$, and a $\mathscr{V}$-transformation ω as displayed in the diagram

(1.5)
$$\begin{tikzcd}
\mathscr{X}_1 \times \cdots \times \mathscr{X}_k \arrow[r, "E"] \arrow[d, "H_1\times\cdots\times H_k"'] \arrow[dr, phantom, "\Downarrow\omega"] & \mathscr{Y} \arrow[d, "J"] \\
\mathscr{X}_1' \times \cdots \times \mathscr{X}_k' \arrow[r, "F"'] & \mathscr{Y}'.
\end{tikzcd}$$

Codifying the coherence properties satisfied in this example will codify a precise definition of a double multicategory.

Our examples will have forgetful multifunctors to $\mathbf{Mult}(\mathscr{K})$ or to analogues of it. In particular, in $\mathbf{Mult}(\mathscr{O})$ as defined below, the objects have underlying $\mathscr{V}$-categories, the k-morphisms have underlying $\mathscr{V}$-functors, and the 2-cells have underlying $\mathscr{V}$-transformations. They will have additional structure that must be suitably respected by 2-cells ω as in (1.5). Since we shall not consider composites of 2-cells, we shall not spell out a formal definition of a double multicategory. The function of such 2-cells ω in our topological applications is that they give categorical input that produces homotopies between maps defined on smash products of spectra (or G-spectra) upon processing by our multiplicative infinite loop space machine; in effect, we are ignoring composites of homotopies.

2. Pseudo-commutative operads

2.1. **The intrinsic pairing of an operad.** We later assume familiarity with operads in $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$, as discussed in [9], but our first definition applies to any reduced operad $\mathscr{O}$ in any cartesian monoidal category $\mathscr{W}$. Reduced means that $\mathscr{O}(0)$ is the unit object of $\mathscr{V}$ and is assumed throughout. Surprisingly, the following elementary structure implicit in the definition of an operad with composition γ is central to our work.

Definition 2.1. The *intrinsic pairing* $\wedge\colon (\mathscr{O},\mathscr{O}) \longrightarrow \mathscr{O}$ on the operad $\mathscr{O}$ is given by the composites

$$\mathscr{O}(m) \times \mathscr{O}(n) \xrightarrow{\mathrm{id}\times\Delta} \mathscr{O}(m) \times \mathscr{O}(n)^m \xrightarrow{\gamma} \mathscr{O}(mn),$$

where $m \geq 0$ and $n \geq 0$.

We are thinking of γ as specifying additive structure, and then the "product" $\wedge$ is taking seriously that $mn = m + \cdots + m$. Thus the intrinsic pairing is an operadic manifestation of the grade school lesson that multiplication is a special case of addition. It is convenient to think of $\mathscr{O}$ as a graded category, with $\mathscr{O}(n)$ its nth category. We write $\underline{\mathscr{O}}$ when we think of it that way.

Proposition 2.2. *The category* $\underline{\mathscr{O}} = \coprod_{n\geq 0} \mathscr{O}(n)$ *is strict monoidal under the intrinsic pairing* $\wedge$. *Its unit is the unit object* $\mathrm{id} \in \mathscr{O}(1)$ *and it has a zero object* $* \in \mathscr{O}(0)$.

Proof. The unit properties of an operad are $\gamma(\mathrm{id}; x) = x$ and $\gamma(x; \mathrm{id}^n) = x$ for $x \in \mathscr{O}(n)$. These say that id is a unit object of $\underline{\mathscr{O}}$. The associativity of the pairing is an easy diagram chase from the following special case of the associativity diagram for γ in the definition of an operad.

$$\begin{tikzcd}
\mathscr{O}(m)\times\mathscr{O}(n)^m\times\mathscr{O}(p)^{mn} \arrow[r, "\gamma\times\mathrm{id}"] \arrow[d, "\pi"'] & \mathscr{O}(mn)\times\mathscr{O}(p)^{mn} \arrow[dr, "\gamma"] & \\
& & \mathscr{O}(mnp) \\
\mathscr{O}(m)\times(\mathscr{O}(n)\times\mathscr{O}(p)^n)^m \arrow[r, "\mathrm{id}\times\gamma^m"'] & \mathscr{O}(m)\times\mathscr{O}(np)^m \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Since $\mathscr{O}$ is reduced, $* \in \mathscr{O}(0)$ is a zero object.

Remark 2.3. The intrinsic pairing is *not* a pairing of operads in the sense originally defined in [16, 1.4]. For many operads occurring naturally in topology, such as the little cubes or Steiner operads, it appears to be of no real interest. However, it appears naturally when trying to construct a multicategory from an operad.

It is natural to ask when the category $\underline{\mathscr{O}}$ is permutative, but that turns out not to be the right question. Recall the permutativity operad $\mathscr{P}$ from [9]. It is the chaotic categorification of the associativity operad **Assoc**. There is another way to think about $\mathscr{P}$. Recall that $\mathscr{F}$ has objects the based sets $\mathbf{n} = \{0, 1, \cdots, n\}$ with base object 0. It is bipermutative under wedge sum and smash product. Ignoring pairings, its subcategory Σ of isomorphisms (in which we can ignore the basepoints) is the domain category for symmetric sequences in any category, and Σ inherits a bipermutative structure from $\mathscr{F}$. If we think of the Σ_j as categories with a single object and thus think of **Assoc** as an operad in **Cat**, then $\underline{\mathbf{Assoc}} = \Sigma$.

To be pedantically precise about **Assoc** and $\mathscr{F}$, remember that when we say that Σ_n is the symmetric group on n letters, we have in mind the ordered set $\underline{n} = \{1, \cdots, n\}$, so that $\sigma \in \Sigma_n$ is a bijection $\underline{n} \longrightarrow \underline{n}$. Changing the order changes the specification of the group. For example, when we apply $\oplus$ to permutations $\sigma \in \Sigma_j$ and $\tau \in \Sigma_k$ and regard the result as a permutation of the $j+k$ letters $\underline{j+k} = \{1, \cdots, j+k\}$, we are implicitly applying the evident isomorphism

$$\zeta_{j,k} \colon \underline{j+k} \longrightarrow \underline{j} \vee \underline{k},$$

then taking the disjoint union of σ and τ, and then applying $\zeta_{j,k}^{-1}$. That is, $\sigma \oplus \tau$ is defined by the commutative diagram

$$\begin{tikzcd}
\underline{j+k} \arrow[r, "\sigma\oplus\tau"] \arrow[d, "\zeta_{j,k}"'] & \underline{j+k} \\
\underline{j} \amalg \underline{k} \arrow[r, "\sigma\vee\tau"'] & \underline{j} \amalg \underline{k} \arrow[u, "\zeta_{j,k}^{-1}"']
\end{tikzcd} \tag{2.4}$$

Similarly, define

$$\lambda = \lambda_{j,k} \colon \underline{jk} \longrightarrow \underline{j} \wedge \underline{k}$$

to be the order-preserving bijection, where $\underline{j} \wedge \underline{k}$ is ordered lexicographically. The intrinsic pairing $\wedge$ on $\underline{\mathbf{Assoc}}$ is just the standard lexicographic product on Σ. Explicitly, $\mu \wedge \nu$ is defined by the commutative diagram

$$\begin{tikzcd} \underline{jk} \arrow[r, "\mu\wedge\nu"] \arrow[d, "\lambda_{j,k}"'] & \underline{jk} \\ \underline{j}\wedge\underline{k} \arrow[r, "\mu\wedge\nu"'] & \underline{j}\wedge\underline{k} \arrow[u, "\lambda_{j,k}^{-1}"'] \end{tikzcd} \tag{2.5}$$

In particular, $e_j \wedge e_k = e_{jk}$. Since $\wedge$ is a group homomorphism, it is equivariant in the sense that $\mu\sigma \wedge \nu\tau = (\mu \wedge \nu)(\sigma \wedge \tau)$. Clearly $e \wedge \nu = \nu$ and $\mu \wedge e = \mu$. The permutativity of Σ under $\wedge$ is just a commutativity relation among permutations.

Definition 2.6. Let $\tau_{j,k} \in \Sigma_{jk}$ be the permutation specified by the composite

$$\underline{jk} \xrightarrow{\lambda_{j,k}} \underline{j}\wedge\underline{k} \xrightarrow{t} \underline{k}\wedge\underline{j} \xrightarrow{\lambda_{k,j}^{-1}} \underline{kj} = \underline{jk}$$

It reorders the set $\underline{jk}$ from lexicographic ordering to reverse lexicographic ordering. Note that $\tau_{j,k}^{-1} = \tau_{k,j}$ and that $\tau_{1,n} = e = \tau_{n,1}$.

For $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$, we have the commutative diagram

$$\begin{tikzcd} \underline{jk} \arrow[r, "\lambda_{j,k}"] \arrow[d, "\tau_{j,k}"'] & \underline{j}\wedge\underline{k} \arrow[r, "\mu\wedge\nu"] \arrow[d, "t"'] & \underline{j}\wedge\underline{k} \arrow[r, "\lambda_{jk}^{-1}"] \arrow[d, "t"] & \underline{jk} \arrow[d, "\tau_{j,k}"] \\ \underline{kj} \arrow[r, "\lambda_{k,j}"'] & \underline{k}\wedge\underline{j} \arrow[r, "\nu\wedge\mu"'] & \underline{k}\wedge\underline{j} \arrow[r, "\lambda_{k,j}^{-1}"'] & \underline{kj} \end{tikzcd}$$

That is,

$$\tau_{j,k}(\mu\wedge\nu) = (\nu\wedge\mu)\tau_{j,k} \quad \text{or equivalently} \quad (\mu\wedge\nu)\tau_{k,j} = \tau_{k,j}(\nu\wedge\mu). \tag{2.7}$$

2.2. **The definition of pseudo-commutative operads.** The following remark clarifies something that confused the senior author for quite some time. It will lead us to the definition of a pseudo-commutative operad (cf [2]).

Remark 2.8. The equation (2.7) admits three closely related interpretations after categorification to $\mathscr{P}$. The elements of the symmetric groups are *morphisms* of Σ and *objects* of $\mathscr{P}$. Let $\gamma_{j,k}\colon \Sigma_{jk} \longrightarrow \Sigma_{jk}$ denote conjugation by $\tau_{j,k}$, so that $\gamma_{j,k}(\xi) = \tau_{j,k}\xi\tau_{j,k}^{-1}$. Then (2.7) says that $\gamma(\mu\wedge\nu) = \nu\wedge\mu$ for $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$. The functions $\gamma_{j,k}$ are the object functions of functors $\gamma\colon \mathscr{P}(jk) \longrightarrow \mathscr{P}(kj)$, and the first interpretation is that the following diagram of functors commutes.

$$\begin{tikzcd} \mathscr{P}(j)\times\mathscr{P}(k) \arrow[r, "\wedge"] \arrow[d, "t"'] & \mathscr{P}(jk) \arrow[d, "\gamma"] \\ \mathscr{P}(k)\times\mathscr{P}(j) \arrow[r, "\wedge"'] & \mathscr{P}(kj) \end{tikzcd}$$

For the second interpretation, observe that $\gamma_{j,k}(\xi)$ can be viewed as the unique morphism $\beta\colon \xi \longrightarrow \tau_{j,k}\xi\tau_{j,k}^{-1}$ in the category $\mathscr{P}(jk)$. Together these morphisms β specify a natural isomorphism of functors $\beta\colon \wedge \longrightarrow \wedge\circ t$. This means that $\underline{\mathscr{P}}$ is a permutative category with its intrinsic pairing and β as symmetry isomorphism. For the third interpretation, we separate out the left and right actions of the permutations $\tau_{j,k}$. We view the right actions by the $\tau_{j,k}$ or (a bit more conveniently) $\tau_{k,j}$, as specifying a functor $\tau\colon \mathscr{P}(jk) \longrightarrow \mathscr{P}(jk)$. Then the left actions by the same elements can be viewed as specifying a natural isomorphism $\alpha\colon \tau \circ \wedge \Longrightarrow \wedge \circ t$.

Rather surprisingly, it is the third interpretation that is relevant to our work. In a general operad $\mathscr{O}$, left multiplication by τ makes no sense (moving the right multiplication left is no answer), and Remark 2.8 leads us to the following definition, which first appeared in [2, Theorem 4.4]

Definition 2.9. A pseudo-commutative structure on an operad $\mathscr{O}$ is a collection of natural isomorphisms, one for each (m, n), of the form

$$\begin{tikzcd}
\mathscr{O}(m) \times \mathscr{O}(n) \arrow[r, "\wedge"] \arrow[d, "t"'] \arrow[dr, phantom, "\Swarrow \alpha_{m,n}"] & \mathscr{O}(mn) \arrow[d, "\tau_{n,m}"] \\
\mathscr{O}(n) \times \mathscr{O}(m) \arrow[r, "\wedge"'] & \mathscr{O}(nm).
\end{tikzcd}$$

We require the component of $\alpha_{1,n}$ at (id, y) to be the identity map, and we require the composite

$$\begin{tikzcd}
\mathscr{O}(m) \times \mathscr{O}(n) \arrow[r, "\wedge"] \arrow[d, "t"'] \arrow[dr, phantom, "\Swarrow \alpha_{m,n}"] & \mathscr{O}(mn) \arrow[d, "\tau_{n,m}"] \\
\mathscr{O}(n) \times \mathscr{O}(m) \arrow[r, "\wedge"] \arrow[d, "t"'] \arrow[dr, phantom, "\Swarrow \alpha_{n,m}"] & \mathscr{O}(nm) \arrow[d, "\tau_{m,n}"] \\
\mathscr{O}(m) \times \mathscr{O}(n) \arrow[r, "\wedge"'] & \mathscr{O}(mn).
\end{tikzcd}$$

to be the identity natural transformation. We also require a compatibility between α and operadic composition.

Remark 2.10. The last condition is fairly elaborate and is spelled out in detail in Corner and Gurski [2, 4.4]. In their language, our condition that $\alpha_{n,m} \circ \alpha_{m,n} = \mathrm{id}$ states that we require pseudo-commutativity structures to be symmetric [2, 4.6].

All conditions are automatically satisfied when $\mathscr{O}$ is chaotic since functors with target a chaotic category are determined by their object functions [9, Lemma 1.18], and we have the following result.

Lemma 2.11. *(*[2, Corollary 4.9]*) A chaotic operad has a unique pseudo-commutative structure.*

3. The multicategory of $\mathscr{O}$-pseudoalgebras

3.1. **The definition of Mult($\mathscr{O}$).** Hyland and Power [11] show that there is a multicategory of algebras over a pseudo-commutative monad, and Corner and Gurski [2] show that the monad corresponding to a pseudo-commutative operad is pseudo-commutative in the sense of [11]. We follow these sources to describe the multicategory of pseudoalgebras over a pseudo-commutative operad. The reader may prefer to restrict attention to algebras, rather than pseudoalgebras, but the gain in simplicity is minimal.[3]

[3]We want the more general context in [8], where the definitions here are given conceptually simpler 2-monadic reinterpretations.

Let $\mathscr{O}$ be a (reduced) pseudo-commutative operad in $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$ in this section. We will later specialize to chaotic operads. We think of an object $x \in \mathscr{O}(n)$ as specifying an n-fold sum F on an $\mathscr{O}$-algebra, and the key point of the following definition is to make sense of a parametrized distributivity law of the general form

$$F(a, b_1 + \cdots + b_n) \simeq F(a, b_1) + \cdots + F(a, b_n).$$

The symbol $\simeq$ is made precise by natural transformations. Diagonal maps enter since a appears once on the left and n times on the right. The definition contains a number of schematic coherence diagrams to the effect that whenever two natural transformations have a chance to be equal they are equal. We shall explain the diagrams after giving the definition. The following maps t_i play a key role.[4]

Notation 3.1. Let $\mathscr{A}_i$, $1 \le i \le k$, be $\mathscr{V}$-categories and let $n \ge 0$. Define t_i to be the composite $\mathscr{V}$-functor displayed in the diagram

$$\begin{tikzcd}
\mathscr{A}_1 \times \cdots \times \mathscr{A}_{i-1} \times \mathscr{O}(n) \times \mathscr{A}_i^n \times \mathscr{A}_{i+1} \times \cdots \times \mathscr{A}_k \arrow[r, "t_i"] \arrow[d, "t"', "\cong"] & \mathscr{O}(n) \times (\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^n \\
\mathscr{O}(n) \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_{i-1} \times \mathscr{A}_i^n \times \mathscr{A}_{i+1} \times \cdots \times \mathscr{A}_k. \arrow[r, "\mathrm{id} \times \Delta"'] & \mathscr{O}(n) \times \mathscr{A}_1^n \times \cdots \times \mathscr{A}_k^n \arrow[u, "\cong", "\mathrm{id} \times \lambda"']
\end{tikzcd}$$

Here t is the evident transposition, Δ is obtained by applying the diagonal maps $\mathscr{A}_j \longrightarrow \mathscr{A}_j^n$ for $j \ne i$, and λ is obtained by transposing from a product of nth powers to an nth power of a product.

Definition 3.2. We define the (double) multicategory $\mathbf{Mult}(\mathscr{O})$ of $\mathscr{O}$-pseudo-algebras. Its underlying category is $\mathscr{O}$-$\mathbf{PsAlg}$, so its objects and morphisms are the $\mathscr{O}$-pseudoalgebras and $\mathscr{O}$-pseudomorphisms of [9, Section 2.3]. Its k-morphisms are the tuples (F, δ_i), where

$$F \colon \mathscr{A}_1 \times \cdots \times \mathscr{A}_k \longrightarrow \mathscr{B}$$

is a $\mathscr{V}$-functor such that $F(a_1, \cdots, a_k) = 0$ in $\mathscr{B}$ if any object $a_i = 0$ in $\mathscr{A}_i$, and the δ_i, $1 \le i \le k$, are sequences of invertible $\mathscr{V}$-transformations

$$\delta_i(n) \colon \theta_n \circ (\mathrm{id} \times F^n) \circ t_i \longrightarrow F \circ (\mathrm{id} \times \theta_n)$$

(given by maps in $\mathscr{B}$) as indicated in the following diagram.

$$\begin{tikzcd}
\mathscr{O}(n) \times (\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^n \arrow[rr, "\mathrm{id} \times F^n"] & & \mathscr{O}(n) \times \mathscr{B}^n \arrow[dd, "\theta_n"] \\
\mathscr{A}_1 \times \cdots \times \mathscr{O}(n) \times \mathscr{A}_i^n \times \cdots \times \mathscr{A}_k \arrow[u, "t_i"] \arrow[d, "\mathrm{id} \times \theta_n \times \mathrm{id}"'] \arrow[r, phantom, "\Downarrow \delta_i(n)"] & {} & \\
\mathscr{A}_1 \times \cdots \times \mathscr{A}_k \arrow[rr, "F"'] & & \mathscr{B}
\end{tikzcd} \tag{3.3}$$

We require $\delta_i(0) = \mathrm{id}_0$. For $n \ge 1$, we require the $\delta_i(n)$ to satisfy the following axioms. In the diagrams of (i), (ii), and (iv) below, we have omitted from the

[4]The elementary maps t_i correspond to the "strengths" t_i in Hyland and Power [11, p. 156]. In their categorical treatment, the existence of t_i with suitable properties is an axiom, although they do make the strengths explicit in the case of permutative categories.

notations the inactive variables $\mathscr{A}_j$ for $j \neq i$ and the concomitant diagonal maps and transpositions implicit in the strengths t_i. Similarly, in the diagrams of (v), we have omitted the inactive variables $\mathscr{A}_h$ for $h \neq i$ or j. In (iv) and (v), we have written μ for composites of the form

$$\mathscr{O}(n) \times \textstyle\prod_r \left(\mathscr{O}(m_r) \times \mathscr{A}^{m_r}\right) \xrightarrow{\pi} \mathscr{O}(n) \times \left(\prod_r \mathscr{O}(m_r)\right) \times \mathscr{A}^m \xrightarrow{\gamma \times \mathrm{id}} \mathscr{O}(m) \times \mathscr{A}^m$$

where $m = \sum m_r$, π is the evident permutation, and γ is a structure map of $\mathscr{O}$. If $\mathscr{A}$ is an $\mathscr{O}$-pseudoalgebra [9, Definition 2.14], it comes equipped with pseudofunctors

$$\begin{tikzcd}
\mathscr{O}(n) \times \prod_r \left(\mathscr{O}(m_r) \times \mathscr{A}^{m_r}\right) \arrow[r, "\mathrm{id} \times \prod_r \theta_{m_r}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Downarrow \phi"] & \mathscr{O}(n) \times \mathscr{A}^n \arrow[d, "\theta_n"] \\
\mathscr{O}(m) \times \mathscr{A}^m \arrow[r, "\theta_m"'] & \mathscr{A}.
\end{tikzcd}$$

(i) (Equivariance) For any permutation $\rho \in \Sigma_n$, the 2-cell

$$\begin{tikzcd}
\mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times \rho"] \arrow[d, "\rho \times \mathrm{id}"'] & \mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times F^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\Downarrow \delta_i(n)"] & \mathscr{O}(n) \times \mathscr{B}^n \arrow[d, "\theta_n"] \\
\mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\theta_n"'] & \mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

is equal to the 2-cell

$$\begin{tikzcd}
\mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times F^n"] \arrow[d, "\rho \times \mathrm{id}"'] & \mathscr{O}(n) \times \mathscr{B}^n \arrow[d, "\rho \times \mathrm{id}"] \\
\mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times F^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\Downarrow \delta_i(n)"] & \mathscr{O}(n) \times \mathscr{B}^n \arrow[d, "\theta_n"] \\
\mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}.
\end{tikzcd}$$

(ii) (Unit Object) $\delta_i(n) = \mathrm{id}_0$ if, in a given object of the domain of $\delta_i(n)$, either $a_j \in \mathscr{A}_j$ is 0 for some $j \neq i$ or all coordinates $a_{i,r}$ of the ith object $a_i \in \mathscr{A}_i^n$ are 0. Moreover, for $1 \leq r \leq n$, the 2-cell

$$\begin{tikzcd}
\mathscr{O}(n) \times \mathscr{A}_i^{n-1} \arrow[r, "\mathrm{id} \times \sigma_r"] \arrow[d, "\sigma_r \times \mathrm{id}"'] & \mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times F^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\Downarrow \delta_i(n)"] & \mathscr{O}(n) \times \mathscr{B}^n \arrow[d, "\theta_n"] \\
\mathscr{O}(n-1) \times \mathscr{A}_i^{n-1} \arrow[r, "\theta_{n-1}"'] & \mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

is equal to the 2-cell

$$\begin{tikzcd}
\mathscr{O}(n) \times \mathscr{A}_i^{n-1} \arrow[r, "\mathrm{id} \times F^{n-1}"] \arrow[d, "\sigma_r \times \mathrm{id}"'] & \mathscr{O}(n) \times \mathscr{B}^{n-1} \arrow[d, "\sigma_r \times \mathrm{id}"] \\
\mathscr{O}(n-1) \times \mathscr{A}_i^{n-1} \arrow[r, "\mathrm{id} \times F^{n-1}"] \arrow[d, "\theta_{n-1}"'] \arrow[dr, phantom, "\Downarrow \delta_i(n-1)"] & \mathscr{O}(n-1) \times \mathscr{B}^{n-1} \arrow[d, "\theta_{n-1}"] \\
\mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}.
\end{tikzcd}$$

(iii) (Operadic Identity): The component of $\delta_i(1)$ at an object

$$(a_1, \ldots, a_{i-1}, (\mathrm{id}, a_i), a_{i+1}, \ldots, a_k)$$

is the identity map, where $\mathrm{id} \in \mathscr{O}(1)$ is the identity object of the operad $\mathscr{O}$.

(iv) (Operadic Composition): Consider the composite 2-cells

$$\begin{tikzcd}
\mathscr{O}(n) \times \prod_r \left(\mathscr{O}(m_r) \times \mathscr{A}_i^{m_r}\right) \arrow[r, "\mathrm{id} \times \prod_r (\mathrm{id} \times F^{m_r})"] \arrow[d, "\mathrm{id} \times (\prod_r \theta_{m_r})"'] \arrow[dr, phantom, "\Downarrow \mathrm{id} \times (\prod_r \delta_i(m_r))"] & \mathscr{O}(n) \times \prod_r \left(\mathscr{O}(m_r) \times \mathscr{B}^{m_r}\right) \arrow[d, "\mathrm{id} \times (\prod_r \theta_{m_r})"] \\
\mathscr{O}(n) \times \mathscr{A}_i^n \arrow[r, "\mathrm{id} \times F^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\Downarrow \delta_i(n)"] & \mathscr{O}(n) \times \mathscr{B}^n \arrow[d, "\theta_n"] \\
\mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{O}(n) \times \prod_r \left(\mathscr{O}(m_r) \times \mathscr{A}_i^{m_r}\right) \arrow[r, "\mathrm{id} \times \prod_r (\mathrm{id} \times F^{m_r})"] \arrow[d, "\mu"'] & \mathscr{O}(n) \times \prod_r \left(\mathscr{O}(m_r) \times \mathscr{B}^{m_r}\right) \arrow[d, "\mu"] \\
\mathscr{O}(m) \times \mathscr{A}_i^m \arrow[r, "\mathrm{id} \times F^m"] \arrow[d, "\theta_m"'] \arrow[dr, phantom, "\Downarrow \delta_i(m)"] & \mathscr{O}(m) \times \mathscr{B}^m \arrow[d, "\theta_m"] \\
\mathscr{A}_i \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

If the $\mathscr{A}_i$ and $\mathscr{B}$ are $\mathscr{O}$-algebras, these 2-cells must be equal. If the $\mathscr{A}_i$ and $\mathscr{B}$ are $\mathscr{O}$-pseudoalgebras, then transformations ϕ for $\mathscr{A}_i$ and $\mathscr{B}$ map the left and right vertical composites of the first 2-cell to the left and right vertical composites of the second 2-cell. The 2-cell obtained by pasting $\phi_{\mathscr{B}}$ to the right of the first 2-cell must then be equal to the 2-cell obtained by pasting $\phi_{\mathscr{A}_i}$ to the left of the second 2-cell.

(v) (Commutation of δ_i and δ_j): For $i \neq j$, the 2-cell

$$\begin{tikzcd}
\mathscr{O}(m)\times\mathscr{A}_i^m\times\mathscr{O}(n)\times\mathscr{A}_j^n \arrow[rr, "\mathrm{id}\times\theta_n"] \arrow[d, "t_i"'] & & \mathscr{O}(m)\times\mathscr{A}_i^m\times\mathscr{A}_j \arrow[r, "\theta_m\times\mathrm{id}"] \arrow[d, "t_i"'] & \mathscr{A}_i\times\mathscr{A}_j \arrow[ddd, "F"] \\
\mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{O}(n)\times\mathscr{A}_j^n)^m \arrow[rr, "\mathrm{id}\times(\mathrm{id}\times\theta_n)^m"] \arrow[d, "\mathrm{id}\times t_j^m"'] \arrow[drr, phantom, "\mathrm{id}\times\delta_j(n)^m\ \Uparrow"] & & \mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{A}_j)^m \arrow[d, "\mathrm{id}\times F^m"'] \arrow[ur, phantom, "\delta_i(m)\ \Nearrow"] & \\
\mathscr{O}(m)\times(\mathscr{O}(n)\times(\mathscr{A}_i\times\mathscr{A}_j)^n)^m \arrow[r, "\mathrm{id}\times(\mathrm{id}\times F^n)^m"'] \arrow[d, "\mu"'] & \mathscr{O}(m)\times(\mathscr{O}(n)\times\mathscr{B}^n)^m \arrow[r, "\mathrm{id}\times\theta_n^m"'] \arrow[d, "\mu"'] \arrow[drr, phantom, "\phi^{-1}\ \Uparrow"] & \mathscr{O}(m)\times\mathscr{B}^m \arrow[dr, "\theta_m"'] & \\
\mathscr{O}(mn)\times_{\Sigma_{mn}}(\mathscr{A}_i\times\mathscr{A}_j)^{mn} \arrow[r, "\mathrm{id}\times F^{mn}"'] & \mathscr{O}(mn)\times_{\Sigma_{mn}}\mathscr{B}^{mn} \arrow[rr, "\theta(mn)"'] & & \mathscr{B}
\end{tikzcd}$$

is equal to the 2-cell obtained by pasting the 2-cell

$$\begin{tikzcd}
& \mathscr{O}(m)\times\mathscr{A}_i^m\times\mathscr{O}(n)\times\mathscr{A}_j^n \arrow[dl, "t_i"'] \arrow[dr, "t_j"] \arrow[dd, phantom, "\alpha_{m,n}\ \Uparrow"] & \\
\mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{O}(n)\times\mathscr{A}_j^n)^m \arrow[d, "\mathrm{id}\times(t_j)^m"'] & & \mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{O}(n)\times\mathscr{A}_j^n)^m \arrow[d, "\mathrm{id}\times(t_i)^n"] \\
\mathscr{O}(m)\times(\mathscr{O}(n)\times(\mathscr{A}_i\times\mathscr{A}_j)^n)^m \arrow[dr, "\mu"'] & & \mathscr{O}(n)\times(\mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{A}_j)^m)^n \arrow[dl, "\mu"] \\
& \mathscr{O}(mn)\times_{\Sigma_{nm}}(\mathscr{A}_i\times\mathscr{A}_j)^{mn} &
\end{tikzcd}$$

to the left column of the 2-cell

$$\begin{tikzcd}
\mathscr{O}(m)\times\mathscr{A}_i^m\times\mathscr{O}(n)\times\mathscr{A}_j^n \arrow[rr, "\theta_m\times\mathrm{id}"] \arrow[d, "t_j"'] & & \mathscr{A}_i\times\mathscr{O}(n)\times\mathscr{A}_j^n \arrow[r, "\mathrm{id}\times\theta_n"] \arrow[d, "t_j"'] & \mathscr{A}_i\times\mathscr{A}_j \arrow[ddd, "F"] \\
\mathscr{O}(n)\times(\mathscr{O}(m)\times\mathscr{A}_i^m\times\mathscr{A}_j)^n \arrow[rr, "\mathrm{id}\times(\theta_m\times\mathrm{id})^n"] \arrow[d, "\mathrm{id}\times t_i^n"'] \arrow[drr, phantom, "\mathrm{id}\times\delta_i(m)^n\ \Uparrow"] & & \mathscr{O}(n)\times(\mathscr{A}_i\times\mathscr{A}_j)^n \arrow[d, "\mathrm{id}\times F^n"'] \arrow[ur, phantom, "\delta_j(n)\ \Nearrow"] & \\
\mathscr{O}(n)\times(\mathscr{O}(m)\times(\mathscr{A}_i\times\mathscr{A}_j)^m)^n \arrow[r, "\mathrm{id}\times(\mathrm{id}\times F^m)^n"'] \arrow[d, "\mu"'] & \mathscr{O}(n)\times(\mathscr{O}(m)\times\mathscr{B}^m)^n \arrow[r, "\mathrm{id}\times\theta_m^n"'] \arrow[d, "\mu"'] \arrow[drr, phantom, "\phi^{-1}\ \Uparrow"] & \mathscr{O}(n)\times\mathscr{B}^n \arrow[dr, "\theta_n"'] & \\
\mathscr{O}(nm)\times_{\Sigma_{nm}}(\mathscr{A}_i\times\mathscr{A}_j)^{nm} \arrow[r, "\mathrm{id}\times F^{nm}"'] & \mathscr{O}(nm)\times_{\Sigma_{nm}}\mathscr{B}^{nm} \arrow[rr, "\theta(nm)"'] & & \mathscr{B}.
\end{tikzcd}$$

We generally abbreviate (F,δ_i) to F. The right action of Σ_k on the k-morphisms sends $F\colon \mathscr{A}_1\times\cdots\times\mathscr{A}_k \longrightarrow \mathscr{B}$ to $F\sigma\colon \mathscr{A}_{\sigma(1)}\cdots,\mathscr{A}_{\sigma(k)}\longrightarrow\mathscr{B}$ for $\sigma\in\Sigma_k$, where[5]

$$(F\sigma)(a_{\sigma(1)},\cdots,a_{\sigma(k)}) = F(a_1,\cdots,a_k).$$

Permuting the indices, the δ_i for $F\sigma$ are inherited from the δ_i for F. Precisely, $\delta_{\sigma^{-1}(i)}(n)$ for $F\sigma$ is induced from $\delta_i(n)$ for F by whiskering the defining diagram (3.3) with the following diagram, which is easily checked to be commutative.

$$\begin{tikzcd}
\mathscr{O}(n)\times(\mathscr{A}_{\sigma(1)}\times\cdots\times\mathscr{A}_{\sigma(k)})^n \arrow[r, "\mathrm{id}\times\sigma^n"] & \mathscr{O}(n)\times(\mathscr{A}_1\times\cdots\times\mathscr{A}_k)^n \\
\mathscr{A}_{\sigma(1)}\times\cdots\times\mathscr{O}(n)\times\mathscr{A}_i^n\times\cdots\times\mathscr{A}_{\sigma(k)} \arrow[u, "t_{\sigma^{-1}(i)}"'] \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}\times\theta_n\times\mathrm{id}"'] & \mathscr{A}_1\times\cdots\times\mathscr{O}(n)\times\mathscr{A}_i^n\times\cdots\times\mathscr{A}_k \arrow[u, "t_i"'] \arrow[d, "\mathrm{id}\times\theta_n\times\mathrm{id}"] \\
\mathscr{A}_{\sigma(1)}\times\cdots\times\mathscr{A}_{\sigma(k)} \arrow[r, "\sigma"'] & \mathscr{A}_1\times\cdots\times\mathscr{A}_k
\end{tikzcd}$$

Since $i=\sigma\sigma^{-1}(i)$, the term $\mathscr{O}(n)\times\mathscr{A}_i^n$ appears in the $\sigma^{-1}(i)$th factor of the term on the middle left.

The identity functor of $\mathscr{A}$ gives the unit element $\mathrm{id}_a\in\mathbf{Mult}(\mathscr{O})(\mathscr{A};\mathscr{A})$. The composition multiproduct

$$\begin{tikzcd}
\mathbf{Mult}(\mathscr{O})(\mathscr{B}_\mathbf{1},\cdots,\mathscr{B}_\mathbf{k});\mathscr{C})\times\times_{\mathbf{1}\le\mathbf{q}\le\mathbf{k}}\mathbf{Mult}(\mathscr{O})(\mathscr{A}_{\mathbf{q},\mathbf{1}},\cdots,\mathscr{A}_{\mathbf{q},\mathbf{j_q}};\mathscr{B}_\mathbf{q}) \arrow[d, "\gamma"] \\
\mathbf{Mult}(\mathscr{O})(\times_{\mathbf{1}\le\mathbf{q}\le\mathbf{k},\mathbf{1}\le\mathbf{r}\le\mathbf{j_q}}\mathscr{A}_{\mathbf{q},\mathbf{r}};\mathscr{C})
\end{tikzcd}$$

is given by

$$\gamma(F;E_1,\cdots,E_k) = F\circ(E_1\times\cdots\times E_k). \tag{3.4}$$

To see that this composite $\mathscr{V}$-functor is j-linear, where $j=j_1+\cdots j_k$, we observe that the composite of the qth distributivity map δ_q for F and $F(\delta_r)$, where δ_r is the rth distributivity map for E_q gives the needed (q,r)th distributivity map for $\gamma(F;E_1,\cdots,E_k)$, as illustrated in the pasting diagram

$$\begin{tikzcd}
\mathscr{O}(n)\times\mathscr{A}_{q,r}^n \arrow[r, "\mathrm{id}\times E_q^n"] \arrow[d, "\theta_n"'] \arrow[dr, phantom, "\Downarrow\delta_r^{E_q}(n)"] & \mathscr{O}(n)\times\mathscr{B}_q^n \arrow[r, "\mathrm{id}\times F^n"] \arrow[d, "\theta_n"] \arrow[dr, phantom, "\Downarrow\delta_q^F(n)"] & \mathscr{O}(n)\times\mathscr{C}^n \arrow[d, "\theta_n"] \\
\mathscr{A}_{q,r} \arrow[r, "E_q"'] & \mathscr{B}_q \arrow[r, "F"'] & \mathscr{C}
\end{tikzcd}$$

Finally, we must specify the morphisms between k-morphisms that prescribe the double structure, as in (1.5). Let $(E,\delta_i^E(n))$ and $(F,\delta_i^F(n))$ be k-linear maps $\underline{\mathscr{A}}\longrightarrow\mathscr{B}$ and $\underline{\mathscr{C}}\longrightarrow\mathscr{D}$, where $\underline{\mathscr{A}}=\mathscr{A}_1\times\cdots\times\mathscr{A}_k$ and $\underline{\mathscr{C}}=\mathscr{C}_1\times\cdots\times\mathscr{C}_k$. As in (1.5), a vertical map $(\underline{H},J,\omega)\colon E\longrightarrow F$ consists of $\mathscr{O}(n)$-pseudomorphisms $H_i\colon\mathscr{A}_i\longrightarrow\mathscr{C}_i$ and $J\colon\mathscr{B}\longrightarrow\mathscr{D}$ together with a $\mathscr{V}$-transformation $\omega\colon J\circ E\longrightarrow F\circ\underline{H}$ such that the pasting diagram (with product symbols generally omitted)

[5] Note that $(F\sigma)\tau = F(\sigma\tau)$, both mapping $\mathscr{A}_{\sigma\tau(1)}\times\cdots\times\mathscr{A}_{\sigma\tau(k)}$ to $\mathscr{B}$.

$$
\begin{tikzcd}
\mathscr{A}_1\cdots\mathscr{A}_{i-1}(\mathscr{O}(n)\times\mathscr{A}_i^n)\mathscr{A}_{i+1}\cdots\mathscr{A}_k \arrow[r, "\mathrm{id}\times\theta_n\times\mathrm{id}"] \arrow[d, "H_1\cdots H_{i-1}(\mathrm{id}\times H_i^n)H_{i+1}\cdots H_k"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}\times(\mathrm{id}\times\zeta_n^{-1})\times\mathrm{id}"] & \underline{\mathscr{A}} \arrow[r, "E"] \arrow[d, "\underline{H}"'] \arrow[dr, phantom, "\swarrow\omega"] & \mathscr{B} \arrow[d, "J"] \\
\mathscr{C}_1\cdots\mathscr{C}_{i-1}(\mathscr{O}(n)\times\mathscr{C}_i^n)\mathscr{C}_{i+1}\cdots\mathscr{C}_k \arrow[r, "\mathrm{id}\times\theta_n\times\mathrm{id}"'] & \underline{\mathscr{C}} \arrow[r, "F"'] & \mathscr{D}
\end{tikzcd}
$$

is equal to the pasting diagram

$$
\begin{tikzcd}
\mathscr{A}_1\cdots\mathscr{A}_{i-1}(\mathscr{O}(n)\times\mathscr{A}_i^n)\mathscr{A}_{i+1}\cdots\mathscr{A}_k \arrow[rr, "\mathrm{id}\times\theta_n\times\mathrm{id}"] \arrow[dr, "t_i"] \arrow[ddd, "\underline{H}"'] & & \underline{\mathscr{A}} \arrow[rr, "E"] \arrow[d, phantom, "\Downarrow \delta_i^E(n)^{-1}"] & & \mathscr{B} \arrow[ddd, "J"] \\
& \mathscr{O}(n)\times\underline{\mathscr{A}}^n \arrow[rr, "\mathrm{id}\times E^n"] \arrow[d, "\mathrm{id}\times\underline{H}^n"'] \arrow[drr, phantom, "\Downarrow \mathrm{id}\times\omega^n"] & {} & \mathscr{O}(n)\times\mathscr{B}^n \arrow[ur, "\theta_n"'] \arrow[d, "\mathrm{id}\times J^n"'] \arrow[dr, phantom, "\Downarrow \zeta_n^{-1}"] & \\
& \mathscr{O}(n)\times\underline{\mathscr{C}}^n \arrow[rr, "\mathrm{id}\times F^n"'] & {} & \mathscr{O}(n)\times\mathscr{D}^n \arrow[dr, "\theta_n"'] & {} \\
\mathscr{C}_1\cdots\mathscr{C}_{i-1}(\mathscr{O}(n)\times\mathscr{C}_i^n)\mathscr{C}_{i+1}\cdots\mathscr{C}_k \arrow[rr, "\mathrm{id}\times\theta_n\times\mathrm{id}"'] \arrow[ur, "t_i"'] & & \underline{\mathscr{C}} \arrow[rr, "F"'] \arrow[u, phantom, "\Downarrow \delta_i^F(n)"] & & \mathscr{D}
\end{tikzcd}
$$

The ζ_n are structure maps of $\mathscr{O}$-pseudomorphisms, as in [9, Definition 2.23].

The definition requires explanation. The axioms encode the idea that whenever the δ_i combine to give two transformations with the same source functor and the same target functor, both with target category $\mathscr{B}$, then they are equal. There is an implicit coherence theorem saying that the diagrams we display generate all others. In all of our diagrams, the interior subdiagrams unoccupied by a 2-cell $\Rightarrow$ commute either by the definition of an operad or by a naturality diagram. To make sense of these diagrams as pasting diagrams, we often compose ("whisker") a natural transformation with a functor. Note that the only axioms that differ in their statements for $\mathscr{O}$-algebras and $\mathscr{O}$-pseudoalgebras are (iv) and (v).

Axioms (i) and (ii) correspond to the Equivariance and Unit Object Axioms in the definition, [9, Definition 2.14], of an $\mathscr{O}$-pseudoalgebra. We must check that the source and target functors of the two diagrams displayed in each are equal. In both, the target functors agree trivially. The source functors in (i) agree by the Equivariance Axiom and the fact that $F^n\circ\rho = \rho\circ F^n$. The source functors in (ii) agree by the Unit Object Axiom and the fact that $F^n\circ\sigma_r = \sigma_r\circ F^{n-1}$. Axiom (iii) corresponds to the Operadic Identity Axiom and requires no explanation.

In Axiom (v), we pass to orbits over the action of Σ_{mn} in the targets of these maps. Here the target functors of the first and third diagrams agree trivially but their source functors do not; their left vertical composites differ. After pasting the second diagram to the third, the source functors of the first diagram and the composite agree trivially, but it is not at all obvious that the pseudo-commutativity isomorphism $\alpha_{m,n}$ inserted in the second diagram mediates between its source and target functors, and in fact it does not. This only becomes true after whiskering $\alpha_{m,n}$ with the common bottom composite $\theta_{mn}\circ(\mathrm{id}\times F^{mn})$ in the first and third diagrams. Passage to orbits is essential to make this true.

To see this, let a_q, $1\le q\le m$, and b_r, $1\le r\le n$ be objects (or morphisms) of $\mathscr{A}_i$ and $\mathscr{A}_j$, respectively, and let μ and ν be objects (or morphisms) of $\mathscr{O}(m)$ and

$\mathscr{O}(n)$. Traversing the source functor of the second diagram, starting from

$$(\mu, a_1, \cdots, a_m, \nu, b_1, \cdots, b_n)$$

and remembering that $\mu \wedge \nu = \gamma(\mu; \nu^m)$, we obtain

$$(\mu \wedge \nu, (a_1, b_1), \cdots, (a_1, b_n), \cdots, (a_m, b_1), \cdots, (a_m, b_n)).$$

Traversing the target functor of the second diagram, we obtain

$$(\nu \wedge \mu; (a_1, b_1), \cdots, (a_m, b_1), \cdots, (a_1, b_n), \cdots, (a_m, b_n)).$$

In the first we see reverse lexicographic ordering, and in the second we see lexicographic ordering. Composing the first of these with $\theta_{mn} \circ (\mathrm{id} \times F^{mn})$ gives

$$\theta_{mn}(\mu \wedge \nu, \tau_{n,m}(F(a_1, b_1), \cdots, F(a_m, b_1), \cdots, F(a_1, b_n), \cdots, F(a_m, b_n)).$$

Since we have passed to orbits over Σ_{mn} and θ_{mn} is equivariant, this is equal to

$$\theta_{mn}((\mu \wedge \nu)\tau_{n,m}, F(a_1, b_1), \cdots, F(a_m, b_1), \cdots, F(a_1, b_n), \cdots, F(a_m, b_n).$$

Composing the second with $\theta_{mn} \circ (\mathrm{id} \times F^{mn})$ gives

$$\theta_{mn}(\nu \wedge \mu, F(a_1, b_1), \cdots, F(a_m, b_1), \cdots, F(a_1, b_n), \cdots, F(a_m, b_n)).$$

Remembering that $\tau_{n,m}$ in the diagram defining $\alpha_{n,m}$ in Definition 2.9 is right multiplication by $\tau_{n,m}$, we see that $\theta_{mn} \circ (\mathrm{id} \times F^{mn}) \circ \alpha_{m,n}$ does indeed map the source functor of the first diagram to the source functor of the third.

In the pasting diagram describing the δ's for multiproducts, we omitted the variables $\mathscr{A}_{p,s}$ for $p \neq q$ and $s \neq r$ from the notation. Thus E_q^n in the two left horizontal maps is implicitly its product with the E_p for $p \neq q$. Similarly, the transformation on the left is really of the form $\mathrm{id} \times \delta_r^{E_q}(n) \times \mathrm{id}$.

There are verifications to be made that this all really does specify a multicategory. For example, the diagram in Definition 2.9 expressing the condition $\alpha_{m,n} = \alpha_{n,m}^{-1}$ is used in the verification that $F\sigma$ satisfies Axiom (v) when F does.

However, further details are both unilluminating and unnecessary. Although we have also generalized from algebras to pseudoalgebras, we have basically just translated the axioms of Hyland and Power to our operadic setting. Their [11, Proposition 18] applies to show that $\mathbf{Mult}(\mathscr{O})$ is a multicategory enriched in $\mathbf{Cat}$, and we have gone a bit further by prescribing a double multicategory structure. In particular, we learned from them the central role played by pseudo-commutativity.

3.2. **Variants of the definition.** There are several visible variants of the multicategory $\mathbf{Mult}(\mathscr{O})$. We defined it using pseudoalgebras over $\mathscr{O}$, which are symmetric monoidal categories when $\mathscr{O} = \mathscr{P}$. We obtain a submulticategory $\mathbf{Mult}_{\mathbf{St}}(\mathscr{O})$ by restricting to strict algebras over $\mathscr{O}$, which are permutative categories when $\mathscr{O} = \mathscr{P}$. We have required the distributivity transformations δ_i to all be isomorphisms. We could relax that, but our applications do not require lax transformations and the proof of the key strictification theorem would be a little more demanding.

When we restrict from pseudoalgebras to algebras, the multicategory $\mathbf{Mult}(\mathscr{P})$ obtained from the permutativity operad $\mathscr{P}$ is equivalent to the multicategory of permutative (alias symmetric strict monoidal) categories defined by Hyland and Power [11] and to the multicategory of permutative categories defined by Elmendorf and Mandell [3], with one caveat.

Scholium 3.5. In [3], the distributivity arrows δ_i point the other way. Since we require them to be isomorphisms, that is irrelevant to us. The difference matters if we relax the isomorphism requirement. For example, the strengths t_i of Notation 3.1 would no longer be relevant with the opposite choice, so the definition in [3] would no longer be a specialization of [11] and would not be compatible with the conventions of Corner and Gurski [2]. When considering symmetric bimonoidal and bipermutative categories, it would also be incompatible with LaPlaza's coherence theory [12, 13] and would lead to some erroneous conclusions [17, Scholium 12.3].

The work of [3] uses the classical biased definition of permutative categories rather than its unbiased operadic equivalent, and that simplifies the details of the construction of $\mathbf{Mult}(\mathscr{P})$ and of comparisons of operadic algebraic structures with their classical biased equivalents.

With our unbiased operadic reformulation, the equivariant generalization is immediate. We just replace $\mathscr{V}$ by the category $G\mathscr{V}$ of G-objects in $\mathscr{V}$ and use the operad $\mathscr{P}_G$, viewed via [9, 1.4] as an operad in $\mathbf{Cat(G\mathscr{V})}$. Thus we obtain the multicategory $\mathbf{Mult}(\mathscr{P}_G)$ of genuine symmetric monoidal G-$\mathscr{V}$-categories and the multicategory $\mathbf{Mult_{St}}(\mathscr{P}_G)$ of genuine permutative G-categories. We reiterate that the only definition of a genuine permutative or genuine symmetric monoidal G-category we have is that it is a $\mathscr{P}_G$–algebra or a $\mathscr{P}_G$–pseudoalgebra [4, 9] so that working operadically is imperative for the equivariant generalization.

4. Pseudo-commutative categories of operators

4.1. **The definition of pseudo-commutative categories of operators.** In analogy with our definition of pseudo-commutativity of an operad, we define a compatible notion of pseudo-commutativity of a reduced category of operators $\mathscr{D}$. Reduced means that $\mathscr{D}(\mathbf{m},\mathbf{n}) = *$ when either $m = 0$ or $n = 0$ and we assume that $\mathscr{D}$ is reduced throughout.

We take for granted the preliminaries on $\mathscr{V}$-2-categories, that is categories enriched in $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$, that are given in [9, Section 1.1] and [5, Section 1.1]. Recall that $\mathscr{F}$ is the category of finite based sets $\mathbf{n}$ and Π is its subcategory of projections, injections, and permutations. Both are permutative under the smash product $\wedge$ of finite based sets, defined using lexicographical ordering. Their symmetry isomorphisms τ are given by conjugation with the permutations $\tau_{m,n}$ of Definition 2.6, which reorder the sets $\mathbf{mn}$ from lexicographic to reversed lexicographic ordering; compare Remark 2.8. As in [9, Remark 1.4] and [5, Remark 2.2], we regard Π and $\mathscr{F}$ as $\mathscr{V}$-categories and as $\mathscr{V}$-2-categories with only identity 2-cells by applying the functor from sets to $\mathscr{V}$ obtained by taking coproducts over elements of copies of $*$.

Recall from [5, Definition 2.4] that a category of operators, abbreviated $\mathscr{K}$-**CO**, is a $\mathscr{V}$-2-category over $\mathscr{F}$ and under Π, with objects the finite based sets $\mathbf{n}$. With Π and $\mathscr{F}$ replaced by Π_G and $\mathscr{F}_G$, everything we say applies equally well equivariantly to categories of operators over $\mathscr{F}_G$, as we explain in Section 6.

We start with an elementary preliminary definition.

Definition 4.1. A monoidal structure on a $\mathscr{V}$-2-category $\mathscr{D}$ is a (normal[6]) $\mathscr{V}$-pseudofunctor

$$\wedge\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}$$

[6]Normal requires strict preservation of identity functors; see [5, Definition 1.2].

that is strictly associative and strictly unital with respect to a unit $\eta\colon * \longrightarrow \mathscr{D}$. The monoidal structure is strict if $\wedge$ is a $\mathscr{V}$-functor.

Definition 4.2. A $\mathscr{K}$-**CO** $\mathscr{D}$ over $\mathscr{F}$ is pseudo-commutative if it is strict monoidal under a $\mathscr{V}$-pseudofunctor $\wedge\colon \mathscr{D}\times\mathscr{D} \longrightarrow \mathscr{D}$ and has a symmetry given by a $\mathscr{V}$-pseudotransformation

$$\begin{tikzcd}
\mathscr{D}\times\mathscr{D} \arrow[rr, "t"] \arrow[dr, "\wedge"'] & \arrow[d, phantom, "\overset{\tau}{\Rightarrow}"] & \mathscr{D}\times\mathscr{D} \arrow[dl, "\wedge"] \\
& \mathscr{D}. &
\end{tikzcd}$$

These data are required to satisfy the following properties.

(i) The $\mathscr{V}$-2-functors $\iota\colon \Pi \longrightarrow \mathscr{D}$ and $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ commute with $\wedge$ and units and preserve the symmetry.

(ii) The following composites are $\mathscr{V}$-2-functors rather than just $\mathscr{V}$-pseudofunctors.

$$\mathscr{D}\times\Pi \xrightarrow{\mathrm{id}\times\iota} \mathscr{D}\times\mathscr{D} \xrightarrow{\wedge} \mathscr{D} \qquad \Pi\times\mathscr{D} \xrightarrow{\iota\times\mathrm{id}} \mathscr{D}\times\mathscr{D} \xrightarrow{\wedge} \mathscr{D}$$

(iii) The composite $\tau\circ\tau$ is the identity pseudotransformation.

Say that a map $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of categories of operators over $\mathscr{F}$ is pseudo-commutative if there is a pseudo-natural isomorphism

$$\begin{tikzcd}
\mathscr{D}\times\mathscr{D} \arrow[r, "\nu\times\nu"] \arrow[d, "\wedge"'] \arrow[dr, phantom, "\Downarrow_{\mu}"] & \mathscr{E}\times\mathscr{E} \arrow[d, "\wedge"] \\
\mathscr{D} \arrow[r, "\nu"'] & \mathscr{E}.
\end{tikzcd}$$

such that the following equality of pasting diagrams holds.

$$\begin{tikzcd}
\mathscr{D}\times\mathscr{D} \arrow[r, "\nu\times\nu"] \arrow[dd, "\wedge"'] \arrow[ddr, phantom, "\Downarrow_{\mu}"] & \mathscr{E}\times\mathscr{E} \arrow[dr, "t"] \arrow[dd, "\wedge"] & \\
& \arrow[r, phantom, "\underset{\tau}{\Leftarrow}"] & \mathscr{E}\times\mathscr{E} \arrow[dl, "\wedge"] \\
\mathscr{D} \arrow[r, "\nu"'] & \mathscr{E}. &
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{D}\times\mathscr{D} \arrow[rr, "\nu\times\nu"] \arrow[dr, "t"] \arrow[dd, "\wedge"'] & & \mathscr{E}\times\mathscr{E} \arrow[dr, "t"] & \\
\arrow[r, phantom, "\underset{\tau}{\Leftarrow}"] & \mathscr{D}\times\mathscr{D} \arrow[rr, "\nu\times\nu"] \arrow[dl, "\wedge"] & \arrow[d, phantom, "\Downarrow_{\mu}"] & \mathscr{E}\times\mathscr{E} \arrow[dl, "\wedge"] \\
\mathscr{D} \arrow[rr, "\nu"'] & & \mathscr{E} &
\end{tikzcd}$$

Condition (i) implies that $\wedge$ for $\mathscr{D}$ is given on objects by $\mathbf{m}\wedge\mathbf{n} = \mathbf{mn}$ and has unit $\iota\circ\eta$, where η is the unit of Π. Given four objects of $\mathscr{D}$, there are $\mathscr{V}$-functors

$$\wedge\colon \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{p},\mathbf{q}) \to \mathscr{D}(\mathbf{mp},\mathbf{nq}). \tag{4.3}$$

The product $\wedge$ is strictly associative and unital, and it preserves composition up to coherent invertible 2-cells. When $\mathscr{D}$ is Π or $\mathscr{F}$, these 2-cells are identity maps. In diagram form, condition (ii) says that the following diagrams commute.

$$\begin{tikzcd}
\mathscr{D}(\mathbf{n},\mathbf{p})\times\mathscr{D}(\mathbf{r},\mathbf{s})\times\Pi(\mathbf{m},\mathbf{n})\times\Pi(\mathbf{q},\mathbf{r}) \arrow[r, "\wedge\times\wedge"] \arrow[d, "(\circ\times\circ)\circ(\mathrm{id}\times t\times\mathrm{id})"'] & \mathscr{D}(\mathbf{nr},\mathbf{ps})\times\Pi(\mathbf{mq},\mathbf{nr}) \arrow[d, "\circ"] \\
\mathscr{D}(\mathbf{m},\mathbf{p})\times\mathscr{D}(\mathbf{q},\mathbf{s}) \arrow[r, "\wedge"'] & \mathscr{D}(\mathbf{mq},\mathbf{ns})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Pi(\mathbf{n},\mathbf{p}) \times \Pi(\mathbf{r},\mathbf{s}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{D}(\mathbf{q},\mathbf{r}) \arrow[r, "\wedge\times\wedge"] \arrow[d, "(\circ\times\circ)\circ(\mathrm{id}\times t\times\mathrm{id})"', "\cong"] & \Pi(\mathbf{nr},\mathbf{ps}) \times \mathscr{D}(\mathbf{mq},\mathbf{nr}) \arrow[d, "\circ"] \\
\mathscr{D}(\mathbf{m},\mathbf{p}) \times \mathscr{D}(\mathbf{q},\mathbf{s}) \arrow[r, "\wedge"'] & \mathscr{D}(\mathbf{mq},\mathbf{ns})
\end{tikzcd}$$

This implies among other things that $\wedge$ is equivariant with respect to permutations of each of its four variables in (4.3) and that 0 is a strict zero for $\wedge$ in the sense that $\wedge$ takes the value 0 if either of its inputs is 0. Condition (i) determines the 1-cell component of τ as the image of the symmetry in Π.

4.2. **The pseudo-commutativity of $\mathscr{D}(\mathscr{O})$.** In this section we prove the first part of Theorem 0.4, showing that if $\mathscr{O}$ is a pseudo-commutative operad, then $\mathscr{D} = \mathscr{D}(\mathscr{O})$ is a pseudo-commutative category of operators. The verification is essentially combinatorial bookkeeping and is painstaking rather than hard.

We break the proof into several lemmas. For a morphism $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ of $\mathscr{F}$ and $1 \le j \le n$, we write $\phi_j = |\phi^{-1}(j)|$. Then

$$\mathscr{D}(\mathbf{m},\mathbf{n}) = \coprod_{\phi} \prod_{j} \mathscr{O}(\phi_j),$$

where we take it as understood that $\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})$ and $1 \le j \le n$.

We first define the pseudofunctor

$$\wedge\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}$$

on morphism $\mathscr{V}$-categories. On objects, $\mathbf{m} \wedge \mathbf{n} = \mathbf{mn}$, and on hom categories its components

$$\mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{D}(\mathbf{p},\mathbf{q}) \longrightarrow \mathscr{D}(\mathbf{mp},\mathbf{nq})$$

are the composite $\mathscr{V}$-functors

$$\begin{tikzcd}
\coprod_{\phi} \prod_{j} \mathscr{O}(\phi_j) \times \coprod_{\psi} \prod_{k} \mathscr{O}(\phi_k) \arrow[d, "\cong"] \\
\coprod_{\phi,\psi} \prod_{j,k} \mathscr{O}(\phi_j) \times \mathscr{O}(\psi_k) \arrow[d, "\amalg\wedge"] \\
\coprod_{\phi\wedge\psi} \prod_{(j,k)} \mathscr{O}((\phi\wedge\psi)_{(j,k)}).
\end{tikzcd}$$

At the bottom, the (j,k) are understood to be elements of $\mathbf{nq}$, ordered lexicographically. The definition makes sense since

$$(\phi\wedge\psi)_{(j,k)} = |(\phi\wedge\psi)^{-1}(j,k)| = |\phi^{-1}(j)||\psi^{-1}(k)| = \phi_j\psi_k.$$

It is easy to check that $\wedge$ is strictly associative and unital, these properties following from the corresponding properties of the canonical pairing on $\mathscr{O}$.

For this to define a $\mathscr{V}$-pseudofunctor, we must prove the following result, which is the heart of the proof that $\mathscr{D}$ is pseudo-commutative and is central to our work.

Lemma 4.4. *The following diagram of $\mathscr{V}$-functors relating $\wedge$ to composition commutes up to a $\mathscr{V}$-transformation λ.*

$$
\begin{tikzcd}
\mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{r},\mathbf{s}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{D}(\mathbf{q},\mathbf{r}) \arrow[r, "\wedge\times\wedge"] \arrow[d, "(\circ\times\circ)\circ(\mathrm{id}\times t\times\mathrm{id})"'] \arrow[dr, phantom, "\lambda\Uparrow"] & \mathscr{D}(\mathbf{nr},\mathbf{ps}) \times \mathscr{D}(\mathbf{mq},\mathbf{nr}) \arrow[d, "\circ"] \\
\mathscr{D}(\mathbf{m},\mathbf{p}) \times \mathscr{D}(\mathbf{q},\mathbf{s}) \arrow[r, "\wedge"'] & \mathscr{D}(\mathbf{mq},\mathbf{ps})
\end{tikzcd}
$$

Proof. Since Π and $\mathscr{F}$ are permutative categories regarded as $\mathscr{V}$-2-categories, the diagram clearly commutes when $\mathscr{D} = \Pi$ or $\mathscr{D} = \mathscr{F}$. That is, fixing morphisms

$$\phi\colon \mathbf{m} \longrightarrow \mathbf{n}, \quad \psi\colon \mathbf{n} \longrightarrow \mathbf{p} \quad \mu\colon \mathbf{q} \longrightarrow \mathbf{r}, \quad \nu\colon \mathbf{r} \longrightarrow \mathbf{s}$$

in $\mathscr{F}$, we have

$$(\psi \wedge \nu) \circ (\phi \wedge \mu) = (\psi \circ \phi) \wedge (\nu \circ \mu). \tag{4.5}$$

It is immediate from the definition that $\wedge$ commutes with ι and ξ, as required for property (i) of Definition 4.2.

The essential combinatorial claim is that the pseudo-commutativity isomorphisms α of $\mathscr{O}$ from Definition 2.9 assemble to give the required invertible $\mathscr{V}$-transformations λ. The strategy is to express the results of the right-down and down-left compositions in such a way that the natural isomorphism between them becomes obvious. The expressions that follow may look impenetrable, but all we are doing is using the associativity and equivariance formulas from the definition of an operad to massage the two composites into comparable form.

It suffices to restrict the diagram to the summands of the coproducts in the displayed morphism $\mathscr{V}$-categories. For our fixed morphisms ϕ, ψ, μ, and ν in $\mathscr{F}$, as above, let

$$1 \le j \le n, \quad 1 \le k \le p, \quad 1 \le h \le r, \quad 1 \le i \le s.$$

Looking at the definitions of $\circ$ in [5, Definition 3.1] and of $\wedge$, we see that to chase the diagram starting on the top left with

$$\prod_k \mathscr{O}(\psi_k) \times \prod_i \mathscr{O}(\nu_i) \times \prod_j \mathscr{O}(\phi_j) \times \prod_h \mathscr{O}(\mu_h),$$

it suffices to fix k and i, project to

$$\mathscr{O}(\psi_k) \times \mathscr{O}(\nu_i) \times \prod_{\psi(j)=k} \mathscr{O}(\phi_j) \times \prod_{\nu(h)=i} \mathscr{O}(\mu_h),$$

and then chase, provided that we remember the original ordering of variables. The permutations of the form $\rho(\psi,\phi)_k$ in [5, Definition 3.1] keep track of the ordering.

Writing in terms of elements (objects and morphisms of categories, so implicitly working in the underlying category of $\mathscr{V}$) to better apply formulas rather than chase large diagrams, let

$$c \in \mathscr{O}(\psi_k), \quad e \in \mathscr{O}(\nu_i), \quad d_j \in \mathscr{O}(\phi_j), \quad f_h \in \mathscr{O}(\mu_h).$$

Going right, and down, we rewrite the source as

$$\prod_{j,h} \mathscr{O}(\psi_k) \times \mathscr{O}(\nu_i) \times \mathscr{O}(\phi_j) \times \mathscr{O}(\mu_h),$$

where we restrict to those j and h such that $\psi(j) = k$ and $\nu(h) = i$. We then apply $\prod_{j,h}(\wedge \times \wedge)$ to land in

$$\prod_{j,h} \mathscr{O}((\psi \wedge \nu)_{(k,i)}) \times \mathscr{O}((\phi \wedge \mu)_{(j,h)}),$$

which we rewrite as

$$\mathscr{O}((\psi \wedge \nu)_{(k,i)}) \times \prod_{j,h} \mathscr{O}((\phi \wedge \mu)_{(j,h)}).$$

Finally, we apply $\rho(\psi \wedge \nu, \phi \wedge \mu)_{(k,i)} \circ \gamma$ to land in

$$\mathscr{O}(((\psi \wedge \nu) \circ (\phi \wedge \mu))_{(k,i)}).$$

Remembering that $\wedge$ on $\mathscr{O}$ is given by composites of the form $\gamma \circ (\mathrm{id} \times \Delta)$, we see that, applied to $(c, e, \times_j d_j, \times_h f_h)$, this gives

$$\gamma\big(\gamma(c; e^{\psi_k}); \prod_{(\psi\wedge\nu)(j,h)=(k,i)} \gamma(d_j; f_h^{\phi_j})\big)\, \rho(\psi \wedge \nu, \phi \wedge \mu)_{(k,i)}, \tag{4.6}$$

where $e^{\psi_k} = (e, \cdots, e)$, ψ_k factors, and similarly for $f_h^{\phi_j}$.

Going down and right, we first permute in the middle to map the source to

$$\mathscr{O}(\psi_k) \times \prod_{\psi(j)=k} \mathscr{O}(\phi_j) \times \mathscr{O}(\nu_i) \times \prod_{\nu(h)=i} \mathscr{O}(\mu_h).$$

We then apply

$$(\rho(\psi, \phi)_k \circ \gamma) \times (\rho(\nu, \mu)_i \circ \gamma)$$

to land in

$$\mathscr{O}((\psi \circ \phi)_k) \times \mathscr{O}((\nu \circ \mu)_i).$$

Finally, we apply $\wedge$ to land in

$$\mathscr{O}(((\psi \circ \phi) \wedge (\nu \circ \mu))_{(k,i)}).$$

The targets are the same by (4.5). Applied to $(c, e, \times_j d_j, \times_h f_h)$, this gives

$$\gamma\big(\gamma(c; \times_{\psi(j)=k} d_j)\rho(\psi, \phi)_k;\ (\gamma(e; \times_{\nu(h)=i} f_h)\rho(\nu, \mu)_i)^{(\psi\circ\phi)_k}\big). \tag{4.7}$$

Using the associativity from the definition of an operad twice and abbreviating $\rho(\psi \wedge \nu, \phi \wedge \mu)_{(k,i)} = \rho(k, i)$, we identify the expression (4.6) as follows.

$$\begin{aligned}
&\gamma\big(\gamma(c; e^{\psi_k}); \prod_{(\psi\wedge\nu)(j,h)=(k,i)} \gamma(d_j; f_h^{\phi_j})\big)\, \rho_{(k,i)} \\
= \quad & \gamma\big(c;\ \times_{\psi(j)=k} \gamma\big(e; \times_{\nu(h)=i} \gamma(d_j; f_h^{\phi_j})\big)\big)\rho_{(k,i)} \\
= \quad & \gamma\big(c;\ \times_{\psi(j)=k} \gamma\big(\gamma(e; d_j^{\nu_i}); \times_{\nu(h)=i} f_h^{\phi_j}\big)\big)\rho_{(k,i)}
\end{aligned}$$

Using the two equivariance formulas from the definition of an operad, abbreviating $\rho(\psi, \phi)_k = \rho_k$ and $\rho(\nu, \mu)_i = \rho_i$, and writing σ^t for the block sum of t copies of a permutation σ, we identify the expression (4.7) as follows.

$$\begin{aligned} & \gamma\big(\gamma(c; \times_{\psi(j)=k} d_j)\rho_k;\ (\gamma(e; \times_{\nu(h)=i} f_h)\rho_i)^{(\psi\circ\phi)_k}\big) \\ = \quad & \gamma\big(\gamma(c; \times_{\psi(j)=k} d_j)\rho_k;\ \big(\gamma(e; \times_{\nu(h)=i} f_h)\big)^{(\psi\circ\phi)_k}\big)\big)\rho_i^{(\psi\circ\phi)_k} \\ = \quad & \gamma\big(\gamma(c; \times_{\psi(j)=k} d_j);\ \big(\gamma(e; \times_{\nu(h)=i} f_h)\big)^{(\psi\circ\phi)_k}\big)\big)\rho_k\rho_i^{(\psi\circ\phi)_k} \end{aligned}$$

The permutation $\rho_k \in \Sigma_{(\psi\circ\phi)_k}$, when pulled outside γ at the bottom, is regarded as a permutation in $\Sigma_{(\psi\circ\phi)_k(\nu\circ\mu)_i}$ by identifying it with the permutation which permutes $(\psi\circ\phi)_k$ blocks of letters, each block having $(\nu\circ\mu)_i$ letters, as ρ_k permutes $(\psi\circ\phi)_k$ letters. Again using the associativity for an operad, we further identify this last expression as:

$$\begin{aligned} = \quad & \gamma\big(c;\ \times_{\psi(j)=k}\gamma(d_j, \gamma(e; \times_{\nu(h)=i} f_h)^{\phi_j}\big)\rho_k\rho_i^{(\psi\circ\phi)_k} \\ = \quad & \gamma\big(c;\ \times_{\psi(j)=k}\gamma\big(\gamma(d_j, e^{\phi_j}); (\times_{\nu(h)=i} f_h)^{\phi_j}\big)\big)\rho_k\rho_i^{(\psi\circ\phi)_k} \end{aligned}$$

The similarity between these reinterpretations of (4.6) and (4.7) is clear. We use the α's from Definition 2.9 to build a natural isomorphism from the expression (4.6) to the expression (4.7), identified as above. This will specify λ^{-1}. For legibility, we omit the indices on the α's. From the pseudo-commutativity of $\mathscr{O}$, for fixed i and k we have an isomorphism

$$\alpha\colon \gamma(e; d_j^{\nu_i})\tau_{\phi_j,\nu_i} \longrightarrow \gamma(d_j; e^{\phi_j}).$$

It induces the second isomorphism in the following composite.

$$\begin{array}{c} \gamma\big(\gamma(e, d_j^{\nu_i});\ \times_{\nu(h)=i} f_h^{\phi_j}\big)\tau_{\phi_j,\nu_i} \\ \Big\downarrow{=} \\ \gamma\big(\gamma(e; d_j^{\nu_i})\tau_{\phi_j,\nu_i};\ \times_{\nu(h)=i} f_h^{\phi_j}\big) \\ \Big\downarrow{\scriptstyle \gamma(\alpha;\mathbf{id}^{\phi_j})} \\ \gamma\big(\gamma(d_j; e^{\phi_j});\ \times_{\nu(h)=i} f_h^{\phi_j}\big) \end{array}$$

The first equality is given by the equivariance formula for γ; the permutation τ_{ϕ_j,ν_i} in the source is interpreted as the permutation of $\phi_j \cdot \sum_{\mu(h)=i}\mu_{(h)} = \phi_j \cdot (\nu\circ\mu)_i$ elements which permutes the $\phi_j \cdot \nu_i$ blocks of lengths μ_h, where h runs through the set $\nu^{-1}(i)$.

By applying $\gamma(c;-)$ to the composite above, we obtain the second isomorphism in the composite

$$\begin{array}{c}
\gamma\big(c;\ \times_{\psi(j)=k}\gamma\big(\gamma(e;\ d_j^{\nu_i});\times_{\nu(h)=i}f_h^{\phi_j}\big)\big)\oplus_{\psi(j)=k}\tau_{\phi_j,\nu_i} \\
\big\downarrow = \\
\gamma\big(c;\times_{\psi(j)=k}\gamma\big(\gamma(e;d_j^{\nu_i});\times_{\nu(h)=i}f_h^{\phi_j}\big)\tau_{\phi_j,\nu_i}\big) \\
\big\downarrow \\
\gamma\big(c;\times_{\psi(j)=k}\gamma\big(\gamma(d_j;e^{\phi_j});\ (\times_{\nu(h)=i}f_h^{\phi_j}\big)\big)
\end{array}$$

The first equality is again given by the equivariance of γ.

A straightforward but tedious computation, which we omit, gives that

$$\rho_{(k,i)}=(\oplus_{\psi(j)=k}\tau_{\phi_j,\nu_i})\cdot(\rho_k\rho_i^{(\psi\circ\phi)_k}).$$

Therefore, multiplying the above isomorphism by $\rho_k\rho_i^{(\psi\circ\phi)_k}$ yields the desired isomorphism

$$\begin{array}{c}
\gamma\big(c;\ \times_{\psi(j)=k}\gamma\big(\gamma(e;\ d_j^{\nu_i});\times_{\nu(h)=i}f_h^{\phi_j}\big)\big)\rho_{(k,i)} \\
\big\downarrow \\
\gamma\big(c;\ \times_{\psi(j)=k}\gamma\big(\gamma(d_j;e^{\phi_j});\ (\times_{\nu(h)=i}f_h^{\phi_j}\big)\big)\rho_k\rho_i^{(\psi\circ\phi)_k}
\end{array}$$

from (4.6)) to (4.7)). Interpreting the proof diagrammatically shows that we have a $\mathscr{V}$-isomorphism making our diagram commute, and this completes the proof that $\wedge\colon\mathscr{D}\times\mathscr{D}\longrightarrow\mathscr{D}$ is a $\mathscr{V}$-pseudofunctor. □

The following observation shows that $\wedge$ satisfies condition Definition 4.2(ii).

Lemma 4.8. *The $\mathscr{V}$-pseudofunctor $\wedge\colon\mathscr{D}\times\mathscr{D}\longrightarrow\mathscr{D}$ restricts to $\mathscr{V}$-2-functors*

$$\mathscr{D}\times\Pi\xrightarrow{\mathrm{id}\times\iota}\mathscr{D}\times\mathscr{D}\xrightarrow{\wedge}\mathscr{D}\qquad\Pi\times\mathscr{D}\xrightarrow{\iota\times\mathrm{id}}\mathscr{D}\times\mathscr{D}\xrightarrow{\wedge}\mathscr{D}$$

Proof. The statement means that the diagram of Lemma 4.4 commutes if we replace $\mathscr{D}$ by Π when considering the variables $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{p}$ or if we replace $\mathscr{D}$ by Π when considering the variables $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$. This holds since the image of ι sees only $*\in\mathscr{O}(0)$ and $\mathrm{id}\in\mathscr{O}(1)$. The verification is a specialization of the previous proof.

In more detail, the inclusion $\iota\colon\Pi\longrightarrow\mathscr{D}$ sends $\phi\colon\mathbf{m}\longrightarrow\mathbf{n}$ to (ϕ,c), where $c_i=\mathrm{id}\in\mathscr{O}(1)$ if $\phi(i)=1$ and $c_i=*\in\mathscr{O}(0)$ if $\phi(i)=0$. Therefore, when we restrict $\wedge$ to $\mathscr{D}\times\Pi$, e and the f_h in the previous proof are all $*$ or id, and when we restrict to $\Pi\times\mathscr{D}$, c and the d_j are all $*$ or id. If either e or d_j is $*$, then the isomorphism

$$\alpha\colon\gamma(e;d_j^{\nu_i})\tau_{\phi_j,\nu_i}\longrightarrow\gamma(d_j;e^{\phi_j}).$$

of the previous proof is the identity map of $*$. If $e=\mathrm{id}\in\mathscr{O}(1)$, then $\tau_{\phi_j,1}=\mathrm{id}$,

$$\gamma(e;d_j)=d_j=\gamma(d_j;e^{\phi_j}),$$

and $\alpha = \mathrm{id}$. Similarly, if $d_j = \mathrm{id} \in \mathscr{O}(1)$, then $\tau_{1,\phi_j} = \mathrm{id}$,

$$\gamma(e; d_j^{\nu_i}) = e = \gamma(d_j; e^{\phi_j}),$$

and again $\alpha = \mathrm{id}$. In these cases, the natural isomorphism induced by α in the previous proof is the identity map. $\square$

For condition (i) of Definition 4.2, we have already observed that $\wedge$ commutes with ι and ξ. Therefore the following lemma completes the proof that $\mathscr{D}$ is pseudo-commutative.

Lemma 4.9. *The $\mathscr{V}$-pseudofunctor $\wedge\colon \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}$ has a strong symmetry $\mathscr{V}$-transformation τ such that the strict monoidal $\mathscr{V}$-2-functors $\iota\colon \Pi \longrightarrow \mathscr{D}$ and $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ preserve the symmetry.*

Proof. The $\mathscr{V}$-pseudofunctors we are comparing have the same object functions. Given objects $\mathbf{m}$ and $\mathbf{p}$, the 1-cell component $\tau_{\mathbf{m},\mathbf{p}}\colon \mathbf{mp} \longrightarrow \mathbf{pm}$ is given by $\tau_{m,p}$, thought of as a morphism in $\Pi \subset \mathscr{D}$. With $F = \wedge$ and $G = \wedge \circ t$, both being $\mathscr{V}$-pseudofunctors $\mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}$, we need $\mathscr{V}$-pseudotransformations

$$\tau\colon \tau_{n,q} \circ \wedge \longrightarrow \wedge \circ t \circ \tau_{m,p}.$$

As in the previous proofs, we can restrict to the components of $\mathscr{D}(\mathbf{m},\mathbf{n})$ and $\mathscr{D}(\mathbf{p},\mathbf{q})$ indexed on morphisms $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $\psi\colon \mathbf{p} \longrightarrow \mathbf{q}$ of $\mathscr{F}$ and start with $\prod_{j,k} \mathscr{O}(\phi_j) \times \mathscr{O}(\phi_k)$, where $1 \le j \le n$ and $1 \le k \le q$. Again we ignore $\mathscr{V}$ for simplicity and work with elements (objects and morphisms) $c_j \in \mathscr{O}(\phi_j)$ and $d_k \in \mathscr{O}(\psi_k)$, writing $c = \times_j c_j$ and $d = \times_k d_k$. Thus consider the following diagram.

$$\begin{tikzcd}
\mathbf{mp} \arrow[r, "{(\phi,c)\wedge(\psi,d)}"] \arrow[d, "\tau_{m,p}"'] \arrow[dr, phantom, "\Uparrow"] & \mathbf{nq} \arrow[d, "\tau_{n,q}"] \\
\mathbf{pm} \arrow[r, "{(\psi,d)\wedge(\phi,c)}"'] & \mathbf{qn}
\end{tikzcd}$$

Considering permutations as morphisms of $\Pi \subset \mathscr{D}$ and using the definition of composition in $\mathscr{D}$, [5, Definition 2.8], we find that the down-right composite is equal to

$$((\psi \wedge \phi) \circ \tau_{m,p};\ \times_{j,k}(d_k \wedge c_j)\tau_{\phi_j,\psi_k})$$

and the right-down composite is equal to

$$(\tau_{n,q} \circ (\phi \wedge \psi);\ \times_{k,j} c_j \wedge d_k),$$

where the product here is in reverse lexicographical order. As maps in $\mathscr{F}$, we have

$$(\psi \wedge \phi) \circ \tau_{m,p} = \tau_{n,q} \circ (\phi \wedge \psi).$$

Applying a product of maps α to the second variable (mapping the (j,k)th factor to the (k,j)th factor) gives a natural isomorphism as indicated in the diagram. It is quite tedious to check compatibility with composition, and we shall not bore the reader with the details. $\square$

If $\mathscr{O} \longrightarrow \mathscr{O}'$ is a map of chaotic operads, then $\mathscr{D}(\mathscr{O}) \longrightarrow \mathscr{D}(\mathscr{O}')$ is a map of pseudocommutative categories of operators over $\mathscr{F}$.

5. The multicategory of $\mathscr{D}$-pseudoalgebras

5.1. **The definition of $\mathbf{Mult}(\mathscr{D})$.** Let $\mathscr{D}$ be a (reduced) pseudo-commutative category of operators over $\mathscr{F}$ in this section. Recall the definition of the category $\mathscr{D}$-**PsAlg** of $\mathscr{D}$-pseudoalgebras from [5, Definition 2.8]. Briefly, a $\mathscr{D}$-pseudoalgebra $\mathscr{X}$ is a $\mathscr{V}$-pseudofunctor $\mathscr{X}: \mathscr{D} \longrightarrow \mathscr{K}$ satisfying certain assumptions. In adjoint form, it is given by $\mathscr{V}$-pseudofunctors

$$\theta\colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n}).$$

We require $\mathscr{X}$ to be reduced, so that $\mathscr{X}(\mathbf{0}) = *$. The morphisms $\zeta\colon \mathscr{X} \longrightarrow \mathscr{Y}$ are $\mathscr{V}$-pseudotransformations. These pseudo notions are spelled out in detail in [5, Definitions 1.9 and 1.10].

We here define the multicategory $\mathbf{Mult}(\mathscr{D})$ of $\mathscr{D}$-pseudoalgebras. The essential point is to describe coherence conditions on the k-morphisms.[7] We give the basic definitions, but we omit details of the verification that the multicategory axioms are satisfied. A more conceptual monadic description in the sequel [8, Theorem 0.11] recasts the definition in a form that makes the verifications transparent.

Recall that $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$ is a closed cartesian monoidal 2-category. We denote its symmetry isomorphism by τ. Perhaps the most conceptual definition of $\mathbf{Mult}(\mathscr{D})$ uses the hom objects in $\mathscr{V}$. However, to fill in the coherence data and to work with the definition, we give it in an adjoint form that precisely parallels the definition of $\mathbf{Mult}(\mathscr{O})$. This is in line with the definition of a $\mathscr{D}$-pseudoalgebra in adjoint form given in [5, Section 1.2 and Definition 2.8]. As there, we write

$$\theta\colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n})$$

for the evaluation $\mathscr{V}$-functors of a $\mathscr{D}$-pseudoalgebra $\mathscr{X}$.

For sequences $(n_1, \cdots, n_k)$, we abbreviate notation by setting $n = n_1 \cdots n_k$. (Note that this implicitly hides a lexicographic ordering.)

Definition 5.1. For objects $\mathscr{X}_i$, $1 \le i \le k$, and $\mathscr{Y}$ of $\mathscr{D}$-**PsAlg**, a k-morphism $(F, \delta)\colon \underline{\mathscr{X}} \longrightarrow \mathscr{Y}$ consists of $\mathscr{V}$-functors

$$F\colon \mathscr{X}_1(\mathbf{n}_1) \times \cdots \times \mathscr{X}_k(\mathbf{n}_k) \longrightarrow \mathscr{Y}(\mathbf{n})$$

that together define a natural transformation of functors $\Pi^k \longrightarrow \mathscr{K}$ together with invertible $\mathscr{V}$-transformations δ in the following diagrams, in which $1 \le i \le k$.

$$\begin{tikzcd}
\prod_i \mathscr{D}(\mathbf{m}_i, \mathbf{n}_i) \times \prod_i \mathscr{X}_i(\mathbf{m}_i) \arrow[r, "\mathrm{id} \times F"] \arrow[d, "t"'] & \prod_i \mathscr{D}(\mathbf{m}_i, \mathbf{n}_i) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\wedge \times \mathrm{id}"] \\
\prod_i \mathscr{D}(\mathbf{m}_i, \mathbf{n}_i) \times \mathscr{X}_i(\mathbf{m}_i) \arrow[d, "\prod_i \theta"'] \arrow[r, phantom, "\Downarrow \delta"] & \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\theta"] \\
\prod_i \mathscr{X}_i(\mathbf{n}_i) \arrow[r, "F"'] & \mathscr{Y}(\mathbf{n}).
\end{tikzcd} \tag{5.2}$$

Composition is given by composites of products of natural transformations.

[7] We remark that the Hyland-Power axiomatization using strengths [11] is no longer available, since here we do not see how to define the strengths they assume.

Since we require $\mathscr{D}$ and the $\mathscr{X}_i$ and $\mathscr{Y}$ to be reduced, both composites are the trivial map if any $n_i = 0$ and the 0 map if any $m_i = 0$, so the diagram then commutes trivially with δ the identity. We also require δ to be the identity if all but one factor $\mathscr{D}(\mathbf{m_i}, \mathbf{n_i})$ is restricted to $\Pi(\mathbf{m_i}, \mathbf{n_i})$ (in particular if all factors are restricted to Π). This already incorporates the analogues of axioms (i)–(iii) of Definition 3.2, and we require the following analogue of the operadic composition axiom (iv).

(Categorical Composition Axiom) Write θ^k for the left vertical composite

$$\textstyle\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \prod_j \mathscr{X}_j(m_j) \xrightarrow{\ t\ } \prod_j \big(\mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{X}_j(m_j)\big) \xrightarrow{\ \theta^k\ } \prod_j \mathscr{X}(n_j)$$

in (5.2)). Analogously, write C for the k-fold composition

$$\textstyle\prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \xrightarrow{\ t\ } \prod_j \big(\mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j)\big) \xrightarrow{\ \circ^k\ } \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{p}_j).$$

The right vertical composite

$$\textstyle\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{Y}(\mathbf{m}) \xrightarrow{\wedge\times\mathrm{id}} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \xrightarrow{\ \theta\ } \mathscr{Y}(\mathbf{n})$$

in (5.2)) is the action Θ_k. The maps named θ^k and Θ_k then play analogous conceptual roles. With these notations, the following pasting diagrams are equal. In these diagrams, the unlabeled squares commute, the ϕ denote the coherence $\mathscr{V}$-pseudotransformations for the given $\mathscr{D}$-pseudoalgebras $\mathscr{X}_j$ and $\mathscr{Y}$, and λ is an iterated coherence $\mathscr{V}$-pseudotransformation from Lemma 4.4.

$$\begin{tikzcd}
\prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{Y}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\wedge\times\mathrm{id}"] & \prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[d, phantom, "\Downarrow \mathrm{id}\times\delta"] & \prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \mathscr{Y}(\mathbf{n}) \arrow[d, "\wedge\times\mathrm{id}"] \\
\prod_j \mathscr{D}(\mathbf{n_j}, \mathbf{p_j}) \times \prod_{\mathbf{j}} \mathscr{D}(\mathbf{m_j}, \mathbf{n_j}) \times \prod_{\mathbf{j}} \mathscr{X}_{\mathbf{j}}(\mathbf{m_j}) \arrow[u, "\mathrm{id}\times F"] \arrow[r, "\mathrm{id}\times\theta^k"] \arrow[d, "C\times\mathrm{id}"] \arrow[dr, phantom, "\Downarrow \prod_j \phi"] & \prod_j \mathscr{D}(\mathbf{n_j}, \mathbf{p_j}) \times \prod_{\mathbf{j}} \mathscr{X}_{\mathbf{j}}(\mathbf{n_j}) \arrow[ur, "\mathrm{id}\times F"] \arrow[d, "\theta^k"] \arrow[dr, phantom, "\Downarrow \delta"] & \mathscr{D}(\mathbf{n}, \mathbf{p}) \times \mathscr{Y}(\mathbf{n}) \arrow[d, "\theta"] \\
\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{p}_j) \times \prod_j \mathscr{X}_j(m_j) \arrow[r, "\theta^k"'] & \prod_j \mathscr{X}_j(p_j) \arrow[r, "F"'] & \mathscr{Y}(\mathbf{p})
\end{tikzcd}$$

$$\begin{tikzcd}
\prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{Y}(\underline{m}) \arrow[r, "\mathrm{id}\times\wedge\times\mathrm{id}"] \arrow[ddr, "C\times\mathrm{id}"] \arrow[dr, phantom, "\lambda\Uparrow"] & \prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[d, "\wedge\times\mathrm{id}"] & \prod_j \mathscr{D}(\mathbf{n}_j, \mathbf{p}_j) \times \mathscr{Y}(\mathbf{n}) \arrow[d, "\wedge\times\mathrm{id}"] \\
\prod_j \mathscr{D}(\mathbf{n_j}, \mathbf{p_j}) \times \prod_{\mathbf{j}} \mathscr{D}(\mathbf{m_j}, \mathbf{n_j}) \times \prod_{\mathbf{j}} \mathscr{X}_{\mathbf{j}}(\mathbf{m_j}) \arrow[u, "\mathrm{id}\times F"] \arrow[dd, "C\times\mathrm{id}"] & \mathscr{D}(\mathbf{n}, \mathbf{p}) \times \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[dr, "C\times\mathrm{id}"] & \mathscr{D}(\mathbf{n}, \mathbf{p}) \times \mathscr{Y}(\mathbf{n}) \arrow[dd, bend left=60, "\theta"] \arrow[d, phantom, "\Downarrow\phi"] \\
 & \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{p}_j) \times \mathscr{Y}(\underline{m}) \arrow[r, "\wedge\times\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\delta"] & \mathscr{D}(\mathbf{m}, \mathbf{p}) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\theta"] \\
\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{p}_j) \times \prod_j \mathscr{X}_j(m_j) \arrow[r, "\theta^k"'] \arrow[ur, "\mathrm{id}\times F"] & \prod_j \mathscr{X}_j(\mathbf{p}_j) \arrow[r, "F"'] & \mathscr{Y}(\underline{p})
\end{tikzcd}$$

The two unmarked squares in the bottom diagram commute trivially.

The action of Σ_k on $\mathbf{Mult_k}$ is obtained by permuting the variables $\mathscr{X}_r$ of multifunctors $\underline{\mathscr{X}} \longrightarrow \mathscr{Y}$, noting that this entails a lexicographic reordering in the target. More precisely, define $(F,\delta)\sigma$ by first defining $F\sigma$ by commutativity of the diagrams

(5.3)
$$\begin{tikzcd}
\mathscr{X}_{\sigma(1)}(\mathbf{n}_{\sigma(1)}) \times \cdots \times \mathscr{X}_{\sigma(k)}(\mathbf{n}_{\sigma(k)}) \arrow[r, "F\sigma"] \arrow[d, "\sigma^{-1}"'] & \mathscr{Y}(\mathbf{n}) \\
\mathscr{X}_1(\mathbf{n}_1) \times \cdots \times \mathscr{X}(\mathbf{n}_k) \arrow[r, "F"'] & \mathscr{Y}(\mathbf{n}) \arrow[u, "\mathscr{Y}(\tau_\sigma)"']
\end{tikzcd}$$

where τ_σ is the permutation

(5.4)
$$\mathbf{n} \xrightarrow{\lambda} \mathbf{n}_1 \wedge \cdots \wedge \mathbf{n}_k \xrightarrow{t} \mathbf{n}_{\sigma(1)} \wedge \cdots \wedge \mathbf{n}_{\sigma(k)} \xrightarrow{\lambda^{-1}} \mathbf{n}$$

obtained by iteration from Definition 2.6. Then define $\delta\sigma$ by whiskering the δ of (5.2), but using the pseudocommutativity of $\mathscr{D}$. Schematically, we construct $\delta_{|si}$ by the following schematic pasting diagram, where the inner hexagon is (5.2) and the outer hexagon is the corresponding diagram for $F\sigma$; we have abbreviated $\tau_\sigma = \mathscr{Y}(\tau_\sigma)$ on both $\mathscr{Y}(\mathbf{m})$ and $\mathscr{Y}(\mathbf{n})$ and similarly on $\mathscr{D}(\mathbf{m},\mathbf{n})$.

$$\begin{tikzcd}
\bullet \arrow[rrr, "\mathrm{id}\times F\sigma"] \arrow[dr, "\sigma^{-1}\times\sigma^{-1}"] \arrow[dd, "t"'] & & & \bullet \arrow[dd, "\wedge\times\mathrm{id}"] \\
& \bullet \arrow[r, "\mathrm{id}\times F"] \arrow[d, "t"] & \bullet \arrow[ur, "\sigma\times\tau_\sigma"'] \arrow[d, "\wedge\times\mathrm{id}"'] \arrow[r, phantom, "\Swarrow \tau\times\mathrm{id}"] & \\
\bullet \arrow[r, "\sigma^{-1}"] \arrow[dd, "\prod_i \theta"'] & \bullet \arrow[d, "\prod_i \theta"] \arrow[r, phantom, "\Swarrow \delta"] & \bullet \arrow[r, "\tau_\sigma\times\tau_\sigma"] \arrow[d, "\theta"'] & \bullet \arrow[dd, "\theta"] \\
& \bullet \arrow[r, "F"'] & \bullet \arrow[dr, "\tau_\sigma"] & \\
\bullet \arrow[rrr, "F\sigma"'] \arrow[ur, "\sigma^{-1}"'] & & & \bullet
\end{tikzcd}$$

The top and bottom trapepoids commute by the definition of $F\sigma$. The left two trapezoids commute trivially. The bottom right trapezoid commutes by the equivariance of θ, as encoded in [5, Definition 2.8]. The top right trapezoid must be filled by a $\mathscr{V}$-transformation τ given by the pseudocommutativity of $\mathscr{D}$.

The double structure extends the double structure on the multicategory associated to the underlying symmetric monoidal 2-category $\mathscr{K}$.

5.2. **The proof that $\mathbb{R}$ is a multifunctor.** Let $\mathscr{O}$ be a pseudo-commutative operad in $\mathscr{K}$ with associated pseudo-commutative category of operators $\mathscr{D}$. We must prove that the 2-functor $\mathbb{R}\colon \mathscr{O}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}$ extends to a multifunctor $\mathbf{Mult}(\mathscr{O}) \longrightarrow \mathbf{Mult}(\mathscr{D})$.

Recall that $\mathbb{R}$ is given on an $\mathscr{O}$-pseudoalgebra $\mathscr{A}$ by $(\mathbb{R}\mathscr{A})(\mathbf{n}) = \mathscr{A}^n$. A conceptual definition of how the action $\mathscr{V}$-functors $\theta_n\colon \mathscr{O}(n) \times \mathscr{A}^n \longrightarrow \mathscr{A}$ induce action $\mathscr{V}$-functors $\theta\colon \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{A}^m \longrightarrow \mathscr{A}^n$ is given in [5, Section 3.2]. Explicitly, we want action maps

$$\theta\colon \coprod_{\phi\colon \mathbf{m}\to\mathbf{n}} \prod_{1\le j\le n} \mathscr{O}(|\phi^{-1}(j)|) \times \mathscr{A}^m \longrightarrow \mathscr{A}^n.$$

Fixing the map $\phi \in \mathscr{F}$, let $m_j = |\phi^{-1}(j)|$, where $0 \le j \le n$. Then $m = \sum m_j$. We obtain a map

$$\pi\colon \mathscr{A}^m \longrightarrow \prod_{1\le j\le n} \mathscr{A}^{m_j}$$

by projecting out the coordinates such that $\phi(i) = 0$ (the basepoint) and permuting to collect together the coordinates such that $\phi(i) = j$. Then, permuting operad coordinates past powers of $\mathscr{A}$, π induces a map

$$\prod_{1\le j\le n} \mathscr{O}(m_j) \times \mathscr{A}^m \longrightarrow \prod_{1\le j\le n} \mathscr{O}(m_j) \times \mathscr{A}^{m_j}$$

Composing with $\prod_{1\le j\le n} \theta_{m_j}$ gives the component at ϕ of the action map θ.

Now let (F, δ_i),

$$F\colon \mathscr{A}_1 \times \cdots \times \mathscr{A}_k \longrightarrow \mathscr{B},$$

be a k-morphism in $\mathbf{Mult}(\mathscr{O})$. Here δ_i is given by $\mathscr{V}$-transformations $\delta_i(n)$ as in (3.3). We must construct a k-morphism $(\mathbb{R}\mathscr{A}_1, \cdots, \mathbb{R}\mathscr{A}_k) \longrightarrow \mathbb{R}\mathscr{B}$ as in Definition 5.1, which we will denote by $\tilde{F}$.

As a right adjoint, the 2-functor $\mathbb{R}$ preserves products, but that is misleading: we must regard the product $\mathbb{R}\mathscr{A}_1 \times \cdots \times \mathbb{R}\mathscr{A}_k$ *externally* as a functor defined on $\mathscr{D}^k$, rather than internally as a functor defined on $\mathscr{D}$. Thus we must construct a lax $\mathscr{V}$-transformation

$$\tilde{F}\colon \mathbb{R}\mathscr{A}_1 \times \cdots \times \mathbb{R}\mathscr{A}_k \longrightarrow \mathbb{R}\mathscr{B} \circ \wedge^k$$

comparing the displayed $\mathscr{V}$-pseudofunctors $\mathscr{D}^k \longrightarrow \mathbf{Cat}(\mathscr{V})$. That the source and target are $\mathscr{V}$-pseudofunctors follows from [5, Theorem 0.3] and the first part of Theorem 0.4. The composition 2-cells of the source are products of the composition 2-cells ϕ_j of the $\mathbb{R}\mathscr{A}_j$. The evident composite $\mathscr{V}$-functors

$$\mathscr{A}_1^{n_1} \times \cdots \times \mathscr{A}_k^{n_k} \xrightarrow{\ \lambda\ } (\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^{n_1\cdots n_k} \xrightarrow{F^{n_1\cdots n_k}} \mathscr{B}^{n_1\cdots n_k}$$

specify $\tilde{F}$ on 1-cells. The map λ uses lexicographical ordering and projections in the familiar way: the $(q_1, \cdots, q_k)$th coordinate is $(a_{1,q_1}, \cdots, a_{k,q_k})$, where $(a_{j,1}, \cdots, a_{j,n_j})$ are the coordinates in $\mathscr{A}_j^{n_j}$. Morphisms in $\mathscr{D}^k$ are composites of morphisms that are the identity on all but one variable, and to specify $\tilde{F}$ on 2-cells we must relate such composites to the smash product $\wedge\colon \mathscr{D}^k \longrightarrow \mathscr{D}$ that is used in the target of $\tilde{F}$. From here, it is an inspection of definitions to see that the δ_i that are part of the data of F being a multifunctor in $\mathscr{O}$-$\mathbf{PsAlg}$ can be used to construct invertible $\mathscr{V}$-transformations δ in the following specialization of the diagram (5.2).

$$\begin{tikzcd}
\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \prod_j \mathscr{A}_j^{m_j} \arrow[r, "\mathrm{id}\times\tilde{F}"] \arrow[d, "t"'] & \prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{B}^{\underline{m}} \arrow[d, "\wedge\times\mathrm{id}"] \\
\prod_j \mathscr{D}(\mathbf{m}_j, \mathbf{n}_j) \times \mathscr{A}^{m_j} \arrow[r, phantom, "\Swarrow\delta"] \arrow[d, "\prod_j \theta"'] & \mathscr{D}(\underline{\mathbf{m}}, \mathbf{n}) \times \mathscr{B}^{\underline{m}} \arrow[d, "\theta"] \\
\prod_j \mathscr{A}^{n_j} \arrow[r, "F"'] & \mathscr{B}^{\underline{n}}.
\end{tikzcd} \tag{5.5}$$

To give the idea, we will abuse notation and work with "elements" as objects or morphisms in our categories. Using the definition of $\mathscr{D}$ in terms of $\mathscr{O}$, we see that an element in the source of the two composites in (5.5)) consists of the following data:

$$(\phi_j : m_j \longrightarrow n_j;\ x_j^1, \ldots, x_j^{n_j};\ a_j^1, \ldots, a_j^{m_j})_{1 \le j \le k},$$

where each x_j^i is in $\mathscr{O}(|\phi_j^{-1}(i)|)$, and the a_j^i's are in $\mathscr{A}_j$.

We specify where this element maps to in $\mathscr{B}^{n_1 \cdots n_k}$ along the right-down and the down-right composites. Remember that $\underline{n}$ denotes $n_1 \times \cdots \times n_k$ (with lexicographic ordering). For $1 \le i_j \le n_j$, the $(i_1, \ldots, i_k)$th entry of the right-down composite is

$$F\big(\theta(x_1^{i_1};\ (a_1^i)_{i \in \phi_1^{-1}(i_1)}),\ \theta(x_2^{i_2};\ (a_2^i)_{i \in \phi_2^{-1}(i_2)}),\ \ldots,\ \theta(x_k^{i_k};\ (a_k^i)_{i \in \phi_k^{-1}(i_k)})\big),$$

and the $(i_1, \ldots, i_k)$th entry of the down-right composite is

$$\theta\big(x_1^{i_1} \wedge \ldots \wedge x_k^{i_k};\ \prod_{(l_1, \ldots, l_k) \in (\phi_1 \wedge \ldots \wedge \phi_k)^{-1}(i_1, \ldots, i_k)} F(a_1^{l_1}, \ldots, a_k^{l_k})\big).$$

We need to construct a map from the first expression to the second. We show how to do so in the case $k = 2$ for the sake of transparency. The general case is just an iteration of this. We have a sequence of maps

$$\begin{array}{c}
F\big(\theta(x_1^{i_1};\ (a_1^i)_{i \in \phi_1^{-1}(i_1)}),\ \theta(x_2^{i_2};\ (a_2^i)_{i \in \phi_2^{-1}(i_2)})\big) \\
\Big\downarrow{\scriptstyle \delta_i} \\
\theta\big(x_1^{i_1};\ \prod_{l_1 \in \phi_1^{-1}(i_1)} \theta(x_2^{i_2};\ \prod_{l_2 \in \phi_2^{-1}(i_2)} F(a_1^{l_1}, a_2^{l_2}))\big) \\
\Big\downarrow{\scriptstyle =} \\
\theta\big(x_1^{i_1};\ \prod_{\substack{l_1 \in \phi_1^{-1}(i_1) \\ l_2 \in \phi_2^{-1}(i_2)}} \theta(x_2^{i_2};\ F(a_1^{l_1}, a_2^{l_2}))\big) \\
\Big\downarrow \\
\theta\big(\gamma(x_1^{i_1};\ x_2^{i_2}, \ldots, x_2^{i_2});\ \prod_{\substack{l_1 \in \phi_1^{-1}(i_1) \\ l_2 \in \phi_2^{-1}(i_2)}} F(a_1^{l_1}, a_2^{l_2})\big) \\
\Big\downarrow{\scriptstyle =} \\
\theta\big(x_1^{i_1} \wedge x_2^{i_2};\ \prod_{(l_1, l_2) \in (\phi_1 \wedge \phi_2)^{-1}(i_1, i_2)} F(a_1^{l_1}, a_2^{l_2})\big)
\end{array}$$

The third map, like the second and fourth, would be an equality if $\mathscr{B}$ were an $\mathscr{O}$-algebra, but it is an invertible 2-cell if $\mathscr{B}$ is just a pseudoalgebra. Thus the desired transformation $\zeta_{m_1, \ldots, m_k; n_1, \ldots, m_k}$ is built out of the δ_i that are part of the data of F being a multifunctor in $\mathscr{O}$-**PsAlg**, and the transformations that appear in the definition of a pseudoalgebra.

From here, we see that the commutative coherence diagrams satisfied by the δ_i imply the commutative coherence diagrams required of δ by lengthy, but routine, diagram chases starting from the descriptions of the coherence diagrams in Section 3.1 and Section 5.1. This is analogous to the comparison of diagrams in the proof in Section 4.2 that $\mathscr{D}(\mathscr{O})$ is pseudo-commutative when $\mathscr{O}$ is so.

6. An equivariant variant

6.1. **Pseudocommutative categories of operators over $\mathscr{F}_G$.** So far, $\mathscr{V}$ has been any bicomplete cartesian closed category. In this section we let G be a finite group and consider the category $G\mathscr{V}$ of G-objects in $\mathscr{V}$ and G-maps between them.

Notations 6.1. To emphasize G, we let $\mathscr{K}_G$ denote the 2-category $\mathbf{Cat}(\mathbf{G}\mathscr{V})$ of categories internal to $G\mathscr{V}$.

We call an operad $\mathscr{O}$ in $\mathscr{K}_G$ a G-operad and we call a category of operators $\mathscr{D}$ in $\mathscr{K}_G$ a G-category of operators over $\mathscr{F}$, abbreviated G-**CO** over $\mathscr{F}$; its hom objects are in $\mathscr{K}_G$. When $\mathscr{O}$ and $\mathscr{D}$ are pseudo-commutative, we have associated multicategories $\mathbf{Mult}(\mathscr{O})$ and $\mathbf{Mult}(\mathscr{D})$. For a G-operad $\mathscr{O}$, we have the associated G-**CO** $\mathscr{D}(\mathscr{O})$ over $\mathscr{F}$, and we have the symmetric multifunctor

$$\mathbb{R}\colon \mathbf{Mult}(\mathscr{O}) \longrightarrow \mathbf{Mult}(\mathscr{D}(\mathscr{O})).$$

Thus far, we have just specialized the general theory, hence equivariance has been fully incorporated in the fact that the ground category we start from is $G\mathscr{V}$. Nothing more needs to be said about it.

However, as developed in detail for G-spaces in [19, Section 4], we have a more specifically equivariant variant of everything we have done. Let $\mathscr{F}_G$ be the category of finite based G-sets $(\mathbf{n}, \alpha)$, where $\alpha\colon G \longrightarrow \Sigma_n$ is a homomorphism that fixes an action of G on $\mathbf{n}$; maps are just based functions $\mathbf{m} \longrightarrow \mathbf{n}$, and G acts by conjugation on hom sets. We let Π_G be the G-subcategory of $\mathscr{F}_G$ with the same objects and those morphisms ϕ such that $|\phi^{-1}(j)|$ is 0 or 1. Just as we regard Π and $\mathscr{F}$ as internal categories in $\mathscr{V}$, we regard Π_G and $\mathscr{F}_G$ as internal categories in $G\mathscr{V}$.

Definition 6.2. A category $\mathscr{D}_G$ of operators over $\mathscr{F}_G$, abbreviated G-**CO** over $\mathscr{F}_G$, is a $\mathscr{K}_G$-category which has the same objects as $\mathscr{F}_G$ and has $\mathscr{K}_G$-functors $\iota\colon \Pi_G \longrightarrow \mathscr{D}_G$ and $\xi\colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$ such that $\xi \circ \iota$ is the inclusion of Π_G in $\mathscr{F}_G$. We require $\mathscr{D}_G$ to be reduced, meaning that $\mathscr{D}_G(\mathbf{0}, (\mathbf{n}, \beta)) = *$ and $\mathscr{D}_G((\mathbf{m}, \alpha), \mathbf{0}) = *$.

As in [19, Lemma 4.6], regarding the based sets $\mathbf{n}$ as based G-sets with trivial G-action, we see that the full subcategory $\mathscr{D}$ of $\mathscr{D}_G$ with objects $\mathbf{n}$ is a G-**CO** over $\mathscr{F}$. Conversely, [19, Construction 4.7] applies verbatim in the present context to show how to prolong a G-**CO** $\mathscr{D}$ over $\mathscr{F}$ to a G-**CO** $\mathscr{D}_G$ over $\mathscr{F}_G$. We define

$$\mathscr{D}_G((\mathbf{m}, \alpha), (\mathbf{n}, \beta)) = \mathscr{D}(\mathbf{m}, \mathbf{n}) \tag{6.3}$$

with action by G specified by conjugation. That is, for $g \in G$, the action of g on $\mathscr{D}_G((\mathbf{m}, \alpha), (\mathbf{n}, \beta))$ is the following composite in $\mathscr{D}$. Here and below, we write $\mathscr{D}(\mu, \mathrm{id}) = \mu^*$ and $\mathscr{D}(\mathrm{id}, \nu) = \nu_*$ for morphisms μ and ν in Π.

$$\mathscr{D}(\mathbf{m}, \mathbf{n}) \xrightarrow{\alpha(g^{-1})^*} \mathscr{D}(\mathbf{m}, \mathbf{n}) \xrightarrow{g\cdot} \mathscr{D}(\mathbf{m}, \mathbf{n}) \xrightarrow{\beta(g)_*} \mathscr{D}(\mathbf{m}, \mathbf{n}). \tag{6.4}$$

We restrict attention to those G-COs $\mathscr{D}_G$ over $\mathscr{F}_G$ that are constructed in this way from their underlying G-COs $\mathscr{D}$ over $\mathscr{F}$. Then, as just illustrated, all structure that we see in $\mathscr{D}_G$ is defined explicitly in $\mathscr{D}$.

Now assume that $\mathscr{D}$ is pseudo-commutative, as in Definition 4.2, with product denoted $\wedge_{\mathscr{D}}$. Then $\wedge_{\mathscr{D}}$ is G-equivariant, where G acts diagonally on products. We view the given product

$$\wedge_{\mathscr{D}}\colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{D}(\mathbf{p}, \mathbf{q}) \longrightarrow \mathscr{D}(\mathbf{mp}, \mathbf{nq})$$

as defining the smash product

$$\wedge_{\mathscr{D}_G} \colon \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \times \mathscr{D}_{\mathbf{G}}((\mathbf{p},\alpha),(\mathbf{q},\beta)) \longrightarrow \mathscr{D}_{\mathbf{G}}((\mathbf{mp},\alpha\wedge\gamma),(\mathbf{nq},\beta\wedge\delta)).$$

Here $\alpha\wedge\gamma$ is specified on $g \in G$ as the permutation given by the following commutative diagram, in which $\lambda_{m,p} \colon \mathbf{mp} \longrightarrow \mathbf{m}\wedge\mathbf{p}$ is the lexicographic isomorphism of finite based sets.

$$\begin{tikzcd} \mathbf{mp} \arrow[r, "(\alpha\wedge_{\mathscr{D}}\gamma)(g)"] \arrow[d, "\lambda_{m,p}"'] & \mathbf{mp} \\ \mathbf{m}\wedge\mathbf{p} \arrow[r, "\alpha(g)\wedge\gamma(g)"'] & \mathbf{m}\wedge\mathbf{p}. \arrow[u, "\lambda^{-1}_{m,p}"'] \end{tikzcd} \tag{6.5}$$

Of course, $\beta\wedge\delta$ is defined in the same way. Then $\wedge_{\mathscr{D}_G}$ is G-equivariant since $\wedge_{\mathscr{D}}$ is G-equivariant. In view of (6.4), this means that the following diagram commutes in $\mathscr{D}$.

$$\begin{tikzcd} & \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{p},\mathbf{q}) \arrow[r, "g\cdot\times g\cdot"] \arrow[ddd, "\wedge_{\mathscr{D}}"] & \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{p},\mathbf{q}) \arrow[ddd, "\wedge_{\mathscr{D}}"] \arrow[dr, "\beta(g)_*\times\delta(g)_*"] & \\ \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{p},\mathbf{q}) \arrow[ur, "\alpha^{-1}(g)^*\times\gamma^{-1}(g)^*"] \arrow[d, "\wedge_{\mathscr{D}}"] & & & \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{p},\mathbf{q}) \arrow[d, "\wedge_{\mathscr{D}}"] \\ \mathscr{D}(\mathbf{mp},\mathbf{nq}) \arrow[dr, "(\alpha\wedge_{\mathscr{D}}\gamma)(g^{-1})^*"'] & & & \mathscr{D}(\mathbf{mp},\mathbf{nq}) \\ & \mathscr{D}(\mathbf{mp},\mathbf{nq}) \arrow[r, "g\cdot"'] & \mathscr{D}(\mathbf{mp},\mathbf{nq}) \arrow[ur, "(\beta\wedge_{\mathscr{D}}\delta)(g)_*"'] & \end{tikzcd}$$

Note that permutations in $\mathscr{D}$ are actions by elements of Π. Therefore the left and right trapezoids commute by condition (ii) of Definition 4.2, as the diagrams spelling out that condition make clear. The rectangle commutes since $\wedge_{\mathscr{D}}$ is G-equivariant. Similarly, the symmetry isomorphism τ on $\mathscr{D}$ gives an equivariant symmetry isomorphism on $\mathscr{D}_G$. We summarize with the following definition and theorem.

Definition 6.6. Say that a G-**CO** $\mathscr{D}_G$ over $\mathscr{F}_G$ is pseudo-commutative if it satisfies the conditions of Definition 4.2 with $\mathscr{F}$ and Π replaced by $\mathscr{F}_G$ and Π_G and with all structure G-equivariant.

Theorem 6.7. *The $\mathscr{K}_G$-**CO** $\mathscr{D}_G$ over $\mathscr{F}_G$ associated to a $\mathscr{K}$-**CO** $\mathscr{D}$ over $\mathscr{F}$ is pseudocommutative.*

6.2. **The comparison between $\mathscr{D}$-PsAlg and $\mathscr{D}_G$-PsAlg.** Let $\mathscr{D}_G$ be the pseudo-commutative G-**CO** over $\mathscr{F}_G$ associated to a pseudo-commutative G-**CO** $\mathscr{D}$ over $\mathscr{F}$. We define $\mathscr{D}_G$-algebras and $\mathscr{D}_G$-pseudoalgebras in exactly the same way as we defined $\mathscr{D}$-algebras and $\mathscr{D}$-pseudoalgebras in [5, Definition 2.8], again just replacing $\mathscr{F}$ and $\mathscr{D}$ by $\mathscr{F}_G$ and Π_G and requiring all structure to be G-equivariant. Defining $\mathscr{D}_G$-pseudomorphisms and $\mathscr{D}_G$-modifications as there, we have the 2-category $\mathscr{D}_G$-**PsAlg** of $\mathscr{D}_G$-pseudoalgebras.

The inclusion $\mathscr{D} \subset \mathscr{D}_G$ induces a forgetful 2-functor

$$\mathbb{U} \colon \mathscr{D}_G\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}.$$

Just as for G-spaces in [19, Section 4], U has a right adjoint prolongation 2-functor

$$\mathbb{P}\colon \mathscr{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}_{\mathbf{G}}\text{-}\mathbf{PsAlg}.$$

For G-spaces, we constructed $\mathbb{P}$ conceptually, using tensor products of functors. However, [19, Lemma 4.12] gave an explicit concrete description in terms of $\mathscr{D}$ that readily adapts to the present categorical context. In line with that result, for a $\mathscr{D}$-pseudoalgebra $\mathscr{X}$, we define

$$(\mathbb{P}\mathscr{X})(\mathbf{n},\alpha) = \mathscr{X}(\mathbf{n},\alpha),$$

where $\mathscr{X}(\mathbf{n},\alpha)$ denotes the underlying object $\mathscr{X}(\mathbf{n})$ in $\mathscr{K}$ but with a new G-action specified in terms of the one that $\mathscr{X}(\mathbf{n})$ starts with as the composite

$$\mathscr{X}(\mathbf{n}) \xrightarrow{g} \mathscr{X}(\mathbf{n}) \xrightarrow{\mathscr{X}(\alpha(g))} X(\mathbf{n}). \tag{6.8}$$

Since $\mathscr{X}(\alpha(g))\colon \mathscr{X}(\mathbf{n}) \longrightarrow \mathscr{X}(\mathbf{n})$ is G-equivariant, we can write the G-action equivalently as the composite

$$\mathscr{X}(\mathbf{n}) \xrightarrow{\mathscr{X}(al(g))} \mathscr{X}(\mathbf{n}) \xrightarrow{g\cdot} X(\mathbf{n}). \tag{6.9}$$

As we shall elaborate, all structure that we see starting in $\mathscr{D}$ lifts equivariantly to $\mathscr{D}_G$ because G-actions in $\mathscr{D}$ are given explicitly in terms of permutations, permutations are in Π, and all structure involving Π behaves strictly with respect to all of the pseudofunctors and pseudotransformations relevant to structure in $\mathscr{D}$-$\mathbf{PsAlg}$.

In more detail, for a $\mathscr{D}$-pseudoalgebra $\mathscr{X}$,

$$\mathscr{X} : \mathscr{D}(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{K}_{\mathbf{G}}(\mathscr{X}(\mathbf{m}),\mathscr{X}(\mathbf{n}))$$

must be G-equivariant, where G acts by conjugation on the target. We work throughout with the adjoint action map

$$\theta\colon \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n}),$$

and θ must be G-equivariant, where G acts diagonally on the source. Since $\mathscr{X}$ is a $\mathscr{D}$-pseudoalgebra, we have a $G\mathscr{V}$-transformation

$$\begin{tikzcd} \mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\mathrm{id}\times\theta"] \arrow[d, "\circ\times\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \mathscr{D}(\mathbf{n},\mathbf{p}) \times \mathscr{X}(\mathbf{n}) \arrow[d, "\theta"] \\ \mathscr{D}(\mathbf{m},\mathbf{p}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\theta"'] & \mathscr{X}(\mathbf{p}). \end{tikzcd} \tag{6.10}$$

subject to coherence properties. For a $\mathscr{D}$-algebra, ϕ is the identity. In general, ϕ must be the identity when $\mathscr{D}(\mathbf{n},\mathbf{p})$ is restricted to $\Pi(\mathbf{n},\mathbf{p})$ or when $\mathscr{D}(\mathbf{m},\mathbf{n})$ is restricted to $\Pi(\mathbf{m},\mathbf{n})$.

We must show that that the same conditions with $\mathscr{D}$ replaced by $\mathscr{D}_G$ hold for the $\mathscr{D}_G$-pseudoalgebra $\mathbb{P}\mathscr{X}$. As a map of $\mathscr{K}$-categories, we define the action map

$$\theta_G\colon \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \times \mathscr{X}(\mathbf{m},\alpha) \longrightarrow \mathscr{X}(\mathbf{n},\beta)$$

to be the action map

$$\theta\colon \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \longrightarrow \mathscr{X}(\mathbf{n}).$$

Similarly, we define ϕ_G for $\mathscr{D}_G$ to be the relevant instances of ϕ for $\mathscr{D}$. The crucial point is that θ_G and ϕ_G are then G-equivariant since θ and ϕ are G-equivariant. The following commutative diagram shows this for θ_G.

$$\begin{tikzcd}[column sep=small]
 & \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "g\times g"] \arrow[d, "\theta"'] & \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[r, "\beta(g)_*\times\mathrm{id}"] \arrow[d, "\theta"'] & \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[d, "\theta"'] \\
\mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m}) \arrow[ur, "\alpha(g^{-1})^*\times\mathscr{X}(\alpha(g))"] \arrow[r, "\theta"'] & \mathscr{X}(\mathbf{n}) \arrow[r, "g"'] & \mathscr{X}(\mathbf{n}) \arrow[r, "\mathscr{X}(\beta(g))"'] & \mathscr{X}(\mathbf{n})
\end{tikzcd}$$

The triangle and the right square commute since they are implicitly defined using actions by morphisms in Π. The middle square commutes since θ is G-equivariant. The top composite gives the diagonal action of G on $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)\times\mathscr{X}(\mathbf{m},\alpha)$ and the bottom row gives the action of G on $\mathscr{X}(\mathbf{n},\beta)$. The G-equivariance of ϕ is checked by chasing an analogous whiskering of the G-equivariant diagram (6.10). That is, action by $g\in G$ maps the boundary of (6.10) to itself, and the relation $g\phi=\phi g$ is an equality of pasting diagrams; whiskering that diagram gives the required analogous equality $g\phi_G=\phi_G g$ of pasting diagrams.

It is proven in [19, Theorem 4.11] that prolongation and the forgetful functor specify an adjoint equivalence between the categories of $\mathscr{D}$-G-spaces and of $\mathscr{D}_G$-G-spaces. The categorical analogue is also true. Modulo language, the proof is essentially the same and will be omitted.

Theorem 6.11. *The adjoint pair of 2-functors*

$$\mathscr{D}\text{-}\mathbf{PsAlg} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{D}_G\text{-}\mathbf{PsAlg}$$

specifies an adjoint equivalence of 2-categories.

6.3. **The comparison between $\mathbf{Mult}(\mathscr{D})$ and $\mathbf{Mult}(\mathscr{D}_\mathbf{G})$.** From here, we can argue exactly as in Section 5.1 to construct $\mathbf{Mult}(\mathscr{D}_\mathbf{G})$ from $\mathscr{D}_G$-$\mathbf{PsAlg}$. We record the required modification of Definition 5.1. For sequences $(\mathbf{n}_i,\alpha_i)$, $1\le i\le k$, of finite G-sets, we abbreviate notation by setting

$$(\mathbf{n},\alpha)=(\mathbf{n}_1\cdots\mathbf{n}_k,\alpha_1\wedge\cdots\wedge\alpha_k).$$

Definition 6.12. For objects $\mathscr{X}_i$, $1\le i\le k$, and $\mathscr{Y}$ of $\mathscr{D}_G$-$\mathbf{PsAlg}$, a k-morphism $(F,\delta)\colon \underline{\mathscr{X}}\longrightarrow\mathscr{Y}$ consists of $G\mathscr{V}$-functors

$$F\colon \mathscr{X}_1(\mathbf{n}_1,\alpha_1)\times\cdots\times\mathscr{X}_k(\mathbf{n}_k,\alpha_k)\longrightarrow\mathscr{Y}(\underline{\mathbf{n}},\alpha) \tag{6.13}$$

that together define a natural transformation of functors $\Pi_G^k\longrightarrow\mathscr{K}_G$ together with invertible $G\mathscr{V}$-transformations δ in the following diagrams, in which $1\le i\le k$.

(6.14)

$$\begin{tikzcd}
\prod_i \mathscr{D}((\mathbf{m}_i,\alpha_i),(\mathbf{n}_i,\beta_i))\times\prod_i\mathscr{X}_i(\mathbf{m}_i,\alpha_i) \arrow[r, "\mathrm{id}\times F"] \arrow[d, "t"] & \prod_i \mathscr{D}((\mathbf{m}_i,\alpha_i),(\mathbf{n}_i,\beta_i))\times\mathscr{Y}(\mathbf{m},\alpha) \arrow[d, "\wedge\times\mathrm{id}"] \\
\prod_i \mathscr{D}((\mathbf{m}_i,\alpha_i),(\mathbf{n}_i,\beta_i))\times\mathscr{X}_i(\mathbf{m}_i,\alpha_i) \arrow[d, "\prod_i\theta"] \arrow[r, phantom, "\Downarrow\delta"] & \mathscr{D}((\mathbf{m},\alpha),(\mathbf{n},\beta))\times\mathscr{Y}(\mathbf{m},\alpha) \arrow[d, "\theta"] \\
\prod_i \mathscr{X}_i(\mathbf{n}_i,\beta_i) \arrow[r, "F"'] & \mathscr{Y}(\mathbf{n},\beta).
\end{tikzcd}$$

Composition is given by composites of products of natural transformations.

Up to notation, the rest of the details of the definition of $\mathbf{Mult}(\mathscr{D}_{\mathbf{G}})$ are the same as the details in the rest of the definition of $\mathbf{Mult}(\mathscr{D})$ in Section 5.1. To compare these multicategories, start with objects $\mathscr{X}_i$ and $\mathscr{Y}$ of $\mathscr{D}$-**PsAlg** and replace the $\mathscr{X}_i$ and $\mathscr{Y}$ of Definition 6.12 with the $\mathbb{P}\mathscr{X}_i$ and $\mathbb{P}\mathscr{Y}$. A k-morphism $(F,\delta)\colon \underline{\mathscr{X}} \longrightarrow \mathscr{Y}$ in $\mathscr{D}$ consists of $G\mathscr{V}$-functors

$$F\colon \mathscr{X}_1(\mathbf{n}_1)\times\cdots\times\mathscr{X}_k(\mathbf{n}_k)\longrightarrow\mathscr{Y}(\mathbf{n})$$

together with invertible $G\mathscr{V}$-transformations δ as in (5.2). Ignoring group actions, we define $(\mathbb{P}F,\mathbb{P}\delta)$ to be equal to (F,δ) on underlying $\mathscr{V}$-categories. We must show that these data are equivariant when interpreted in terms of the group actions specified on $\mathscr{D}_G$ by (6.4) and on the $\mathbb{P}\mathscr{X}$ by (6.8) or, equivalently, (6.9). For the equivariance of $\mathbb{P}F$, we must show that the following diagram commutes for every k-tuple $(\mathbf{n}_i,\alpha_i)$ in $\mathscr{D}_G$.

$$\begin{tikzcd}
\prod_i \mathscr{X}_i(\mathbf{n}_i) \arrow[r, "F"] \arrow[d, "\prod \mathscr{X}_i(\alpha_i(g))"'] & \mathscr{Y}(\mathbf{m}) \arrow[d, "\mathscr{Y}(\alpha(g))"] \\
\prod_i \mathscr{X}_i(\mathbf{n}_i) \arrow[r, "F"] \arrow[d, "\prod g"'] & \mathscr{Y}(\mathbf{m}) \arrow[d, "g"] \\
\prod_i \mathscr{X}_i(\mathbf{n}_i) \arrow[r, "F"] & \mathscr{Y}(\mathbf{m})
\end{tikzcd}$$

The top square commutes since F restricts to a strict transformation of Π-functors and the bottom square commutes since F was assumed to be equivariant. The equivariance of $\mathbb{P}\delta$ is checked by chasing whiskerings of the equality of pasting diagrams $g\delta=\delta g$ obtained by applying $g\in G$ to the diagram (5.2), remembering that we have changed ground 2-categories to $G\mathscr{V}$, and again using that F restricts to a strict transformation of Π-functors.

7. The free $\mathscr{O}$-algebra functor $\mathbb{O}$

7.1. **The extension of $\mathbb{O}$ to a multifunctor.** To illustrate definitions, we here relate free $\mathscr{O}$-algebras to $\mathbf{Mult}_{\mathbf{st}}(\mathscr{O})$, where $\mathscr{O}$ is chaotic. We do not claim that the multifunctor we construct is symmetric.

An operad $\mathscr{O}$ in any symmetric monoidal category $\mathscr{W}$ has an associated monad $\mathbb{O}$ specified on objects $X\in\mathscr{W}$ by

$$\mathbb{O}X=\coprod_{j\geq 0}\mathscr{O}(j)\times_{\Sigma_j}X^j.$$

As explained in [18], this is a monad on unbased objects in $\mathscr{W}$ whose algebras are the same as those of a similar monad on based objects in $\mathscr{W}$ that is defined using base object identifications. For the comparison in this section and its applications, we prefer to avoid such identifications, and we now take $\mathscr{W}$ to be $\mathscr{K}=\mathbf{Cat}(\mathscr{V})$. Thus fix a chaotic operad $\mathscr{O}$ in $\mathscr{K}$ in this section.

Definition 7.1. For $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, define

$$\omega\colon \mathbb{O}\mathscr{A}\times\mathbb{O}\mathscr{B}\longrightarrow\mathbb{O}(\mathscr{A}\times\mathscr{B})$$

by passage to orbits from the maps

$$\mathscr{O}(j) \times \mathscr{A}^j \times \mathscr{O}(k) \times \mathscr{B}^k \xrightarrow{t} \mathscr{O}(j) \times \mathscr{O}(k) \times \mathscr{A}^j \times \mathscr{B}^k \xrightarrow{\wedge \times \lambda} \mathscr{O}(jk) \times (\mathscr{A} \times \mathscr{B})^{jk}.$$

To specify λ, we use the lexicographic identification $\mathbf{j} \times \mathbf{k} \cong \mathbf{jk}$; thus the (q,r)th coordinate of λ is the product of the qth coordinate in $\mathscr{A}^j$ and the rth coordinate in $\mathscr{B}^k$. Since the intrinsic pairing is associative and unital, so is the pairing ω; for the unital condition, we take $j = 1$ or $k = 1$, restrict to $\mathrm{id} \in \mathscr{O}(1)$, and take $\mathscr{A}$ or $\mathscr{B}$ to be the trivial category.

Observe the implicit role of the diagonal $\Delta\colon \mathscr{A}^j \longrightarrow (\mathscr{A}^j)^k \cong \mathscr{A}^{jk}$ and similarly for $\mathscr{B}$. As noted, the product $\wedge$ uses the lexicographic ordering. We have a second, twisted, product which uses reverse lexicographic ordering. Comparison of the two is implicit in the pseudo-commutativity of $\mathscr{O}$. More precisely, a little diagram chase from Definition 2.9 shows that the pseudo-commutativity isomorphism of $\mathscr{O}$ induces a natural isomorphism

$$\begin{tikzcd} \mathbb{O}\mathscr{A} \times \mathbb{O}\mathscr{B} \arrow[r, "\omega"] \arrow[d, "t"'] \arrow[dr, phantom, "\Downarrow \alpha"] & \mathbb{O}(\mathscr{A} \times \mathscr{B}) \arrow[d, "\mathbb{O}(t)"] \\ \mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A} \arrow[r, "\omega"'] & \mathbb{O}(\mathscr{B} \times \mathscr{A}), \end{tikzcd} \tag{7.2}$$

where t denotes transposition.

We repeat and prove Theorem 0.7.

Theorem 7.3. *For a chaotic operad $\mathscr{O}$ in $\mathscr{K}$, the functor $\mathbb{O}$ from $\mathscr{V}$-categories to $\mathscr{O}$-algebras extends to a multifunctor*

$$\mathbb{O}\colon \mathbf{Mult}(\mathscr{K}) \longrightarrow \mathbf{Mult}_{\mathbf{st}}(\mathscr{O}).$$

Proof. In $\mathbf{Mult}(\mathscr{K})$, a k-linear map $F\colon \mathscr{A}_1 \times \cdots \times \mathscr{A}_k \longrightarrow \mathscr{B}$ is just a $\mathscr{V}$-functor between $\mathscr{V}$-categories. We must give the composite functor

$$\mathbb{O}(\mathscr{A}_1) \times \cdots \times \mathbb{O}(\mathscr{A}_k) \xrightarrow{\omega} \mathbb{O}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k) \xrightarrow{\mathbb{O}F} \mathbb{O}\mathscr{B}$$

the structure of a k-linear map. From there, easy verifications show that these composites $\mathbb{O}F \circ \omega$ on k-linear maps specify the required extension of $\mathbb{O}$ to a multifunctor. We must construct the maps δ_i for $\mathbb{O}F \circ \omega$, with the $\mathscr{A}_i$ and $\mathscr{B}$ in Definition 3.2 replaced by $\mathbb{O}\mathscr{A}_i$ and $\mathbb{O}\mathscr{B}$ here. Since $\mathbb{O}F$ is a map of $\mathscr{O}$-algebras and the composite of such a map with a k-linear map is again a k-linear map, it suffices to give ω a structure of a k-linear map. We claim that we can take the $\delta_1(n)$ to be identity transformations. That is, we claim that the following diagram commutes.

$$\begin{tikzcd} \mathscr{O}(n) \times (\mathbb{O}\mathscr{A}_1 \times \cdots \times \mathbb{O}\mathscr{A}_k)^n \arrow[rr, "\mathrm{id} \times \omega^n"] & & \mathscr{O}(n) \times \mathbb{O}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^n \arrow[dd, "\theta_n"] \\ \mathscr{O}(n) \times (\mathbb{O}\mathscr{A}_1)^n \times \mathbb{O}\mathscr{A}_2 \times \cdots \times \mathbb{O}\mathscr{A}_k \arrow[u, "t_1"] \arrow[d, "\theta_n \times \mathrm{id}"'] & & \\ \mathbb{O}\mathscr{A}_1 \times \cdots \times \mathbb{O}\mathscr{A}_k \arrow[rr, "\omega"'] & & \mathbb{O}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k) \end{tikzcd}$$

To see this and to construct the $\delta_i(n)$ for $i \geq 2$, observe that, by the definition of the monad $\mathbb{O}$ and the product ω, we can first bring all the operad variables to the left and all the category variables $\mathscr{A}_j$ to the right and then pass to orbits. For $i = 1$, the variables in the $\mathscr{A}_j$ are arranged lexicographically under either composite. The variables in the operad agree under the composite by iterated application of the associativity diagram for the structure maps γ, as in Proposition 2.2. For $i > 1$, we have the analogous diagram

$$\begin{tikzcd}
\mathscr{O}(n) \times (\mathbb{O}\mathscr{A}_1 \times \cdots \times \mathbb{O}\mathscr{A}_k)^n \arrow[rr, "\mathrm{id} \times \omega^n"] & & \mathscr{O}(n) \times \mathbb{O}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^n \arrow[dd, "\theta_n"] \\
\mathbb{O}\mathscr{A}_1 \times \cdots \times \mathscr{O}(n) \times (\mathbb{O}\mathscr{A}_i)^n \times \cdots \times \mathbb{O}\mathscr{A}_k \arrow[u, "t_i"] \arrow[d, "\mathrm{id} \times \theta_n \times \mathrm{id}"'] \arrow[r, phantom, "\Downarrow \delta_i(n)"] & & \\
\mathbb{O}\mathscr{A}_1 \times \cdots \times \mathbb{O}\mathscr{A}_k \arrow[rr, "\omega"'] & & \mathbb{O}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)
\end{tikzcd}$$

Expanding the diagram by use of the definition of $\mathbb{O}$ and then moving the operad variables to the left, we see that [9, Lemma 1.16] applies to construct a unique $\delta_i(n)$ as required since $\mathscr{O}$ is chaotic. A more explicit construction can be obtained by comparing the top composites and the bottom composites in the two diagrams above, then using (7.2) and that the first diagram commutes to obtain $\delta_i(n)$ in the second diagram. In more detail, let ρ_i be the permutation that transposes 1 and i, leaving the other j fixed. Then ρ_i transposes the upper map t_1 to an instance of t_i in the second diagram and transposes the lower map $\theta_n \times \mathrm{id}$ to an instance of $\mathrm{id} \times \theta_n \times \mathrm{id}$ in the second diagram. $\square$

7.2. **Symmetry and the free $\mathscr{O}$-algebra functor.** To understand how the symmetry behaves under the multifunctor of Theorem 0.7, consider the case $k = 2$, letting t be the non identity permutation in Σ_2. The general case works similarly. As in the proof of Theorem 0.7, δ_1 is the identity but δ_2 is not. Renaming $\mathscr{A}_1 = \mathscr{A}$ and $\mathscr{A}_2 = \mathscr{B}$ for notational convenience, the relationship between δ_1 and δ_2 vis-a-vis t is that the composite 2-cells in the following two pasting diagrams are equal. Both diagrams utilize (7.2)).

$$\begin{tikzcd}
 & \mathscr{O}(n) \times \mathbb{O}(\mathscr{B} \times \mathscr{A})^n \arrow[dr, "\mathrm{id} \times \mathbb{O}(t)^n"] \arrow[d, phantom, "\Downarrow \mathrm{id} \times \mathbb{O}(\alpha)^n"] & \\
\mathscr{O}(n) \times (\mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A})^n \arrow[ur, "\mathrm{id} \times \omega^n"] \arrow[r, "\mathrm{id} \times t^n"'] & \mathscr{O}(n) \times (\mathbb{O}\mathscr{A} \times \mathbb{O}\mathscr{B})^n \arrow[r, "\mathrm{id} \times \omega^n"'] & \mathscr{O}(n) \times \mathbb{O}(\mathscr{A} \times \mathscr{B})^n \arrow[dd, "\theta_n"] \\
\mathscr{O}(n) \times (\mathbb{O}\mathscr{B})^n \times \mathbb{O}\mathscr{A} \arrow[u, "t_1"] \arrow[d, "\theta_n \times \mathrm{id}"'] \arrow[r, "t"] & \mathbb{O}\mathscr{A} \times \mathscr{O}(n) \times (\mathbb{O}\mathscr{B})^n \arrow[u, "t_2"'] \arrow[d, "\mathrm{id} \times \theta_n"'] \arrow[r, phantom, "\Downarrow \delta_2(n)"] & \\
\mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A} \arrow[r, "t"'] & \mathbb{O}\mathscr{A} \times \mathbb{O}\mathscr{B} \arrow[r, "\omega"'] & \mathbb{O}(\mathscr{A} \times \mathscr{B})
\end{tikzcd}$$

The right square is an instance of the general diagram displaying $\delta_i(n)$ in the proof of Theorem 0.7. The two left squares commute as an instance of a general diagram in Definition 3.2 and, as there, the composite whiskering of $\delta_2(n)$ displays $\delta_1(n)$ for $\omega \circ t$. Since this composite is not the identity, our multifunctor is not symmetric. We compare this pasting diagram with the following one, which has the same periphery.

$$
\begin{tikzcd}
\mathscr{O}(n) \times (\mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A})^n \arrow[rr, "\mathrm{id} \times \omega^n"] & & \mathscr{O}(n) \times \mathbb{O}(\mathscr{B} \times \mathscr{A})^n \arrow[rr, "\mathrm{id} \times \mathbb{O}(t)^n"] \arrow[dd, "\theta_n"] & & \mathscr{O}(n) \times \mathbb{O}(\mathscr{A} \times \mathscr{B})^n \arrow[dd, "\theta_n"] \\
\mathscr{O}(n) \times (\mathbb{O}\mathscr{B})^n \times \mathbb{O}\mathscr{A} \arrow[u, "t_1"] \arrow[d, "\theta_n \times \mathrm{id}"'] & & & & \\
\mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A} \arrow[rr, "\omega"] \arrow[drr, "t"'] & & \mathbb{O}(\mathscr{B} \times \mathscr{A}) \arrow[rr, "\mathbb{O}(t)"] \arrow[d, phantom, "\Downarrow \alpha"] & & \mathbb{O}(\mathscr{A} \times \mathscr{B}) \\
& & \mathbb{O}\mathscr{A} \times \mathbb{O}\mathscr{B} \arrow[urr, "\omega"'] & &
\end{tikzcd}
$$

The left square commutes because $\delta_1(n)$ for $\omega\colon \mathbb{O}\mathscr{B} \times \mathbb{O}\mathscr{A} \longrightarrow \mathbb{O}(\mathscr{B} \times \mathscr{A})$ is the identity, and the right square commutes by naturality. The equality of the interior 2-cells shows that the pseudo-commutativity isomorphism α mediates between $\delta_1(n)$ for $\omega \circ t$ and the identity map. An alternative formulation is that the composite 2-cell in the pasting diagram

$$
\begin{tikzcd}
& \mathscr{O}(n)\times\mathbb{O}(\mathscr{B}\times\mathscr{A})^n \arrow[dr, "\mathrm{id} \times \mathbb{O}(t)^n"] \arrow[d, phantom, "\Downarrow \mathrm{id} \times \mathbb{O}(\alpha)^n"] & & \\
\mathscr{O}(n)\times(\mathbb{O}\mathscr{B}\times\mathbb{O}\mathscr{A})^n \arrow[ur, "\mathrm{id} \times \omega^n"] \arrow[r, "\mathrm{id} \times t^n"'] & \mathscr{O}(n)\times(\mathbb{O}\mathscr{A}\times\mathbb{O}\mathscr{B})^n \arrow[r, "\mathrm{id} \times \omega^n"'] & \mathscr{O}(n)\times\mathbb{O}(\mathscr{A}\times\mathscr{B})^n \arrow[r, "\mathrm{id} \times \mathbb{O}(t)^n"] \arrow[dd, "\theta_n"] & \mathscr{O}(n)\times\mathbb{O}(\mathscr{B}\times\mathscr{A})^n \arrow[dd, "\theta_n"] \\
\mathscr{O}(n)\times(\mathbb{O}\mathscr{B})^n\times\mathbb{O}\mathscr{A} \arrow[u, "t_1"] \arrow[d, "\theta_n \times \mathrm{id}"'] \arrow[r, "\sigma"] & \mathbb{O}\mathscr{A}\times\mathscr{O}(n)\times(\mathbb{O}\mathscr{B})^n \arrow[u, "t_2"'] \arrow[d, "\mathrm{id} \times \theta_n"'] \arrow[r, phantom, "\swarrow \delta_2(n)"] & & \\
\mathbb{O}\mathscr{B}\times\mathbb{O}\mathscr{A} \arrow[r, "t"'] & \mathbb{O}\mathscr{A}\times\mathbb{O}\mathscr{B} \arrow[r, "\omega"] \arrow[dr, "t"'] & \mathbb{O}(\mathscr{A}\times\mathscr{B}) \arrow[r, "\mathbb{O}(t)"] \arrow[d, phantom, "\Downarrow \alpha"] & \mathbb{O}(\mathscr{B}\times\mathscr{A}) \\
& & \mathbb{O}\mathscr{B}\times\mathbb{O}\mathscr{A} \arrow[ur, "\omega"'] &
\end{tikzcd}
$$

is the identity. With a suitable formal definition of a pseudo-symmetric multifunctor between symmetric multicategories, the analogous equalities of composite 2-cells in the appropriate pasting diagrams for k-linear maps and any $\rho \in \Sigma_k$ show that $\mathbb{O}$ is a pseudo-symmetric multifunctor.

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