THE 2-MONADIC THEORY OF OPERADIC MULTICATEGORIES

ABSTRACT. In this paper we show how to simultaneously strictify and change multicategories associated to categories of operators. It starts from symmetric monoidal graded 2-categories and their multicateogries. In the 1-category case, these implicitly pervade stable homotopy theory. In the 2-category case that is our focus, it gives a conceptual reinterpretation of the multicategory associated to a category of operators $\mathscr{D}$ in terms of a graded multicategory of 2-monads. From here we use Lack's theory of codescent objects to show how to simultaneously transport structures from $\mathscr{D}$ to the category $\mathscr{F}$ of finite based sets and to strictify 2-categorical structure to 1-categorical structure. The original motivation comes from equivariant infinite loop space theory. In a short sequel, which is relatively unencumbered by categorical language, we will show that it leads to a conceptual proof of the equivariant multiplicative Barratt-Priddy-Quillen theorem needed to justify the description of the category of G-spectra as the presheaf category of spectral Mackey functors. Everything here works in a general context that is likely to have other applications.

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Introduction and statement of results

In [8], we constructed multicategories associated to operads and to categories of operators. We show here how to define and strictify such 2-categorical multiplicative structure. Such structure is general and applies in many contexts, but our original motivation came from equivariant stable homotopy theory, where we show how to construct ring, module, and algebra G-spectra, G-maps, and G-homotopies from general 2-categorical input. We apply the theory in that context in [7], but we are confident that it will have other applications. We fix a bicomplete cartesian closed category $\mathscr{V}$ and work in the 2-category $\mathbf{Cat}(\mathscr{V})$ of categories internal to $\mathscr{V}$ throughout this paper. We fix notation as in [8].

Notation 0.1. Throughout this paper, let $\mathscr{K}$ denote the 2-category $\mathbf{Cat}(\mathscr{V})$.

The motivating example is $\mathscr{V} = G\mathscr{U}$, the category of G-spaces, but equivariance plays no role in the relevant category theory (see Remark 0.19). We start with an operad $\mathscr{O}$ in $\mathscr{K}$. In [9, §1.2], we described the 2-category $\mathscr{O}$-$\mathbf{PsAlg}$ of $\mathscr{O}$-pseudoalgebras and $\mathscr{O}$-pseudomorphisms together with its sub 2-category $\mathscr{O}$-$\mathbf{AlgSt}$ of (strict) $\mathscr{O}$-algebras and (strict) $\mathscr{O}$-morphisms, where $\mathscr{O}$ is an operad in $\mathscr{K}$. We think of $\mathscr{O}$-$\mathbf{PsAlg}$ as providing underlying additive structure.

We assume familiarity with multicategories, which we take to be symmetric throughout. A multicategory with one object is an operad. We showed in [8] that any operad $\mathscr{O}$ in any symmetric monoidal category has a certain intrinsic pairing $\wedge_{\mathscr{O}}$. When that pairing has a suitable symmetry property, called pseudo-commutativity, we have the following result.

Theorem 0.2. [8, Theorem 0.2] *For any pseudo-commutative operad $\mathscr{O}$ in $\mathbf{Cat}(\mathscr{V})$, there is a multicategory $\mathbf{Mult}(\mathscr{O})$ of $\mathscr{O}$-pseudoalgebras and multilinear k-functors, and there is a submulticategory $\mathbf{Mult_{st}}(\mathscr{O})$ of (strict) $\mathscr{O}$-algebras and multilinear k-functors.*

We emphasize that although the 1-cells of the underlying 2-category $\mathscr{O}$-$\mathbf{AlgSt}$ of $\mathbf{Mult_{st}}(\mathscr{O})$ are 2-functors, the k-morphisms for $k \geq 2$ are only pseudofunctors. They depend on pseudofunctors that give the pseudo-commutativity of $\wedge_{\mathscr{O}}$.

Categories of operators are discussed in the present categorical framework in [5]. There we described the 2-category $\mathscr{D}$-$\mathbf{PsAlg}$ of $\mathscr{D}$-pseudoalgebras and $\mathscr{D}$-pseudomorphisms together with its sub 2-category $\mathscr{D}$-$\mathbf{AlgSt}$ of (strict) $\mathscr{D}$-algebras and (strict) $\mathscr{D}$-morphisms, where $\mathscr{D}$ is a category of operators in $\mathscr{K}$. In [8], we defined pseudo-commutative categories of operators and we proved the following companion to Theorem 0.2.

Theorem 0.3. [8, Theorem 0.3] *For any pseudo-commutative category of operators $\mathscr{D}$ in $\mathbf{Cat}(\mathscr{V})$, there is a multicategory $\mathbf{Mult}(\mathscr{D})$ of $\mathscr{D}$-pseudoalgebras and multilinear k-functors and there is a sub multicategory $\mathbf{Mult_{st}}(\mathscr{D})$ of $\mathscr{D}$-algebras and multilinear k-functors.*

Here again, we emphasize that although the 1-cells of the underlying 2-category $\mathscr{D}$-$\mathbf{AlgSt}$ of $\mathbf{Mult_{st}}(\mathscr{D})$ are 2-functors, the k-morphisms for $k \geq 2$ are only pseudofunctors since they depend on the product $\wedge_{\mathscr{D}}$ that defines the pseudo-commutative pairing on $\mathscr{D}$.

In [5], we constructed a category of operators $\mathscr{D}(\mathscr{O})$ from an operad $\mathscr{O}$ in $\mathscr{K}$. We are only interested in categories of operators of this form. With $\mathscr{D} = \mathscr{D}(\mathscr{O})$, we also constructed an adjoint pair of 2-functors $(\mathbb{L}, \mathbb{R})$ between $\mathscr{O}$-**PsAlg** and $\mathscr{D}$-**PsAlg** that restricts to an adjoint pair of 2-functors between $\mathscr{O}$-**AlgSt** and $\mathscr{D}$-**AlgSt**. We proved the following result in [8].

Theorem 0.4. [8, Theorem 0.4] *Let $\mathscr{O}$ be a pseudo-commutative operad in $\mathbf{Cat}(\mathscr{V})$ and let $\mathscr{D} = \mathscr{D}(\mathscr{O})$. Then $\mathscr{D}$ is a pseudo-commutative category of operators and the 2-functor*

$$\mathbb{R} \colon \mathscr{O}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{PsAlg}$$

extends to a multifunctor $\mathbf{Mult}(\mathscr{O}) \longrightarrow \mathbf{Mult}(\mathscr{D})$ that restricts to a multifunctor $\mathbf{Mult_{st}}(\mathscr{O}) \longrightarrow \mathbf{Mult_{st}}(\mathscr{D})$.

In [9, Theorem 0.4] and [5, Theorem 0.2], we showed how to strictify our underlying additive 2-categories. We constructed 2-adunctions $(\mathbb{S}t, \mathbb{J})$ between $\mathscr{O}$-**PsAlg** and $\mathscr{O}$-**AlgSt** and between $\mathscr{D}$-**PsAlg** and $\mathscr{D}$-**AlgSt**. We proved that these adjunctions are compatible under $\mathbb{R}$ in [5, Theorem 0.3]. Here $\mathbb{J}$ denotes both of the evident inclusions of 2-functors. Their left adjoint strictification 2-functors $\mathbb{S}t$ are constructed using beautiful 2-monadic theory due to Power and Lack [11, 15].

For the case of $\mathscr{D}$, we constructed a 2-monad $\mathbb{D}$ from $\mathscr{D}$ such that $\mathscr{D}$-**PsAlg** is isomorphic to the 2-category $\mathbb{D}$-**PsAlg** of $\mathbb{D}$-pseudoalgebras and $\mathbb{D}$-pseudomorphisms and $\mathscr{D}$-**AlgSt** is isomorphic to the 2-category $\mathbb{D}$-**AlgSt** of $\mathbb{D}$-algebras and $\mathbb{D}$-morphisms. We then applied the general 2-monadic strictification theorem [9, Theorem 2.14] of Power and Lack. This monadic translation is our starting point here. While 2-monads have been important in category theory since the 1970s, they are unfamiliar to most topologists and we avoided using them in [8] for simplicity.

We give a conceptual monadic interpretation of the multicategories in sight and show how to simultaneously strictify and transfer structure from $\mathscr{D}$-pseudoalgebras to strict $\mathscr{F}$-algebras, where $\mathscr{F}$ is the category of finite based sets. The proofs rely on several notions of general interest, independent of their use here, and on some beautiful 2-category theory, primarily due to Lack [11].

There is a standard functorial construction of a multicategory $\mathbf{Mult}(\mathscr{C})$ associated to any symmetric monoidal category $\mathscr{C}$. In Section 1.1, we introduce[1] the notion of a symmetric monoidal *graded* 2-category $\mathscr{C}_*$. It is given by a sequence of 2-categories $\mathscr{C}_k$ and pairings $\mathscr{C}_j \times \mathscr{C}_k \longrightarrow \mathscr{C}_{j+k}$ that are unital, associative, and commutative up to coherent natural isomorphism. Such structure has been used implicitly in stable homotopy for over four decades. Specializing a more general construction, for any (small) category Ψ we shall obtain a symmetric monoidal graded 2-category $\mathscr{C}_*^{\Psi}$ such that $\mathscr{C}_k^{\Psi}$ is the 2-category $\mathscr{C}^{\Psi^k}$ of functors $\Psi^k \longrightarrow \mathscr{C}$.

We show in Section 1.2 that when Ψ is permutative, but not in general otherwise, we can associate a multicategory $\mathbf{Mult}(\mathscr{C}_*^{\mathbf{\Psi}})$ to $\mathscr{C}_*^{\Psi}$. Here $\mathscr{C}^{\Psi}$ is symmetric monoidal under Day convolution, and the universal property of Day convolution implies that the multicategories $\mathbf{Mult}(\mathscr{C}^{\Psi})$ and $\mathbf{Mult}(\mathscr{C}_*^{\Psi})$ are isomorphic. We are mainly interested in the case $\mathscr{C} = \mathscr{K}$ and $\Psi = \mathscr{F}$, where $\mathscr{F}$ has the permutative structure given by the smash product of finite based sets. We may embed $\mathscr{F}$ in $\mathbf{Cat}(\mathscr{V})$ as in [9, §1.1].

Notation 0.5. Note that $\mathscr{F}$-**AlgSt** $= \mathscr{K}^{\mathscr{F}}$. We shall generally use the notation $\mathscr{F}$-**AlgSt** when thinking of this just as a 2-category and shall use the notation $\mathscr{K}^{\mathscr{F}}$ when thinking of it as a symmetric monoidal 2-category.

[1] Actually, the senior author defined this notion in the early 1970's, but never published it.

Remark 0.6. The category $\mathscr{F}$ is itself the category of operators associated to the commutativity operad, and we have the multicategories $\mathbf{Mult}(\mathscr{F})$ and $\mathbf{Mult_{st}}(\mathscr{F})$. These both have k-morphisms defined in terms of pseudofunctors for $k \geq 2$. We also have the multicategory $\mathbf{Mult}(\mathscr{K}^{\mathscr{F}})$. We emphasize that it is defined solely in terms of functors, with no pseudofunctors in sight. The multicategory $\mathbf{Mult}(\mathscr{F})$ gives one starting point for a multiplicative Segalic infinite loop space machine. Our strictification results apply with $\mathscr{D} = \mathscr{F}$, where they show how to strictify from $\mathbf{Mult}(\mathscr{F})$ to $\mathbf{Mult}(\mathscr{K}^{\mathscr{F}})$. However, in contrast with [8], in this paper we shall *never* consider pseudofunctors defined on $\mathscr{F}$.

Categories of operators come with a map $\xi \colon \mathscr{D} \longrightarrow \mathscr{F}$ of categories of operators as part of their structure. In [5, Theorem 0.4], we constructed a 2-functor $\xi_* \colon \mathscr{D}\text{-}\mathbf{AlgSt} \longrightarrow \mathscr{F}\text{-}\mathbf{AlgSt}$ that is left 2-adjoint to the pullback 2-functor ξ^* associated to ξ. The following theorem gives a quick summary of where we are headed.

Notations 0.7. Define $\xi_\# \colon \mathscr{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{F}\text{-}\mathbf{AlgSt}$ to be the composite $\xi_* \circ \mathbb{S}t$.

Theorem 0.8. *For any pseudo-commutative category of operators $\mathscr{D}$, there is a multifunctor*

$$\xi_\# \colon \mathbf{Mult}(\mathscr{D}) \longrightarrow \mathbf{Mult}(\mathscr{K}^{\mathscr{F}})$$

that restricts to $\xi_\# \colon \mathscr{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{F}\text{-}\mathbf{AlgSt}$ on underlying 2-categories.

We shall prove this using 2-monads and Lack's theory of codescent objects [11].

Notations 0.9. Recall that Π is the subcategory of maps ϕ in $\mathscr{F}$ such that $|\phi^{-1}(j)| \leq 1$ for $1 \leq j \leq k$. Specializing and abbreviating from the general notations above, we now define $\mathscr{K}_k$ to be the 2-category $\mathscr{K}^{\Pi^k}$. We then have the symmetric monoidal graded 2-category $\mathscr{K}_*$. Since Π is permutative under the smash product, we have the associated multicategory $\mathbf{Mult}(\mathscr{K}_*)$.

In Section 2.1, we define a notion of a map between 2-monads that are defined on different ground 2-categories.[2] In Section 2.2, we construct a sequence of 2-monads $\mathbb{D}_k$ associated to a category of operators $\mathscr{D}$. The ground 2-category of $\mathbb{D}_k$ is $\mathscr{K}_k$. We also define 2-isomorphisms of 2-monads $\pi_{j,k} \colon \mathbb{D}_{j+k} \longrightarrow \mathbb{D}_j \times \mathbb{D}_k$.

Definition 0.10. We define $\mathbb{D}_*\text{-}\mathbf{PsAlg}$ to be the graded 2-category with kth 2-category $\mathbb{D}_k\text{-}\mathbf{PsAlg}$ and we define $\mathbb{D}_*\text{-}\mathbf{AlgSt}$ to be its sub graded 2-category with kth 2-category $\mathbb{D}_k\text{-}\mathbf{AlgSt}$.

We use the $\pi_{j,k}$ to prove that these are symmetric monoidal graded categories. A punch line is that these constructions make no use of $\wedge_{\mathscr{D}}$, so that $\mathbb{D}_*\text{-}\mathbf{AlgSt}$ is defined entirely without use of pseudofunctors. Using $\wedge_{\mathscr{D}}$, we see that $\mathbb{D}_*\text{-}\mathbf{PsAlg}$ has an associated multicategory, and in Section 2.3, we prove the following result, which in principle is just a comparison of definitions.

Theorem 0.11. *The multicategories $\mathbf{Mult}(\mathscr{D})$ and $\mathbf{Mult}(\mathbb{D}_*\text{-}\mathbf{PsAlg})$ are isomorphic.*

As we observe in Section 1.3, the following simpler analogue puts together more elementary comparisons already summarized.

Theorem 0.12. *The multicategories $\mathbf{Mult}(\mathscr{K}^{\mathscr{F}})$ and $\mathbf{Mult}(\mathscr{K}_*^{\mathscr{F}})$ are isomorphic.*

[2]Our definition differs from the one that is standard in the categorical literature.

In Section 3.1, we use [5, Theorem 2.14] and a check of compatibility with pairings to prove the following theorem, which a priori has nothing to do with multicategories.

Theorem 0.13. *For any pseudo-commutative category of operators $\mathscr{D}$, the inclusion of symmetric monoidal graded 2-categories*

$$\mathbb{J}\colon \mathbb{D}_*\text{-}\mathbf{AlgSt} \longrightarrow \mathbb{D}_*\text{-}\mathbf{PsAlg}$$

has a degree-wise left adjoint strictification 2-functor

$$\mathbb{S}t\colon \mathbb{D}_*\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{D}_*\text{-}\mathbf{AlgSt}.$$

For each $\mathscr{X}$ in each $\mathbb{D}_k$-$\mathbf{PsAlg}$, the unit $\mathscr{X} \longrightarrow \mathbb{J}\mathbb{S}t\mathscr{X}$ of the adjunction is an (internal) equivalence in $\mathbb{D}_k$-$\mathbf{PsAlg}$.

In Section 3.2, we define and study maps of 2-monads $\xi_k\colon \mathbb{D}_k \longrightarrow \mathbb{F}_k$. These induce pullback 2-functors ξ_k^* from strict $\mathbb{F}_k$-algebras to strict $\mathbb{D}_k$-algebras. Specializing a general construction in [5, §4], these 2-functors have left adjoints ξ_*^k. We prove the following companion to Theorem 0.13. As noted in [5], it is weaker in that the analogue of the last statement of Theorem 0.13 is false in general.

Theorem 0.14. *The left adjoints ξ_*^k together give a map*

$$\xi_*\colon \mathbb{D}_*\text{-}\mathbf{AlgSt} \longrightarrow \mathbb{F}_*\text{-}\mathbf{AlgSt}$$

of symmetric monoidal graded 2-categories.

We do not consider the multicategory associated to $\mathbb{D}_*$-$\mathbf{AlgSt}$ since its definition requires use of $\wedge_{\mathscr{D}}$ and thus of pseudomorphisms. Instead, we regard $\mathbb{D}_*$-$\mathbf{AlgSt}$ as a convenient intermediary between $\mathbb{D}_*$-$\mathbf{PsAlg}$ and $\mathbb{F}_*$-$\mathbf{AlgSt}$.

Notations 0.15. Define $\xi_{\#}^k\colon \mathbb{D}_k\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{F}_{\mathbf{k}}\text{-}\mathbf{AlgSt}$ to be $\xi_*^k \circ \mathbb{S}t$. The $\xi_{\#}^k$ give a map $\xi_{\#}\colon \mathbb{D}_*\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{F}_*\text{-}\mathbf{AlgSt}$ of symmetric monoidal graded 2-categories.

To complete the picture, we must get a grip on the image of $\mathbf{Mult}(\mathscr{D})$ under $\xi_{\#}$. Just as $\mathscr{F}$-$\mathbf{AlgSt}$ is another name for $\mathscr{K}^{\mathscr{F}}$, $\mathbb{F}_*$-$\mathbf{AlgSt}$ is another name for $\mathscr{K}_*^{\mathscr{F}}$. The following diagram helps visualize the picture. Here $\mathscr{D} = \mathscr{D}(\mathscr{O})$.

$$\begin{tikzcd}
\mathbf{Mult}(\mathscr{O}) \arrow[r, "\mathbb{R}"] & \mathbf{Mult}(\mathscr{D}) \arrow[r, "\cong"] \arrow[d, dashed, "\xi_{\#}"'] & \mathbf{Mult}(\mathbb{D}_*\text{-}\mathbf{PsAlg}) \arrow[d, dashed, "\xi_{\#}"] \\
& \mathbf{Mult}(\mathscr{K}^{\mathscr{F}}) \arrow[r, "\cong"] & \mathbf{Mult}(\mathscr{K}_*^{\mathscr{F}})
\end{tikzcd} \tag{0.16}$$

In Section 3.4, we use codescent objects to give an equivalent alternative 2-categorical description of the composites $\xi_{\#}^k$ and use that to prove the following result, which gives the right arrow $\xi_{\#}$. Defining the left arrow $\xi_{\#}$ by commutativity of the square, this completes the proof of Theorem 0.8.

Theorem 0.17. *The composite $\xi_{\#} = \xi_* \circ \mathbb{S}t\colon \mathscr{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{F}\text{-}\mathbf{AlgSt}$ is the underlying map of 2-categories of a multifunctor*

$$\xi_{\#}\colon \mathbf{Mult}(\mathbb{D}_*\text{-}\mathbf{PsAlg}) \longrightarrow \mathbf{Mult}(\mathscr{K}_*^{\mathscr{F}}).$$

Since the published treatment of codescent objects is quite concise and seems to us to obscure their relationship to reflexive coequalizers and to simplicial objects, we review their general theory and its restriction to 2-monads in Section 4.

The formal theory described so far has one major defect. When we take $\mathscr{V} = \mathscr{U}$ or $\mathscr{V} = G\mathscr{U}$ and apply the classifying space functor, we lose homotopical control when we pass from $\mathscr{D}$ to $\mathscr{F}$ because of the already noted weakness of Theorem 0.14.

The key problem in the passage from categorical data to G-spectra is to control the homotopical behavior of formal constructions, such as $\xi_{G\#}$, while retaining the good formal properties in so far as possible. We give a construction in our present general 2-categorical framework. In the space level theory of [14, 6], we used two-sided bar constructions to give such homotopical control. The categorical precursor of the bar construction is a certain double $\mathscr{V}$-category construction, which we call a Grothendieck category[3] and describe in Section 5. The idea is to change the ground 2-category from $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$ to the 2-category $\mathbf{Cat^2}(\mathscr{V})$ of double $\mathscr{V}$-categories. We define a graded symmetric monoidal 2-category $\mathscr{D}_*$-$\mathbf{AlgSt^2}$ of "weak double $\mathscr{D}_*$-algebras" , which plays an intermediate role analogous to that of $\mathbb{D}_*$-$\mathbf{AlgSt}$ above, and we construct a monoidal, but not symmetric monoidal, 2-functor

$$\mathbb{G}r\colon \mathscr{D}_*\text{-}\mathbf{PsAlg} \longrightarrow \mathscr{D}\text{-}\mathbf{AlgSt^2}.$$

Theorem 0.12 extends to prove that $\mathscr{F}$-$\mathbf{AlgSt^2}$ is isomorphic to the multicategory associated to the symmetric monoidal 2-category $\mathbf{Cat^2}(\mathscr{V})^{\mathscr{F}}$, and we prove the following analog of Theorem 0.8.

Theorem 0.18. *For any pseudo-commutative category of operators $\mathscr{D}$, there is a (non-symmetric) multifunctor*

$$\xi_\#\mathbb{G}r\colon \mathbf{Mult}(\mathscr{D}) \longrightarrow \mathbf{Mult}\big(\mathbf{Cat^2}(\mathscr{V})^{\mathscr{F}}\big)$$

that restricts to $\xi_\#\mathbb{G}r\colon \mathscr{D}$-$\mathbf{PsAlg} \longrightarrow \mathbf{Cat^2}(\mathscr{V})^{\mathscr{F}}$ on underlying 2-categories.

Remark 0.19. Let G be a finite group. We can do everything that we have discussed starting in the category $G\mathscr{V}$ of G-objects in $\mathscr{V}$ since it satisfies our original hypotheses on $\mathscr{V}$. Our ground 2-category then becomes $\mathbf{Cat}(\mathbf{G}\mathscr{V})$. As indicated in [5, Remark 2.6] and explained in detail in [8, Section 6], we then start with operads $\mathscr{O}$ in $\mathbf{Cat}(\mathbf{G}\mathscr{V})$ and their associated G-categories of operators $\mathscr{D}_G$ over the G-category $\mathscr{F}_G$ of finite G-sets. With no changes other than notation, all of the results above, without exception, apply verbatim in that equivariant context.

Problem 0.20. *While the theory here is more than enough to lead to a quick proof [7] of what is needed to make [4] rigorous, the loss of symmetry is a very serious defect. What conditions on a $\mathbb{D}$-pseudoalgebra $\mathscr{X}$ might ensure that the unit $\mathscr{X} \longrightarrow \xi^*\xi_*\mathscr{X}$ is a (weak) equivalence of underlying Π-categories? In topological situations, we mean that it induces an equivalence on passage to classifying spaces.*

1. Symmetric monoidal graded 2-categories and their multicategories

In Section 2.3, we will identify $\mathbf{Mult}(\mathscr{D})$ as the multicategory associated to the symmetric monoidal graded 2-category $\mathbb{D}_*$-$\mathbf{PsAlg}$. This preliminary section establishes the context of symmetric monoidal graded 2-categories and their associated multicategories.

[3]This conflicts with the usual notion of an abelian Grothendieck category.

These structures appear in many contexts, for example in stable homotopy theory. Although details were never published, since the early 1970's the senior author has understood Lewis-May spectra [12] indexed on the powers U^k of a universe U, with their external smash product, as an example of a symmetric monoidal graded category with an implicit associated multicategory. Operadic internalization of the external smash product is the starting point of [2].

Analogously, there are evident external smash products that give symmetric monoidal graded categories that are precursors of the symmetric monoidal categories of symmetric and orthogonal spectra. In those examples, the internalization is given by Day convolution [13]. The present context is tailor made to relate external products to their internalization via Day convolution, as we shall explain.

1.1. **Symmetric monoidal graded 2-categories.** A graded 2-category $\mathscr{C}_*$ is just a sequence of 2-categories $\mathscr{C}_k$, $k \geq 0$. A pairing $\mathbb{P}$ on $\mathscr{C}_*$ is a set of 2-functors

$$\mathbb{P}_{j,k} \colon \mathscr{C}_j \times \mathscr{C}_k \longrightarrow \mathscr{C}_{j+k}.$$

We say that $\mathbb{P}$ is monoidal if $\mathbb{P}$ is associative and unital up to coherent 2-natural isomorphisms. The unit requires a unit 0-cell in $\mathscr{C}_0$, which we think of as a 2-functor $* \longrightarrow \mathscr{C}_0$ where $*$ is the trivial 2-category. In our examples, $\mathscr{C}_0 = *$. We shall suppress detailed discussion of associativity constraints in the interests of clarity, and we shall use the following notations.

Notations 1.1. For a partition $(j_1, \cdots, j_k)$ of j, we write

$$\mathbb{P}_{j_1,\cdots,j_k} \colon \mathscr{C}_{j_1} \times \cdots \times \mathscr{C}_{j_k} \longrightarrow \mathscr{C}_j$$

for the composite 2-functor obtained by iteration of the $\mathbb{P}_{j,k}$. When each $j_s = 1$, we abbreviate $\mathbb{P}_{1,\cdots,1}$ to

$$\mathbb{P}_k \colon \mathbb{C}_1^k \longrightarrow \mathbb{C}_k.$$

Notice that associativity implies that $\mathbb{P}_j$ is isomorphic to the composite

$$\mathscr{C}_1^j = \mathscr{C}_1^{j_1} \times \cdots \times \mathscr{C}_1^{j_k} \xrightarrow{\mathbb{P}_{j_1} \times \cdots \times \mathbb{P}_{j_k}} \mathscr{C}_{j_1} \times \cdots \times \mathscr{C}_{j_k} \xrightarrow{\mathbb{P}_{j_1,\cdots,j_k}} \mathscr{C}_j.$$

For symmetry, we require $\mathscr{C}_*$ to be a (right) symmetric sequence of 2-categories, so that the symmetric group Σ_k acts from the right on $\mathscr{C}_k$ via 2-functors. We require $\mathbb{P}_{j,k}$ to be a $\Sigma_j \times \Sigma_k$-map, where we use the usual embedding of $\Sigma_j \times \Sigma_k$ in Σ_{j+k} to define the action of $\Sigma_j \times \Sigma_k$ on $\mathscr{C}_{j+k}$. In addition, we require the diagrams

$$\begin{tikzcd}
\mathscr{C}_j \times \mathscr{C}_k \arrow[r, "\tau"] \arrow[d, "\mathbb{P}_{j,k}"'] & \mathscr{C}_k \times \mathscr{C}_j \arrow[d, "\mathbb{P}_{k,j}"] \\
\mathscr{C}_{j+k} \arrow[r, "\tau_{k,j}"'] & \mathscr{C}_{j+k}
\end{tikzcd}$$

to commute up to coherent 2-natural isomorphisms, where τ is the transposition and $\tau_{k,j}$ transposes the first k and last j letters.[4] The coherence requires graded versions of the usual unit, pentagon, and octahedral axioms.

We introduce notations for the kind of examples that will serve as ground 2-categories for our 2-monads $\mathbb{D}_k$.

[4]To see why $\tau_{k,j}$ and not $\tau_{j,k}$ think of conjugation.

Notations 1.2. In this subsection and the next, let $(\mathscr{C}, \otimes)$ be any symmetric monoidal 2-category and let Ψ be any (small) category. In later sections, we shall specialize to cartesian monoidal 2-categories and in particular to $\mathscr{C} = \mathbf{Cat}(\mathscr{V})$ and to the cases $\Psi = \Pi$ and $\Psi = \mathscr{F}$. We regard Ψ as a 2-category with no non-identity 2-cells. Let $\mathscr{C}^\Psi$ denote the 2-category of 2-functors $\Psi \longrightarrow \mathscr{C}$. Let $(\mathscr{C}^\Psi)^k$ denote its k-fold product 2-category and let $\mathscr{C}^\Psi_k$ denote the 2-category $\mathscr{C}^{\Psi^k}$ of 2-functors $\Psi^k \longrightarrow \mathscr{C}$. In all of these 2-categories, the 0-cells are functors, the 1-cells are natural transformations, and the 2-cells are modifications. Since the Ψ^k have no non-identity 2-cells, the compatibility condition for the behavior of modifications on 2-cells trivializes.

Lemma 1.3. *For any symmetric monoidal* 2*-category* $(\mathscr{C}, \otimes)$ *and any (small) category* Ψ*, the graded* 2*-category* $\mathscr{C}^\Psi_*$ *with* $\mathscr{C}^\Psi_k = \mathscr{C}^{\Psi^k}$ *has a symmetric monoidal pairing* $\mathbb{P}$*, which is denoted by* $\mathscr{X} \,\overline{\times}\, \mathscr{Y}$ *for functors* $\mathscr{X} \colon \Psi^j \longrightarrow \mathscr{C}$ *and* $\mathscr{Y} \colon \Psi^k \longrightarrow \mathscr{C}$.

Proof. The required pairing $\mathbb{P}_{j,k} \colon \mathscr{C}^\Psi_j \times \mathscr{C}^\Psi_k \longrightarrow \mathscr{C}^\Psi_{j+k}$ is given by the composite

$$\mathscr{C}^{\Psi^j} \times \mathscr{C}^{\Psi^k} \xrightarrow{\times} (\mathscr{C} \times \mathscr{C})^{\Psi^j \times \Psi^k} \xrightarrow{\cong} (\mathscr{C} \times \mathscr{C})^{\Psi^{j+k}} \xrightarrow{\otimes^{\mathrm{Id}}} \mathscr{C}^{\Psi^{j+k}}.$$

It sends a pair $(\mathscr{X}, \mathscr{Y})$ to the functor

$$\mathscr{X} \,\overline{\otimes}\, \mathscr{Y} \colon \Psi^{j+k} = \Psi^j \times \Psi^k \longrightarrow \mathscr{C}.$$

The action of Σ_k on $\mathscr{C}_k$ is given by permutations of the coordinates of Ψ^k. The coherence isomorphisms are induced from those of the symmetric monoidal category $\mathscr{C}$. For example, for the symmetry, the squares commute in the following diagram and the symmetry isomorphism in $\mathscr{C}$ fills the triangle.

$$\begin{tikzcd}
\Psi^{k+j} \arrow[rr, "\tau_{k,j}"] \arrow[d, equal] & & \Psi^{j+k} \arrow[d, equal] \\
\Psi^k \times \Psi^j \arrow[rr, "\tau"] \arrow[d, "\mathscr{Y} \times \mathscr{X}"'] & & \Psi^j \times \Psi^k \arrow[d, "\mathscr{X} \times \mathscr{Y}"] \\
\mathscr{C} \times \mathscr{C} \arrow[rr, "\tau"] \arrow[dr, "\otimes"'] & \Uparrow & \mathscr{C} \times \mathscr{C} \arrow[dl, "\otimes"] \\
& \mathscr{C} &
\end{tikzcd}$$

The notation $\overline{\otimes}$ indicates that the product is *external*, defined on the evident product of categories. It does not depend on any possible product on Ψ, and that is vital to our theory. We turn to multicategories to internalize, and we focus on the external iterated product

$$\mathbb{P}_k(\mathscr{X}_1, \cdots, \mathscr{X}_k) = \mathscr{X}_1 \overline{\otimes} \cdots \overline{\otimes} \mathscr{X}_k$$

of 0-cells $\mathscr{X}_r$ in $\mathscr{C}^\Psi$.

1.2. **The multicategory associated to $\mathscr{C}^\Psi_*$.** Keeping the notations of Lemma 1.3, we now assume further that Ψ is permutative, and we denote its product by $\wedge_\Psi$. The notation is chosen since the examples of primary interest are Π and $\mathscr{F}$ with their smash products $\wedge_\Pi$ and $\wedge_{\mathscr{F}}$. We defer bringing the non-example $\mathscr{D}$ into the picture until Section 2.3. We shall define a multicategory $\mathbf{Mult}(\mathscr{C}_*^{\mathbf{\Psi}})$, where $\mathscr{C}_k^{\Psi} = \mathscr{C}^{\Psi^k}$.

We could define a larger multicategory than the one we specify, but it seems clearest to focus on $\mathscr{C}^{\Psi} = \mathscr{C}_1^{\Psi}$. For now, we focus on functors rather than pseudo-functors, but we will generalize later. We start with additional structure in $\mathscr{C}_*^{\Psi}$.

Definition 1.4. Let $\wedge_\Psi^k$ denote the k-fold product $\Psi^k \longrightarrow \Psi$. For $k \geq 0$, define a 2-functor $\mathbb{L}_k \colon \mathscr{C}_1^{\Psi} \longrightarrow \mathscr{C}_k^{\Psi}$ by

$$\mathbb{L}_k \mathscr{Y} = \mathscr{Y} \circ \wedge_\Psi^k \colon \Psi^k \longrightarrow \mathscr{C}$$

for a functor $\mathscr{Y} \colon \Psi \longrightarrow \mathscr{C}$, that is for a 0-cell of $\mathscr{C}_1^{\Psi}$. Composition with $\wedge_\Psi^k$ also defines $\mathbb{L}_k$ on 1-cells and 2-cells. For a partition $(j_1, \cdots, j_k)$ of j, define

$$\mathbb{L}_{j_1,\cdots,j_k} \colon \mathscr{C}_k^{\Psi} \longrightarrow \mathscr{C}_j^{\Psi}$$

by sending $\mathscr{W} \colon \Psi^k \longrightarrow \mathscr{C}$ to the composite

$$\Psi^j = \Psi^{j_1} \times \cdots \times \Psi^{j_k} \xrightarrow{\wedge_\Psi^{j_1} \times \cdots \times \wedge_\Psi^{j_k}} \Psi^k \xrightarrow{\mathscr{W}} \mathscr{C}.$$

Notice that the associativity of $\wedge_\Psi$ implies that $\mathbb{L}_j$ is isomorphic to the composite

$$\mathscr{C}_1 \xrightarrow{\mathbb{L}_k} \mathscr{C}_k \xrightarrow{\mathbb{L}_{j_1,\cdots,j_k}} \mathscr{C}_1^j.$$

Construction 1.5. We associate a multicategory $\mathbf{Mult}(\mathscr{C}_*^{\mathbf{\Psi}})$ to the symmetric monoidal graded category $\mathscr{C}_*^{\Psi}$. Recall Notations 1.1 and Definition 1.4. Define the category $\mathbf{Mult_k}(\mathscr{C}_*^{\mathbf{\Psi}})$ of k-morphisms to have objects the tuples $(\mathscr{X}_1, \cdots, \mathscr{X}_k; \mathscr{Y})$ of objects of $\mathscr{C}_1^{\Psi}$ and k-morphisms the morphisms (natural transformations)

$$\mathscr{X}_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \mathscr{X}_k = \mathbb{P}_k(\mathscr{X}_1, \cdots, \mathscr{X}_k) \longrightarrow \mathbb{L}_k \mathscr{Y}$$

in $\mathscr{C}_k^{\Psi}$. The action of Σ_k on the k-morphisms is by permutation of the variables $\mathscr{X}_r$. Schematically, the composition is just like that in the multicategory associated to a symmetric monoidal category, and we offer two explications. Suppose given a k-morphism

$$F \colon \mathscr{Y}_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \mathscr{Y}_k \longrightarrow \mathbb{L}_k \mathscr{Z}$$

and j_r-morphisms

$$E_r \colon \mathscr{X}_{r,1} \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \mathscr{X}_{r,j_r} \longrightarrow \mathbb{L}_{j_r} \mathscr{Y}_r$$

for $1 \leq r \leq k$. Then their composite, written $F \circ (E_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} E_k)$ is the composite morphism in $\mathscr{C}_j$ displayed in the following diagram, where $j = j_1 + \cdots j_k$.

$$\begin{array}{c}
\overline{\bigotimes}_{1\leq r\leq k} \overline{\bigotimes}_{1\leq s\leq j_k} \mathscr{X}_{r,s} \\
\Big\downarrow {\scriptstyle \overline{\otimes}_{1\leq r\leq k} E_r} \\
\overline{\bigotimes}_{1\leq r\leq k} \mathbb{L}_{j_r} \mathscr{Y}_r \\
\Big\downarrow {\scriptstyle \cong} \\
\mathbb{L}_{j_1,\cdots,j_k} \big(\overline{\bigotimes}_{1\leq r\leq k} \mathscr{Y}_r \big) \\
\Big\downarrow {\scriptstyle \mathbb{L}_{j_1,\cdots,j_k} F} \\
\mathbb{L}_{j_1,\cdots,j_k} \mathbb{L}_k \mathscr{Z} \cong \mathbb{L}_j \mathscr{Z}.
\end{array}$$

A perhaps more conceptual description writes this in terms of the 2-functors $\mathbb{P}_k$ and $\mathbb{L}_k$, so we change notations by starting with

$$F\colon \mathbb{P}_k(\mathscr{Y}_1,\cdots,\mathscr{Y}_k) \longrightarrow \mathbb{L}_k\mathscr{Z}$$

and

$$E_r\colon \mathbb{P}_{j_r}(\mathscr{X}_{r,1},\cdots,\mathscr{X}_{r,j_r}) \longrightarrow \mathbb{L}_{j_r}\mathscr{Y}_r,$$

$1 \leq r \leq k$. Write

$$\underline{\mathscr{Y}} = (\mathscr{Y}_1,\cdots,\mathscr{Y}_k) \quad \text{and} \quad \underline{\mathscr{X}}_{j_r} = (\mathscr{X}_{r,1},\cdots,\mathscr{X}_{r,j_r}).$$

We have the sequence

$$\underline{\mathscr{X}} = (\underline{\mathscr{X}}_1,\cdots,\underline{\mathscr{X}_k})$$

of the j-variables $\mathscr{X}_{r,s}$, $1 \leq r \leq k$ and $1 \leq s \leq j_r$. Write

$$\mathbb{L}_{\underline{j}} = \mathbb{L}_{j_1,\cdots,j_k} \quad \text{and} \quad \mathbb{P}_{\underline{j}} = \mathbb{P}_{j_1,\cdots,j_k}.$$

We obtain $F\circ(E_1 \,\overline{\otimes}\, \cdots \,\overline{\otimes}\, E_k)$ by applying the following composite to $(F; E_1,\cdots,E_k)$.

$$\begin{array}{c}
\mathscr{C}_j(\mathbb{P}_k\underline{\mathscr{Y}},\mathbb{L}_k\mathscr{Z}) \times \Big(\mathscr{C}_{j_1}(\mathbb{P}_{j_1}\underline{\mathscr{X}}_1,\mathbb{L}_{j_1}\mathscr{Y}_1) \times \cdots \times \mathscr{C}_{j_k}(\mathbb{P}_{j_k}\underline{\mathscr{X}}_k,\mathbb{L}_{j_k}\mathscr{Y}_k)\Big) \\
\Big\downarrow {\scriptstyle \mathbb{L}_{\underline{j}}\times\mathbb{P}_{\underline{j}}} \\
\mathscr{C}_j\big(\mathbb{L}_{\underline{j}}\mathbb{P}_k\mathscr{Y},\mathbb{L}_j\mathscr{Z}\big) \times \mathscr{C}_j\big(\mathbb{P}_j\underline{\mathscr{X}},\mathbb{P}_{\underline{j}}(\mathbb{L}_{j_1},\cdots,\mathbb{L}_{j_k})(\underline{\mathscr{Y}})\big) \\
\Big\downarrow {\scriptstyle \circ} \\
\mathscr{C}_j(\mathbb{P}_j\underline{\mathscr{X}},\mathbb{L}_j\mathscr{Z})
\end{array} \tag{1.6}$$

We are identifying $\mathbb{L}_{\underline{j}}\mathbb{L}_k$ with $\mathbb{L}_j$ and $\mathbb{P}_{\underline{j}} \circ (\mathbb{P}_{j_1},\cdots,\mathbb{P}_{j_k})$ with $\mathbb{P}_j$, as in Notations Definition 1.4 and Notations 1.1. To make sense of composition, we are also identifying $\mathbb{L}_{\underline{j}}\mathbb{P}_k\mathscr{Y}$ with $\mathbb{P}_{\underline{j}}(\mathbb{L}_{j_1},\cdots,\mathbb{L}_{j_k})\underline{\mathscr{Y}}$. Indeed, by inspection, both can be identified with

$$\mathbb{L}_{j_1}\mathscr{Y}_1 \,\overline{\otimes}\, \cdots \,\overline{\otimes}\, \mathbb{L}_{j_k}\mathscr{Y}_k\colon \Psi^j = \Psi^{j_1} \times \cdots \times \Psi^{j_k} \longrightarrow \mathscr{C}.$$

1.3. **Day convolution and the multicategory $\mathbf{Mult}(\mathscr{K}^{\mathscr{F}})$.** We continue to write $\mathscr{C}_k^{\Psi} = \mathscr{C}^{\Psi^k}$ for a symmetric monoidal category $(\mathscr{C},\otimes)$ and a permutative category $(\Psi,\wedge_\Psi)$. Day convolution internalizes our external k-fold product. It is given by left Kan extension as in the diagram

$$\begin{tikzcd}
\Psi^k \arrow[r, "\mathscr{X}_1 \overline{\otimes} \cdots \overline{\otimes} \mathscr{X}_k"] \arrow[d, "\wedge_\Psi^k"'] & \mathscr{C} \\
\Psi \arrow[ur, "\mathscr{X}_1 \otimes \cdots \otimes \mathscr{X}_k"'] &
\end{tikzcd}$$

where the $\mathscr{X}_r$ are functors $\Psi \longrightarrow \mathscr{C}$. Recall that $\mathbb{L}_k\mathscr{Y} = \mathscr{Y} \circ \wedge_\Psi^k$, where $\mathscr{Y}$ is also a functor $\Psi \longrightarrow \mathscr{C}$. The universal property of left Kan extension gives

$$\mathscr{C}_1(\mathscr{X}_1 \otimes \cdots \otimes \mathscr{X}_k,\mathscr{Y}) \cong \mathscr{C}_k(\mathscr{X}_1 \,\overline{\otimes}\, \cdots \,\overline{\otimes}\, \mathscr{X}_k,\mathbb{L}_k\mathscr{Y}). \tag{1.7}$$

The 2-category $\mathscr{C}_1^{\Psi}$ is symmetric monoidal under Day convolution and therefore has an associated multicategory $\mathbf{Mult}(\mathscr{C}_\mathbf{1}^{\boldsymbol{\Psi}})$, and (1.7) identifies its k-morphisms with the k-morphisms of the multicategory $\mathbf{Mult}(\mathscr{C}_*^{\boldsymbol{\Psi}})$ associated to the symmetric monoidal graded 2-category $\mathscr{C}_*^{\Psi}$. Straightforward verifications give the following result, which says that $\mathbf{Mult}(\mathscr{C}_*^{\boldsymbol{\Psi}})$ is the external avatar of $\mathbf{Mult}(\mathscr{C}_\mathbf{1}^{\boldsymbol{\Psi}})$.

Theorem 1.8. *The multicategories* $\mathbf{Mult}(\mathscr{C}_{\mathbf{1}}^{\mathbf{\Psi}})$ *and* $\mathbf{Mult}(\mathscr{C}_{*}^{\mathbf{\Psi}})$ *are isomorphic.*

This is a general result, and we change notations in the case of greatest interest. We now take $\mathscr{C}$ to be the cartesian monoidal 2-category $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$ and take $\Psi = \mathscr{F}$, so that $\mathscr{C}_1^\Psi = \mathscr{K}^{\mathscr{F}}$ and $\mathscr{C}_k^\Psi = \mathscr{K}^{\mathscr{F}^k}$. With these identifications, Theorem 1.8 specializes to Theorem 0.12.

Remark 1.9. Since $\mathscr{F}$ is a category of operators, it has an associated 2-monad $\mathbb{F}$ such that $\mathscr{K}^{\mathscr{F}} = \mathscr{F}$-**AlgSt** is isomorphic to $\mathbb{F}$-**AlgSt**. Constructions in the next section give monads $\mathbb{F}_k$ such that $\mathbb{F}_*$-**AlgSt** is isomorphic to the symmetric monoidal graded 2-category with kth 2-category $\mathscr{K}^{\mathscr{F}^k}$ and similarly on passage to multicategories.

Warning 1.10. We thought in terms of $\mathscr{K}^{\mathscr{F}^k}$ here. That is reasonable since $\mathscr{F}$ is permutative. For general categories of operators we shall *never* consider $\mathscr{K}^{\mathscr{D}^k}$ and *never* consider Day convolution along $\wedge_{\mathscr{D}}^k$ since the pseudo-commutativity of $\wedge_{\mathscr{D}}$ would defeat our purposes. From now on, as in Notations 0.9, the notation $\mathbb{L}_k$ will refer *only* to precomposition by $\mathscr{K}_k = \wedge_\Pi^k$. Working externally and monadically, we will find a different way to exploit $\wedge_{\mathscr{D}}^k$, using change of monads.

2. Monads and the multicategory $\mathbf{Mult}(\mathscr{D})$

2.1. **Change of monad.** To construct $\mathbb{D}_*$-**PsAlg**, we shall first construct a sequence of monads $\mathbb{D}_k$, where $\mathbb{D}_k$ is defined on the ground 2-category $\mathscr{K}_k$. To describe the relationship between the $\mathbb{D}_k$ as k varies, we insert a definition that explains the idea of change of monad. Recall that a strict map $\xi\colon \mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads defined on the same ground category $\mathscr{K}$ is a 2-natural transformation ξ such that the following diagrams commute.

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta"] \arrow[dr, "\eta"'] & \mathbb{D} \arrow[d, "\xi"] \\ & \mathbb{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}\mathbb{D} \arrow[rr, "\mu"] \arrow[d, "\mathbb{D}\xi"'] & & \mathbb{D} \arrow[d, "\xi"] \\ \mathbb{D}\mathbb{E} \arrow[r, "\xi"'] & \mathbb{E}\mathbb{E} \arrow[r, "\mu"'] & \mathbb{E} \end{tikzcd}$$

A pseudomap ξ allows these diagrams to commute only up to invertible 2-cells. In line with [9, §2.1], we shall only allow normal pseudomaps, for which the unit diagram commutes. We view the 2-cell in the second diagram as mapping $\mu\circ\xi\circ\mathbb{D}\xi$ to $\xi\circ\mu$. We generalize these notions so as to allow change of ground 2-category.

Definition 2.1. Let $\mathbb{D}$ be a 2-monad in a ground 2-category $\mathscr{K}$, let $\mathbb{E}$ be a 2-monad in a ground 2-category $\mathscr{L}$, and let $\mathbb{Q}\colon \mathscr{L} \longrightarrow \mathscr{K}$ be a 2-functor (note the direction of the arrow). A pseudomap $\mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads is a 2-natural transformation $\xi\colon \mathbb{D}\circ\mathbb{Q} \Longrightarrow \mathbb{Q}\circ\mathbb{E}$ of 2-functors $\mathscr{L} \longrightarrow \mathscr{K}$ such that the following diagrams of 2-natural transformations commute, the second up to an invertible modification λ, and such that certain coherence conditions are satisfied. If λ is the identity, we say that ξ is a (strict) map.

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\eta"] \arrow[dr, "\mathbb{Q}\eta"'] & \mathbb{D}\mathbb{Q} \arrow[d, "\xi"] \\ & \mathbb{Q}\mathbb{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}\mathbb{D}\mathbb{Q} \arrow[rr, "\mu"] \arrow[d, "\mathbb{D}\xi"'] & \arrow[d, phantom, "\lambda \Uparrow"] & \mathbb{D}\mathbb{Q} \arrow[d, "\xi"] \\ \mathbb{D}\mathbb{Q}\mathbb{E} \arrow[r, "\xi"'] & \mathbb{Q}\mathbb{E}\mathbb{E} \arrow[r, "\mathbb{Q}\mu"'] & \mathbb{Q}\mathbb{E} \end{tikzcd}$$

Strictly speaking, this defines a normal pseudomap. More generally, we could allow the first diagram to also commute up to an invertible modification, but it does commute in our examples.

There are two coherence conditions connecting monadic composition to unit maps. We restrict to the case when the unit is strict, so that $\xi\eta\mathbb{Q} = \mathbb{Q}\eta \colon \mathbb{Q} \longrightarrow \mathbb{Q}\mathbb{E}$, but the general case is identical. We require both of the pasting diagrams

$$\begin{tikzcd}
\mathbb{D}\mathbb{Q} \arrow[rr, "\xi"] & \arrow[d, phantom, "\lambda \Uparrow"] & \mathbb{Q}\mathbb{E} \\
\mathbb{D}\mathbb{D}\mathbb{Q} \arrow[u, "\mu"'] \arrow[r, "\mathbb{D}\xi"] & \mathbb{D}\mathbb{Q}\mathbb{E} \arrow[r, "\xi"] & \mathbb{Q}\mathbb{E}\mathbb{E} \arrow[u, "\mathbb{Q}\mu"] \\
\mathbb{D}\mathbb{Q} \arrow[u, "\eta"'] \arrow[uu, bend left=40, "\mathrm{id}"] \arrow[r, "\xi"'] & \mathbb{Q}\mathbb{E} \arrow[u, "\eta"'] \arrow[r, "\mathrm{id}"'] & \mathbb{Q}\mathbb{E} \arrow[u, "\mathbb{Q}\eta"] \arrow[uu, bend right=40, "\mathrm{id}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{D}\mathbb{Q} \arrow[rr, "\xi"] & \arrow[d, phantom, "\lambda \Uparrow"] & \mathbb{Q}\mathbb{E} \\
\mathbb{D}\mathbb{D}\mathbb{Q} \arrow[u, "\mu"'] \arrow[r, "\mathbb{D}\xi"] & \mathbb{D}\mathbb{Q}\mathbb{E} \arrow[r, "\xi"] & \mathbb{Q}\mathbb{E}\mathbb{E} \arrow[u, "\mathbb{Q}\mu"] \\
\mathbb{D}\mathbb{Q} \arrow[u, "\mathbb{D}\eta"'] \arrow[uu, bend left=40, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & \mathbb{D}\mathbb{Q} \arrow[u, "\mathbb{D}\mathbb{Q}\eta"'] \arrow[r, "\xi"'] & \mathbb{Q}\mathbb{E} \arrow[u, "\mathbb{Q}\mathbb{E}\eta"] \arrow[uu, bend right=40, "\mathrm{id}"']
\end{tikzcd}$$

to be equal to the identity pasting diagram

$$\begin{tikzcd}
\mathbb{D}\mathbb{Q} \arrow[r, "\xi"] & \mathbb{Q}\mathbb{E} \\
\mathbb{D}\mathbb{Q} \arrow[u, "\mathrm{id}"] \arrow[r, "\xi"'] & \mathbb{Q}\mathbb{E}. \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Finally we have an associativity coherence condition. The following two associativity pasting diagrams must be equal.

$$\begin{tikzcd}
\mathbb{D}\mathbb{D}\mathbb{D}\mathbb{Q} \arrow[r, "\mu"] \arrow[dr, "\mathbb{D}\mu"] \arrow[dd, "\mathbb{D}\mathbb{D}\xi"'] & \mathbb{D}\mathbb{D}\mathbb{Q} \arrow[r, "\mu"] & \mathbb{D}\mathbb{Q} \arrow[r, "\xi"] \arrow[d, phantom, "\lambda \Uparrow"] & \mathbb{Q}\mathbb{E} \\
& \mathbb{D}\mathbb{D}\mathbb{Q} \arrow[ur, "\mu"] \arrow[r, "\mathbb{D}\xi"] \arrow[d, phantom, "\mathbb{D}\lambda \Uparrow"] & \mathbb{D}\mathbb{Q}\mathbb{E} \arrow[dr, "\xi"] & \\
\mathbb{D}\mathbb{D}\mathbb{Q}\mathbb{E} \arrow[r, "\mathbb{D}\xi"'] & \mathbb{D}\mathbb{Q}\mathbb{E}\mathbb{E} \arrow[ur, "\mathbb{D}\mathbb{Q}\mu"] \arrow[r, "\xi"'] & \mathbb{Q}\mathbb{E}\mathbb{E}\mathbb{E} \arrow[r, "\mathbb{Q}\mu"'] & \mathbb{Q}\mathbb{E}\mathbb{E} \arrow[uu, "\mathbb{Q}\mu"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{DDDQ} \arrow[r, "\mu"] \arrow[dd, "\mathbb{DD}\xi"'] & \mathbb{DDQ} \arrow[r, "\mu"] \arrow[dr, "\mathbb{D}\xi"] & \mathbb{DQ} \arrow[r, "\xi"] \arrow[d, phantom, "\lambda\Uparrow"] & \mathbb{QE} \\
 & & \mathbb{DQE} \arrow[dr, "\xi"] \arrow[d, phantom, "\mathbb{D}\lambda\Uparrow"] & \\
\mathbb{DDQE} \arrow[r, "\mathbb{D}\xi"'] \arrow[urr, "\mu"] & \mathbb{DQEE} \arrow[r, "\xi"'] & \mathbb{QEEE} \arrow[r, "\mathbb{Q}\mu"'] & \mathbb{QEE} \arrow[uu, "\mathbb{Q}\mu"']
\end{tikzcd}$$

The point of the definition is the following observation, the proof of which is diagram chasing from the definitions, just as in the case when $\mathbb{Q} = \mathrm{id}$.

Lemma 2.2. *Let $\xi\colon \mathbb{D}\circ\mathbb{Q} \longrightarrow \mathbb{Q}\circ\mathbb{E}$ specify a map $\mathbb{E} \longrightarrow \mathbb{D}$ of 2-monads, as in Definition 2.1. If (X,θ) is an $\mathbb{E}$-algebra, then $\mathbb{Q}X$ is a $\mathbb{D}$-algebra with structure map the composite*

$$\mathbb{DQ}X \xrightarrow{\ \xi\ } \mathbb{QE}X \xrightarrow{\ \mathbb{Q}\theta\ } \mathbb{Q}X.$$

More generally, if ξ is a pseudomap and (X,θ,ϕ) is an $\mathbb{E}$-pseudoalgebra, so that $\phi\colon \theta\circ\mathbb{E}\theta \Longrightarrow \theta\circ\mu$ is an invertible 2-cell, then $(\mathbb{Q}X, \mathbb{Q}\theta\circ\xi, \psi)$ is a $\mathbb{D}$-pseudoalgebra, where ψ denotes the composite in the pasting diagram

$$\begin{tikzcd}
\mathbb{DDQ}X \arrow[rr, "\mu"] \arrow[d, "\mathbb{D}\xi"'] & \arrow[d, phantom, "\lambda\Uparrow"] & \mathbb{DQ}X \arrow[d, "\xi"] \\
\mathbb{DQE}X \arrow[r, "\xi"] \arrow[d, "\mathbb{DQ}\theta"'] & \mathbb{QEE}X \arrow[r, "\mathbb{Q}\mu"] \arrow[d, "\mathbb{QE}\theta"'] \arrow[dr, phantom, "\mathbb{Q}\phi\Uparrow"] & \mathbb{QE}X \arrow[d, "\mathbb{Q}\theta"] \\
\mathbb{DQ}X \arrow[r, "\xi"'] & \mathbb{QE}X \arrow[r, "\mathbb{Q}\theta"'] & \mathbb{Q}X.
\end{tikzcd}$$

Remark 2.3. In the categorical literature, starting (implicitly) with Street [16], a pseudomap $\mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads defined on 2-categories $\mathscr{K}$ and $\mathscr{L}$ is defined by a 2-functor $\mathbb{P}\colon \mathscr{K} \longrightarrow \mathscr{L}$ together with a 2-natural transformation $\zeta\colon \mathbb{EP} \longrightarrow \mathbb{PD}$ such that the analogues of the diagrams in Definition 2.1 commute and equalities of pasting diagrams analogous to those above hold. With that definition, a $\mathbb{D}$-pseudoalgebra X pushes forward to give a $\mathbb{E}$-pseudoalgebra $\mathbb{P}X$. Thus our pseudomaps $\mathbb{D} \longrightarrow \mathbb{E}$ are the category theorists' pseudomaps $\mathbb{E} \longrightarrow \mathbb{D}$. When $\mathbb{Q}$ is the identity 2-functor of a 2-category $\mathscr{K}$, it is our definition that specializes to the standard notion of a morphism of monoids in a functor category.

2.2. **The monads $\mathbb{D}_k$ and the graded 2-category $\mathbb{D}_*$-PsAlg.** From here on, Notations 0.9 apply, so that $\mathscr{K}_k$ denotes $\mathbf{Cat}(\mathscr{V})^{\mathbf{\Pi^k}}$; see Warning 1.10. We shall construct a monad $\mathbb{D}_k$ on $\mathscr{K}_k$ for each $k \geq 0$.

Recall that $\mathscr{D}$ contains the category Π (viewed as a $\mathscr{V}$-category), and recall from [5, §2.2] that for an object $\mathscr{X}$ of $\mathscr{K}$, $\mathbb{D}\mathscr{X}$ is the categorical tensor product of functors $\mathscr{D}\otimes_{\Pi}\mathscr{X}$. More precisely, its nth $\mathscr{V}$-category is the tensor product of the contravariant functor $\mathscr{D}(-,\mathbf{n})\colon \mathbf{\Pi} \longrightarrow \mathbf{Cat}(\mathscr{V})$, where composition and the inclusion of Π in $\mathscr{D}$ give the functoriality, and the covariant functor $\mathscr{X} : \Pi \longrightarrow \mathbf{Cat}(\mathscr{V})$.

Ignoring the projections in Π, which serve to remove superfluous summands when $\mathscr{D} = \mathscr{D}(\mathscr{O})$, $\mathbb{D}\mathscr{X}(\mathbf{n})$ can be viewed as a quotient of

$$\coprod_m \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{X}(\mathbf{m})$$

by equivariance and base object identifications. Although notationally cumbersome, it is helpful to think of $\mathscr{D}$ as a functor $\mathscr{D}(-,-)$ of two variables, contravariant in the first and covariant in the second, to make sense of the constructions in this section. The tensor product of functors uses the contravariant variable, and then the covariant variable gives the functoriality on Π.

Analogously, for $\mathscr{Y}\colon \Pi^k \longrightarrow \mathbf{Cat}(\mathscr{V})$, define $\mathbb{D}_k\mathscr{Y}$ to be the tensor product $\mathscr{D}^k \otimes_{\Pi^k} \mathscr{Y}$. It is the left Kan extension of $\mathscr{Y}$ along the inclusion $\iota^k\colon \Pi^k \longrightarrow \mathscr{D}^k$. More explicitly, its $(n_1,\cdots,n_k)$th $\mathscr{V}$-category $\mathbb{D}_k\mathscr{Y}(\mathbf{n_1},\cdots,\mathbf{n_k})$ can be viewed as a quotient of

$$\coprod_{(m_1,\cdots,m_k)} \big(\prod_{1\le i\le k} \mathscr{D}(\mathbf{m_i},\mathbf{n_i})\big) \times \mathscr{Y}(\mathbf{m_1},\cdots,\mathbf{m_k})\big) \tag{2.4}$$

by equivariance and base object identifications. The monadic unit $\eta\colon \mathscr{Y} \longrightarrow \mathbb{D}_k\mathscr{Y}$ is induced by the product of the units $* \longrightarrow \mathscr{D}(\mathbf{m_j},\mathbf{m_j})$ and the monadic product μ is induced by the composition morphisms of the categories $\mathscr{D}(\mathbf{m},\mathbf{n})$. Conceptually, we have an evident associativity isomorphism

$$\mathscr{D}^k \otimes_{\Pi^k} (\mathscr{D}^k \otimes_{\Pi^k} \mathscr{Y}) \cong (\mathscr{D}^k \otimes_{\Pi^k} \mathscr{D}^k) \otimes_{\Pi^k} \mathscr{Y}$$

and we obtain μ by tensoring the identity of $\mathscr{Y}$ with the composition bifunctor

$$\mathscr{D}^k \otimes_{\Pi^k} \mathscr{D}^k \longrightarrow \mathscr{D}^k.$$

Of course, $\mathbb{D}_1 = \mathbb{D}$. We define $\mathbb{D}_0$ to be the identity monad on the trivial sub 2-category of $\mathbf{Cat}(\mathscr{V})$. We assemble the monads $\mathbb{D}_k$ into a graded structure resembling a comonoid. We could just as well regard it as a graded monoid, but the direction of the relevant isomorphisms matters.

Since $\mathscr{K} = \mathbf{Cat}(\mathscr{V})$ is cartesian monoidal, we write $\overline{\times}$ rather than $\overline{\otimes}$ to indicate external products. Thus consider functors $\mathscr{X}\colon \Pi^j \longrightarrow \mathscr{K}$ and $\mathscr{Y}\colon \Pi^k \longrightarrow \mathscr{K}$, where $j \ge 0$ and $k \ge 0$ (with $\mathscr{X}$ or $\mathscr{Y}$ being $*$ if $j = 0$ or $k = 0$). The evident projections give the coordinates of a 2-natural transformation

$$\pi_{j,k}\colon \mathscr{D}^{j+k} \otimes_{\Pi^{j+k}} (\mathscr{X} \,\overline{\times}\, \mathscr{Y}) \longrightarrow (\mathscr{D}^j \otimes_{\Pi^j} \mathscr{X}) \,\overline{\times}\, (\mathscr{D}^k \otimes_{\Pi^k} \mathscr{Y}).$$

Looking at the expression ((2.4)) for $\mathbb{D}_{j+k}(\mathscr{X} \,\overline{\times}\, \mathscr{Y})$, we see that it can be identified with the product of the corresponding expressions for $\mathbb{D}_j\mathscr{X}$ and $\mathbb{D}_k\mathscr{Y}$. Since the equivariance and base object identifications agree, we conclude that

$$\pi_{j,k}\colon \mathbb{D}_{j+k}(\mathscr{X} \,\overline{\times}\, \mathscr{Y}) \longrightarrow \mathbb{D}_j(\mathscr{X}) \,\overline{\times}\, \mathbb{D}_k(\mathscr{Y})$$

is a 2-natural isomorphism. Now recall Definition 2.1 and the pairings

$$\mathbb{P}_{j,k}\colon \mathscr{K}_j \times \mathscr{K}_k \longrightarrow \mathscr{K}_{j+k}$$

from Section 1.1. Together with a check of diagrams, we find that the isomorphism $\pi_{j,k}$ gives the following result, whose second statement is an application of Lemma 2.2.

Lemma 2.5. *The* 2*-natural isomorphism* $\pi_{j,k}\colon \mathbb{D}_{j+k} \circ \mathbb{P}_{j,k} \longrightarrow \mathbb{P}_{j,k} \circ (\mathbb{D}_j, \mathbb{D}_k)$ *specifies a map of* 2*-monads* $\mathbb{D}_{j+k} \longrightarrow (\mathbb{D}_j, \mathbb{D}_k)$*. Therefore, if* $\mathscr{X}$ *is a* $\mathbb{D}_j$*-pseudoalgebra* *and $\mathscr{Y}$ is a $\mathbb{D}_k$-pseudoalgebra, then $\mathscr{X} \,\overline{\times}\, \mathscr{Y}$ is a $\mathbb{D}_{j+k}$-pseudoalgebra; its action map $\Theta_{j,k}$ is determined by the action maps θ_j of $\mathscr{X}$ and θ_k of $\mathscr{Y}$ as the composite*

$$\mathbb{D}_{j+k}(\mathscr{X} \,\overline{\times}\, \mathscr{Y}) \xrightarrow{\pi_{j,k}} \mathbb{D}_j \mathscr{X} \,\overline{\times}\, \mathbb{D}_k \mathscr{Y} \xrightarrow{\theta_j \,\overline{\times}\, \theta_k} \mathscr{X} \,\overline{\times}\, \mathscr{Y}.$$

The 2-categories $\mathbb{D}_k$-**PsAlg** of $\mathbb{D}_k$-pseudoalgebras give the graded 2-category $\mathbb{D}_*$-**PsAlg**. The lemma gives a system of pairings

$$\mathbb{P}_{j,k} \colon \mathbb{D}_j\text{-}\mathbf{PsAlg} \times \mathbb{D}_\mathbf{k}\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{D}_\mathbf{j+k}\text{-}\mathbf{PsAlg}.$$

On underlying 2-categories, the pairing $\mathbb{P} = \{P_{j,k}\}$ restricts to the pairing $\mathbb{P}$ on $\mathscr{K}_*$ of Section 1.1. With unit, associativity, and symmetry constraints inherited from those of $\mathscr{K}_*^\Pi$, we have the following result.

Proposition 2.6. *The pairing $\mathbb{P}$ gives $\mathbb{D}_*$-**PsAlg** a structure of symmetric monoidal graded 2-category. It restricts to give $\mathbb{D}_*$-**AlgSt** a structure of symmetric monoidal graded 2-category.*

Observe that the monads $\mathbb{D}_k$ do not depend on the product $\wedge_{\mathscr{D}}$ on $\mathscr{D}$. Therefore neither do the symmetric monoidal graded 2-categories of Proposition 2.6. We see that $\mathbb{D}_*$-**AlgSt** is defined entirely in terms of strict categorical data, with no pseudofunctors or pseudotransformations in sight.

2.3. **The monadic reinterpretation of Mult($\mathscr{D}$).** As explained in [5, §2.2], $\mathscr{D}$-pseudoalgebras are the same as $\mathbb{D}$-pseudoalgebras in $\mathscr{K}_1 = \mathscr{K}^\Pi$. We here generalize to obtain an analogous description of the k-morphisms of **Mult**($\mathscr{D}$) for $k \geq 2$. We use that to identify **Mult**($\mathscr{D}$) with **Mult**($\mathbb{D}_*$-**PsAlg**), thus proving Theorem 0.11.

We have the product 2-monad $\mathbb{D}^k = (\mathbb{D}, \cdots, \mathbb{D})$ defined on the kth power $\mathscr{K}_1^k$ of $\mathscr{K}_1$. Thus

$$\mathbb{D}^k(\mathscr{X}_1, \cdots, \mathscr{X}_k) = (\mathbb{D}\mathscr{X}_1, \cdots, \mathbb{D}\mathscr{X}_k).$$

Composition with the iterated product $\mathscr{K}^k \longrightarrow \mathscr{K}$ gives the 2-functor

$$\mathbb{P}_k \colon \mathscr{K}_1^k = (\mathscr{K}^\Pi)^k \longrightarrow \mathscr{K}^{\Pi^k} = \mathscr{K}_k.$$

It sends a k-tuple $(\mathscr{X}_1, \cdots, \mathscr{X}_k)$ of functors $\Pi \longrightarrow \mathscr{K}$ to the single functor

$$\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k \colon \Pi^k \longrightarrow \mathscr{K},$$

where we have again written $\overline{\times}$ for this external iterated cartesian product.

The evident projections specify the coordinates of a 2-natural transformation

$$\pi_k \colon \mathscr{D}^k \otimes_{\Pi^k} (\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k) \longrightarrow (\mathscr{D} \,\overline{\times}\,_\Pi \mathscr{X}_1) \,\overline{\times}\, \cdots \,\overline{\times}\, (\mathscr{D} \otimes_\Pi \mathscr{X}_k).$$

Arguing as above Lemma 2.7, we see that it is a 2-natural isomorphism

$$\pi_k \colon \mathbb{D}_k(\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k) \longrightarrow \mathbb{D}\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathbb{D}\mathscr{X}_k.$$

This leads to the following analog of Lemma 2.5.

Lemma 2.7. *The 2-natural transformation $\pi_k \colon \mathbb{D}_k \circ \mathbb{P}_k \longrightarrow \mathbb{P}_k \circ \mathbb{D}^k$ specifies a map of 2-monads $\mathbb{D}_k \longrightarrow \mathbb{D}^k$. Therefore, if $\mathscr{Y}$ is a $\mathbb{D}^k$-pseudoalgebra, then $\mathbb{P}_k\mathscr{Y}$ is a $\mathbb{D}_k$-pseudoalgebra.*

Thus if $\mathscr{X}_r$ are $\mathbb{D}$-pseudoalgebras, $1 \leq r \leq k$, then their external product $\prod_r \mathscr{X}_i$ inherits a structure of $\mathbb{D}_k$-pseudoalgebra via the composite

$$\mathbb{D}_k(\textstyle\prod_r \mathscr{X}_r) \xrightarrow{\pi_k} \textstyle\prod_r \mathbb{D}\mathscr{X}_r \xrightarrow{\theta^k} \textstyle\prod_r \mathscr{X}_r,$$

which we denote by Θ^k.

We have the 2-functor $\mathbb{L}_k \colon \mathscr{K}_1 \longrightarrow \mathscr{K}_k$ that sends a functor $\mathscr{Y} \colon \Pi \longrightarrow \mathscr{K}$ to $\mathscr{Y} \circ \wedge_\Pi^k \colon \Pi^k \longrightarrow \mathscr{K}$. While Lemmas Lemma 2.5 and Lemma 2.7 gave maps of 2-monads, the following more substantial result only gives a pseudomap.

Proposition 2.8. *There is a* 2*-natural transformation* $\Lambda_k \colon \mathbb{D}_k \circ \mathbb{L}_k \longrightarrow \mathbb{L}_k \circ \mathbb{D}$ *and an invertible modification*

$$\begin{tikzcd}
\mathbb{D}_k\mathbb{D}_k\mathbb{L}_k \arrow[rr, "\mu"] \arrow[d, "\mathbb{D}_k\Lambda_k"'] \arrow[drr, phantom, "\lambda_k \Uparrow"] & & \mathbb{D}_k\mathbb{L}_k \arrow[d, "\Lambda_k"] \\
\mathbb{D}_k\mathbb{L}_k\mathbb{D} \arrow[r, "\Lambda_k"'] & \mathbb{L}_k\mathbb{D}\mathbb{D} \arrow[r, "\mathbb{L}_k\mu"'] & \mathbb{L}_k\mathbb{D}
\end{tikzcd}$$

that together specify a pseudomap of 2*-monads* $\mathbb{D}_k \longrightarrow \mathbb{D}$.

Proof. Let $\mathscr{Y} \in \mathscr{K}^\Pi$. For a sequence $(\mathbf{n_1}, \cdots, \mathbf{n_k})$ of objects of Π, let $n = n_1 \cdots n_k$. Then the object $\mathbf{n}$ of Π is $\wedge_\Pi^k(\mathbf{n_1}, \cdots, \mathbf{n_k})$. Fixing such a sequence and considering varying sequences $(\mathbf{m_1}, \cdots, \mathbf{m_k})$, consider the maps

$$\begin{tikzcd}
\coprod_{(m_1,\cdots,m_k)} \big(\prod_{1\le r\le k} \mathscr{D}(\mathbf{m_r}, \mathbf{n_r})\big) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\amalg \wedge_{\mathscr{D}}^k \times \mathrm{id}"] \\
\coprod_{(m_1,\cdots,m_k)} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \arrow[d, "\cup"] \\
\coprod_m \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}).
\end{tikzcd}$$

The second map is the sum, meaning that it restricts to the identity map from the component of $(m_1, \cdots, m_k)$ to the component of m. Passing to quotients, the composite induces a map

$$(\mathbb{D}_k\mathbb{L}_k\mathscr{Y})(\mathbf{n_1}, \cdots, \mathbf{n_k}) \longrightarrow (\mathbb{L}_\mathbf{k}\mathbb{D}\mathscr{Y})(\mathbf{n_1}, \cdots, \mathbf{n_k}). \tag{2.9}$$

Our conventions on $\mathscr{D}$-pseudoalgebras in [5, Definition 2.8] ensure that the part of the structure of $\mathscr{D}$-pseudofunctors that involves Π is strict, and this ensures that (2.9) is the component at $(\mathbf{n_1}, \cdots, \mathbf{n_k})$ of a 2-cell

$$(\mathbb{D}_k \circ \mathbb{L}_k)(\mathscr{Y}) \longrightarrow (\mathbb{L}_k \circ \mathbb{D})(\mathscr{Y})$$

of $\mathscr{K}_k$. That 2-cell is the component at $\mathscr{Y}$ of the required 2-natural transformation

$$\Lambda_k \colon \mathbb{D}_k \circ \mathbb{L}_k \longrightarrow \mathbb{L}_k \circ \mathbb{D}$$

Of course $\mathbb{L}_1 = \mathrm{id}$ and $\Lambda_1 = \mathrm{id}$, hence we take $\lambda_1 = \mathrm{id}$. Since $\wedge_{\mathscr{D}}$ is a $\mathscr{V}$-pseudofunctor, we have a $\mathscr{V}$-transformation

$$\begin{tikzcd}
\mathscr{D}(\mathbf{n}, \mathbf{p}) \times \mathscr{D}(\mathbf{r}, \mathbf{s}) \times \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{D}(\mathbf{q}, \mathbf{r}) \arrow[r, "\circ"] \arrow[d, "\wedge\times\wedge"'] \arrow[dr, phantom, "\lambda \Uparrow"] & \mathscr{D}(\mathbf{m}, \mathbf{p}) \times \mathscr{D}(\mathbf{q}, \mathbf{s}) \arrow[d, "\wedge"] \\
\mathscr{D}(\mathbf{nr}, \mathbf{ps}) \times \mathscr{D}(\mathbf{mq}, \mathbf{nr}) \arrow[r, "\circ"'] & \mathscr{D}(\mathbf{mq}, \mathbf{ps})
\end{tikzcd}$$

where the top arrow $\circ$ is the product composition $(\circ \times \circ) \circ (\mathrm{id} \times t \times \mathrm{id})$. Factoring $\wedge \times \wedge$ as $(\wedge \times \mathrm{id}) \circ (\mathrm{id} \times \wedge)$, we see that the λ induce the required 2-cell λ_2. Iterated use of the 2-cells λ induce the required 2-cells λ_k for $k \geq 3$. From here, diagram chases prove the result. $\square$

By Lemma 2.2, Proposition 2.8 implies that if $(\mathscr{Y}, \theta)$ is a $\mathbb{D}$-pseudoalgebra, then $\mathbb{L}_k\mathscr{Y}$ is a $\mathbb{D}_k$-pseudoalgebra with action the composite

$$\mathbb{D}_k\mathbb{L}_k\mathscr{Y} \xrightarrow{\Lambda_k} \mathbb{L}_k\mathbb{D}\mathscr{Y} \xrightarrow{\mathbb{L}_k\theta} \mathbb{L}_k\mathscr{Y},$$

which we denote by Θ_k. Specialization of the diagram in Lemma 2.2 shows how to construct its coherence 2-cell from λ_k and ϕ. We emphasize that even if $\mathscr{Y}$ is a $\mathscr{D}$-algebra, $\mathbb{L}_k\mathscr{Y}$ is only a $\mathbb{D}^k$-pseudoalgebra when $k \geq 2$. With these results and notations, we have the following description of the k-morphisms of $\mathbf{Mult}(\mathscr{D})$.

Theorem 2.10. *For $\mathscr{D}$-pseudoalgebras $\mathscr{X}_j$ and $\mathscr{Y}$, a k-morphism*

$$(F, \delta)\colon (\mathscr{X}_1 \times \cdots \times \mathscr{X}_k, \theta^k) \longrightarrow (\mathscr{Y}, \theta)$$

in the multicategory $\mathbf{Mult}(\mathscr{D})$ is the same structure as a $\mathbb{D}_k$-pseudomorphism

$$(\mathscr{X}_1 \mathbin{\overline{\times}} \cdots \mathbin{\overline{\times}} \mathscr{X}_k, \Theta^k) \longrightarrow (\mathbb{L}_k\mathscr{Y}, \Theta_k)$$

between $\mathbb{D}_k$-pseudoalgebras. The 2-category $\mathbf{Mult_k}(\mathscr{D})$ of k-morphisms in $\mathbf{Mult}(\mathscr{D})$ is isomorphic to the 2-category $\mathbb{D}_k$-$\mathbf{PsAlg}$ of such $\mathscr{D}_k$-pseudofunctors.

Proof. The proof is a check of coherence diagrams showing that our definitions on the level of categories of operators in [8, Section 5.1] dovetail with standard definitions on the level of 2-monads. Since the verification is a direct comparison of definitions, which were tailored towards making this result true, we only outline the argument. Ignoring δ, we view F as an underlying map

$$F\colon \mathscr{X}_1 \times \cdots \times \mathscr{X}_k \longrightarrow \mathbb{L}_k\mathscr{Y}$$

of Π^k-$\mathscr{V}$-categories. Both domain and codomain are $\mathbb{D}_k$-pseudoalgebras. To say that (F, δ) is a $\mathbb{D}_k$-pseudomorphism is to say that δ measures the deviation of F from strict compatibility with the structure maps. This is expressed by the following immediate reinterpretation of the Categorical Composition Axiom in [8, Definition 5.1]. It is just the standard coherence requirement [11] for a map of (normal) pseudoalgebras over a 2-monad.

(Monadic Composition Axiom) The following pasting diagrams are equal.

$$\begin{tikzcd}
 & \mathbb{D}_k^2\mathbb{L}_k\mathscr{Y} \arrow[rr, "\mathbb{D}_k\Theta_k"] \arrow[dr, "\mu"] & & \mathbb{D}_k\mathbb{L}_k\mathscr{Y} \arrow[dr, "\Theta_k"] \arrow[d, phantom, "\Downarrow\phi"] & \\
\mathbb{D}_k^2(\prod_j \mathscr{X}_j) \arrow[ur, "\mathbb{D}_k^2F"] \arrow[dr, "\mu"'] & & \mathbb{D}_k\mathbb{L}_k\mathscr{Y} \arrow[rr, "\Theta_k"] & \arrow[d, phantom, "\Downarrow\delta"] & \mathbb{L}_k\mathscr{Y} \\
 & \mathbb{D}_k(\prod_j \mathscr{X}_j) \arrow[ur, "\mathbb{D}_kF"] \arrow[rr, "\theta^k"'] & & \prod_j \mathscr{X}_j \arrow[ur, "F"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{D}_k^2\mathbb{L}_k\mathscr{Y} \arrow[rr, "\mathbb{D}_k\Theta_k"] \arrow[dr, phantom, "\Downarrow \mathbb{D}_k\delta"] & & \mathbb{D}_k\mathbb{L}_k\mathscr{Y} \arrow[dr, "\Theta_k"] & \\
\mathbb{D}_k^2(\prod_j \mathscr{X}_j) \arrow[ur, "\mathbb{D}_k^2 F"] \arrow[rr, "\mathbb{D}_k\theta^k"] \arrow[dr, "\mu"'] & & \mathbb{D}_k(\prod_j \mathscr{X}_j) \arrow[ur, "\mathbb{D}_k F"] \arrow[dr, "\theta^k"] \arrow[rr, phantom, "\Downarrow \delta"] & & \mathbb{L}_k\mathscr{Y} \\
& \mathbb{D}_k(\prod_j \mathscr{X}_j) \arrow[rr, "\theta^k"'] & & \prod_j \mathscr{X}_j \arrow[ur, "F"'] &
\end{tikzcd}$$

□

Proof of Theorem 0.11. The claim is that $\mathbf{Mult}(\mathscr{D})$ and $\mathbf{Mult}(\mathbb{D}_*\textbf{-PsAlg})$ are isomorphic. Theorem 2.10 gives the isomorphism on 2-categories of k-morphisms for all k. We must show that the compositions agree. However, on the one hand, with $\mathscr{D} = \mathscr{D}(\mathscr{O})$, the definition of $\mathbf{Mult}(\mathscr{D})$ in [8, Section 5.1] is forced by consistency with the definition of $\mathbf{Mult}(\mathscr{O})$ in [8, Section 3.1]. On the other hand, the underlying 2-category of $\mathbf{Mult}(\mathbb{D}_*\textbf{-PsAlg})$ is $\mathbb{D}_*\textbf{-PsAlg}$, and its composition is forced by Construction 1.5. With $\Psi = \Pi$ and $\overline{\otimes} = \overline{\times}$, that specifies composition on the level of underlying ground categories. When the $\mathscr{X}_{r,s}$, $\mathscr{Y}_k$, and $\mathscr{Z}$ in Construction 1.5 are $\mathbb{D}$-pseudoalgebras and the E_r and F are $\mathbb{D}_{j_r}$ and $\mathbb{D}_k$-pseudomorphisms, the composition specified there is given by $\mathbb{D}_j$-pseudomorphisms. That specifies composition in $\mathbf{Mult}(\mathbb{D}_*\textbf{-PsAlg})$. Thus, in effect, Construction 1.5 completes the definition of $\mathbf{Mult}(\mathbb{D}_*\textbf{-PsAlg})$. Therefore the proof of Theorem 0.11 is just another check that we have gotten our definitions right. □

3. The strictification theorem and the passage from $\mathscr{D}$ to $\mathscr{F}$

3.1. **The strictification 2-functor $\mathbb{S}t$.** We prove Theorem 0.13 here, but there is little to be done since [9, Theorem 1.14] specializes to give the conclusion for each k separately. The proofs of the following results are straightforward from the proof of [9, Theorem 1.14], which is given in [9, §3.2]. In [9, Definition 2.21] we define two classes of $\mathscr{V}$-functors f between $\mathscr{V}$-categories, those that are bijective on objects (abbreviated $\mathscr{B}\mathscr{O}$) and those that are fully faithful (abbreviated $\mathscr{F}\mathscr{F}$). The definitions apply levelwise to each $\mathscr{K}_k = \mathscr{K}^{\Pi^k}$. External products $\overline{\times}$ are defined in terms of the pairings $\mathbb{P}_{j,k}\colon \mathscr{K}_j \times \mathscr{K}_k \longrightarrow \mathscr{K}_{j+k}$, and the following observation is immediate from the definitions.

Lemma 3.1. *The classes $\mathscr{B}\mathscr{O}$ and $\mathscr{F}\mathscr{F}$ are closed under $\overline{\times}$. That is, if f and g are in $\mathscr{B}\mathscr{O}$ in $\mathscr{K}_j$ and $\mathscr{K}_k$, then $f \,\overline{\times}\, g$ is in $\mathscr{B}\mathscr{O}$ in $\mathscr{K}_{j+k}$, and similarly for $\mathscr{F}\mathscr{F}$.*

We write e generically for functors in $\mathscr{B}\mathscr{O}$, and we write m generically for functors in $\mathscr{F}\mathscr{F}$. Every morphism $f\colon \mathscr{X} \longrightarrow \mathscr{Y}$ in $\mathscr{K}_k$ factors as a composite

$$\mathscr{X} \xrightarrow{\ e\ } \mathbb{I}(f) \xrightarrow{\ m\ } \mathscr{Y},$$

where $e \in \mathscr{B}\mathscr{O}$ and $m \in \mathscr{F}\mathscr{F}$. If $(\mathscr{X}, \theta)$ is a $\mathbb{D}_k$-pseudoalgebra, $\theta\colon \mathbb{D}_k\mathscr{X} \longrightarrow \mathscr{X}$, then the strictification $\mathbb{S}t\mathscr{X}$ is $\mathbb{I}(\theta)$, and it is a strict $\mathbb{D}_k$-algebra.

We need to prove the compatibility of the 2-functors

$$\mathbb{S}t\colon \mathbb{D}_k\textbf{-PsAlg} \longrightarrow \mathbb{D}_{\mathbf{k}}\textbf{-AlgSt}$$

with the pairings $\mathbb{P}_{j,k}$, where $\mathbb{P}_{j,k}(\mathscr{X}, \mathscr{Y}) = \mathscr{X} \,\overline{\times}\, \mathscr{Y}$. We have the following result.

Proposition 3.2. *There is a 2-natural isomorphism*

$$\phi\colon \mathbb{S}t(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \longrightarrow \mathbb{S}t\mathscr{X} \mathbin{\overline{\times}} \mathbb{S}t\mathscr{Y}$$

in $\mathbb{D}_{j+k}$-**AlgSt** *for* $\mathscr{X} \in \mathbb{D}_j$-**PsAlg** *and* $\mathscr{Y} \in \mathbb{D}_k$-**PsAlg**.

Proof. The rectangles commute in the the following two diagrams. In view of Lemma 3.1, [9, Definition 2.9(ii)] gives unique diagonal arrows that make the diagrams commute.

$$\begin{tikzcd}
\mathbb{D}_{j+k}(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \arrow[r, "\pi_{j,k}"] \arrow[d, "e"'] & \mathbb{D}_j\mathscr{X} \mathbin{\overline{\times}} \mathbb{D}_k\mathscr{Y} \arrow[r, "e \mathbin{\overline{\times}} e"] & \mathbb{S}t\mathscr{X} \mathbin{\overline{\times}} \mathbb{S}t\mathscr{Y} \arrow[d, "m \mathbin{\overline{\times}} m"] \\
\mathbb{S}t(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \arrow[urr, "\phi"] \arrow[rr, "m"'] & & \mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_j\mathscr{X} \mathbin{\overline{\times}} \mathbb{D}_k\mathscr{Y} \arrow[r, "\pi_{j,k}^{-1}"] \arrow[d, "e \mathbin{\overline{\times}} e"'] & \mathbb{D}_{j+k}(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \arrow[r, "e"] & \mathbb{S}t(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \arrow[d, "m"] \\
\mathbb{S}t\mathscr{X} \mathbin{\overline{\times}} \mathbb{S}t\mathscr{Y} \arrow[urr, "\phi^{-1}"] \arrow[rr, "m \mathbin{\overline{\times}} m"'] & & \mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}
\end{tikzcd}$$

Noticing their common edges and using that $\pi_{j,k}$ and $\pi_{j,k}^{-1}$ are inverse isomorphisms, we see that ϕ and ϕ^{-1} are inverse isomorphisms by gluing the diagrams together in either order and using the uniqueness clause in [9, Definition 2.9(ii)]. $\square$

Either iterating or mimicking the proof, we obtain the following analog.

Corollary 3.3. *There is a 2-natural isomorphism*

$$\phi\colon \mathbb{S}t(\mathscr{X}_1 \mathbin{\overline{\times}} \cdots \mathbin{\overline{\times}} \mathscr{X}_k) \longrightarrow \mathbb{S}t\mathscr{X}_1 \mathbin{\overline{\times}} \cdots \mathbin{\overline{\times}} \mathbb{S}t\mathscr{X}_k$$

in $\mathbb{D}_k$-**AlgSt** *for* $\mathscr{X}_r \in \mathbb{D}$-**PsAlg**, $1 \le r \le k$.

3.2. **The change of monads 2-functors ξ_*^k.** We have a map $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ of categories of operators, and it induces a map $\xi_k\colon \mathbb{D}_k \longrightarrow \mathbb{F}_k$ of monads in $\mathscr{K}_k$ for each k. Explicitly, it is obtained by applying $\xi\colon \mathscr{D}(\mathbf{m_i}, \mathbf{n_i}) \longrightarrow \mathscr{F}(\mathbf{m_i}, \mathbf{n_i})$ in (2.4), observing that the equivariance and base object identifications are compatible,and that ξ_*^k is compatible with the units and products of our monads. A general construction in [5, §4] gives the left 2-adjoint

$$\xi_*^k\colon \mathbb{D}_k\text{-}\mathbf{AlgSt} \longrightarrow \mathbb{F}_{\mathbf{k}}\text{-}\mathbf{AlgSt}$$

to the evident pullback of action functor ξ_k^*. We have analogues of the results of the previous subsection.

Proposition 3.4. *There is a 2-natural isomorphism*

$$\rho_{j,k}\colon \xi_*^{j+k}(\mathscr{X} \mathbin{\overline{\times}} \mathscr{Y}) \longrightarrow \xi_*^j\mathscr{X} \mathbin{\overline{\times}} \xi_*^k\mathscr{Y}$$

in $\mathbb{F}_{j+k}$-**AlgSt** *for* $\mathscr{X} \in \mathbb{D}_j$-**AlgSt** *and* $\mathscr{Y} \in \mathbb{D}_k$-**AlgSt**.

Proof. The adjunction units $\eta\colon \mathrm{id} \longrightarrow \xi_j^*\xi_*^j$ and $\eta\colon \mathrm{id} \longrightarrow \xi_k^*\xi_*^k$ specialize to give

$$\eta \otimes \eta\colon \mathscr{X} \mathbin{\overline{\times}} \mathscr{Y} \longrightarrow \xi_j^*\xi_*^j\mathscr{X} \mathbin{\overline{\times}} \xi_k^*\xi_*^k\mathscr{Y} = \xi_{j+k}^*(\xi_*^j\mathscr{X} \mathbin{\overline{\times}} \xi_*^k\mathscr{Y})$$

for $\mathscr{X} \in \mathbb{D}_j$-**AlgSt** and $\mathscr{Y} \in \mathbb{D}_k$-**AlgSt**. To see the equality here, observe that for any $\mathbb{F}_j$-algebra $\mathscr{W}$ and any $\mathbb{F}_k$-algebra $\mathscr{Z}$, the square commutes in the following diagram, which exhibits the actions of $\mathbb{F}_{j+k}$ and $\mathbb{D}_{j+k}$ on $\mathscr{W} \,\overline{\times}\, \mathscr{Z}$.

$$\begin{tikzcd}
\mathbb{D}_{j+k}(\mathscr{W} \,\overline{\times}\, \mathscr{Z}) \arrow[r, "\pi_{j,k}"] \arrow[d, "\xi_{j+k}"'] & \mathbb{D}_j\mathscr{W} \,\overline{\times}\, \mathbb{D}_k\mathscr{Z} \arrow[d, "\xi_j \,\overline{\times}\, \xi_k"] & \\
\mathbb{F}_{j+k}(\mathscr{W} \,\overline{\times}\, \mathscr{Z}) \arrow[r, "\pi_{j,k}"'] & \mathbb{F}_j\mathscr{W} \,\overline{\times}\, \mathbb{F}_k\mathscr{Z} \arrow[r, "\theta \,\overline{\times}\, \theta"'] & \mathscr{W} \,\overline{\times}\, \mathscr{Z}.
\end{tikzcd}$$

Here $\pi_{j,k}$ is the isomorphism recorded in Lemma 2.5. We define $\rho_{j,k}$ to be the adjoint of $\eta \,\overline{\times}\, \eta$, and we must prove that it is an isomorphism. Explicitly, $\xi^k_*\mathscr{Y}$ can be constructed as the coequalizer displayed in the diagram

$$\begin{tikzcd}
\mathbb{F}_k\mathbb{D}_k\mathscr{Y} \arrow[r, shift left, "\nu"] \arrow[r, shift right, "\mathbb{F}_k\theta_k"'] & \mathbb{F}_k\mathscr{Y} \arrow[r] & \xi^k_*\mathscr{Y},
\end{tikzcd}$$

where ν is the composite

$$\mathbb{F}_k\mathbb{D}_k \xrightarrow{\mathbb{F}_k\xi_k} \mathbb{F}_k\mathbb{F}_k \xrightarrow{\mu} \mathbb{F}_k$$

and θ_k is the action of $\mathbb{D}_k$ on $\mathscr{Y}$. Writing the corresponding coequalizer diagrams for $\xi^j_*\mathscr{X}$ and for $\xi^{j+k}_*(\mathscr{X} \,\overline{\times}\, \mathscr{Y})$, we see that the external product $\overline{\times}$ defines an isomorphism between the external product of the coequalizer diagrams defining $\xi^j_*\mathscr{X}$ and $\xi^k_*\mathscr{Y}$ and the coequalizer diagram defining $\xi^{j+k}_*(\mathscr{X} \,\overline{\times}\, \mathscr{Y})$. The resulting isomorphism is $\rho^{-1}_{j,k}$. □

Either iterating or mimicking the proof, we obtain the following analog.

Corollary 3.5. *There is a 2-natural isomorphism*

$$\rho_k \colon \xi^k_*(\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k) \longrightarrow \xi_*\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \xi_*\mathscr{X}_k$$

ln $\mathbb{F}_k$-**AlgSt** *for* $\mathscr{X}_r \in \mathbb{D}$-**AlgSt**, $1 \le r \le k$.

Observe that the following diagram clearly commutes by direct comparison of definitions.

(3.6)
$$\begin{tikzcd}
\mathbb{D}_k\mathbb{L}_k \arrow[r, "\xi^k"] \arrow[d, "\Lambda_k"'] & \mathbb{F}_k\mathbb{L}_k \arrow[d, "\Lambda_k"] \\
\mathbb{L}_k\mathbb{D} \arrow[r, "\mathbb{L}_k\xi"'] & \mathbb{L}_k\mathbb{F}
\end{tikzcd}$$

Moreover, the following diagram also commutes since $\mathscr{F}$ is permutative under $\wedge_{\mathscr{F}}$; no λ as in Definition 2.1 is required.

(3.7)
$$\begin{tikzcd}
\mathbb{F}_k\mathbb{F}_k\mathbb{L}_k \arrow[rr, "\mu"] \arrow[d, "\mathbb{F}_k\Lambda_k"'] & & \mathbb{F}_k\mathbb{L}_k \arrow[d, "\Lambda_k"] \\
\mathbb{F}_k\mathbb{L}_k\mathbb{F}_k \arrow[r, "\Lambda_k"'] & \mathbb{L}_k\mathbb{F}_k\mathbb{F}_k \arrow[r, "\mathbb{L}_k\mu"'] & \mathbb{L}_k\mathbb{F}
\end{tikzcd}$$

3.3. **The comparison of $\mathbb{D}_k$-pseudoalgebras to $\mathbb{F}_k$-algebras.** We must define the multifunctor

$$\xi_\# \colon \mathbf{Mult}(\mathbb{D}_*\text{-}\mathbf{PsAlg}) \longrightarrow \mathbf{Mult}(\mathbb{F}_*\text{-}\mathbf{AlgSt})$$

promised in Theorem 0.17. We already have $\xi_\#$ on underlying 2-categories, that is, on objects $\mathscr{X}$ and on 1-morphisms $\mathscr{X} \longrightarrow \mathscr{Y}$. We must define $\xi_\#$ on k-morphisms for $k \geq 2$ in such a way that composition is preserved. For the domain of k-morphisms, Corollaries 3.3 and 3.5 show that $\xi_\#$ commutes with external products. For the target of k-morphisms, we need an analogue, starting from (3.6), that gives a commutation relation between $\xi_\#$ and $\mathbb{L}_k$.

This is where codescent objects enter the picture. By specialization of Section 4, for each $k \geq 1$ we have the codescent object 2-functor

$$\xi^k_\# \colon \mathbb{D}_k\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{F}_{\mathbf{k}}\text{-}\mathbf{AlgSt}.$$

It is isomorphic to the composite $\xi^k_* \circ \mathbb{S}t$. With this reformulation, $\xi^k_\#$ simultaneously strictifies and changes from $\mathscr{D}$ to $\mathscr{F}$ and thereby gets around problems caused by the fact that $\mathbb{L}_k\mathscr{Y}$ is only a $\mathscr{D}_k$-pseudoalgebra when $\mathscr{Y}$ is a $\mathscr{D}$-algebra.

If (Y, θ) is a $\mathbb{D}$-pseudoalgebra, then $\mathbb{L}_k Y$ has the $\mathbb{D}_k$-action $\Theta = \mathbb{L}_k\theta \circ \Lambda_k$. Similarly, if $(\mathscr{Z}, \psi)$ is an $\mathbb{F}$-pseudoalgebra, then $\mathbb{L}_k\mathscr{Z}$ has the $\mathbb{F}_k$ action $\Psi = \mathbb{L}_k\psi \circ \Lambda_k$. However, we are only interested in strict $\mathbb{F}$ and $\mathbb{F}_k$-algebras, not pseudoalgebras. The commutativity of (3.7) ensures that $\mathbb{L}_k\mathscr{Z}$ is a strict $\mathbb{F}_k$-algebra if $\mathscr{Z}$ is a strict $\mathbb{F}$-algebra.The following comparison is crucial, and we give two different proofs.

Proposition 3.8. *For $\mathbb{D}$-pseudoalgebras $\mathscr{Y}$, there is a canonical natural map of $\mathbb{F}_k$-algebras*

$$\omega_k \colon \xi^k_\# \mathbb{L}_k \mathscr{Y} \longrightarrow \mathbb{L}_k \xi_\# \mathscr{Y}.$$

Proof. For the first proof, observe that we have the following commutative diagram.

$$\begin{tikzcd}
 & & \mathbb{D}\text{-}\mathbf{AlgSt} \arrow[dr, "\mathbb{J}"] & & \\
\mathbb{F}\text{-}\mathbf{StAlg} \arrow[urr, "\xi^*"] \arrow[r, "\mathbb{J}"] \arrow[d, "\mathbb{L}_k"] & \mathbb{F}\text{-}\mathbf{PsAlg} \arrow[rr, "\xi^*"] \arrow[d, "\mathbb{L}_k"] & & \mathbb{D}\text{-}\mathbf{PsAlg} \arrow[d, "\mathbb{L}_k"] \\
\mathbb{F}_k\text{-}\mathbf{StAlg} \arrow[r, "\mathbb{J}"] \arrow[drr, "{\xi^k}^*"'] & \mathbb{F}_k\text{-}\mathbf{PsAlg} \arrow[rr, "{\xi^k}^*"] & & \mathbb{D}_k\text{-}\mathbf{PsAlg} \\
 & & \mathbb{F}_k\text{-}\mathbf{AlgSt} \arrow[ur, "\mathbb{J}"'] & &
\end{tikzcd}$$

Here the $\mathbb{J}$ are forgetful inclusion 2-functors and ξ^* and ${\xi^k}^*$ are pullback of action 2-functors induced by ξ and ξ^k. Ignoring the actions, both composites send an $\mathbb{F}$-algebra $\mathscr{X}$ to $\mathbb{L}_k\mathscr{X}$, and the actions agree in view of (3.6). Ignoring the $\mathbb{L}_k$, denote the top and bottom horizontal composites by $\xi^\#$ and ${\xi^k}^\#$. These 2-functors have left adjoints $\xi_\# \cong \xi_* \circ \mathbb{S}t_{\mathbb{F}}$ and $\xi^k_\# \cong \xi^k_* \circ \mathbb{S}t_{\mathbb{F}_k}$, where the isomorphisms compare different explicit constructions of the adjoint. Let η and ε denote units and counits of our adjunctions. We define ω_k to be the composite

$$\xi^k_\# \mathbb{L}_k Y \xrightarrow{\xi^k_\# \mathbb{L}_k \eta} \xi^k_\# \mathbb{L}_k \xi^\# \xi_\# Y = \xi^k_\# {\xi^k}^\# \mathbb{L}_k \xi_\# \mathscr{Y} \xrightarrow{\ \varepsilon\ } \mathbb{L}_k \xi_\# Y.$$

For the second proof, we assume that $\mathbb{L}_k$ preserves codescent objects. We observe as in Remark 4.7 that this holds in our examples. The construction of the codescent object $\xi_\#\mathscr{Y}$ comes with a map $\pi\colon \mathbb{F}\mathscr{Y} \longrightarrow \xi_\#\mathscr{Y}$, and that of the codescent object $\xi^k_\#\mathbb{L}_k\mathscr{Y}$ comes with a map $\rho\colon \mathbb{F}_k\mathbb{L}_k\mathscr{Y} \longrightarrow \xi_\#\mathbb{L}_k\mathscr{Y}$. A little diagram chasing shows that the universal property of ρ gives a natural map $\omega_k\colon \xi^k_\#\mathbb{L}_k\mathscr{Y} \longrightarrow \mathbb{L}_k\xi_\#\mathscr{Y}$ such that the following diagram commutes (on the nose).

$$\begin{tikzcd}
\mathbb{F}_k L_k\mathscr{Y} \arrow[r, "\Lambda_k"] \arrow[d, "\rho"'] & \mathbb{L}_k\mathbb{F}\mathscr{Y} \arrow[d, "\mathbb{L}_k\pi"] \\
\xi^k_\#\mathbb{L}_k\mathscr{Y} \arrow[r, "\omega_k"'] & \mathbb{L}_k\xi_\#\mathscr{Y}.
\end{tikzcd}$$

The preliminaries on codescent objects that are needed to make sense of the second proof may be found in Section 4. In a bit more detail, taking $\psi = \mathbb{L}_k\pi \circ \Lambda_k$ and constructing an invertible 2-cell χ from the given data such that the coherence condition (4.9) is satisfied, the first universal property of ρ applies to construct ω_k. Similarly, constructing an invertible 2-cell α from the given data such that the coherence condition (4.10) is satisfied, the second universal property of ρ applies to construct a canonical invertible 2-cell from our first construction of ω_k to our second construction of ω_k. □

3.4. **The multifunctor $\xi_\#$.** We put things together to prove Theorem 0.8. For a k-morphism $F\colon \mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k \longrightarrow \mathbb{L}_k\mathscr{Y}$ of $\mathbb{D}_k$-pseudoalgebras we define a k-morphism $\xi_\# F$ of (strict) $\mathbb{F}_k$-algebras by commutativity of the diagram

(3.9)
$$\begin{tikzcd}
\xi_\#\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \xi_\#\mathscr{X}_k \arrow[rr, "\xi_\# F"] \arrow[d, "\alpha_k^{-1}"'] & & \mathbb{L}_k\xi_\#\mathscr{Y} \\
\xi^k_\#\big(\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k\big) \arrow[rr, "\xi^k_\# F"'] & & \xi^k_\#(\mathbb{L}_k\mathscr{Y}) \arrow[u, "\omega_k"']
\end{tikzcd}$$

of $\mathbb{F}_k$-algebras. With notations as in Construction 1.5, we then have the following commutative diagram, which shows that $\xi_\#$ preserves composition of multifunctors. To avoid notational clutter, we delete subscripts and superscripts k from the previous notations. The indices below run over $1 \le r \le k$ and $1 \le s \le j_r$. We let $j = j_1 + \cdots + j_k$ and define $\mathbb{L}_{\underline{j}} = \mathbb{L}_{j_1,\cdots,j_r}$ as in Construction 1.5. Composing Corollaries 3.3 and 3.5 gives isomorphisms

$$\alpha\colon \xi_\#(\overline{\times}_r\mathscr{X}_r) \xrightarrow{\cong} \overline{\times}_r\xi_\#(\mathscr{X}_r)$$

for $\mathbb{D}$-pseudoalgebras $\mathscr{X}_r$. Similarly, iterating Proposition 3.8 gives maps

$$\omega\colon \xi_\#\mathbb{L}_{\underline{j}}\,\overline{\times}_r\mathscr{X}_r \longrightarrow \mathbb{L}_{\underline{j}}\xi_\#\mathscr{X}_r.$$

(3.10)

$$\begin{tikzcd}
\overline{\times}_r \overline{\times}_s \xi_\# \mathscr{X}_{r,s} \arrow[r, "\overline{\times}_r \alpha^{-1}"] \arrow[rr, bend left, "\alpha^{-1}"] \arrow[d, "\overline{\times}_r \xi_\# E_r"'] & \overline{\times}_r \xi_\#(\overline{\times}_s \mathscr{X}_{r,s}) \arrow[r, "\alpha^{-1}"] \arrow[d, "\overline{\times}_r \xi_\#(E_r)"] & \xi_\#(\overline{\times}_r \overline{\times}_s \mathscr{X}_{r,s}) \arrow[d, "\xi_\#(\overline{\times}_r E_r)"] \\
\overline{\times}_r \mathbb{L}_{j_r} \xi_\# \mathscr{Y}_r \arrow[d, "\cong"'] & \overline{\times}_r \xi_\#(\mathbb{L}_{j_r} \mathscr{Y}_r) \arrow[l, "\overline{\times}_r \omega"'] & \xi_\#(\overline{\times}_r \mathbb{L}_{j_r} \mathscr{Y}_r) \arrow[l, "\alpha"'] \arrow[d, "\cong"] \\
\mathbb{L}_{\underline{j}} \overline{\times}_r \xi_\# \mathscr{Y}_r \arrow[d, "\mathbb{L}_{\underline{j}} \xi_\# F"'] & \mathbb{L}_{\underline{j}} \xi_\# \overline{\times}_r \mathscr{Y}_r \arrow[l, "\mathbb{L}_{\underline{j}} \alpha"'] \arrow[d, "\mathbb{L}_{\underline{j}} \xi_\#(F)"] & \xi_\# \mathbb{L}_{\underline{j}} \overline{\times}_r \mathscr{Y}_r \arrow[l, "\omega"'] \arrow[d, "\xi_\# \mathbb{L}_{\underline{j}} F"] \\
\mathbb{L}_{\underline{j}} \mathbb{L}_k \xi_\# \mathscr{Z} \arrow[d, "\cong"'] & \mathbb{L}_{\underline{j}} \xi_\# \mathbb{L}_k \mathscr{Z} \arrow[l, "\mathbb{L}_{\underline{j}} \omega"'] & \xi_\# \mathbb{L}_{\underline{j}} \mathbb{L}_k \mathscr{Z} \arrow[l, "\omega"'] \arrow[d, "\cong"] \\
\mathbb{L}_j \xi_\# \mathscr{Z} & & \xi_\# \mathbb{L}_j \mathscr{Z} \arrow[ll, bend left, "\omega"]
\end{tikzcd}$$

The right column displays the 2-functor $\xi^j_\#$ applied to a typical composite in $\mathbf{Mult}(\mathbb{D}_*\text{-}\mathbf{PsAlg})$. The two left hand squares are obtained by applying $\overline{\times}_r$ and $\mathbb{L}_{\underline{j}}$ to instances of (3.9), and the left column displays the composite in $\mathbf{Mult}(\mathbb{F}_*\text{-}\mathbf{AlgSt})$ obtained by first applying $\xi_\#$ to input $\mathbb{D}$-pseudoalgebras and then composing. The periphery displays application of (3.9) to the composite, so commutativity of the diagram gives the required compatibility of $\xi_\#$ with composition. The two right hand squares are naturality diagrams; we see that for the top right square by replacing α^{-1} by α. The middle rectangle commutes by inspection of definitions; it just says that we can apply ω before or after product identifications and obtain the same answer. The top and bottom subdiagrams commute by inspection of how α^{-1} at the top can be factored on iterated products and how ω at the botttom can be factored on iterated 2-functors $\mathbb{L}$.

This proves that $\xi^k_\#$ is a multifunctor. A straightforward diagram chase proves that this multifunctor is symmetric. To see that, consider (3.9) and abbreviate notation by writing

$$\underline{\mathscr{X}} = \mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k, \quad \underline{\xi_\# \mathscr{X}} = \xi_\# \mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \xi_\# \mathscr{X}_k$$

and, for $\sigma \in \Sigma_k$,

$$\underline{\mathscr{X}_\sigma} = \mathscr{X}_{\sigma(1)} \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_{\sigma(k)}, \quad \underline{\xi_\# \mathscr{X}_\sigma} = \xi_\# \mathscr{X}_{\sigma(1)} \,\overline{\times}\, \cdots \,\overline{\times}\, \xi_\# \mathscr{X}_{\sigma(k)}.$$

Then, by [8, (5.3)] but reinterpreted according to Theorem 2.10, $F\sigma$ is the composite

$$\underline{\mathscr{X}_\sigma} \xrightarrow{\sigma^{-1}} \underline{\mathscr{X}} \xrightarrow{F} \mathbb{L}_k \mathscr{Y} \xrightarrow{\tau_\sigma} \mathbb{L}_k \mathscr{Y}$$

where τ_σ is obtained by applying $\mathbb{L}_k \mathscr{Y}$ to block permutations $\tau_\sigma \in \Pi$ as in [8, (5.4)]. The symmetry of $\xi^k_\#$ says that $\xi_\#(F\sigma) = (\xi_\# F)\sigma$, and that holds by commutativity of the following diagram.

$$\begin{tikzcd}
 & \xi_\# \underline{\mathscr{X}_\sigma} \arrow[r, "\xi_\#(F\sigma)"] \arrow[dl, "\sigma^{-1}"'] \arrow[d, "\alpha_k^{-1}"] & \mathbb{L}_k \xi_\# \mathscr{Y} & \\
\xi_\# \underline{\mathscr{X}} \arrow[dr, "\alpha_k^{-1}"'] \arrow[rrr, bend right=60, "\xi_\# F"'] & \xi^k_\#(\underline{\mathscr{X}_\sigma}) \arrow[r, "\xi^k_\#(F\sigma)"] \arrow[d, "\xi^k_\#(\sigma^{-1})"] & \xi^k_\# \mathbb{L}_k \mathscr{Y} \arrow[u, "\omega_k"] & \mathbb{L}_k \xi^k_\# \mathscr{Y} \arrow[ul, "\mathbb{L}_k \xi_\# \tau_\sigma"'] \\
 & \xi^k_\#(\underline{\mathscr{X}}) \arrow[r, "\xi^k_\#(F)"'] & \xi^k_\# \mathbb{L}_k \mathscr{Y} \arrow[u, "\xi^k_\#(\mathbb{L}_k \tau_\sigma)"] \arrow[ur, "\omega_k"'] &
\end{tikzcd}$$

The upper square and the bottom interior diagram (with the long arrow) commute by the definition, (3.9), of $\xi_\#$ on k-morphisms. The middle square commutes since it is obtained by applying $\xi_\#$ to the commutative diagram that defines $F\sigma$. The left triangle commutes by inspection and the right triangle commutes by the naturality of ω_k on morphisms in Π. The peripheral composite defines $(\xi_\# F)\sigma$, which is therefore equal to $\xi_\#(F\sigma)$.

4. Codescent objects and the reinterpretation of $\xi_* \circ \mathbb{S}t$

4.1. **Overview of the codescent description of $\xi_* \circ \mathbb{S}t$.** Returning to the context of 2-monads in a general ground 2-category $\mathscr{K}$, we assume given a map $\xi\colon \mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads defined on $\mathscr{K}$, as at the start of Section 2.1. Then $\mathbb{S}t$ is defined on $\mathbb{D}$-pseudoalgebras and ξ_* is defined on strict $\mathscr{D}$-algebras. We give a description of the composite $\xi_* \circ \mathbb{S}t\colon \mathbb{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{E}\text{-}\mathbf{AlgSt}$ as a single 2-functor $\xi_\#$ that enjoys good properties that are invisible to separate analysis of ξ_* and $\mathbb{S}t$. It takes $\mathbb{D}$-pseudoalgebras $\mathscr{Y}$ and $\mathbb{D}$-pseudomorphisms as input and gives (strict) $\mathbb{E}$-algebras and (strict) $\mathbb{E}$-maps as output, and it satisfies a 2-categorical universal property that we spell out in Section 4.3.

In brief, the coequalizer $\xi_* = \mathbb{E} \otimes_{\mathbb{D}} (-)$ has a more elaborate 2-categorical colimit analogue, called a codescent object, which we shall also denote in two ways:

$$\xi_\# \mathscr{Y} = \mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y}.$$

Formally, $\xi_\#$ is a 2-functor

$$\xi_\# \colon \mathbb{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{E}\text{-}\mathbf{AlgSt}.$$

We explain what codescent objects are in general in Section 4.2 and describe those of interest to us in Section 4.3. By Remarks 4.7 and 4.11, the required codescent objects always exist in our context, and our monads preserve codescent objects.

When ξ is the identity map of $\mathbb{D}$, this gives an alternative, more sophisticated, construction of $\mathbb{S}t$ itself, since Lack proves[5] in [11, Theorem 2.6] that $\mathrm{id}_\#$ is left adjoint to the inclusion $\mathbb{J}$, as is $\mathbb{S}t$. We therefore agree to identify $\mathbb{S}t\mathscr{Y}$ with $\mathbb{D} \boxtimes_{\mathbb{D}} \mathscr{Y}$. Lack's [11, Theorem 44] gives an alternative proof that the unit of the adjunction is given by equivalences of $\mathbb{D}$-pseudoalgebras.

[5] His statement of the theorem is less explicit than its proof.

As a left adjoint, the 2-functor ξ_* preserves colimits and in particular codescent objects. By an immediate inspection, applying $\mathbb{E} \otimes_{\mathbb{D}} (-)$ to the codescent data that define $\mathbb{D} \boxtimes_{\mathbb{D}} \mathscr{Y}$ gives the codescent data displayed in (4.8) below, and that gives the identification $\xi_\# \mathscr{Y} = \xi_* \mathbb{S}t\mathscr{Y}$. Alternatively, we can see this directly by showing that $\xi_\#$ is left adjoint to the composite $\mathbb{J}\xi^*$, using the following pasting diagram.

(4.1)
$$
\begin{tikzcd}
\mathbb{E}\mathbb{D}\mathscr{Y} \arrow[rr, "\mathbb{E}\theta"] \arrow[dr, "\mathbb{E}\mathbb{D}\alpha"'] \arrow[dd, "\mathbb{E}\xi"'] & & \mathbb{E}\mathscr{Y} \arrow[dd, "\mathbb{E}\alpha"] \\
& \mathbb{E}\mathbb{D}\mathscr{Z} \arrow[d, "\mathbb{E}\xi"'] & \\
\mathbb{E}\mathbb{E}\mathscr{Y} \arrow[r, "\mathbb{E}\mathbb{E}\alpha"] \arrow[d, "\mu"'] & \mathbb{E}\mathbb{E}\mathscr{Z} \arrow[r, "\mathbb{E}\theta"] \arrow[d, "\mu"] & \mathbb{E}\mathscr{Z} \arrow[d, "\theta"] \\
\mathbb{E}\mathscr{Y} \arrow[r, "\mathbb{E}\alpha"] & \mathbb{E}\mathscr{Z} \arrow[r, "\theta"] & \mathscr{Z}
\end{tikzcd}
$$

Here $\mathscr{Y}$ is a $\mathbb{D}$-pseudoalgebra, $\mathscr{Z}$ is an $\mathbb{E}$-algebra viewed as the $\mathbb{D}$-pseudoalgebra $\mathbb{J}\xi^*\mathscr{Z}$ and $\alpha\colon \mathscr{Y} \longrightarrow \mathscr{Z}$ is a pseudomorphism of $\mathbb{D}$-pseudoalgebras. That gives 2-cells in the right two subdiagrams of (4.1), and the left two subdiagrams are naturality diagrams. The universal property of codescent objects gives the adjoint map $\tilde{\alpha}\colon \xi_\#\mathscr{Y} \longrightarrow \mathscr{Z}$ of (strict) $\mathbb{E}$-algebras.

We can give another interpretation of the composite $\xi_*\mathbb{S}t$. Observe that the following diagram obviously commutes.

$$
\begin{tikzcd}
\mathbb{D}\text{-}\mathbf{PsAlg} & \mathbb{D}\text{-}\mathbf{AlgSt} \arrow[l, "\mathbb{J}"'] \\
\mathbb{E}\text{-}\mathbf{PsAlg} \arrow[u, "\xi^*"] & \mathbb{E}\text{-}\mathbf{AlgSt}, \arrow[l, "\mathbb{J}"] \arrow[u, "\xi^*"']
\end{tikzcd}
$$

Here the $\mathbb{J}$ are the inclusions and the ξ^* are the forgetful functors. We therefore have a conjugate diagram of right adjoints which, in our 2-categorical setting, commutes up to 2-natural isomorphism.

Thus consider the following diagram, in which the notations $\mathbb{S}t_{\mathbb{D}}$ and $\mathbb{S}t_{\mathbb{E}}$ record the monad to which the strictification is being applied.

(4.2)
$$
\begin{tikzcd}
\mathbb{D}\text{-}\mathbf{PsAlg} \arrow[r, "\mathbb{S}t_{\mathbb{D}}"] \arrow[d, "\xi_*"'] & \mathbb{D}\text{-}\mathbf{AlgSt} \arrow[d, "\xi_*"] \\
\mathbb{E}\text{-}\mathbf{PsAlg} \arrow[r, shift right, "\mathbb{S}t_{\mathbb{E}}"'] & \mathbb{E}\text{-}\mathbf{AlgSt}. \arrow[l, shift right, dashed, "\mathbb{J}"']
\end{tikzcd}
$$

The left adjoint on the right has already been defined as $\xi_* = \mathbb{E} \otimes_{\mathbb{D}} (-)$. We define ξ_* on the left to be the composite $\mathbb{J} \circ \xi_* \circ \mathbb{S}t_{\mathbb{D}}$.[6] Therefore, for a $\mathbb{D}$-pseudoalgebra $\mathscr{Y}$, we have the map of strict $\mathbb{E}$-algebras

(4.3)
$$\mathbb{S}t_E\xi_*\mathscr{Y} = \mathbb{S}t_E\mathbb{J}\xi_*\mathbb{S}t_{\mathbb{D}}\mathscr{Y} \longrightarrow \xi_*\mathbb{S}t_{\mathbb{D}}\mathscr{Y}$$

given by the counit $\mathbb{S}t_E\mathbb{J} \longrightarrow \mathrm{id}$ of the adjunction $(\mathbb{S}t_E, \mathbb{J})$. A priori, this strict map is only an equivalence in $\mathbb{E}$-$\mathbf{PsAlg}$, but we can use "flexibility" as in [9, Remark

[6] According to Mike Shulman (private communication), it is left adjoint to ξ^*.

2.20] to see that it is an equivalence in $\mathbb{E}$-**AlgSt**. Indeed, the composite

$$\mathbb{S}t_{\mathbb{D}}\mathscr{Y} \xrightarrow{\mathbb{S}t_{\mathbb{D}}\eta} \mathbb{S}t_{\mathbb{D}}\mathbb{J}\mathbb{S}t_{\mathbb{D}}\mathscr{Y} \xrightarrow{\varepsilon} \mathbb{S}t_{\mathbb{D}}\mathscr{Y}$$

is the identity. Applying ξ_*, we see that $\xi_*\mathbb{S}t_{\mathbb{D}}\mathscr{Y}$ is a retract in $\mathbb{E}$-**AlgSt**. By [1, Theorem 4.4], that is equivalent to $\xi_*\mathbb{S}t_{\mathbb{D}}\mathscr{Y}$ being flexible, so that the equivalence (4.3) is an equivalence in $\mathbb{E}$-**AlgSt**. Thus both solid arrow composites in (4.2) are equivalent to the codescent object 2-functor $\mathbb{E} \boxtimes_{\mathbb{D}} (-)$. Therefore, for a $\mathbb{D}$-pseudoalgebra $\mathscr{Y}$, we have equivalences of strict $\mathbb{E}$-algebras

$$\mathbb{S}t_{\mathbb{E}}\xi_*\mathscr{Y} \cong \xi_{\#}\mathscr{Y} \cong \xi_*\mathbb{S}t_{\mathbb{D}}\mathscr{Y}.$$

4.2. **Codescent data and codescent objects.** Since the definition of codescent objects given by Lack [11] can be hard reading, we change his notations to highlight the intuition.

Recall that the data for a coequalizer is a pair of arrows, as in the left diagram below. The data for a reflexive coequalizer adds in s_0 such that $d_0 s_0 = \mathrm{id} = d_1 s_0$, as in the right diagram.

$$\begin{tikzcd} K_1 \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] \\ K_0 \end{tikzcd} \qquad\qquad \begin{tikzcd} K_1 \arrow[d, shift right=4, "d_0"'] \arrow[d, shift left=4, "d_1"] \\ K_0 \arrow[u, "s_0"'] \end{tikzcd}$$

The data for a codescent object in a 2-category $\mathscr{K}$ consists of 0-cells K_i, where $i = 0, 1, 2$, and 1-cells between them as displayed in the left diagram below. The data for a reflexive codescent object throws in further 1-cells as in the right diagram.

$$\begin{tikzcd} K_2 \arrow[d, shift right=5, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=5, "d_2"] \\ K_1 \arrow[d, shift right=5, "d_0"'] \arrow[d, shift left=5, "d_1"] \\ K_0 \arrow[u, "s_0"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} K_2 \arrow[d, shift right=10, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=10, "d_2"] \\ K_1 \arrow[u, shift right=5, "s_0"] \arrow[u, shift left=5, "s_1"'] \arrow[d, shift right=5, "d_0"'] \arrow[d, shift left=5, "d_1"] \\ K_0 \arrow[u, "s_0"'] \end{tikzcd}$$

The usual identities for compositions of face and degeneracy operators for the 2-skeleton of a simplicial object are all replaced by prescribed invertible 2-cells, which are an integral part of the codescent data. To avoid a notational morasse, we do not introduce notation for these 2-cells.

The restriction to invertible 2-cells, rather than general ones, is discussed in [11, p. 231], where Lack makes clear that his more general definitions and theorems restrict appropriately.

A codescent object for such codescent data is a pair (k, ζ), where $k \colon K_0 \longrightarrow K$ is a 1-cell and $\zeta \colon k \circ d_0 \Longrightarrow k \circ d_1$ is an invertible 2-cell such that the following equalities of pasting diagrams hold and (k, ζ) is universal with this property.

$$\begin{tikzcd}
& & K_2 \arrow[dll, "d_0"'] \arrow[d, "d_2"] \arrow[drr, "d_1"] & & \\
K_1 \arrow[d, "d_0"'] \arrow[drr, "d_1"] \arrow[r, phantom, "\Downarrow"] & & K_1 \arrow[d, "d_0"] \arrow[drr, "d_1"] \arrow[r, phantom, "\Downarrow"] & & K_1 \arrow[d, "d_1"] \\
K_0 \arrow[drr, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & & K_0 \arrow[d, "k"] \arrow[r, phantom, "\Downarrow\zeta"] & & K_0 \arrow[dll, "k"] \\
& & K & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& K_2 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
K_1 \arrow[d, "d_0"'] \arrow[r, phantom, "\Downarrow"] & & K_1 \arrow[dll, "d_0"'] \arrow[d, "d_1"] \\
K_0 \arrow[dr, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & & K_0 \arrow[dl, "k"] \\
& K &
\end{tikzcd}
\tag{4.4}$$

$$\begin{tikzcd}
& K_0 \arrow[d, "s_0"] & \\
& K_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
K_0 \arrow[dr, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & & K_0 \arrow[dl, "k"] \\
& K &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& K_0 \arrow[dl, "s_0"'] \arrow[dr, "s_0"] & \\
K_1 \arrow[dr, "d_0"'] \arrow[r, phantom, "\Downarrow"] & & K_1 \arrow[dl, "d_1"] \\
& K_0 \arrow[d, "k"] & \\
& K &
\end{tikzcd}
\tag{4.5}$$

The universality means two things. First, given a pair (ℓ, χ), where $\ell\colon K_0 \longrightarrow L$ is a 1-cell and $\chi\colon \ell \circ d_0 \Longrightarrow \ell \circ d_1$ is an invertible 2-cell which make the evident analogs of diagrams (4.4) and (4.5) commute, there is a unique 1-cell $z\colon K \longrightarrow L$ such that $z \circ k = \ell$ and $z \circ \zeta = \chi$. Second, given 1-cells $z_1, z_2\colon K \longrightarrow L$ together with an invertible 2-cell $\alpha\colon z_1 \circ k \Longrightarrow z_2 \circ k$ such that

$$\begin{tikzcd}
& K_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
K_0 \arrow[dr, "z_2\circ k"] \arrow[dr, bend right=60, "z_1\circ k"'] \arrow[r, phantom, "\Downarrow z_2\circ\zeta"] & & K_0 \arrow[dl, "z_2\circ k"'] \\
& L &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& K_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
K_0 \arrow[dr, "z_1\circ k"] \arrow[r, phantom, "\Downarrow z_1\circ\zeta"] & & K_0 \arrow[dl, "z_1\circ k"'] \arrow[dl, bend left=60, "z_2\circ k"] \\
& L, &
\end{tikzcd}
\tag{4.6}$$

there is a unique 2-cell $\beta\colon z_1 \Longrightarrow z_2$ such that $\beta \circ k = \alpha$. Since the existence and uniqueness apply equally well to α^{-1}, β is necessarily invertible.

Remark 4.7. Lack describes how to construct codescent objects in terms of more elementary 2-categorical colimits called "coinserters" and "coequifiers" in [11, Proposition 2.1], and he describes them as weighted colimits in [11, Proposition 2.1]. His [11, Theorem 2.4] gives several conditions that ensure their existence. In particular, he observes that codescent objects are filtered colimits, so that it suffices for $\mathscr{K}$ to have filtered colimits. Although it precedes Lack's introduction of codescent objects, the paper [1] already highlighted the relevance of filtered colimits to the construction of strictification 2-functors. These results ensure that the codescent objects we need do indeed exist, but we have no need to describe their construction. The universal property, or really just its first part, suffices.

However, our ground 2-categories are functor 2-categories $\mathbf{Cat}(\mathscr{V})^{\boldsymbol{\Psi}}$, where Ψ is a category viewed as a $\mathscr{V}$-2-category with no non-identity 2-cells. Their colimits are constructed levelwise in $\mathbf{Cat}(\mathscr{V})$. To be precise, let $\mathscr{O}$ be the discrete subcategory of $\mathscr{C}$ given by its objects and their identity maps and let $\mathscr{K}^{\mathscr{O}} = \mathbf{Cat}(\mathscr{V})^{\mathscr{O}}$. We have a restriction to objects 2-functor $\mathbb{U}\colon \mathscr{L} \longrightarrow \mathscr{L}^{\mathscr{O}}$. It has the effect of looking levelwise at underlying categories in $\mathscr{V}$, ignoring the morphisms in Ψ. Our 2-categorical colimits in $\mathbf{Cat}(\mathscr{V})^{\boldsymbol{\Psi}}$, such as codescent objects, are created in $\mathbf{Cat}(\mathscr{V})^{\mathscr{O}}$. That is, starting from a diagram in $\mathbf{Cat}(\mathscr{V})^{\boldsymbol{\Psi}}$, one takes the colimit in $\mathbf{Cat}(\mathscr{V})^{\mathscr{O}}$, and one then uses the universal property there to build in the morphisms of Ψ. In turn, colimits in $\mathbf{Cat}(\mathscr{V})^{\mathscr{O}}$ are created objectwise in $\mathbf{Cat}(\mathscr{V})$. Colimits in $\mathbf{Cat}(\mathscr{V})$ do not seem to be not well documented, but under very mild hypotheses on $\mathscr{V}$, they are inherited from $\mathscr{V}$. Boundedness in the sense prescribed in [3, 3.1.1] suffices and, as in [3, 3.3.2 and 3.3.6], all $\mathscr{V}$ of interest to us are bounded, by the arguments of [10]. This description of colimits makes it clear that the functor $\mathbb{U}$ preserves codescent objects.

4.3. **The monadic codescent data.** We again assume given a map $\xi\colon \mathbb{D} \longrightarrow \mathbb{E}$ of 2-monads in a ground 2-category $\mathscr{K}$. We emphasize that there are no coherence 2-cells so far. The associativity and unit diagrams for the products μ and units η of our monads commute strictly, and so do the diagrams relating them to ξ. We assume given a $\mathbb{D}$-pseudoalgebra $(\mathscr{Y}, \theta, \phi)$, normal as always so that the unit 2-cell is the identity; ϕ denotes the given invertible 2-cell $\theta \circ \mathbb{D}\theta \Longrightarrow \theta \circ \mu$. We then have the following reflexive codescent data in the 2-category $\mathbb{E}$-**AlgSt**.

(4.8)
$$\begin{tikzcd}[row sep=large]
\mathbb{E}\mathbb{D}^2\mathscr{Y} \arrow[d, shift right=8, "\nu\mathbb{D}"'] \arrow[d, shift right=4, leftarrow, "\mathbb{E}\eta"'] \arrow[d, "\mathbb{E}\mu"] \arrow[d, shift left=4, leftarrow, "\mathbb{E}\mathbb{D}\eta"] \arrow[d, shift left=8, "\mathbb{E}\mathbb{D}\theta"] \\
\mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, shift right=4, "\nu"'] \arrow[d, leftarrow, "\mathbb{E}\eta"] \arrow[d, shift left=4, "\mathbb{E}\theta"] \\
\mathbb{E}\mathscr{Y}
\end{tikzcd}$$

Here, since ξ is a map of 2-monads and $\mathscr{Y}$ is normal, it is easy to see that all but one of the required simplicial identities hold on the nose, so that we can take all but one of the required invertible 2-cells to be the identity. The one that remains $(d_1 \circ d_2 \cong d_1 \circ d_1)$ is

$$\mathbb{E}\phi\colon \mathbb{E}(\theta \circ \mathbb{D}\theta) \longrightarrow \mathbb{E}(\theta \circ \mu).$$

Thus if $\mathscr{Y}$ is a $\mathbb{D}$-algebra, so that $\phi = \mathrm{id}$, we require no non-identity 2-cells.

We write $\mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y}$ for the codescent object of the given codescent data, writing

$$\pi\colon \mathbb{E}\mathscr{Y} \longrightarrow \mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y} \quad \text{and} \quad \zeta\colon \pi \circ \nu \Longrightarrow \pi \circ \mathbb{E}\theta$$

for the cells witnessing the universality. It is worth being explicit about what the universal property says in this case.

First, let $\psi\colon \mathbb{E}\mathscr{Y} \longrightarrow \mathscr{Z}$ be a 1-cell in $\mathscr{K}$ and let $\chi\colon \psi \circ \nu \Longrightarrow \psi \circ \mathbb{E}\theta$ be an invertible 2-cell such that the following specialization of (4.4) holds; the corresponding specialization of (4.5) holds tautologically.

(4.9)
$$\begin{tikzcd}[column sep=small]
& \mathbb{E}\mathbb{D}\mathbb{D}\mathscr{Y} \arrow[dl, "\nu"'] \arrow[d, "\mathbb{E}\mathbb{D}\theta"] \arrow[dr, "\mathbb{E}\mu"] & \\
\mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, "\nu"'] \arrow[dr, "\mathbb{E}\theta"] & \mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, "\nu"] \arrow[dr, "\mathbb{E}\theta"] \arrow[r, phantom, "\Downarrow \mathbb{E}\phi"] & \mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, "\mathbb{E}\theta"] \\
\mathbb{E}\mathscr{Y} \arrow[dr, "\psi"'] \arrow[r, phantom, "\Downarrow \chi"] & \mathbb{E}\mathscr{Y} \arrow[d, "\psi"] \arrow[r, phantom, "\Downarrow \chi"] & \mathbb{E}\mathscr{Y} \arrow[dl, "\psi"] \\
& \mathscr{Z} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
& \mathbb{E}\mathbb{D}\mathbb{D}\mathscr{Y} \arrow[dl, "\nu"'] \arrow[dr, "\mathbb{E}\mu"] & \\
\mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, "\nu"'] & & \mathbb{E}\mathbb{D}\mathscr{Y} \arrow[d, "\mathbb{E}\theta"] \arrow[dll, "\nu"'] \\
\mathbb{E}\mathscr{Y} \arrow[dr, "\psi"'] \arrow[rr, phantom, "\Downarrow \chi"] & & \mathbb{E}\mathscr{Y} \arrow[dl, "\psi"] \\
& \mathscr{Z} &
\end{tikzcd}$$

Then there is a unique 1-cell $\gamma\colon \mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y} \longrightarrow \mathscr{Z}$ such that

$$\gamma \circ \pi = \psi \quad \text{and} \quad \gamma \circ \zeta = \chi.$$

Second, let $\gamma_1, \gamma_2\colon \mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y} \longrightarrow \mathscr{Z}$ be 1-cells together with an invertible 2-cell $\alpha\colon \gamma_1 \circ \pi \Longrightarrow \gamma_2 \circ \pi$ such that

(4.10)
$$\begin{tikzcd}[column sep=small]
& \mathbb{E}\mathbb{D}\mathscr{Y} \arrow[dl, "\nu"'] \arrow[dr, "\mathbb{E}\theta"] & \\
\mathbb{E}\mathscr{Y} \arrow[dr, "\gamma_2\circ\pi"] \arrow[dr, bend right=60, "\gamma_1\circ\pi"', "\Rightarrow\alpha"] \arrow[rr, phantom, "\Downarrow \gamma_2\circ\zeta"] & & \mathbb{E}\mathscr{Y} \arrow[dl, "\gamma_2\circ\pi"'] \\
& L &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
& \mathbb{E}\mathbb{D}\mathscr{Y} \arrow[dl, "\nu"'] \arrow[dr, "\mathbb{E}\theta"] & \\
\mathbb{E}\mathscr{Y} \arrow[dr, "\gamma_1\circ\pi"] \arrow[rr, phantom, "\Downarrow \gamma_1\circ\zeta"] & & \mathbb{E}\mathscr{Y} \arrow[dl, "\gamma_1\circ\pi"'] \arrow[dl, bend left=60, "\gamma_2\circ\pi", "\Downarrow\alpha"'] \\
& L, &
\end{tikzcd}$$

Then there is a unique (invertible) 2-cell $\beta\colon \gamma_1 \Longrightarrow \gamma_2$ such that $\beta \circ \pi = \alpha$. We re-emphasize that the codescent object $\mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y}$ is a strict $\mathbb{E}$-algebra since our codescent data are in $\mathbb{E}$-**AlgSt** and our codescent objects are constructed in that 2-category.

Remark 4.11. Our monads $\mathbb{D}$ in functor 2-categories $\mathbf{Cat}(\mathscr{V})^{\mathbf{\Psi}}$ arise from 2-categories $\mathscr{D}$ that contain Ψ and have the same objects, and $\mathbb{D}$ can be viewed as a composite 2-functor

$$\mathbf{Cat}(\mathscr{V})^{\mathbf{\Psi}} \longrightarrow \mathbf{Cat}(\mathscr{V})^{\mathscr{D}} \longrightarrow \mathbf{Cat}(\mathscr{V})^{\mathbf{\Psi}}.$$

Here the first arrow is a left adjoint prolongation 2-functor and the second arrow is its right adjoint restriction 2-functor. The left adjoint automatically preserves colimits. By a discussion like that in Remark 4.7, the second arrow also preserves colimits since colimits in $\mathscr{K}$ are constructed levelwise. In particular, $\mathbb{D}$ preserves codescent objects.

Remark 4.12. There are evident variants of our basic construction that play a role in our work. We have been considering 2-monads in the same ground 2-category $\mathscr{K}$. If we have a map $\mathbb{Q}\colon \mathscr{L} \longrightarrow \mathscr{K}$ of 2-categories, a monad $\mathbb{D}\colon \mathscr{K} \longrightarrow \mathscr{K}$ in $\mathscr{K}$. a monad $\mathbb{E}\colon \mathscr{L} \longrightarrow \mathscr{L}$ in $\mathscr{L}$ and a pseudomap of monads $\xi\colon \mathbb{D} \circ \mathbb{Q} \Longrightarrow \mathbb{Q} \circ \mathbb{E}$ as specified in Definition 2.1, then, with a slight abuse of notation, $\mathbb{E} \boxtimes_{\mathbb{D}} (-)$ can be constructed as a codescent object in the 2-category of $\mathbb{E}$-algebras in $\mathscr{L}$. This gives a 2-functor $\mathbb{D}$-**PsAlg** $\longrightarrow \mathbb{E}$-**AlgSt**, where the source has ground 2-category $\mathscr{K}$ and the target has ground 2-category $\mathscr{L}$. This idea is used in our formal construction of prolongation functors, where $\mathbb{Q}$ is an inclusion.

5. Weak double $\mathscr{D}$-algebras and the multifunctor $\xi_\# \circ \mathbb{G}r$

5.1. **Weak double $\mathscr{D}$-algebras.** As advertised in the introduction, we here present a 2-categorical construction that on passage to topology allows us to retain most of the formal properties obtained using the theory of 2-monads while maintaining homotopical control. It is convenient to ignore monads in this section.[7]

Recall again that a double category is a category internal to **Cat**. Here we resolutely ignore the standard symmetric reinterpretation that puts the horizontal and vertical morphisms on an equal footing. We have an evident generalization.

Definition 5.1. A double $\mathscr{V}$-category is a category $\mathscr{C}$ internal to $\mathbf{Cat}(\mathscr{V})$. Unravelling, we have $\mathscr{V}$-categories $\mathbf{Ob}\mathscr{C}$ and $\mathbf{Mor}\mathscr{C}$ of objects and morphisms of $\mathscr{C}$ together with source and target $\mathscr{V}$-functors $S, T\colon \mathbf{Mor}\mathscr{C} \longrightarrow \mathbf{Ob}\mathscr{C}$, an identity $\mathscr{V}$-functor $I\colon \mathbf{Ob}\mathscr{C} \longrightarrow \mathbf{Mor}\mathscr{C}$ and a composition $\mathscr{V}$-functor

$$C\colon \mathbf{Mor}\mathscr{C} \times_{\mathbf{Ob}\mathscr{C}} \mathbf{Mor}\mathscr{C} \longrightarrow \mathbf{Mor}\mathscr{C}$$

that satisfy the usual identities for these functions in a category.

Remark 5.2. We may regard a $\mathscr{V}$-category $\mathscr{C}$ as the constant double $\mathscr{V}$-category whose object and morphism $\mathscr{V}$-categories are $\mathscr{C}$, with S, T, I, and C all being the identity functor on $\mathscr{C}$.

Intuitively, the idea in this section is to replace $\mathbf{Cat}(\mathscr{V})$ by the 2-category $\mathbf{Cat}^{\mathbf{2}}(\mathscr{V})$ of double $\mathscr{V}$-categories as the target of our $\mathscr{V}$-functors $\mathscr{X}$ defined on $\mathscr{D}$, where $\mathscr{D}$ is a category of operators. Here we can define a strict $\mathscr{D}$-algebra

$$\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}^{\mathbf{2}}(\mathscr{V})$$

to consist of object and morphism $\mathscr{D}$-algebras

$$\mathbf{Ob}\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V}) \quad \text{and} \quad \mathbf{Mor}\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$$

together with $\mathscr{V}$-transformations S, T, I, and C that make each $\mathscr{X}(\mathbf{n})$ a double $\mathscr{V}$-category. The domain of C, denoted $\mathbf{Dom}\mathscr{X}$, is the pullback displayed in the diagram

$$\begin{tikzcd} \mathbf{Dom}\mathscr{X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbf{Mor}\mathscr{X} \arrow[d, "T"] \\ \mathbf{Mor}\mathscr{X} \arrow[r, "S"'] & \mathbf{Ob}\mathscr{X}. \end{tikzcd} \tag{5.3}$$

It is given the action of $\mathscr{D}$ induced by the action on the first variable $\mathbf{Mor}\mathscr{X}$.

However, that is not quite what we see in our examples since we can weaken structure using $\mathscr{V}$-pseudofunctors and $\mathscr{V}$-pseudotransformations. We can define pseudo double $\mathscr{D}$-algebras via object and morphism $\mathscr{D}$-pseudoalgebras such that S, T, I, and C are $\mathscr{D}$-pseudomorphisms. Our examples are in between, but they are very much closer to the strict than the pseudo version. The following ad hoc definition describes the structures that we shall actually see.

[7] Thinking monadically, we are changing ground 2-categories of the form $\mathscr{K}^\Psi$ to $\mathscr{K}^N$, where N is the set of objects of Ψ; the difference is discussed in topological contexts in [6, §3].

Definition 5.4. A weak double $\mathscr{D}$-algebra $\mathscr{X}$ consists of object and morphism (strict) $\mathscr{D}$-algebras

$$\mathbf{Ob}\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V}) \quad \text{and} \quad \mathbf{Mor}\mathscr{X} : \mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$$

together with (strict) maps of $\mathscr{D}$-algebras S, T, I, and C such that composition is unital and associative, the following diagram commutes

$$\begin{tikzcd} \mathbf{Mor}\mathscr{X} \times_{\mathbf{Ob}\mathscr{X}} \mathbf{Mor}\mathscr{X} \arrow[r, "X"] \arrow[d, "\pi_2"'] & \mathbf{Mor}\mathscr{X} \arrow[d, "S"] \\ \mathbf{Mor}\mathscr{X} \arrow[r, "S"'] & \mathbf{Ob}\mathscr{X} \end{tikzcd} \tag{5.5}$$

and there is a $\mathscr{V}$-transformation λ filling the diagram

$$\begin{tikzcd} \mathbf{Mor}\mathscr{X} \times_{\mathbf{Ob}\mathscr{X}} \mathbf{Mor}\mathscr{X} \arrow[r, "C"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lambda \Uparrow"] & \mathbf{Mor}\mathscr{X} \arrow[d, "T"] \\ \mathbf{Mor}\mathscr{X} \arrow[r, "T"'] & \mathbf{Ob}\mathscr{X}. \end{tikzcd} \tag{5.6}$$

If the diagram (5.6) were to commute, we would have a strict double $\mathscr{D}$-algebra, and we may regard a strict $\mathscr{D}$-algebra $\mathscr{Y}$ as the constant double $\mathscr{D}$-algebra at $\mathscr{Y}$. We are mainly interested in the following example. When $\mathscr{Y}$ is a strict $\mathscr{D}$-algebra, it is a special case of the "Grothendieck category of elements" (e.g. [14, §3.1]). Since we find it a little surprising that the general case differs so little from a strict double $\mathscr{D}$-algebra, we spell out the construction in detail.

Construction 5.7. Let $\mathscr{Y} = (\mathscr{Y}, \theta, \phi)$ be a $\mathscr{D}$-pseudoalgebra. Thus $\mathscr{Y}$ is a $\mathscr{V}$-functor $\mathscr{D} \longrightarrow \mathbf{Cat}(\mathscr{V})$, the action θ is given by $\mathscr{V}$-functors

$$\theta \colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}) \longrightarrow \mathscr{Y}(\mathbf{n}),$$

and ϕ is a $\mathscr{V}$-transformation $\theta \circ (\mathrm{id} \times \theta) \Longrightarrow \theta \circ (C \times \mathrm{id})$. We define a weak double $\mathscr{D}$-algebra $\mathbb{G}r\mathscr{Y}$. We Ignore the action of $\mathscr{D}$ on $\mathscr{Y}$ as far as possible. For each fixed object $\mathbf{n}$ of $\mathscr{D}$, define $\mathscr{V}$-categories

$$\mathbf{Ob}\mathbb{G}\mathbf{r}(\mathscr{Y})(\mathbf{n}) = \coprod_{\mathbf{m}} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m})$$

and

$$\mathbf{Mor}\mathbb{G}r(\mathscr{Y})(\mathbf{n}) = \coprod_{\ell, \mathbf{m}} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{D}(\ell, \mathbf{m}) \times \mathscr{Y}(\ell).$$

Letting n vary, the $\mathscr{V}$-categories $\mathbf{Ob}\mathbb{G}\mathbf{r}(\mathscr{Y})(\mathbf{n})$ and $\mathbf{Mor}\mathbb{G}\mathbf{r}(\mathscr{Y})(\mathbf{n})$ specify strict $\mathscr{D}$-algebras $\mathbf{Ob}\mathbb{G}\mathbf{r}(\mathscr{Y})$ and $\mathbf{Mor}\mathbb{G}\mathbf{r}(\mathscr{Y})$ with actions induced on components by the composition

$$\mathscr{D}(\mathbf{n}, \mathbf{p}) \times \mathscr{D}(\mathbf{m}, \mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{m}, \mathbf{p}) \tag{5.8}$$

in $\mathscr{D}$. Define S and T on the components of $\mathbf{Mor}\mathbb{G}\mathbf{r}(\mathscr{Y})$ by the $\mathscr{V}$-functors

$$S = C \times \mathrm{id} \colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{D}(\ell, \mathbf{m}) \times \mathscr{Y}(\ell) \longrightarrow \mathscr{D}(\ell, \mathbf{n}) \times \mathscr{Y}(\ell)$$

and

$$T = \mathrm{id} \times \theta \colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{D}(\ell, \mathbf{m}) \times \mathscr{Y}(\ell) \longrightarrow \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{Y}(\mathbf{m}).$$

The domain (5.3) of composition can be identified as the coproduct over (k,ℓ,m) of the pullbacks displayed in the evident commutative diagram
(5.9)

$$\begin{tikzcd}
\mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\ell,\mathbf{m})\times\mathscr{D}(\mathbf{k},\ell)\times\mathscr{Y}(\mathbf{k}) \arrow[r, "\pi_2=C\times\mathrm{id}"] \arrow[d, "\pi_1=\mathrm{id}\times T=\mathrm{id}\times\mathrm{id}\times\theta"'] & \mathscr{D}(\ell,\mathbf{n})\times\mathscr{D}(\mathbf{k},\ell)\times\mathscr{Y}(\mathbf{k}) \arrow[d, "T=\mathrm{id}\times\theta"] \\
\mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\ell,\mathbf{m})\times\mathscr{Y}(\ell) \arrow[r, "S=C\times\mathrm{id}"'] & \mathscr{D}(\ell,\mathbf{n})\times\mathscr{Y}(\ell)
\end{tikzcd}$$

The action of $\mathscr{D}$ is again induced componentwise from the composition (5.8) in $\mathscr{D}$. Define I and C in terms of I and C in $\mathscr{D}$. Thus, on the components of the coproducts,

$$I=\mathrm{id}\times I\times\mathrm{id}\colon \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{Y}(\mathbf{m})\longrightarrow\mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{m},\mathbf{m})\times\mathscr{Y}(\mathbf{m})$$

and

$$C=\mathrm{id}\times C\times\mathrm{id}\colon \mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\ell,\mathbf{m})\times\mathscr{D}(\mathbf{k},\ell)\times\mathscr{Y}(\mathbf{k})\longrightarrow\mathscr{D}(\mathbf{m},\mathbf{n})\times\mathscr{D}(\mathbf{k},\mathbf{m})\times\mathscr{Y}(\mathbf{k}).$$

Since $\mathscr{D}$ is a $\mathscr{V}$-category, it is immediate by inspection that C is unital and associative and that I, C, and S are strict $\mathscr{D}$-maps; the diagram (5.9) says that T is also a strict $\mathscr{D}$-map since S and π_2 in that diagram specify the action of $\mathscr{D}$ on $\mathbf{Ob}\mathscr{Y}$ and $\mathbf{Mor}\mathscr{Y}$. If $\mathscr{Y}$ is a strict $\mathscr{D}$-algebra, then these data specify a double $\mathscr{V}$-category of strict $\mathscr{D}$-algebras. In general, $\mathrm{id}\times\phi$ gives the $\mathscr{V}$-transformation λ required in the definition of a weak double $\mathscr{V}$-category.

Remark 5.10. We think of $\mathbb{G}r\mathscr{Y}$ as analogous to a cofibrant approximation of $\mathscr{Y}$. We have $\mathscr{V}$-functors ε from the object, morphism and domain $\mathscr{V}$-functors of $\mathbb{G}r\mathscr{Y}$ to $\mathscr{Y}$ given by θ, $\theta\circ C\times\mathrm{id}$, and $\theta\circ C\times\mathrm{id}\circ C\times\mathrm{id}\times\mathrm{id}$. These are all $\mathscr{D}$-pseudomorphisms; they satisfy $\varepsilon\circ I=\varepsilon$, $\varepsilon\circ C=\varepsilon$, $\varepsilon\circ S=\varepsilon$, and λ maps $\varepsilon\circ T$ to ε. We think of ε as a kind of weak map $\mathbb{G}r\mathscr{Y}\longrightarrow\mathscr{Y}$ of $\mathscr{D}$-pseudoalgebras in $\mathbf{Cat^2}(\mathscr{V})$. Using I on $\mathscr{D}$ in an evident way, we can construct a map $\eta\colon\mathscr{Y}\longrightarrow\mathbb{G}r\mathscr{Y}$ of Π-algebras in $\mathbf{Cat^2}(\mathscr{V})$.[8] such that $\varepsilon\circ\eta=\mathrm{id}$. When $\mathscr{Y}$ is a $\mathscr{D}$-algebra, this structure is all strict.

5.2. **The multifunctor $\xi_\#\circ\mathbb{G}r$.** Formally, we can now define the 2-category $\mathscr{D}$-$\mathbf{AlgSt^2}$ whose 0-cells are the weak double $\mathscr{D}$-algebras. The 1-cells $\zeta\colon\mathscr{X}\Longrightarrow\mathscr{Y}$ are given by triples of $\mathscr{D}$-pseudomorphisms

$$\zeta\colon\mathbf{ob}\mathscr{X}\Longrightarrow\mathbf{Ob}\mathscr{Y},\quad \zeta\colon\mathbf{Mor}\mathscr{X}\Longrightarrow\mathbf{Mor}\mathscr{Y},\quad \textbf{and}\quad \zeta\colon\mathbf{Dom}\mathscr{X}\Longrightarrow\mathbf{Dom}\mathscr{Y}$$

that commute with S, T, I, and C up to $\mathscr{D}$-modifications. Since our examples will be quite explicit in terms of structures already introduced, we omit full details. For example, the essential point in showing that a $\mathscr{D}$-pseudomorphism $\mu\colon\mathscr{X}\longrightarrow\mathscr{Y}$ induces a well-defined map of weak double categories $\mathbb{G}r\mu\colon\mathbb{G}r\mathscr{X}\longrightarrow\mathbb{G}r\mathscr{Y}$ is to construct a cubical pasting diagram from the diagrams (5.6) for $\mathscr{X}$ and for $\mathscr{Y}$; the equality of pasting diagrams in [5, Defn 1.10] gives an equality of pasting diagrams here. All relevant structure can be expressed inside the 2-category $\mathscr{D}$-$\mathbf{PsAlg}$ of $\mathscr{D}$-pseudoalgebras and $\mathscr{D}$-pseudomorphisms and $\mathscr{D}$-modifications of [5, §2] that we have been working in from the start.

Of course, the construction applies verbatim with $\mathscr{D}$ replaced by $\mathscr{D}^k$, giving us the graded 2-category $\mathscr{D}_*$-$\mathbf{AlgSt^2}$. Applying the external product $\overline{\times}$ to object,

[8] This uses that $\mathscr{D}$-pseudoalgebras are required to be normal [5, Definition 1.2].

morphism and domain $\mathscr{D}$-algebras, we see that $\mathscr{D}_*$-$\mathbf{AlgSt^2}$ is symmetric monoidal. The following two analogs of results in Section 3.1 and Section 3.2 are proven by transposition of variables and are easily checked from the definitions.

Proposition 5.11. *There is a 2-natural isomorphism of weak double $\mathscr{D}^{j+k}$-algebras*

$$\beta_{j,k}\colon \mathbb{G}r(\mathscr{X}\,\overline{\times}\,\mathscr{Y}) \longrightarrow \mathbb{G}r\mathscr{X}\,\overline{\times}\,\mathbb{G}r\mathscr{Y}$$

for $\mathscr{X} \in \mathscr{D}_j$-$\mathbf{PsAlg}$ and $\mathscr{Y} \in \mathscr{D}_k$-$\mathbf{PsAlg}$.

Corollary 5.12. *There is a 2-natural isomorphism of weak double $\mathscr{D}^k$-algebras*

$$\beta_k\colon \mathbb{G}r(\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathscr{X}_k) \longrightarrow \mathbb{G}r\mathscr{X}_1 \,\overline{\times}\, \cdots \,\overline{\times}\, \mathbb{G}r\mathscr{X}_k$$

for $\mathscr{X}_r \in \mathbb{D}$-$\mathbf{PsAlg}$, $1 \le r \le k$.

Warning 5.13. These results prove that $\mathbb{G}r\colon \mathscr{D}_*$-$\mathbf{PsAlg} \longrightarrow \mathscr{D}_*$-$\mathbf{AlgSt^2}$ is monoidal, but it is *not* symmetric monoidal. To see that, consider the following commutative diagram, in which we only consider $\mathbf{Ob}\mathbb{G}\mathbf{r}$ with $k = 2$.

$$\begin{tikzcd}
\big(\mathscr{D}(\mathbf{m_1},\mathbf{n_1}) \times \mathscr{D}(\mathbf{m_2},\mathbf{n_2})\big) \times \big(\mathscr{X}_\mathbf{1}(\mathbf{m_1}) \times \mathscr{X}_\mathbf{2}(\mathbf{m_2})\big) \arrow[r, "\beta_2"] \arrow[d, "t\times t"'] & \big(\mathscr{D}(\mathbf{m_1},\mathbf{n_1}) \times \mathscr{X}_\mathbf{1}(\mathbf{m_1})\big) \times \big(\mathscr{D}(\mathbf{m_2},\mathbf{n_2}) \times \mathscr{X}_\mathbf{2}(\mathbf{m_2})\big) \arrow[d, "t"] \\
\big(\mathscr{D}(\mathbf{m_2},\mathbf{n_2}) \times \mathscr{D}(\mathbf{m_1},\mathbf{n_1})\big) \times \big(\mathscr{X}_\mathbf{2}(\mathbf{m_2}) \times \mathscr{X}_\mathbf{1}(\mathbf{m_1})\big) \arrow[r, "\beta_2"'] & \big(\mathscr{D}(\mathbf{m_2},\mathbf{n_2}) \times \mathscr{X}_\mathbf{2}(\mathbf{m_2})\big) \times \big(\mathscr{D}(\mathbf{m_1},\mathbf{n_1}) \times \mathscr{X}_\mathbf{1}(\mathbf{m_1})\big)
\end{tikzcd}$$

Symmetry would require us to replace $t \times t$ by $\mathrm{id} \times t$ on the left, but that makes no sense here.

Remark 5.14. There is a symmetric version of the Grothendieck construction that can be obtained by passage to orbits over appropriate symmetric group actions. However, as far as we know it generally loses control of homotopy on passage to topology in the cases of interest, a manifestation of the fact that categorial colimits are not preserved by the classifying space functor. This idea does work topologically, with $\mathscr{F} = \mathscr{D}$, as is explained and exploited in [6].

Unlike its analog for $\xi_\#$, the proof of the following result is straightforward.

Proposition 5.15. *For $\mathscr{D}$-pseudoalgebras $\mathscr{Y}$, there is a 2-natural pseudomorphism*

$$\nu_k\colon \mathbb{G}r\mathbb{L}_k\mathscr{Y} \longrightarrow \mathbb{L}_k\mathbb{G}r\mathscr{Y}$$

of weak double $\mathscr{V}$-categories of $\mathscr{D}^k$-algebras.

Proof. We define

$$\nu_k\colon \mathbf{Ob}\mathbb{G}\mathbf{r}\mathbb{L}_\mathbf{k}\mathscr{Y} \longrightarrow \mathbf{Ob}\mathbb{L}_\mathbf{k}\mathbb{G}\mathbf{r}\mathscr{Y}$$

at level $(\mathbf{n_1}, \cdots, \mathbf{n_k})$ by taking coproducts over $(\mathbf{m_1}, \cdots, \mathbf{m_k})$ of the $\mathscr{V}$-functors

$$\Lambda^k_{\mathscr{D}} \times \mathrm{id}\colon\ \times_r\, \mathscr{D}(\mathbf{m_r}, \underline{\mathrm{n}}_\mathbf{r}) \times \mathscr{Y}(\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{Y}(\mathbf{n})$$

where $m = m_1 \cdots m_k$ and $n = n_1 \cdots n_k$. We define

$$\nu_k\colon \mathbf{Mor}\mathbb{G}\mathbf{r}\mathbb{L}_\mathbf{k}\mathscr{Y} \longrightarrow \mathbf{Mor}\mathbb{L}_\mathbf{k}\mathbb{G}\mathbf{r}\mathscr{Y}$$

at level $(\mathbf{n_1}, \cdots, \mathbf{n_k})$ by taking coproducts over $(\boldsymbol{\ell}_\mathbf{1}, \cdots, \boldsymbol{\ell}_\mathbf{k})$ and $(\mathbf{m_1}, \cdots, \mathbf{m_k})$ of the $\mathscr{V}$-functors

$$(\Lambda^k_{\mathscr{D}} \times \Lambda^k_{\mathscr{D}}) \circ t \times \mathrm{id}\colon\ \times_r \big(\mathscr{D}(\mathbf{m_r},\mathbf{n_r}) \times \mathscr{D}(\boldsymbol{\ell}_\mathbf{r},\mathbf{m_r})\big) \times \mathscr{Y}(\boldsymbol{\ell}) \longrightarrow \mathscr{D}(\mathbf{m},\mathbf{n}) \times \mathscr{D}(\boldsymbol{\ell},\mathbf{m}) \times \mathscr{Y}(\boldsymbol{\ell})$$

where t is the evident shuffle inclusion

$$\times_r\big(\mathscr{D}(\mathbf{m_r},\mathbf{n_r})\times\mathscr{D}(\ell_\mathbf{r},\mathbf{m_r})\big)\longrightarrow\big(\times_\mathbf{r}\,\mathscr{D}(\mathbf{m_r},\mathbf{n_r})\big)\times\big(\times_\mathbf{r}\,\mathscr{D}(\ell_\mathbf{r},\mathbf{m_r})\big).$$

Indexing over three sequences rather than two, we define

$$\nu_k\colon \mathbf{Dom}\mathbb{G}\mathbf{r}\mathbb{L}_\mathbf{k}\mathscr{Y}\longrightarrow \mathbf{Dom}\mathbb{L}_\mathbf{k}\mathbb{G}\mathbf{r}\mathscr{Y}'$$

the same way. The $\mathscr{V}$-transformations witessing the commutation of these $\mathscr{V}$-functors with the action of $\mathscr{D}$ and with the structure maps S, T, I, and C come directly from the $\mathscr{V}$-transformations witnessing that $\wedge^k_\mathscr{D}$ is a pseudofunctor and that $\mathscr{Y}$ and $\mathscr{Y}'$ are $\mathscr{D}$-pseudoalgebras. Pasting diagrams relating these imply the unstated pasting diagrams in the definition of pseudomaps of weak double $\mathscr{V}$-categories. $\square$

Since all structure in sight in the previous results is given by data in $\mathscr{D}^k$-**PsAlg**, it is transferred to $\mathscr{F}$-**AlgSt** by application of $\xi^k_\#$. We put things together to prove Theorem 0.18. For a k-morphism $F\colon \mathscr{X}_1\,\overline{\times}\cdots\overline{\times}\,\mathscr{X}_k\longrightarrow\mathbb{L}_k\mathscr{Y}$ of $\mathbb{D}_k$-pseudoalgebras we define a k-morphism $\xi_\#\mathbb{G}rF$ of (strict) double $\mathscr{F}_k$-algebras by commutativity of the diagram

(5.16)
$$\begin{tikzcd}
\xi_\#\mathbb{G}r\mathscr{X}_1\,\overline{\times}\,\cdots\,\overline{\times}\,\xi_\#\mathbb{G}r\mathscr{X}_k \arrow[r, "\xi_\#\mathbb{G}rF"] \arrow[d, "\alpha_k^{-1}"'] & \mathbb{L}_k\xi_\#\mathbb{G}r\mathscr{Y} \\
\xi^k_\#\big(\mathbb{G}r\mathscr{X}_1\,\overline{\times}\,\cdots\,\overline{\times}\,\mathbb{G}r\mathscr{X}_k\big) \arrow[d, "\xi^k_\#\beta_k^{-1}"'] & \xi^k_\#\mathbb{L}_k\mathbb{G}r\mathscr{Y} \arrow[u, "\omega_k"'] \\
\xi^k_\#\mathbb{G}r\big(\mathscr{X}_1\,\overline{\times}\cdots\overline{\times}\,\mathscr{X}_k\big) \arrow[r, "\xi^k_\#\mathbb{G}rF"'] & \xi^k_\#\mathbb{G}r\mathbb{L}_k\mathscr{Y} \arrow[u, "\xi^k_\#\nu_k"']
\end{tikzcd}$$

of $\mathbb{F}_k$-algebras. Just as (5.16) is an expansion of the diagram (3.9), so an expansion of the diagram (3.10) shows that $\xi_\#\mathbb{G}r$ preserves composition.

The following two remarks will combine to give homotopical control of $\xi_\#\mathbb{G}r$ in our topological applications [7].

Remark 5.17. Of course, $\xi_\#\circ\mathbb{G}r$ is isomorphic to $\xi_*\circ\mathbb{S}t\circ\mathbb{G}r$. Here $\mathbb{S}t$ and ξ_*, like their composite $\xi_\#$, are applied to the constituent data in the construction of $\mathbb{G}r$. By [9, Remark 2.20], the natural map $\mathbb{S}t\mathbb{G}r(\mathscr{Y})\longrightarrow\mathbb{G}r\mathscr{Y}$ is given by compatible equivalences of strict $\mathscr{D}$-algebras. The natural map $\mathbb{S}t\mathscr{Y}\longrightarrow\mathscr{Y}$ of $\mathscr{D}$-pseudoalgebras is an equivalence [9, Theorem 2.15] and therefore, by inspection, the induced map $\mathbb{G}r\mathbb{S}t\mathscr{Y}\longrightarrow\mathbb{G}r\mathscr{Y}$ is given by equivalences on its constituent $\mathscr{D}$-algebras. Thus we have equivalences

$$\mathbb{S}t\mathbb{G}r\mathscr{Y}\longrightarrow\mathbb{G}r\mathscr{Y}\longleftarrow\mathbb{G}r\mathbb{S}t\mathscr{Y}.$$

Remark 5.18. The construction of $\mathbb{G}r\mathscr{Y}$ can be thought of as the special case $\mathbb{G}r(\mathscr{D},\mathscr{D},\mathscr{Y})$ of a more general construction $\mathbb{G}r(\mathscr{E},\mathscr{D},\mathscr{Y})$, where $\mathscr{D}$ acts suitably on $\mathscr{E}$. In particular, we can replace the first variable $\mathscr{D}(\mathbf{m},\mathbf{n})$ in Construction 5.7 by $\mathscr{F}(\mathbf{m},\mathbf{n})$ to obtain $\mathbb{G}r(\mathscr{F},\mathscr{D},\mathscr{Y})$. When $\mathscr{Y}$ is a strict $\mathscr{D}$-algebra, so that λ is the identity, we can apply the strict 2-functor ξ_* to all data to obtain a strict double $\mathscr{F}$-algebra $\mathbb{G}r(\mathscr{F},\mathscr{D},\mathscr{Y})$. Then, comparing coequalizer diagrams, we obtain an isomorphism

$$\xi_*\mathbb{G}r\mathscr{Y}\cong\mathbb{G}r(\mathscr{F},\mathscr{D},\mathscr{Y}).$$

Applying this together with the previous remark we see that, for any $\mathscr{D}$-pseudoalgebra $\mathscr{Y}$, $\xi_{\#}\mathbb{G}r\mathscr{Y}$ is equivalent to $\mathbb{G}r(\mathscr{F}, \mathscr{D}, \mathbb{S}t\mathscr{Y})$.

5.3. **Remarks on $\mathscr{D}$-transformations.** It is classical that the classifying space functor takes categories, functors and natural transformations to spaces, maps, and homotopies. For the last, natural transformations are functors $\mathscr{X} \times \mathscr{I} \longrightarrow \mathscr{Y}$, where $\mathscr{I}$ is the category with two objects [0] and [1] and one non-identity morphism $[0] \longrightarrow [1]$. The functor B commutes with products, and it satisfies $B\mathscr{I} = I$. Thinking about structured categories and functors, in particular the $\mathscr{D}$-pseudoalgebras here, it is worth thinking about how they behave with respect to products with categories, such as $\mathscr{I}$, thought of as in $\mathbf{Cat}(\mathscr{V})$ as in [9, Section 1.1]. The remarks here are elementary, but they lead to proofs that structured transformations on the category level lead to homotopies between spectra in infinite loop space theory; see [7, Section 3.3].

For a small category $\mathscr{C}$ and a $\mathscr{D}$-pseudoalgebra $\mathscr{Y}$, $\mathscr{Y} \times \mathscr{C}$ is a $\mathscr{D}$-pseudoalgebra with

$$(\mathscr{Y} \times \mathscr{C})(\mathbf{n}) = \mathscr{Y}(\mathbf{n}) \times \mathscr{C}.$$

It is then obvious from the definition of $\mathbb{G}r$ that we have a natural identification

$$\mathbb{G}r(\mathscr{Y} \times \mathscr{C}) \cong \mathbb{G}r(\mathscr{Y}) \times \mathscr{C}.$$

A similar remark applies to $\mathbf{S}t$ and to ξ_*. At this point we think monadically. The argument is the same starting in $\mathbf{Cat}(\mathscr{V})$ or in $\mathbf{Cat}^2(\mathscr{V})$. The 2-monad $\mathbb{D}$ on Π-categories $\mathscr{X}$ is given by the tensor product of functors

$$(\mathbb{D}\mathscr{X})(\mathbf{n}) = \mathscr{D}(-, \mathbf{n}) \otimes_{\mathbf{\Pi}} \mathscr{X}(-) = \coprod_{\mathbf{m}} \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{X}(\mathbf{m})/(\sim).$$

Using diagonal maps on $\mathscr{C}$, we obtain functors

$$\Delta\colon \mathscr{D}(\mathbf{m}, \mathbf{n}) \times \mathscr{X}(\mathbf{m}) \times \mathscr{C} \longrightarrow \mathscr{D}_{\mathbf{G}}(\mathbf{m}, \mathbf{n}) \times \mathscr{X}(\mathbf{m}) \times \mathscr{C}^{\mathbf{m}}$$

that induce maps of Π-categories

$$\Delta\colon (\mathbb{D}\mathscr{X}) \times \mathscr{C} \longrightarrow \mathbb{D}(\mathscr{X} \times \mathscr{C}).$$

As explained in [9, 5]), we construct $\mathbf{S}t$ by factoring the structure map $\theta\colon \mathbb{D}_G\mathscr{Y} \longrightarrow \mathscr{Y}$ of a $\mathscr{D}$-pseudoalgebra $\mathscr{Y}$ as a composite

$$\mathbb{D}\mathscr{Y} \xrightarrow{e} \mathbb{S}t\mathscr{Y} \xrightarrow{m} \mathscr{Y}$$

where e is bijective on objects and m is fully faithful. By specialization of [9, Definition 2.9(ii)], applied as in [9, (3.6)], there is a map α of $\mathscr{D}$-algebras that makes the following diagram commute.

$$\begin{tikzcd}
(\mathbb{D}\mathscr{Y}) \times \mathscr{C} \arrow[r, "\Delta"] \arrow[d, "e\times\mathrm{id}"'] & \mathbb{D}(\mathscr{Y} \times \mathscr{C}) \arrow[r, "e"] & \mathbb{S}t(\mathscr{Y} \times \mathscr{C}) \arrow[d, "m"] \\
(\mathbb{S}t\mathscr{Y}) \times \mathscr{C} \arrow[urr, "\alpha"] \arrow[rr, "m\times\mathrm{id}"'] & & \mathscr{Y} \times \mathscr{C}
\end{tikzcd}$$

The left adjoint ξ_* is constructed via reflexive coequalizer diagrams

$$\begin{tikzcd}
\mathbb{F}\mathbb{D}\mathscr{Y} \arrow[r, shift left, "\mu\circ\mathbb{F}\xi"] \arrow[r, shift right, "\mathbb{F}\theta"'] & \mathbb{F}Y \arrow[r] & \xi_* Y,
\end{tikzcd}$$

where the reflexivity is given by the section $\mathbb{F}\eta\colon \mathbb{F}\mathscr{Y} \longrightarrow \mathbb{F}\mathbb{D}\mathscr{Y}$. Since reflexive coequalizers commute with finite products, a comparison of coequalizer diagrams gives a natural map

$$\beta\colon \xi_*(\mathscr{Y}) \times \mathscr{C} \longrightarrow \xi_*(\mathscr{Y} \times \mathscr{C})$$

of $\mathscr{F}$-algebras for $\mathscr{D}$-algebras $\mathscr{Y}$. Taking $\mathscr{C} = \mathscr{I}$, these comparisons say that all of the constructions used in passing from $\mathscr{P}$-pseudoalgebras to $\mathscr{F}$-algebras preserve $\mathscr{D}$-transformations. We think of this as saying that they are "homotopy preserving".

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