COMPLETIONS IN ALGEBRA AND TOPOLOGY

J.P.C. GREENLEES AND J.P. MAY

Contents

Introduction

Localization and completion are among the fundamental first tools in commutative algebra. They play a correspondingly fundamental role in algebraic topology. Localizations and completions of spaces and spectra have been central tools since the 1970's. Some basic references are [3, 17, 24]. These constructions start from ideals in the ring of integers and are very simple algebraically since $\mathbb{Z}$ is a principal ideal domain. Localizations and completions that start from ideals in the representation ring or the Burnside ring of a compact Lie group play a correspondingly central role in equivariant topology. These rings are still relatively simple algebraically since, when G is finite, they are Noetherian and of Krull dimension one. A common general framework starts from ideals in the coefficient ring of a generalized cohomology theory. We shall explain some old and new algebra that arises in this context, and we will show how this algebra can be mimicked topologically. The topological constructions require the foundations described in the previous article, which deals with the algebraically familiar theory of localization at multiplicatively closed subsets. We here explain the deeper and less familiar theory of completion, together with an ideal theoretic variant of localization. There is a still more general theory of localization of spaces and spectra at spectra, due to Bousfield [1, 2], and we shall see how our theory of localizations and completions with respect to ideals in coefficient rings fits into this context.

Consider an ideal I in a commutative ring A and the completions $M^{\wedge}_I = \lim M/I^kM$ of R-modules M. The algebraic fact that completion is not exact in general forces topologists to work with the derived functors of completion, and we shall explain how topological completions of spectra mimic an algebraic description of these derived functors in terms of "local homology groups". These constructs are designed for the study of cohomology theories, and we will describe dual constructs that are designed for the study of homology theories and involve Grothendieck's local cohomology groups. There are concomitant notions of "Čech homology and cohomology groups", which fit into algebraic fibre sequences that we shall mimic by interesting fibre sequences of spectra. These lead to a theory of localizations of spectra away from ideals. When specialized to MU-module spectra, these new localizations shed considerable conceptual light on the chromatic filtration that is at the heart of the study of periodic phenomena in stable homotopy theory.

1. Algebraic definitions: Local and Čech cohomology and homology

Suppose to begin with that A is a commutative Noetherian ring and that $I = (\alpha_1, \ldots, \alpha_n)$ is an ideal in A. There are a number of cases of topological interest where we must deal with non-Noetherian rings and infinitely generated ideals, but in these cases we attempt to follow the Noetherian pattern.

We shall be concerned especially with two naturally occurring functors on A-modules: the I-power torsion functor and the I-adic completion functor.

The I-power torsion functor Γ_I is defined by

$$M \longmapsto \Gamma_I(M) = \{x \in M \mid I^k x = 0 \text{ for some positive integer } k\}.$$

It is easy to see that the functor Γ_I is left exact.

We say that M is an I-power torsion module if $M = \Gamma_I M$. This admits a useful reinterpretation. Recall that the support of M is the set of prime ideals $\wp$ of A such that the localization $M_\wp$ is non-zero. We say that M is supported over I if every prime in the support of M contains I. This is equivalent to the condition that $M[1/\alpha] = 0$ for each $\alpha \in I$. It follows that M is an I-power torsion module if and only if the support of M lies over I.

The I-adic completion functor is defined by

$$M \longmapsto M^{\wedge}_I = \lim_k M/I^k M,$$

and M is said to be I-adically complete if the natural map $M \longrightarrow M^{\wedge}_I$ is an isomorphism. The Artin-Rees lemma states that I-adic completion is exact on finitely generated modules, but it is neither right nor left exact in general.

Since the functors that arise in topology are exact functors on triangulated categories, it is essential to understand the algebraic functors at the level of the derived category, which is to say that we must understand their derived functors. The connection with topology comes through one particular way of calculating the derived functors $R^*\Gamma_I$ of Γ_I and L^I_* of I-adic completion. Moreover, this particular method of calculation provides a connection between the two sets of derived functors and makes available various inductive proofs.

In this section, working in an arbitrary commutative ring A, we use our given finite set $\{\alpha_1, \dots , \alpha_n\}$ of generators of I to define various homology groups. We shall explain why a different set of generators gives rise to isomorphic homology groups, but we postpone the conceptual interpretations of our definitions until the next section.

For a single element α, we may form the flat stable Koszul cochain complex

$$K^\bullet(\alpha) = (A \longrightarrow A[1/\alpha])\,,$$

where the non-zero modules are in cohomological degrees 0 and 1. The word stable is included since this complex is the colimit over s of the unstable Koszul complexes

$$K^\bullet_s(\alpha) = (\alpha^s : A \longrightarrow A).$$

When defining local cohomology, it is usual to use the complex $K^\bullet(\alpha)$ of flat modules.

However, we shall need a complex of projective A modules in order to define certain dual local homology modules. Accordingly, we take a projective approximation $PK^\bullet(\alpha)$ to $K^\bullet(\alpha)$. A good way of thinking about this is that, instead of taking the colimit of the $K^\bullet_s(\alpha)$, we take their telescope [13, p.447]. This places the algebra in the form relevant to the topology. However, we shall use the model for $PK^\bullet(\alpha)$ displayed as the upper row in the quasi-isomorphism

$$\begin{array}{ccc} A \oplus A[x] & \xrightarrow{\langle 1, \alpha x - 1\rangle} & A[x] \\ {\scriptstyle \langle 1,0\rangle}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \longrightarrow & A[1/\alpha], \end{array}$$

where $g(x^i) = 1/\alpha^i$, because, like $K^\bullet(\alpha)$, this choice of $PK^\bullet(\alpha)$ is non-zero only in cohomological degrees 0 and 1.

The Koszul cochain complex for a sequence $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is obtained by tensoring together the complexes for the elements, so that

$$K^\bullet(\boldsymbol{\alpha}) = K^\bullet(\alpha_1) \otimes \cdots \otimes K^\bullet(\alpha_n),$$

and similarly for the projective complex $PK^\bullet(\boldsymbol{\alpha})$.

Lemma 1.1. *If β is in the ideal $I = (\alpha_1, \alpha_2, \dots , \alpha_n)$, then $K^\bullet(\boldsymbol{\alpha})[1/\beta]$ is exact.*

Proof. Since homology commutes with colimits, it suffices to show that some power of β acts as zero on the homology of $K^\bullet_s(\boldsymbol{\alpha}) = K^\bullet_s(\alpha_1)\otimes\cdots\otimes K^\bullet_s(\alpha_n)$. However, $(\alpha_i)^s$ annihilates $H^*(K^\bullet_s(\alpha_i))$, and it follows easily that $(\alpha_i)^{2s}$ annihilates $H^*(K^\bullet_s(\boldsymbol{\alpha}))$. Writing β as a linear combination of the n elements α_i, we see that β^{2sn} is a linear combination of elements each of which is divisible by some $(\alpha_i)^{2s}$, and the conclusion follows. □

Note that, by construction, we have an augmentation map

$$\varepsilon : K^\bullet(\boldsymbol{\alpha}) \longrightarrow A.$$

Corollary 1.2. *Up to quasi-isomorphism, the complex $K^\bullet(\boldsymbol{\alpha})$ depends only on the ideal I.*

Proof. The lemma implies that the augmentation $K^\bullet(\boldsymbol{\alpha},\beta) \longrightarrow K^\bullet(\boldsymbol{\alpha})$ is a quasi-isomorphism if $\beta \in I$. It follows that we have quasi-isomorphisms

$$K^\bullet(\boldsymbol{\alpha}) \longleftarrow K^\bullet(\boldsymbol{\alpha}) \otimes K^\bullet(\boldsymbol{\alpha}') \longrightarrow K^\bullet(\boldsymbol{\alpha}')$$

if $\boldsymbol{\alpha}'$ is a second set of generators for I. □

We therefore write $K^\bullet(I)$ for $K^\bullet(\boldsymbol{\alpha})$. Observe that $K^\bullet(\boldsymbol{\alpha})$ is unchanged if we replace the elements α_i by powers $(\alpha_i)^k$. Thus $K^\bullet(I)$ depends only on the radical of the ideal I. Since $PK^\bullet(\boldsymbol{\alpha})$ is a projective approximation to $K^\bullet(\boldsymbol{\alpha})$, it too depends only on the radical of I. We also write $K^\bullet_s(I) = K^\bullet_s(\alpha_1) \otimes \cdots \otimes K^\bullet_s(\alpha_n)$, but this is an abuse of notation since its homology groups do depend on the choice of generators.

The local cohomology and homology of an A-module M are then defined by

$$H^*_I(A; M) = H^*(PK^\bullet(I) \otimes M)$$

and

$$H^I_*(A; M) = H_*(\mathrm{Hom}(PK^\bullet(I), M)).$$

We usually omit the ring A from the notation. In particular, we write $H^*_I(A) = H^*_I(A; A)$. Note that we could equally well use the flat stable Koszul complex in the definition of local cohomology, as is more common. It follows from Lemma 1.1 that $H^*_I(M)$ is supported over I and is thus an I-power torsion module.

We observe that local cohomology and homology are invariant under change of base ring. While the proof is easy enough to leave as an exercise, the conclusion is of considerable calculational value.

Lemma 1.3. *If $A \longrightarrow A'$ is a ring homomorphism, I' is the ideal $I \cdot A'$ and M' is an A'-module regarded by pullback as an A-module, then*

$$H^*_I(A; M') \cong H^*_{I'}(A'; M') \quad \textit{and} \quad H^I_*(A; M') \cong H^{I'}_*(A'; M') \quad \square$$

We next define the Čech cohomology and homology of the A-module M. We will motivate the name at the end of the next section. Observe that $\varepsilon : K^\bullet(\boldsymbol{\alpha}) \longrightarrow A$ is an isomorphism in degree zero and define the flat Čech complex $\check{C}^\bullet(I)$ to be the complex $\Sigma(\ker \varepsilon)$. Thus, if $i \geq 0$, then $\check{C}^i(I) = K^{i+1}(I)$. For example, if $I = (\alpha, \beta)$, then

$$\check{C}^\bullet(I) = (\ A[1/\alpha] \oplus A[1/\beta] \longrightarrow A[1/(\alpha\beta)]\ )\,.$$

The differential $K^0(I) \longrightarrow K^1(I)$ specifies a chain map $A \longrightarrow \check{C}^\bullet(I)$ whose fibre is exactly $K^\bullet(I)$; see [11, pp.439-440]. Thus we have a fibre sequence

$$\boxed{K^\bullet(I) \longrightarrow A \longrightarrow \check{C}^\bullet(I)}\,.$$

We define the projective version $P\check{C}^\bullet(I)$ similarly, using the kernel of the composite of ε and the quasi-isomorphism $PK^\bullet(I) \longrightarrow K^\bullet(I)$; note that $P\check{C}^\bullet(I)$ is non-zero in cohomological degree -1.

The Čech cohomology and homology of an A-module M are then defined by

$$\check{C}H^*_I(A; M) = H^*(P\check{C}^\bullet(I) \otimes M)$$

and

$$\check{C}H_*^I(A; M) = H_*(\mathrm{Hom}(P\check{C}^\bullet(I), M)).$$

The Čech cohomology can also be defined by use of the flat Čech complex and is zero in negative degrees, but the Čech homology is usually non-zero in degree -1.

The fibre sequence $PK^\bullet(I) \longrightarrow A \longrightarrow P\check{C}^\bullet(I)$ gives rise to long exact sequences relating local and Čech homology and cohomology,

$$0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow \check{C}H^0_I(M) \longrightarrow H^1_I(M) \longrightarrow 0$$

and

$$0 \longrightarrow H_1^I(M) \longrightarrow \check{C}H_0^I(M) \longrightarrow M \longrightarrow H_0^I(M) \longrightarrow \check{C}H_{-1}^I(M) \longrightarrow 0,$$

together with isomorphisms

$$H^i_I(M) \cong \check{C}H^{i-1}_I(M) \text{ and } H_i^I(M) \cong \check{C}H_{i-1}^I(M) \text{ for } i \geq 2.$$

Using the Čech theory, we may splice together local homology and local cohomology to define "local Tate cohomology" $\hat{H}^*_I(A; M)$, which has attractive formal properties; we refer the interested reader to [9].

2. Connections with derived functors; calculational tools

We gave our definitions in terms of specific chain complexes, but we gave our motivation in terms of derived functors. The meaning of the definitions appears in the following two theorems.

Theorem 2.1 (Grothendieck [15])**.** *If A is Noetherian, then the local cohomology groups calculate the right derived functors of the left exact functor $M \longmapsto \Gamma_I(M)$. In symbols,*

$$H^n_I(A;M) = (R^n\Gamma_I)(M). \quad \square$$

Since $\Gamma_I(M)$ is clearly isomorphic to $\mathrm{colim}_r\,(\mathrm{Hom}(A/I^r, M)$, these right derived functors can be expressed in more familiar terms:

$$(R^n\Gamma_I)(M) \cong \mathrm{colim}_r\ \mathrm{Ext}^n_A(A/I^r, M).$$

Theorem 2.2 (Greenlees-May [13])**.** *If A is Noetherian, then the local homology groups calculate the left derived functors of the (not usually right exact) I-adic completion functor $M \longmapsto M^\wedge_I$. In symbols,*

$$H^I_n(A;M) = (L_n(\cdot)^\wedge_I)(M). \quad \square$$

The conclusion of Theorem 2.2 is proved in [13] under much weaker hypotheses. There is a notion of "pro-regularity" of a sequence $\boldsymbol{\alpha}$ for a module M [13, (1.8)], and [13, (1.9)] states that local homology calculates the left derived functors of completion provided that A has bounded α_i torsion for all i and $\boldsymbol{\alpha}$ is pro-regular for A. Moreover, if this is the case and if $\boldsymbol{\alpha}$ is also pro-regular for M, then $H^I_0(A;M) = M^\wedge_I$ and $H^I_i(A;M) = 0$ for $i > 0$. We shall refer to a module for which the local homology is its completion concentrated in degree zero as *tame*. By the Artin-Rees lemma, any finitely generated module over a Noetherian ring is tame.

The conclusion of Theorem 2.1 is also true under similar weakened hypotheses [10].

An elementary proof of Theorem 2.1 can be obtained by induction on the number of generators of I. This uses the spectral sequence

$$H^*_I(H^*_J(M)) \Longrightarrow H^*_{I+J}(M)$$

that is obtained from the isomorphism $PK^\bullet(I+J) \cong PK^\bullet(I) \otimes PK^\bullet(J)$. This means that it is only necessary to prove the result when I is principal and to verify that if Q is injective then $\Gamma_I Q$ is also injective. The proof of Theorem 2.2 can also be obtained like this, although it is more complicated because the completion of a projective module will usually not be projective.

One is used to the idea that I-adic completion is often exact, so that L^I_0 is the most significant of the left derived functors. However, it is the top non-vanishing right derived functor of Γ_I that is the most significant. Some idea of the shape of these derived functors can be obtained from the following result. Observe that the complex $PK^\bullet(\boldsymbol{\alpha})$ is non-zero only in cohomological degrees between 0 and n. This shows immediately that local homology and cohomology are zero above dimension n. A result of Grothendieck usually gives a much better bound. Recall that the

Krull dimension of a ring is the length of its longest strictly ascending sequence of prime ideals and that the I-depth of a module M is the length of the longest regular M-sequence in I.

Theorem 2.3 (Grothendieck [14])**.** *If A is Noetherian of Krull dimension d, then*

$$H^n_I(M) = 0 \quad \text{and} \quad H_n^I(M) = 0 \quad \text{if } n > d.$$

Let $depth_I(M) = m$. With no assumptions on A and M,

$$H^i_I(M) = 0 \text{ if } i < m.$$

If A is Noetherian, M is finitely generated, and $IM \neq M$, then

$$H^m_I(M) \neq 0.$$

Proof. The vanishing theorem for local cohomology above degree d follows from the fact that we can re-express the right derived functors of Γ_I in terms of algebraic geometry and apply a vanishing theorem that results from geometric considerations. Indeed, if $X = Spec(A)$ is the affine scheme defined by A, Y is the closed subscheme determined by I with underlying space $V(I) = \{\wp | \wp \supset I\} \subset X$, and $\tilde{M}$ is the sheaf over X associated to M, then $\Gamma_I(M)$ can be identified with the space $\Gamma_Y(\tilde{M})$ of sections of $\tilde{M}$ with support in Y. For sheaves $\mathscr{F}$ of Abelian groups over X, the cohomology groups $H^*_Y(X; \mathscr{F})$ are defined to be the right derived functors $(R^*\Gamma_Y)(\mathscr{F})$, and we conclude that

$$H^*_I(A; M) \cong H^*_Y(X; \tilde{M}).$$

The desired vanishing of local cohomology groups is now a consequence of a general result that can be proven by using flabby sheaves to calculate sheaf cohomology: for any sheaf $\mathscr{F}$ over any Noetherian space of dimension d, $H^n(X; \mathscr{F}) =$ for $n > d$ [14, 3.6.5] (or see [16, III.2.7]). The vanishing result for local homology follows from that for local cohomology by use of the universal coefficient theorem that we shall discuss shortly.

The vanishing of local cohomology below degree m is elementary, but we give the proof since we shall later make a striking application of this fact. We proceed by induction on m. The statement is vacuous if $m = 0$. Choose a regular sequence $\{\beta_1, \dots, \beta_m\}$ in I. Consider the long exact sequence of local cohomology groups induced by the short exact sequence

$$0 \longrightarrow M \xrightarrow{\beta_1} M \longrightarrow M/\beta_1 M \longrightarrow 0.$$

Since $\{\beta_2, \dots, \beta_m\}$ is a regular sequence for $M/\beta_1 M$, the induction hypothesis gives that $H^i_I(M/\beta_1 M) = 0$ for $i < m - 1$. Therefore multiplication by β_1 is a monomorphism on $H^i_I(M)$ for $i < m$. Since $H^i_I(M)[1/\beta_1] = 0$, by Lemma 1.1, this implies that $H^i_I(M) = 0$. The fact that $H^m_I(M) \neq 0$ under the stated hypotheses follows from standard alternative characterizations of depth and local cohomology in terms of Ext [20, §16]. □

It follows directly from the chain level definitions that there is a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^s(H_I^{-t}(A), M) \Longrightarrow H^I_{-t-s}(A; M), \tag{2.4}$$

with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$. This generalises Grothendieck's local duality spectral sequence [15]; see [13] for details.

We record a consequence of the spectral sequence that is implied by the vanishing result of Theorem 2.3. Recall that the nicest local rings are the regular local rings, whose maximal ideals are generated by a regular sequence; Cohen-Macaulay local rings, which have depth equal to their dimension, are more common. The following result applies in particular to such local rings.

Corollary 2.5. *If A is Noetherian and* $\mathrm{depth_I(A) = dim(A) = d}$*, then*

$$L^I_s M = \mathrm{Ext^{d-s}(H^d_I(A), M)}. \quad \square$$

For example if $A = \mathbb{Z}$ and $I = (p)$, then $H^*_{(p)}(\mathbb{Z}) = \mathbb{H}^{\Vdash}_{(\shortmid)}(\mathbb{Z}) = \mathbb{Z}/\shortmid^{\infty}$. Therefore the corollary states that

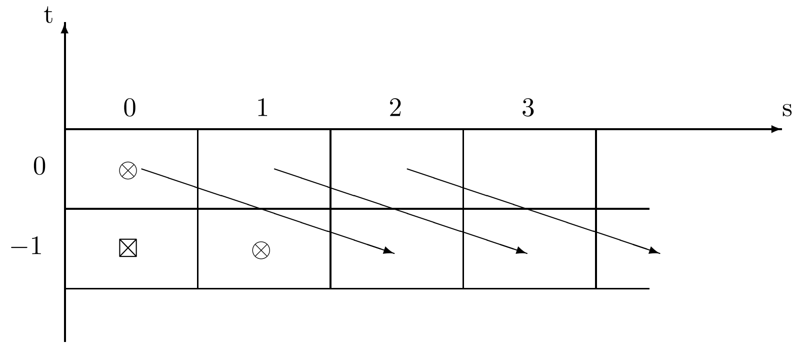
$$L_0^{(p)} M = \mathrm{Ext}(\mathbb{Z}/\shortmid^{\infty}, \mathbb{M}) \quad \text{and} \quad \mathbb{L}_{\Vdash}^{(\shortmid)}\mathbb{M} = \mathrm{Hom}(\mathbb{Z}/\shortmid^{\infty}, \mathbb{M}),$$

as was observed in Bousfield-Kan [3, VI.2.1].

There is a precisely similar universal coefficient theorem for calculating Čech homology from Čech cohomology. Together with Theorem 2.3, this implies vanishing theorems for the Čech theories.

Corollary 2.6. *If A is Noetherian of Krull dimension $d \geq 1$, then $\check{C}H^i_I(M)$ is only non-zero if $0 \leq i \leq d-1$ and $\check{C}H_i^I(M)$ is only non-zero if $-1 \leq i \leq d-1$. If $d = 0$ the Čech cohomology may be non-zero in degree 0 and the Čech homology may be non-zero in degrees 0 and -1.* □

When R is of dimension one, the spectral sequence (2.4) can be pictured as follows:



Here the two boxes marked $\otimes$ contribute to H_0^I, and that marked $\boxtimes$ is H_1^I. Since there is no local homology in negative degrees, the first of the d_2 differentials must be an epimorphism and the remaining d_2 differentials must be isomorphisms. Thus we find an exact sequence

$$0 \to \mathrm{Ext}^1(\mathrm{H}^1_\mathrm{I}(\mathrm{A}), \mathrm{M}) \to \mathrm{H}^\mathrm{I}_0(\mathrm{M}) \to \mathrm{Hom}(\mathrm{H}^0_\mathrm{I}(\mathrm{A}), \mathrm{M}) \to \mathrm{Ext}^2(\mathrm{H}^1_\mathrm{I}(\mathrm{A}), \mathrm{M}) \to 0$$

and an isomorphism

$$H_1^I(M) \cong \mathrm{Hom}(\mathrm{H}^1_\mathrm{I}(\mathrm{A}), \mathrm{M}).$$

Another illuminating algebraic fact is that local homology and cohomology are invariant under the completion $M \longrightarrow M_I^\wedge$ of a tame module M. This can be used in conjunction with completion of A and I in view of Lemma 1.3. However, all that is relevant to the proof is the vanishing of the higher local homology groups, not the identification of the zeroth group.

Proposition 2.7. *If $H_q^I(M) = 0$ for $q > 0$, then the natural map $M \longrightarrow H_0^I(M)$ induces isomorphisms on application of $H_I^*(\cdot)$ and $H_*^I(\cdot)$.*

Proof. The natural map $\varepsilon^* : M \longrightarrow \mathrm{Hom}(PK^\bullet(I), M)$ induces a quasi-isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(PK^\bullet(I), M) & \longrightarrow & \mathrm{Hom}(PK^\bullet(I), \mathrm{Hom}(PK^\bullet(I), M)) \\ & \cong & \mathrm{Hom}(PK^\bullet(I) \otimes PK^\bullet(I), M) \end{array}$$

since the projection $PK^\bullet(I) \otimes PK^\bullet(I) \longrightarrow PK^\bullet(I)$ is a quasi-isomorphism of projective complexes by Corollary 1.2. We obtain a collapsing spectral sequence converging from $E^2_{p,q} = H_p^I(H_q^I(M))$ to the homology of the complex in the middle, and the invariance statement in local homology follows.

For local cohomology we claim that ε also induces a quasi-isomorphism

$$K^\bullet(I) \otimes M \longrightarrow K^\bullet(I) \otimes \mathrm{Hom}(PK^\bullet(I), M).$$

The right side is a double complex, and there will result a collapsing spectral sequence that converges from $E_2^{p,q} = H_I^p((H_{-q}^I(M))$ to its homology. This will give the invariance statement in local cohomology. The fibre of the displayed map is $K^\bullet(I) \otimes \mathrm{Hom}(P\check{C}^\bullet(I), M)$, and we must show that this complex is exact. However $K^\bullet(I)$ is a direct limit of the finite self-dual unstable Koszul complexes $K_s^\bullet(I)$ so it is enough to see that $\mathrm{Hom}(P\check{C}^\bullet(I) \otimes K_s^\bullet(I), M)$ is exact. Since the complex $P\check{C}^\bullet(I) \otimes K_s^\bullet(I)$ is projective, it suffices to show that it is exact. However, it is quasi-isomorphic to $\check{C}^\bullet(I) \otimes K_s^\bullet(I)$, which has a finite filtration with subquotients $A[1/\beta] \otimes K_s^\bullet(I)$ with $\beta \in I$. We saw in the proof of Lemma 1.1 that some power of β annihilates the homology of $K_s^\bullet(I)$. Therefore the homology of $A[1/\beta] \otimes K_s^\bullet(I)$ is zero and the conclusion follows. □

We must still explain why we called $\check{C}^\bullet(I)$ a Čech complex. In fact, this complex arises by using the Čech construction to calculate cohomology from a suitable open cover. More precisely, let Y be the closed subscheme of $X = Spec(A)$ determined by I, as in the proof of Theorem 2.3. The space $V(I) = \{\wp | \wp \supset I\}$ decomposes as $V(I) = V(\alpha_1) \cap \ldots \cap V(\alpha_n)$, and there results an open cover of the open subscheme $X - Y$ as the union of the complements $X - Y_i$ of the closed subschemes Y_i determined by the principal ideals (α_i). However, $X - Y_i$ is isomorphic to the affine scheme $Spec(A[1/\alpha_i])$. Since affine schemes have no higher cohomology,

$$H^*(Spec(A[1/\alpha_i])) = H^0(Spec(A[1/\alpha_i])) = A[1/\alpha_i].$$

Thus the E_1 term of the Mayer-Vietoris spectral sequence for this cover collapses to the chain complex $\check{C}^\bullet(I)$, and

$$H^*(X - Y; \tilde{M}) \cong \check{C}H_I^*(M).$$

3. Topological analogs of the algebraic definitions

We suppose given a commutative S-algebra R, where S is the sphere spectrum. (As explained in [7], this is essentially the same thing as an E_∞ ring spectrum, but adapted to a more algebraically precise topological setting.) We imitate the algebraic definitions of Section 1 in the category of R-modules to construct a variety of useful spectra. Here we understand R-modules in the point-set level sense discussed in the preceding article [7]. The discussion in this section and the next is exactly like that first given for the equivariant sphere spectrum in [11], before the appropriate general context of modules was available.

For $\beta \in \pi_* R$, we define the Koszul spectrum $K(\beta)$ by the fibre sequence

$$K(\beta) \longrightarrow R \longrightarrow R[1/\beta].$$

Here $R[1/\beta] = \mathrm{hocolim}(R \xrightarrow{\beta} R \xrightarrow{\beta} \ldots)$ is a module spectrum and the inclusion of R is a module map, hence $K(\beta)$ is an R-module. Analogous to the filtration at the chain level, we obtain a filtration of the R-module $K(\beta)$ by viewing it as

$$\Sigma^{-1}(R[1/\beta] \cup CR).$$

Next we define the Koszul spectrum for the sequence $\beta_1, \ldots, \beta_n$ by

$$K(\beta_1, \ldots, \beta_n) = K(\beta_1) \wedge_R \cdots \wedge_R K(\beta_n).$$

The topological analogue of Lemma 1.1 states that if $\gamma \in J$ then

$$K(\beta_1, \ldots, \beta_n)[1/\gamma] \simeq *;$$

this follows from Lemma 1.1 and the spectral sequence (3.2) below (or from Lemma 3.6). We may now use precisely the same proof as in the algebraic case to conclude that the homotopy type of $K(\beta_1, \ldots, \beta_n)$ depends only on the radical of the ideal $J = (\beta_1, \cdots, \beta_n)$. We therefore write $K(J)$ for $K(\beta_1, \ldots, \beta_n)$.

We should remark that we are now working over the graded ring $R_* = \pi_*(R)$. All of the algebra in the previous two sections applies without change in the graded setting, but all of the functors defined there are now bigraded, with an internal degree coming from the grading of the given ring and its modules. As usual, we write $M_q = M^{-q}$.

With motivation from Theorems 2.1 and 2.2, we define the homotopical J-power torsion (or local cohomology) and homotopical completion (or local homology) modules associated to an R-module M by

(3.1) $$\Gamma_J(M) = K(J) \wedge_R M \quad \text{and} \quad M^\wedge_J = F_R(K(J), M).$$

In particular, $\Gamma_J(R) = K(J)$.

Because the construction follows the algebra so precisely, it is easy give methods of calculation for the homotopy groups of these R-modules. We use the product of the filtrations of the $K(\beta_i)$ given above and obtain spectral sequences

(3.2) $$E^2_{s,t} = H_J^{-s,-t}(R_*; M_*) \Rightarrow \pi_{s+t}(\Gamma_J M)$$

with differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and

(3.3) $$E_2^{s,t} = H^J_{-s,-t}(R^*; M^*) \Rightarrow \pi_{-(s+t)}(M^\wedge_J)$$

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$.

Similarly, we define the Čech spectrum by the cofibre sequence

(3.4) $$\boxed{K(J) \longrightarrow R \longrightarrow \check{C}(J)}\,.$$

With motivation deferred until Section 5, we define the homotopical localization (or Čech cohomology) and Čech homology modules associated to an R-module M by

(3.5) $$M[J^{-1}] = \check{C}(J) \wedge_R M \quad \text{and} \quad \Delta^J(M) = F_R(\check{C}(J), M).$$

In particular, $R[J^{-1}] = \check{C}(J)$. Once again, we have spectral sequences for calculating their homotopy groups from the analogous algebraic constructions.

We can now give topological analogues of some basic pieces of algebra that we used in Section 1. Recall that the algebraic Koszul complex $K^\bullet(J)$ is a direct limit of unstable complexes $K^\bullet_s(J)$ that are finite complexes of free R_*-modules with homology annihilated by a power of J. We remind the reader that, in contrast with $K^\bullet(J)$, the homology of the modules $K^\bullet_s(J)$ depends on the choice of generators we use. We say that an R-module M is a J-power torsion module if its R_*-module M_* of homotopy groups is a J-power torsion module; equivalently, M_* must have support over J.

Lemma 3.6. *The R-module $K(J)$ is a homotopy direct limit of finite R-modules $K_s(J)$, each of which has homotopy groups annihilated by some power of J. Therefore $K(J)$ is a J-power torsion module.*

Proof. It is enough to establish the result in the principal ideal case and then take smash products over R. Let

$$K_s(\beta) = \Sigma^{-1} R/\beta^s$$

denote the fibre of $\beta^s : R \longrightarrow R$, and observe that its homotopy groups are annihilated by β^{2s}. Now observe that

$$(R \longrightarrow R[1/\beta]) = \mathrm{hocolim}_s \left(R \xrightarrow{\beta^s} R \right),$$

and so their fibres are also equivalent:

$$K(\beta) \simeq \mathrm{hocolim}_s \, K_s(\beta). \quad \square$$

The following lemma is an analogue of the fact that $\check{C}^\bullet(J)$ is a chain complex which is a finite sum of modules $R[1/\beta]$ for $\beta \in J$.

Lemma 3.7. *The R-module $\check{C}(J)$ has a finite filtration by R-submodules with subquotients that are suspensions of modules of the form $R[1/\beta]$ with $\beta \in J$.* □

These lemmas are useful in combination.

Corollary 3.8. *If M is a J-power torsion module then $M \wedge_R \check{C}(J) \simeq *$; in particular $K(J) \wedge_R \check{C}(J) \simeq *$.*

Proof. Since $M[1/\beta] \simeq *$ for $\beta \in J$, Lemma 3.7 gives the conclusion for M. □

We remark that the corollary leads via [9, B.2] to the construction of a topological J-local Tate cohomology module $t_J(M)$ that has formal properties like those of its algebraic counterpart studied in [9].

4. Completion at ideals and Bousfield localization

As observed in the proof of Lemma 3.6, we have $K(\beta) = \mathrm{hocolim}_s\, \Sigma^{-1}R/\beta^s$ and therefore

$$M^{\wedge}_{(\beta)} = F_R(\mathrm{hocolim}_s\, \Sigma^{-1}R/\beta^s, M) = \mathrm{holim}_s\, M/\beta^s.$$

If $J = (\beta, \gamma)$, then

$$M^{\wedge}_J = F_R(K(\beta) \wedge_R K(\gamma), M) = F_R(K(\beta), F_R(K(\gamma), M)) = (M^{\wedge}_{(\gamma)})^{\wedge}_{(\beta)},$$

and so on inductively. This should help justify the notation $M^{\wedge}_J = F_R(K(J), M)$.

When $R = S$ is the sphere spectrum and $p \in \mathbb{Z} \cong \pi_{\nvdash}(\mathbb{S})$, $K(p)$ is a Moore spectrum for $\mathbb{Z}/{}_{|}^{\infty}$ in degree -1 and we recover the usual definition

$$X^{\wedge}_p = F(S^{-1}/p^{\infty}, X)$$

of p-completions of spectra as a special case. The standard short exact sequence for the calculation of the homotopy groups of $X^{\wedge}_p$ in terms of 'Ext completion' and 'Hom completion' follows directly from Corollary 2.5.

Since p-completion has long been understood to be an example of a Bousfield localization, our next task is to show that completion at J is a Bousfield localization in general. The arguments are the same as in [11, §2], which dealt with the (equivariant) case $R = S$.

We must first review definitions. They are usually phrased homologically, but we shall give the spectrum level equivalents so that the translation to other contexts is immediate. Fix a spectrum E. A spectrum A is *E-acyclic* if $A \wedge E \simeq *$; a map $f : X \longrightarrow Y$ is an *E-equivalence* if its cofibre is E-acyclic. A spectrum X is *E-local* if $E \wedge T \simeq *$ implies $F(T, X) \simeq *$. A map $Y \longrightarrow L_E Y$ is a *Bousfield E-localization* of Y if it is an E-equivalence and $L_E Y$ is E-local. This means that $Y \longrightarrow L_E Y$ is terminal among E-equivalences with domain Y, and the Bousfield localization is therefore unique if it exists. Bousfield has proved that $L_E Y$ exists for all E and Y, but we shall construct the localizations that we need directly.

We shall need two variations of the definitions. First, we work in the category of R-modules, so that $\wedge$ and $F(\cdot, \cdot)$ are replaced by $\wedge_R$ and $F_R(\cdot, \cdot)$. It is proven in [8] that Bousfield localizations always exist in this setting. Second, we allow E to be replaced by a class $\mathscr{E}$ of R-modules, so that our conditions for fixed E are replaced by conditions for each $E \in \mathscr{E}$. When the class $\mathscr{E}$ is a set, it is equivalent to work with the single module given by the wedge of all $E \in \mathscr{E}$. Bousfield localizations at classes need not always exist, but the language will be helpful in explaining the conceptual meaning of our examples. The following observation relates the spectrum level and module level notions of local spectra.

Lemma 4.1. *Let $\mathscr{E}$ be a class of R-modules. If an R-module N is $\mathscr{E}$-local as an R-module, then it is $\mathscr{E}$-local as a spectrum.*

Proof. Let $\mathbb{F}$ be the free functor from spectra to R-modules. If $E \wedge T \simeq *$ for all E, then $E \wedge_R \mathbb{F}\mathbb{T} \simeq *$ for all E and therefore $F(T, N) \simeq F_R(\mathbb{F}\mathbb{T}, \mathbb{N}) \simeq *$. □

The class that will concern us most is the class J-**Tors** of finite J-power torsion R-modules M. Thus M must be a finite cell R-module, and its R_*-module M_* of homotopy groups must be a J-power torsion module.

Theorem 4.2. *For any finitely generated ideal J of R_* the map $M \longrightarrow M_J^\wedge$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $\Gamma_J(R) = K(J)$.*
(ii) *with respect to the class J-**Tors** of finite J-power torsion R-modules.*
(iii) *with respect to the R-module $K_s(J)$ for any $s \geq 1$.*

Furthermore, the homotopy groups of the completion are related to local homology groups by a spectral sequence

$$E^2_{s,t} = H^J_{s,t}(M_*) \Longrightarrow \pi_{s+t}(M_J^\wedge).$$

If R_* is Noetherian, the E^2 term consists of the left derived functors of J-adic completion: $H^J_s(M_*) = L^J_s(M_*)$.

Proof. The statements about calculations are repeated from (3.3) and Theorem 2.2. We prove (i). Since

$$F_R(T, M_J^\wedge) \simeq F_R(T \wedge_R K(J), M),$$

it is immediate that $M_J^\wedge$ is $K(J)$-local. We must prove that the map $M \longrightarrow M_J^\wedge$ is a $K(J)$-equivalence. The fibre of this map is $F(\check{C}(J), M)$, so we must show that

$$F(\check{C}(J), M) \wedge_R K(J) \simeq *.$$

By Lemma 3.6, $K(J)$ is a homotopy direct limit of terms $K_s(J)$. Each $K_s(J)$ is in J-**Tors**, and we see by their definition in terms of cofibre sequences and smash products that their duals $D_R K_s(J)$ are also in J-**Tors**, where $D_R(M) = F_R(M, R)$. Since $K_s(J)$ is a finite cell R-module,

$$F_R(\check{C}(J), M) \wedge_R K_s(J) = F_R(\check{C}(J) \wedge_R D_R K_s(J), M),$$

and $\check{C}(J) \wedge_R D_R K_s(J) \simeq *$ by Corollary 3.8. Parts (ii) and (iii) are similar but simpler. For (iii), observe that we have a cofibre sequence $R/\beta^s \longrightarrow R/\beta^{2s} \longrightarrow R/\beta^s$, so that all of the $K_{js}(J)$ may be constructed from $K_s(J)$ using a finite number of cofibre sequences. □

5. Localization away from ideals and Bousfield localization

Bousfield localizations include both completions at ideals and localizations at multiplicatively closed sets, but one may view these Bousfield localizations as falling into the types typified by completion at p and localization away from p. Thinking in terms of $Spec(R_*)$, this is best viewed as the distinction between localization at a closed set and localization at the complementary open subset. We dealt with the closed sets in the previous section, and we deal with the open sets in this one. Observe that, when $J = (\beta)$, $M[J^{-1}]$ is just $R[\beta^{-1}] \wedge_R M = M[\beta^{-1}]$. However, the non-vanishing of higher Čech cohomology groups gives the construction for general finitely generated ideals a quite different algebraic flavour, and $M[J^{-1}]$ is generally not a localization of M at a multiplicatively closed subset of R_*. To characterize this construction as a Bousfield localization, we consider the class J-**Inv** of R-modules M for which there is an element $\beta \in J$ such that $\beta : M \longrightarrow M$ is an equivalence.

Theorem 5.1. *For any finitely generated ideal $J = (\beta_1, \ldots, \beta_n)$ of R_*, the map $M \longrightarrow M[J^{-1}]$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $R[J^{-1}] = \check{C}(J)$.*
(ii) *with respect to the class J-**Inv**.*
(iii) *with respect to the set $\{R[1/\beta_1], \ldots, R[1/\beta_n]\}$.*

Furthermore, the homotopy groups of the localization are related to Čech cohomology groups by a spectral sequence

$$E^2_{s,t} = \check{C}H^{-s,-t}_J(M_*) \Longrightarrow \pi_{s+t}(M[J^{-1}]).$$

If R_* is Noetherian, the E^2 term can be viewed as the cohomology of $Spec(R_*)\backslash V(J)$ with coefficients in the sheaf associated to M_*.

Proof. The spectral sequence is immediate from the construction of $M[J^{-1}]$, and the last paragraph of Section 2 gives the final statement.

To see that $M[J^{-1}]$ is local, suppose that $T \wedge_R \check{C}(J) \simeq *$. We must show that $F_R(T, M[J^{-1}]) \simeq *$. By the cofibre sequence defining $\check{C}(J)$ and the hypothesis, it suffices to show that $F_R(K(J) \wedge_R T, M[J^{-1}]) \simeq *$. By Lemma 3.6,

$$F_R(K(J) \wedge_R T, M[J^{-1}]) \simeq \mathrm{holim}_s\, F_R(K_s(J) \wedge_R T, \check{C}(J) \wedge_R M).$$

Observing that

$$F_R(K_s(J) \wedge_R T, \check{C}(J) \wedge_R M) \simeq F_R(T, D_R K_s(J) \wedge_R \check{C}(J) \wedge_R M),$$

we see that the conclusion follows from Corollary 3.8. The map $M \longrightarrow M[J^{-1}]$ is a $\check{C}(J)$-equivalence since its fibre is $\Gamma_J(M) = K(J) \wedge_R M$ and $K(J) \wedge_R \check{C}(J) \simeq *$ by Corollary 3.8. Parts (ii) and (iii) are proved similarly. $\square$

Translating the usual terminology, we say that a localization L on R-modules is *smashing* if $L(N) = N \wedge_R L(R)$ for all R-modules N. The following fact is obvious.

Lemma 5.2. *Localization away from J is smashing.* □

It is also clear that completion at J will not usually be smashing.

We complete the general theory with an easy, but tantalizing, result that will specialize to give part of the proof of the Chromatic Convergence Theorem of Hopkins-Ravenel [23]. It well illustrates how the algebraic information in Section 2 can have non-obvious topological implications. Observe that if $J' = J + (\beta)$, we have an augmentation map $\varepsilon : K(J') \simeq K(J) \wedge_R K(\beta) \longrightarrow K(J)$ over R. Applying $F_R(\cdot, M)$, we obtain an induced map

$$M^{\wedge}_J \longrightarrow M^{\wedge}_{J'}.$$

A comparison of cofibre sequences in the derived category of R-modules gives a dotted arrow ζ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\Gamma_{J'}(M) & \longrightarrow & M & \longrightarrow & M[J'^{-1}] & \longrightarrow & \Sigma\Gamma_{J'}(M) \\
{\scriptstyle\varepsilon}\downarrow & & \downarrow & & \vdots\,{\scriptstyle\zeta} & & \downarrow \\
\Gamma_J(M) & \longrightarrow & M & \longrightarrow & M[J^{-1}] & \longrightarrow & \Sigma\Gamma_J(M).
\end{array}$$

Here the cofibre of ε is $\Gamma_J(M)[\beta^{-1}]$ and the cofibre of ζ is $\Sigma\Gamma_J(M)[\beta^{-1}]$. If an ideal $\mathscr{J}$ is generated by a countable sequence $\{\beta_i\}$ and J_n is the ideal generated by the first n generators, we may define

$$M^{\wedge}_{\mathscr{J}} = \mathrm{hocolim}_n\, M^{\wedge}_{J_n} \qquad \text{and} \qquad M[\mathscr{J}^{-}] = \mathrm{holim}\, \mathscr{M}[\mathscr{J}^{-}].$$

We say that $\mathscr{J}$ is of infinite depth if $\mathit{depth}_{J_n}(R_*) \longrightarrow \infty$; this holds, for example, if $\{\beta_i\}$ is a regular sequence.

Proposition 5.3. *If M is a finite cell R-module and $\mathscr{J}$ is of infinite depth, then $M \simeq M[\mathscr{J}^{-}]$.*

Proof. It suffices to prove that $\mathrm{holim}_n\, \Gamma_{J_n}(M) \simeq *$, and, since M is finite, it is enough to prove this when $M = R$. We show that the system of homotopy groups $\pi_*(K(J_n))$ is pro-zero. This just means that, for any n, there exists $q > n$ such that the map $K(J_q) \longrightarrow K(J_n)$ induces zero on homotopy groups, and it implies that both $\lim_n \pi_*(K(J_n)) = 0$ and $\lim^1_n \pi_*(K(J_n)) = 0$. By the $\lim^1$ exact sequence for the computation of the homotopy groups of a homotopy inverse limit, this will give the conclusion. Since J_n is finitely generated, there is a d such that $H^i_{J_n}(R_*) = 0$ for $i \geq d$. By hypothesis, we may choose q such that $\mathit{depth}_{J_q}(R_*) > d$. Then, by Theorem 2.3, $H^i_{J_q}(R_*) = 0$ for $i \leq d$. Now the spectral sequence (3.2) for $\pi_*(K(J_n))$ is based on the filtration

$$\cdots \subseteq F_{-s} \subseteq F_{-s+1} \subseteq \cdots F_1 \subseteq F_0 = \pi_*(K(J_n))$$

in which F_{-s} is the group of elements arising from $H^i_{J_n}(R_*)$ for $i > s$. The map $K(J_q) \longrightarrow K(J_n)$ is filtration preserving, hence the filtration corresponding to $s = d$ is mapped to 0. By the choice of q, this filtration is all of $\pi_*(K(J_q))$. □

6. The specialization to ideals in MU_*

We specialize to the commutative S-algebra $R = MU$ in this section, taking [7, §11] as our starting point. Recall that $MU_* = \mathbb{Z}[x_i \mid i \geq 1]$, where $deg\, x_i = 2i$, and that MU_* contains elements v_i of degree $2(p^i - 1)$ that map to the Hazewinkel generators of $BP_* = \mathbb{Z}_{(p)}[v_i \mid i \geq 1]$. We let I_n denote the ideal $(v_0, v_1, \dots, v_{n-1})$ in $\pi_*(MU)$, where $v_0 = p$; we prefer to work in MU rather than BP because of its canonical S-algebra structure. As explained in [7, §11], BP is an MU-ring spectrum whose unit $MU \longrightarrow BP$ factors through the canonical retraction $MU_{(p)} \longrightarrow BP$. We also have MU-ring spectra $E(n)$ such that $E(0)_* = \mathbb{Q}$ and

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$$

if $n > 0$. The Bousfield localization functor $L_n = L_{E(n)}$ on spectra plays a fundamental role in the "chromatic" scheme for the inductive study of stable homotopy theory, and we have the following result.

Theorem 6.1. *When restricted to MU-modules M, the functor L_n coincides with localization away from I_{n+1}:*

$$L_n M \simeq M[I_{n+1}^{-1}].$$

Proof. By [23, 7.3.2], localization at $E(n)$ is the same as localization at $BP[(v_n)^{-1}]$ or at the wedge of the $K(i)$ for $0 \leq i \leq n$. This clearly implies that localization at $E(n)$ is the same as localization at the wedge of the $BP[(v_i)^{-1}]$ for $0 \leq i \leq n$, and this is the same as localization at the wedge of the $MU[(v_i)^{-1}]$ for $0 \leq i \leq n$. By Lemma 4.1, we conclude that $M[I_{n+1}^{-1}]$ is $E(n)$-local. To see that the localization $M \longrightarrow M[I_{n+1}^{-1}]$ is an $MU[(v_i)^{-1}]$-equivalence for $0 \leq i \leq n$, note that its fibre is $\Gamma_{I_{n+1}}(M)$ and $\Gamma_{I_{n+1}}(M)[w^{-1}] \simeq *$ for any $w \in I_{n+1}$. Consider $MU_*(MU) = (MU \wedge MU)_*$ as a left MU_*-module, as usual, and recall from [23, B.5.15] that the right unit $MU_* \longrightarrow (MU \wedge MU)_*$ satisfies

$$\eta_R(v_i) \equiv v_i \;\; mod \;\; I_i \cdot MU_*(MU), \quad \text{hence} \quad \eta_R(v_i) \in I_{i+1} \cdot MU_*(MU).$$

We have

$$\Gamma_{I_{n+1}}(M) \wedge MU \simeq \Gamma_{I_{n+1}}(M) \wedge_{MU} (MU \wedge MU)$$

and can deduce inductively that $\Gamma_{I_{n+1}}(M) \wedge MU[w^{-1}] \simeq *$ for any $w \in I_{n+1}$ since $\Gamma_{I_{n+1}}(M)[w^{-1}] \simeq *$ for any such w. □

When $M = BP$, this result is essentially a restatement in our context of Ravenel's theorem [23, 8.1.1] (see also [21, §§5-6] and [22]) on the geometric realization of the chromatic resolution for the calculation of stable homotopy theory. To explain the connection between our constructions and his, we offer the following dictionary:

$$N_nBP \simeq \Sigma^n\Gamma_{I_n}BP.$$

$$M_nBP \simeq \Sigma^n\Gamma_{I_n}BP[(v_n)^{-1}]$$

$$L_nBP \simeq BP[I_{n+1}^{-1}]$$

In fact, for any spectrum X, Ravenel defines M_nX and N_nX inductively by

$$N_0X = X, \quad M_nX = L_nN_nX,$$

and the cofibre sequences

$$N_nX \longrightarrow M_nX \longrightarrow N_{n+1}X. \tag{6.2}$$

He also defines C_nX to be the fibre of the localization $X \longrightarrow L_nX$ (where, to start inductions, $L_{-1}X = *$ and $C_{-1}X = X$). Elementary formal arguments given in [21, 5.10] show that the definition of Bousfield localization, the cofibrations in the definitions just given, and the fact that $L_mL_n = L_m$ for $m \le n$ [21, 2.1] imply that

$$N_nX = \Sigma^nC_{n-1}X$$

and there is a cofibre sequence

$$\Sigma^{-n}M_nX \longrightarrow L_nX \longrightarrow L_{n-1}X. \tag{6.3}$$

The claimed identifications follow inductively from our description of L_nBP and the fact (implied by Lemma 1.1) that, for any MU-module M,

$$\Gamma_{I_n}(M)[I_{n+1}^{-1}] \simeq \Gamma_{I_n}(M)[(v_n)^{-1}].$$

In fact, the evident cofibrations of MU-modules

$$\Sigma^n\Gamma_{I_n}BP \longrightarrow \Sigma^n\Gamma_{I_n}BP[{v_n}^{-1}] \longrightarrow \Sigma^{n+1}\Gamma_{I_{n+1}}(BP)$$

and

$$\Gamma_{I_n}BP[(v_n)^{-1}] \longrightarrow BP[(I_{n+1})^{-1}] \longrightarrow BP[(I_n)^{-1}]$$

realize the case $X = BP$ of the cofibrations displayed in (6.2) and (6.3). Moreover, it is immediate from our module theoretic constructions that the homotopy groups are given inductively by

$$(N_0BP)_* = BP_*, \;\; (M_nBP)_* = (N_nBP)_*[(v_n)^{-1}],$$

and the short exact sequences

$$0 \longrightarrow (N_nBP)_* \longrightarrow (M_nBP)_* \longrightarrow (N_{n+1}BP)_* \longrightarrow 0. \tag{6.4}$$

Ravenel's original arguments were substantially more difficult because, not having the new category of MU-modules to work in, he had to work directly in the classical stable homotopy category.

Although BP is not a finite cell MU-module, the retraction from $MU_{(p)}$ makes it clear that the proof of Proposition 5.3 applies to give the following conclusion.

Proposition 6.5. *Let $\mathscr{I}$ be generated by $\{v_i | i \geq 0\}$. Then*

$$BP \simeq BP[\mathscr{I}^-] \simeq \operatorname{holim} \mathscr{LBP}. \quad \square$$

The chromatic filtration theorem of Hopkins and Ravenel [23, 7.5.7] asserts that a finite p-local spectrum X is equivalent to $\operatorname{holim} L_n X$; the previous result plays a key role in the proof (in the guise of [23, 8.6.5]).

We close with a result about completions. We have the completion $M \longrightarrow M^\wedge_{I_n}$ on the category of MU-modules M. There is another construction of a completion at I_n which extends to all p-local spectra, and the two constructions agree when both are defined. We recall the other construction. For a sequence $\mathbf{i} = (i_0, i_1, \ldots, i_{n-1})$, we may attempt to construct generalized Toda-Smith spectra

$$M_{\mathbf{i}} = M(p^{i_0}, v_1^{i_1}, \ldots v_{n-1}^{i_{n-1}})$$

inductively, starting with S, continuing with the cofibre sequence

$$M(p^{i_0}) \longrightarrow S \xrightarrow{p^{i_0}} S,$$

and, given $L = M_{(i_0, i_1, \ldots, i_{n-2})}$, concluding with the cofibre sequence

$$M_{\mathbf{i}} \longrightarrow L \xrightarrow{v_{n-1}^{i_{n-1}}} L.$$

Here $M_{\mathbf{i}}$ is a finite complex of type n and hence admits a v_n-self map by the Nilpotence Theorem [5, 17, 19], and $v_n^{i_n}$ is shorthand for such a map. These spectra do not exist for all sequences $\mathbf{i}$, but they do exist for a cofinal set of sequences, and Devinatz has shown [4] that there is a cofinal collection all of which are ring spectra. These spectra are not determined by the sequence, but it follows from the Nilpotence Theorem that they are asymptotically unique in the sense that $\operatorname{hocolim}_{\mathbf{i}} M_{\mathbf{i}}$ is independent of all choices. Hence we may define a completion for all p-local spectra X by

$$X^\wedge_{I_n} = F(\operatorname{hocolim}_{\mathbf{i}} M_{\mathbf{i}}, X).$$

We shall denote the spectrum $\operatorname{hocolim}_{\mathbf{i}} M_{\mathbf{i}}$ by $\Gamma_{I_n}(S)$, although its construction is considerably more sophisticated than that of our local cohomology spectra.

Proposition 6.6. *Localize all spectra at p. Then there is an equivalence of MU-modules*

$$MU \wedge \Gamma_{I_n}(S) \simeq \Gamma_{I_n}(MU).$$

Therefore, for any MU-module M, there is an equivalence of MU-modules between the two completions $M^\wedge_{I_n}$.

Proof. [Sketch] It is proven in [8] that localization at p, and indeed any other Bousfield localization, preserves commutative S-algebras. The second statement follows from the first since

$$F_{MU}(MU \wedge \Gamma_{I_n}(S), M) \simeq F(\Gamma_{I_n}(S), M)$$

as MU-modules. It suffices to construct compatible equivalences

$$MU \wedge M_{\mathbf{i}} \simeq MU/p^{i_0} \wedge_{MU} MU/v_1^{i_1} \wedge_{MU} \dots \wedge_{MU} MU/v_{n-1}^{i_{n-1}}.$$

By [7, 9.9], the right side is equivalent to $MU/I_{\mathbf{i}}$, where $I_{\mathbf{i}} = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \subset I_n$. A v_n-self map $v : X \longrightarrow X$ on a type n finite complex X can be characterized as a map such that, for some i, $BP_*(v^i) : BP_*(X) \longrightarrow BP_*(X)$ is multiplication by v_n^j for some j. Since $MU_*(X) = MU_* \otimes_{BP_*} MU_*(X)$, we can use MU instead of BP. Using MU, we conclude that the two maps of spectra $\mathrm{id} \wedge v^i$ and $v_n^j \wedge \mathrm{id}$ from $MU \wedge X$ to itself induce the same map on homotopy groups. The cofibre of the first is $MU \wedge Cv^i$ and the cofibre of the second is $MU/(v_n^j) \wedge X$. In the case of our generalized Moore spectra, a nilpotence technology argument based on results in [19] shows that some powers of these two maps are homotopic, hence the cofibres of these powers are equivalent. The conclusion follows by induction. □

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The University of Sheffield, Sheffield, S3 7RH UK
E-mail address: j.greenlees@sheffield.ac.uk

The University of Chicago, Chicago, IL 60637 USA
E-mail address: may@math.uchicago.edu