DEFINITIONS: OPERADS, ALGEBRAS AND MODULES

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Let $\mathscr{S}$ be a symmetric monoidal category with product $\otimes$ and unit object κ.

Definition 1. An operad $\mathscr{C}$ in $\mathscr{S}$ consists of objects $\mathscr{C}(j)$, $j \geq 0$, a unit map $\eta : \kappa \to \mathscr{C}(1)$, a right action by the symmetric group Σ_j on $\mathscr{C}(j)$ for each j, and product maps

$$\gamma : \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \to \mathscr{C}(j)$$

for $k \geq 1$ and $j_s \geq 0$, where $\sum j_s = j$. The γ are required to be associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where $\sum j_s = j$ and $\sum i_t = i$; we set $g_s = j_1 + \cdots + j_s$, and $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{CD}
\mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(j_s)) \otimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) @>{\gamma\otimes \mathrm{id}}>> \mathscr{C}(j) \otimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) \\
@V{\text{shuffle}}VV @VV{\gamma}V \\
@. \mathscr{C}(i) \\
@. @AA{\gamma}A \\
\mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} (\mathscr{C}(j_s) \otimes (\bigotimes_{q=1}^{j_s} \mathscr{C}(i_{g_{s-1}+q})))) @>>{\mathrm{id}\otimes(\otimes_s \gamma)}> \mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(h_s)).
\end{CD}$$

(b) The following unit diagrams commute:

$$\begin{array}{ccc}
\mathscr{C}(k) \otimes (\kappa)^k & \xrightarrow{\cong} & \mathscr{C}(k) \\
{\scriptstyle \mathrm{id}\otimes \eta^k}\downarrow & \nearrow{\scriptstyle \gamma} & \\
\mathscr{C}(k) \otimes \mathscr{C}(1)^k & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\kappa \otimes \mathscr{C}(j) & \xrightarrow{\cong} & \mathscr{C}(j) \\
{\scriptstyle \eta\otimes \mathrm{id}}\downarrow & \nearrow{\scriptstyle \gamma} & \\
\mathscr{C}(1) \otimes \mathscr{C}(j). & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$, the permutation $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ permutes k blocks of letter as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ is the block sum:

$$\begin{CD}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) @>{\sigma\otimes\sigma^{-1}}>> \mathscr{C}(k) \otimes \mathscr{C}(j_{\sigma(1)}) \otimes \cdots \otimes \mathscr{C}(j_{\sigma(k)}) \\
@V{\gamma}VV @VV{\gamma}V \\
\mathscr{C}(j) @>{\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})}>> \mathscr{C}(j)
\end{CD}$$

and

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) & \xrightarrow{\mathrm{id}\otimes\tau_1\otimes\cdots\otimes\tau_k} & \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathscr{C}(j) & \xrightarrow{\tau_1\oplus\cdots\oplus\tau_k} & \mathscr{C}(j).
\end{array}$$

The $\mathscr{C}(j)$ are to be thought of as objects of parameters for "j-ary operations" that accept j inputs and produce one output. Thinking of elements as operations, we think of $\gamma(c\otimes d_1\otimes\cdots\otimes d_k)$ as the composite of the operation c with the $\otimes$-product of the operations d_s.

Let X^j denote the j-fold $\otimes$-power of an object X, with Σ_j acting on the left. By convention, $X^0 = \kappa$.

Definition 2. Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is an object A together with maps

$$\theta : \mathscr{C}(j)\otimes A^j \to A$$

for $j \geq 0$ that are associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where $j = \sum j_s$:

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k)\otimes A^j & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathscr{C}(j)\otimes A^j \\
 & & \downarrow{\scriptstyle\theta} \\
\downarrow{\scriptstyle\text{shuffle}} & & A \\
 & & \uparrow{\scriptstyle\theta} \\
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes A^{j_1}\otimes\cdots\otimes\mathscr{C}(j_k)\otimes A^{j_k} & \xrightarrow{\mathrm{id}\otimes\theta^k} & \mathscr{C}(k)\otimes A^k.
\end{array}$$

(b) The following unit diagram commutes:

$$\begin{array}{ccc}
\kappa\otimes A & \xrightarrow{\cong} & A \\
\downarrow{\scriptstyle\eta\otimes\mathrm{id}} & \nearrow{\scriptstyle\theta} & \\
\mathscr{C}(1)\otimes A. & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma\in\Sigma_j$:

$$\begin{array}{ccc}
\mathscr{C}(j)\otimes A^j & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}(j)\otimes A^j \\
{\scriptstyle\gamma}\searrow & & \swarrow{\scriptstyle\gamma} \\
 & A. &
\end{array}$$

Definition 3. Let $\mathscr{C}$ be an operad and A be a $\mathscr{C}$-algebra. An A-module is an object M together with maps

$$\lambda : \mathscr{C}(j)\otimes A^{j-1}\otimes M \to M$$

for $j \geq 1$ that are associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where $j = \sum j_s$:

$$\begin{array}{ccc}
(\mathscr{C}(k)\otimes(\bigotimes_{s=1}^{k}\mathscr{C}(j_s)))\otimes A^{j-1}\otimes M & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathscr{C}(j)\otimes A^{j-1}\otimes M \\
 & & \downarrow{\scriptstyle\lambda} \\
\downarrow{\scriptstyle\text{shuffle}} & & M \\
 & & \uparrow{\scriptstyle\lambda} \\
\mathscr{C}(k)\otimes(\bigotimes_{s=1}^{k-1}(\mathscr{C}(j_s)\otimes A^{j_s}))\otimes(\mathscr{C}(j_k)\otimes A^{j_k-1}\otimes M) & \xrightarrow[\mathrm{id}\otimes\theta^{k-1}\otimes\lambda]{} & \mathscr{C}(k)\otimes A^{k-1}\otimes M.
\end{array}$$

(b) The following unit diagram commutes:

$$\begin{array}{ccc}
\kappa\otimes M & \xrightarrow{\cong} & M \\
\downarrow{\scriptstyle\eta\otimes\mathrm{id}} & \nearrow{\scriptstyle\lambda} & \\
\mathscr{C}(1)\otimes M. & &
\end{array}$$

(c) The following equivariance diagram commutes, where $\sigma\in\Sigma_{j-1}\subset\Sigma_j$:

$$\begin{array}{ccc}
\mathscr{C}(j)\otimes A^{j-1}\otimes M & \xrightarrow{\sigma\otimes\sigma^{-1}\otimes\mathrm{id}} & \mathscr{C}(j)\otimes A^{j-1}\otimes M \\
{\scriptstyle\lambda}\searrow & & \swarrow{\scriptstyle\lambda} \\
 & M. &
\end{array}$$

Maps of operads, of algebras over an operad, and of modules over an algebra over an operad are defined in the evident ways: all structure must be preserved.

Variants 4. (i) *Non-Σ (or non-symmetric) operads.* When modelling non-commutative algebras, it is often useful to omit the permutations from the definition, giving the notion of a non-Σ operad. An operad is a non-Σ operad by neglect of structure.

(ii) *Unital operads.* The object $\mathscr{C}(0)$ parametrizes "0-ary operations". When concerned with unital algebras A, the unit "element" $1\in A$ is defined by a map $\kappa\to A$, and it is sensible to insist that $\mathscr{C}(0)=\kappa$. We then say that $\mathscr{C}$ is a unital operad. For types of algebras without units (e.g. Lie algebras) it is sensible to set $\mathscr{C}(0)=0$ (categorically, an initial object).

(iii) *Augmentations.* If $\mathscr{C}$ is unital, the $\mathscr{C}(j)$ have the "augmentations"

$$\epsilon=\gamma:\mathscr{C}(j)\cong\mathscr{C}(j)\otimes\mathscr{C}(0)^j\to\mathscr{C}(0)=\kappa$$

and the "degeneracy maps" $\sigma_i:\mathscr{C}(j)\to\mathscr{C}(j-1)$, $1\le i\le j$, given by the composites

$$\mathscr{C}(j)\cong\mathscr{C}(j)\otimes\kappa^j\longrightarrow\mathscr{C}(j)\otimes\mathscr{C}(1)^{i-1}\otimes\mathscr{C}(0)\otimes\mathscr{C}(1)^{j-i}\xrightarrow{\gamma}\mathscr{C}(j-1),$$

where the first map is determined by the unit map $\eta:\kappa\longrightarrow\mathscr{C}(1)$.

Example 5. Assume that $\mathscr{S}$ has an internal Hom functor. Define the endomorphism operad of an object X by

$$\mathrm{End}(X)(j)=\mathrm{Hom}(X^j,X).$$

The unit is given by the identity map $X \to X$, the right actions by symmetric groups are given by their left actions on $\otimes$-powers, and the maps γ are given by the following composites, where $\sum j_s = j$:

$$\begin{array}{c} \mathrm{Hom}(X^k, X) \otimes \mathrm{Hom}(X^{j_1}, X) \otimes \cdots \otimes \mathrm{Hom}(X^{j_k}, X) \\ \Big\downarrow \scriptstyle \mathrm{id} \otimes (k\text{-fold } \otimes\text{-product of maps}) \\ \mathrm{Hom}(X^k, X) \otimes \mathrm{Hom}(X^j, X^k) \\ \Big\downarrow \scriptstyle \text{composition} \\ \mathrm{Hom}(X^j, X). \end{array}$$

Conditions (a)-(c) of the definition of an operad are forced by direct calculation. In adjoint form, an action of $\mathscr{C}$ on A is a morphism of operads $\mathscr{C} \to \mathrm{End}(A)$, and conditions (a)-(c) of the definition of a $\mathscr{C}$-algebra are also forced by direct calculation.

Example 6. The operad $\mathscr{M}$ has $\mathscr{M}(j) = \kappa[\Sigma_j]$, the coproduct of a copy of κ for each element of Σ_j; the maps γ are determined by the formulas defining an operad. An $\mathscr{M}$-algebra A is a monoid in $\mathscr{S}$ and an A-module in the operadic sense is an A-bimodule in the classical sense of commuting left and right actions $A \otimes M \longrightarrow M$ and $M \otimes A \longrightarrow M$.

Example 7. The operad $\mathscr{N}$ has $\mathscr{N}(j) = \kappa$; the maps γ are canonical isomorphisms. An $\mathscr{N}$-algebra A is a commutative monoid in $\mathscr{S}$ and an A-module in the operadic sense is a left A-module in the classical sense. If we regard $\mathscr{N}$ as a non-Σ operad, then an $\mathscr{N}$-algebra A is a monoid in $\mathscr{S}$ and an A-module in the operadic sense is a left A-module in the classical sense. A unital operad $\mathscr{C}$ has the augmentation $\varepsilon : \mathscr{C} \longrightarrow \mathscr{N}$; an $\mathscr{N}$-algebra is a $\mathscr{C}$-algebra by pullback along ε.

There are important alternative formulations of some of the definitions. First, there is a conceptual reformulation of operads as monoids in a certain category of functors. Assume that $\mathscr{S}$ has finite colimits. These allow one to make sense of passage to orbits from group actions.

Definition 8. Let Σ denote the category whose objects are the finite sets $\mathbf{n} = \{1, \cdots, n\}$ and their isomorphisms, where $\mathbf{0}$ is the empty set. Define a Σ-object in $\mathscr{S}$ to be a contravariant functor $\mathscr{C} : \Sigma \longrightarrow \mathscr{S}$. Thus $\mathscr{C}(\mathbf{j})$ is an object of $\mathscr{S}$ with a right action by Σ_j; by convention, $\mathscr{C}(\mathbf{0}) = \kappa$. Define a product $\circ$ on the category of Σ-objects by setting

$$(\mathscr{B} \circ \mathscr{C})(\mathbf{j}) = \coprod_{k, j_1, \ldots, j_k} \mathscr{B}(\mathbf{k}) \otimes_{\kappa[\Sigma_k]} ((\mathscr{C}(\mathbf{j_1}) \otimes \cdots \otimes \mathscr{C}(\mathbf{j_k})) \otimes_{\kappa[\Sigma_{j_1 \times \cdots \times j_k}]} \kappa[\Sigma_j]),$$

where $k \geq 0$, $j_r \geq 0$, and $\sum j_r = j$. The implicit right action of $\kappa[\Sigma_{j_1 \times \cdots \times j_k}]$ on $\mathscr{C}(\mathbf{j_1}) \otimes \cdots \otimes \mathscr{C}(\mathbf{j_k})$ and left action of Σ_k on $(\mathscr{C}(\mathbf{j_1}) \otimes \cdots \otimes \mathscr{C}(\mathbf{j_k})) \otimes_{\kappa[\Sigma_{j_1 \times \cdots \times j_k}]} \kappa[\Sigma_j]$ should be clear from the equivariance formulas in the definition of an operad. The right action of Σ_j required of a contravariant functor is given by the right action of Σ_j on itself. The product $\circ$ is associative and has the two-sided unit I specified by $I(\mathbf{1}) = \kappa$ and $I(\mathbf{j}) = \phi$ (an initial object of $\mathscr{S}$) for $j \neq 1$.

A trivial inspection gives the following reformulation of the definition of an operad.

Lemma 9. *Operads in $\mathscr{S}$ are monoids in the monoidal category of Σ-objects in $\mathscr{S}$.*

Similarly, using the degeneracy maps σ_i of Variant 4(iii), if Λ denotes the category of finite sets $\mathbf{n}$ and all injective maps, then a unital operad is a monoid in the monoidal category of contravariant functors $\Lambda \longrightarrow \mathscr{S}$.

These observations are closely related to the comparison of algebras over operads to algebras over an associated monad that led me to invent the name "operad".

Definition 10. Define a functor $C : \mathscr{S} \longrightarrow \mathscr{S}$ associated to a Σ-object $\mathscr{C}$ by

$$CX = \coprod_{j \geq 0} \mathscr{C}(\mathbf{j}) \otimes_{\kappa[\Sigma_j]} X^j,$$

where $\mathscr{C}(\mathbf{0}) \otimes_{\kappa[\Sigma_0]} X^0 = \kappa$.

By inspection of definitions, the functor associated to $\mathscr{B} \circ \mathscr{C}$ is the composite BC of the functors B and C associated to $\mathscr{B}$ and $\mathscr{C}$. Therefore a monoid in the monoidal category of Σ-objects in $\mathscr{S}$ determines a monad (C, μ, η) in $\mathscr{S}$. This leads formally to the following result; it will be expanded in my paper "Operads, algebras, and modules" later in this volume (which gives background, details, and references for most of the material summarized here).

Proposition 11. *An operad $\mathscr{C}$ in $\mathscr{S}$ determines a monad C in $\mathscr{S}$ such that the categories of algebras over $\mathscr{C}$ and of algebras over C are isomorphic.*

There is also a combinatorial reformulation of the definition of operads that is expressed in terms of "$\circ_i$-products".

Definition 12. Let $\mathscr{C}$ be an operad in $\mathscr{S}$. Define the product

$$\circ_i : \mathscr{C}(p) \otimes \mathscr{C}(q) \longrightarrow \mathscr{C}(p+q-1)$$

to be the composite

$$\begin{array}{c} \mathscr{C}(p) \otimes \mathscr{C}(q) \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \eta^{i-1} \otimes \mathrm{id} \otimes \eta^{p-i}} \\ \mathscr{C}(p) \otimes \mathscr{C}(1)^{i-1} \otimes \mathscr{C}(q) \otimes \mathscr{C}(1)^{p-i} \\ \Big\downarrow {\scriptstyle \gamma} \\ \mathscr{C}(p+q-1). \end{array}$$

These products satisfy certain associativity, unity, and equivariance formulas that can be read off from the definition of an operad. Conversely, the structure maps γ can be read off in many different ways from the $\circ_i$-products. In fact, just the first one suffices. By use of the associativity and unity diagrams, we find that the following composite coincides with γ:

$$\begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \\ \Big\downarrow {\scriptstyle \circ_1 \otimes \mathrm{id}} \\ \mathscr{C}(k + j_1 - 1) \otimes \mathscr{C}(j_2) \otimes \cdots \otimes \mathscr{C}(j_k) \\ \Big\downarrow {\scriptstyle \circ_1 \otimes \mathrm{id}} \\ \cdot \end{array}$$

$$\begin{array}{c} \vdots \\ \Big\downarrow {\scriptstyle \circ_1 \otimes \mathrm{id}} \\ \mathscr{C}(k + j_1 + \cdots + j_{k-1} - (k-1)) \otimes \mathscr{C}(j_k) \\ \Big\downarrow {\scriptstyle \circ_1} \\ \mathscr{C}(j_1 + \cdots + j_k). \end{array}$$

We deduce that operads can be redefined in terms of $\circ_i$-products. This leads to another useful variant of the notion of an operad. If we are given Σ_j-objects $\mathscr{C}(j)$ for $j \geq 1$ and $\circ_i$-products that satisfy the associativity and equivariance laws, but not the unit laws, that are satisfied by the $\circ_i$ operations of an operad, we arrive at the notion of an "operad without identity" (analogous to a ring without identity). Such structures arise naturally in some applications related to string theory.

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