DEFINITIONS: OPERADS, ALGEBRAS AND MODULES

J. P. MAY

Let \( \mathcal{C} \) be a symmetric monoidal category with product \( \otimes \) and unit object \( \kappa \).

**Definition 1.** An operad \( \mathcal{C} \) in \( \mathcal{C} \) consists of objects \( \mathcal{C}(j), j \geq 0 \), a unit map \( \eta: \kappa \to \mathcal{C}(1) \), a right action by the symmetric group \( \Sigma_j \) on \( \mathcal{C}(j) \) for each \( j \), and product maps

\[
\gamma: \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j)
\]

for \( k \geq 1 \) and \( j_s \geq 0 \), where \( \sum j_s = j \). The \( \gamma \) are required to be associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where \( \sum j_s = j \) and \( \sum i_t = i \); we set \( g_s = j_1 + \cdots + j_s \), and \( h_s = i_{g_s-1+1} + \cdots + i_{g_s} \) for \( 1 \leq s \leq k \):

\[
\begin{array}{c}
\mathcal{C}(k) \otimes (\bigotimes_{s=1}^{j} \mathcal{C}(j_s)) \otimes (\bigotimes_{r=1}^{j} \mathcal{C}(i_r)) \\
\downarrow \text{shuffle} \\
\mathcal{C}(k) \otimes (\bigotimes_{s=1}^{j} \mathcal{C}(j_s)) \otimes (\bigotimes_{q=1}^{i} \mathcal{C}(i_{g_s+q})) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

(b) The following unit diagrams commute:

\[
\begin{array}{c}
\mathcal{C}(k) \otimes (\kappa)^k \\
\downarrow \text{id} \otimes \eta^k \\
\mathcal{C}(k) \otimes \mathcal{C}(1)^k \\
\downarrow \gamma \\
\mathcal{C}(k)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \\
\downarrow \eta \otimes \text{id} \\
\mathcal{C}(1) \otimes \mathcal{C}(j) \\
\downarrow \gamma \\
\mathcal{C}(1)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \\
\downarrow \text{id} \otimes \eta \\
\mathcal{C}(k) \otimes \mathcal{C}(1) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(1) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

(c) The following equivariance diagrams commute, where \( \sigma \in \Sigma_k, \tau_s \in \Sigma_j \), the permutation \( \sigma(j_1, \ldots, j_k) \in \Sigma_j \) permutes \( k \) blocks of letter as \( \sigma \) permutes \( k \) letters, and \( \tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j \) is the block sum:

\[
\begin{array}{c}
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \otimes \mathcal{C}(j_{\sigma(1)} \cdots j_{\sigma(k)}) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \otimes \mathcal{C}(j_{\sigma(1)} \cdots j_{\sigma(k)}) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}(k) \otimes \mathcal{C}(j_{\sigma(1)} \cdots j_{\sigma(k)}) \\
\downarrow \gamma \\
\mathcal{C}(j)
\end{array}
\]
and
\[
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \xrightarrow{\text{id} \otimes \tau_1 \otimes \cdots \otimes \tau_k} \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \xrightarrow{\gamma} \mathcal{C}(j).
\]

The \( \mathcal{C}(j) \) are to be thought of as objects of parameters for "\( j \)-ary operations" that accept \( j \) inputs and produce one output. Thinking of elements as operations, we think of \( \gamma(c \otimes d_1 \otimes \cdots \otimes d_k) \) as the composite of the operation \( c \) with the \( \otimes \)-product of the operations \( d_s \).

Let \( X^j \) denote the \( j \)-fold \( \otimes \)-power of an object \( X \), with \( \Sigma_j \) acting on the left.

By convention, \( X^0 = \kappa \).

**Definition 2.** Let \( \mathcal{C} \) be an operad. A \( \mathcal{C} \)-algebra is an object \( A \) together with maps
\[
\theta : \mathcal{C}(j) \otimes A^j \rightarrow A
\]
for \( j \geq 0 \) that are associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where \( j = \sum j_s \):
\[
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \otimes A^j \xrightarrow{\gamma \otimes \text{id}} \mathcal{C}(j) \otimes A^j \xrightarrow{\theta} A
\]
\[
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes A^{j_1} \otimes \cdots \otimes \mathcal{C}(j_k) \otimes A^{j_k} \xrightarrow{\text{id} \otimes \theta^k} \mathcal{C}(k) \otimes A^k.
\]

(b) The following unit diagram commutes:
\[
\kappa \otimes A \xrightarrow{\cong} A
\]
\[
\mathcal{C}(1) \otimes A.
\]

(c) The following equivariance diagrams commute, where \( \sigma \in \Sigma_j \):
\[
\mathcal{C}(j) \otimes A^j \xrightarrow{\sigma \otimes \sigma^{-1}} \mathcal{C}(j) \otimes A^j \xrightarrow{\gamma} A.
\]

**Definition 3.** Let \( \mathcal{C} \) be an operad and \( A \) be a \( \mathcal{C} \)-algebra. An \( A \)-module is an object \( M \) together with maps
\[
\lambda : \mathcal{C}(j) \otimes A^{j-1} \otimes M \rightarrow M
\]
for \( j \geq 1 \) that are associative, unital, and equivariant in the following senses.

(a) The following associativity diagrams commute, where \( j = \sum j_s \):
\[
\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \xrightarrow{\text{id} \otimes \tau_1 \otimes \cdots \otimes \tau_k} \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \xrightarrow{\gamma} \mathcal{C}(j).
\]
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\[(\mathcal{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{C}(j_s))) \otimes A^{j-1} \otimes M \xrightarrow{\gamma \otimes \text{id}} \mathcal{C}(j) \otimes A^{j-1} \otimes M\]

(b) The following unit diagram commutes:

\[\kappa \otimes M \xrightarrow{\cong} M \]

\[\eta \otimes \text{id} \quad \lambda\]

\[\mathcal{C}(1) \otimes M.\]

(c) The following equivariance diagram commutes, where \(\sigma \in \Sigma_{j-1} \subset \Sigma_j\):

\[\mathcal{C}(j) \otimes A^{j-1} \otimes M \xrightarrow{\sigma \otimes \sigma^{-1} \otimes \text{id}} \mathcal{C}(j) \otimes A^{j-1} \otimes M\]

\[\lambda \quad \lambda\]

\[M.\]

Maps of operads, of algebras over an operad, and of modules over an algebra over an operad are defined in the evident ways: all structure must be preserved.

**Variants 4.**

(i) **Non-\(\Sigma\) (or non-symmetric) operads.** When modelling non-commutative algebras, it is often useful to omit the permutations from the definition, giving the notion of a non-\(\Sigma\) operad. An operad is a non-\(\Sigma\) operad by neglect of structure.

(ii) **Unital operads.** The object \(\mathcal{C}(0)\) parametrizes “0-ary operations”. When concerned with unital algebras \(A\), the unit “element” \(1 \in A\) is defined by a map \(\kappa \to A\), and it is sensible to insist that \(\mathcal{C}(0) = \kappa\). We then say that \(\mathcal{C}\) is a unital operad. For types of algebras without units (e.g. Lie algebras) it is sensible to set \(\mathcal{C}(0) = 0\) (categorically, an initial object).

(iii) **Augmentations.** If \(\mathcal{C}\) is unital, the \(\mathcal{C}(j)\) have the “augmentations”

\[\epsilon = \gamma : \mathcal{C}(j) \cong \mathcal{C}(j) \otimes \mathcal{C}(0)^j \to \mathcal{C}(0) = \kappa\]

and the “degeneracy maps” \(\sigma_i : \mathcal{C}(j) \to \mathcal{C}(j-1), 1 \leq i \leq j\), given by the composites

\[\mathcal{C}(j) \cong \mathcal{C}(j) \otimes \kappa^j \to \mathcal{C}(j) \otimes \mathcal{C}(1)^{j-1} \otimes \mathcal{C}(0) \otimes \mathcal{C}(1)^{j-2} \mathcal{C}(j-1),\]

where the first map is determined by the unit map \(\eta : \kappa \to \mathcal{C}(1)\).

**Example 5.** Assume that \(\mathcal{F}\) has an internal Hom functor. Define the endomorphism operad of an object \(X\) by

\[\text{End}(X)(j) = \text{Hom}(X^j, X).\]
The unit is given by the identity map $X \rightarrow X$, the right actions by symmetric groups are given by their left actions on $\otimes$-powers, and the maps $\gamma$ are given by the following composites, where $\sum j_s = j$:

$$\begin{align*}
\text{Hom}(X^k, X) \otimes \text{Hom}(X^j, X) & \otimes \cdots \otimes \text{Hom}(X^j, X) \\
\downarrow \text{id} \otimes (k\text{-fold} \otimes \text{-product of maps}) & \\
\text{Hom}(X^k, X) \otimes \text{Hom}(X^j, X^k) & \\
\downarrow \text{composition} & \\
\text{Hom}(X^j, X).
\end{align*}$$

Conditions (a)-(c) of the definition of an operad are forced by direct calculation. In adjoint form, an action of $\mathcal{C}$ on $A$ is a morphism of operads $\mathcal{C} \rightarrow \text{End}(A)$, and conditions (a)-(c) of the definition of a $\mathcal{C}$-algebra are also forced by direct calculation.

**Example 6.** The operad $\mathcal{M}$ has $\mathcal{M}(j) = \kappa[\Sigma_j]$, the coproduct of a copy of $\kappa$ for each element of $\Sigma_j$; the maps $\gamma$ are determined by the formulas defining an operad. An $\mathcal{M}$-algebra $A$ is a monoid in $\mathcal{F}$ and an $A$-module in the operadic sense is an $A$-bimodule in the classical sense of commuting left and right actions $A \otimes M \rightarrow M$ and $M \otimes A \rightarrow M$.

**Example 7.** The operad $\mathcal{N}$ has $\mathcal{N}(j) = \kappa$; the maps $\gamma$ are canonical isomorphisms. An $\mathcal{N}$-algebra $A$ is a commutative monoid in $\mathcal{F}$ and an $A$-module in the operadic sense is a left $A$-module in the classical sense. If we regard $\mathcal{N}$ as a non-$\Sigma$ operad, then an $\mathcal{N}$-algebra $A$ is a monoid in $\mathcal{F}$ and an $A$-module in the operadic sense is a left $A$-module in the classical sense. A unital operad $\mathcal{C}$ has the augmentation $\varepsilon : \mathcal{C} \rightarrow \mathcal{N}$; an $\mathcal{N}$-algebra is a $\mathcal{C}$-algebra by pullback along $\varepsilon$.

There are important alternative formulations of some of the definitions. First, there is a conceptual reformulation of operads as monoids in a certain category of functors. Assume that $\mathcal{F}$ has finite colimits. These allow one to make sense of passage to orbits from group actions.

**Definition 8.** Let $\Sigma$ denote the category whose objects are the finite sets $n = \{1, \ldots, n\}$ and their isomorphisms, where $0$ is the empty set. Define a $\Sigma$-object in $\mathcal{F}$ to be a contravariant functor $\mathcal{F} : \Sigma \rightarrow \mathcal{F}$. Thus $\mathcal{F}(j)$ is an object of $\mathcal{F}$ with a right action by $\Sigma_j$; by convention, $\mathcal{F}(0) = \kappa$. Define a product $\circ$ on the category of $\Sigma$-objects by setting

$$(\mathcal{B} \circ \mathcal{C})(j) = \prod_{k, j_1, \ldots, j_k} \mathcal{B}(k) \otimes_{\kappa[\Sigma_k]} (\mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k)) \otimes_{\kappa[\Sigma_{j_1} \times \cdots \times j_k]} \kappa[\Sigma_j],$$

where $k \geq 0$, $j_r \geq 0$, and $\sum j_r = j$. The implicit right action of $\kappa[\Sigma_{j_1} \times \cdots \times j_k]$ on $\mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k)$ and left action of $\Sigma_k$ on $(\mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k)) \otimes_{\kappa[\Sigma_{j_1} \times \cdots \times j_k]} \kappa[\Sigma_j]$ should be clear from the equivariance formulas in the definition of an operad. The right action of $\Sigma_j$ required of a contravariant functor is given by the right action of $\Sigma_j$ on itself. The product $\circ$ is associative and has the two-sided unit $I$ specified by $I(1) = \kappa$ and $I(j) = \phi$ (an initial object of $\mathcal{F}$) for $j \neq 1$.

A trivial inspection gives the following reformulation of the definition of an operad.
Lemma 9. Operads in \( \mathcal{I} \) are monoids in the monoidal category of \( \Sigma \)-objects in \( \mathcal{I} \).

Similarly, using the degeneracy maps \( \sigma_i \) of Variant 4(iii), if \( \Lambda \) denotes the category of finite sets \( n \) and all injective maps, then a unital operad is a monoid in the monoidal category of contravariant functors \( \Lambda \to \mathcal{I} \).

These observations are closely related to the comparison of algebras over operads to algebras over an associated monad that led me to invent the name “operad”.

Definition 10. Define a functor \( C : \mathcal{I} \to \mathcal{I} \) associated to a \( \Sigma \)-object \( C \) by

\[
CX = \prod_{j \geq 0} C(j) \otimes_{\kappa[\Sigma_j]} X^j,
\]

where \( C(0) \otimes_{\kappa[\Sigma_0]} X^0 = \kappa \).

By inspection of definitions, the functor associated to \( B \circ C \) is the composite \( BC \) of the functors \( B \) and \( C \) associated to \( B \) and \( C \). Therefore a monoid in the monoidal category of \( \Sigma \)-objects in \( \mathcal{I} \) determines a monad \( (C, \mu, \eta) \) in \( \mathcal{I} \). This leads formally to the following result; it will be expanded in my paper “Operads, algebras, and modules” later in this volume (which gives background, details, and references for most of the material summarized here).

Proposition 11. An operad \( C \) in \( \mathcal{I} \) determines a monad \( C \) in \( \mathcal{I} \) such that the categories of algebras over \( C \) and of algebras over \( C \) are isomorphic.

There is also a combinatorial reformulation of the definition of operads that is expressed in terms of “\( \circ_i \)-products”.

Definition 12. Let \( C \) be an operad in \( \mathcal{I} \). Define the product

\[
\circ_i : C(p) \otimes C(q) \to C(p + q - 1)
\]
to be the composite

\[
\begin{array}{ccc}
C(p) \otimes C(q) & \xrightarrow{id \otimes \eta^{-1} \otimes id \otimes \eta^{-1}} & C(p) \otimes C(1)^{i-1} \otimes C(q) \otimes C(1)^{p-i} \\
& \downarrow \gamma & \downarrow \\
& C(p + q - 1). & \\
\end{array}
\]

These products satisfy certain associativity, unity, and equivariance formulas that can be read off from the definition of an operad. Conversely, the structure maps \( \gamma \) can be read off in many different ways from the \( \circ_i \)-products. In fact, just the first one suffices. By use of the associativity and unity diagrams, we find that the following composite coincides with \( \gamma \):

\[
\begin{array}{ccc}
C(k) \otimes C(j_1) \otimes \cdots \otimes C(j_k) & \xrightarrow{\circ_1 \otimes id} & C(k + j_1 - 1) \otimes C(j_2) \otimes \cdots \otimes C(j_k) \\
& \downarrow \circ_1 \otimes id & \\
& & \\
\end{array}
\]
We deduce that operads can be redefined in terms of $\circ_i$-products. This leads to another useful variant of the notion of an operad. If we are given $\Sigma_j$-objects $\mathcal{C}(j)$ for $j \geq 1$ and $\circ_i$-products that satisfy the associativity and equivariance laws, but not the unit laws, that are satisfied by the $\circ_i$ operations of an operad, we arrive at the notion of an “operad without identity” (analogous to a ring without identity). Such structures arise naturally in some applications related to string theory.

The University of Chicago, Chicago, IL 60637

E-mail address: may@math.uchicago.edu