OPERADS, ALGEBRAS AND MODULES

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There are many different types of algebra: associative, associative and commutative, Lie, Poisson, etc., etc. Each comes with an appropriate notion of a module. As is becoming more and more important in a variety of fields, it is often necessary to deal with algebras and modules of these sorts "up to homotopy". I shall give a very partial overview, concentrating on algebra, but saying a little about the original use of operads in topology.

The development of abstract frameworks in which to study such algebras has a long history. As this conference attests, it now seems to be widely accepted that, for many purposes, the most convenient setting is that given by operads and their actions. While the notion was first written up in a purely topological framework [19], it was thoroughly understood by 1971 [12] that the basic definitions apply equally well in any underlying symmetric monoidal (= tensor) category.

The definitions and ideas had many precursors. I will indicate those that I was aware of at the time.

- Algebraists such as Kaplansky, Herstein, and Jacobson systematically studied algebras defined by different kinds of identities.
- Lawvere [15] formalized algebraic theories as a way of codifying different kinds of algebraic structures.
- Adams and MacLane [16] developed certain chain level concepts, PROP's and PACT's, with a view towards understanding the algebraic structure of the singular chain complex.
- Stasheff [23] introduced A_∞ spaces and constructed their classifying spaces, using associahedra and what in retrospect was an example of an operad.
- Milgram [22] proved an approximation theorem for iterated loop spaces, using what are now known as permutahedra.
- Boardman and Vogt [3] switched from algebraic to topological PROP's and used PROP's to prove a recognition principle in infinite loop space theory.
- Beck [2] pointed out the relevance of monads to infinite loop space theory.
- Dyer and Lashof [6] systematized homology operations for iterated loop spaces as analogs of Steenrod operations in the cohomology of spaces.

It was my extraordinary good fortune to have had close mathematical contact with all of the people whom I have mentioned. I learned from Steenrod at Princeton and from Jacobson at Yale. Remarkably, all of the rest were at Chicago, or were there when the relevant work was done, or were regular visitors there.

I wanted a notion that carried the combinatorial structure familiar to me from a paper that I had written [18] that gave a general algebraic approach to Steenrod operations. The diagrams in the definition of an operad action are generalizations of diagrams that were used there to prove the Cartan formula and Adem relations. I wanted the notion to be so intimately related to monads that one could easily go back and forth between locally and globally defined structures. I had a view towards using monads to obtain a new approximation theorem for iterated loop spaces and towards using this approximation theorem together with a monadic bar construction to prove a new recognition principle for infinite loop spaces.

To these ends, I consciously sacrificed all-embracing generality: many types of algebras defined by identities are deliberately excluded. The name "operad" is a word that I coined myself, spending a week thinking about nothing else. Besides having a nice ring to it, the name is meant to bring to mind both operations and monads. Incidentally, I persuaded MacLane to discard the term "triple" in favor of "monad" in his book "Categories for the working mathematician" [17][1], which was being written about the same time. I was convinced that the notion of an operad was an important one, and I wanted the names to mesh.

What I did not foresee was just how flexible the notion would be, how many essentially different mathematical contexts there are in which it would play a natural role, how many philosophically different ways it could be exploited. Things have gone so far that I feel quite incompetent to give a thoroughgoing survey. It would be a little like surveying the use of groups in mathematics. Maybe I will feel a little more competent by the end of the conference.

1. The definitions of operads, algebras, and modules

To work. Let $\mathscr{S}$ be any symmetric monoidal category, with product $\otimes$ and unit object κ. In this talk, I will focus on algebraic examples and their relationship with some basic examples in topology, but there is no point in specializing the general definitions. Details are given in the handout[2]. While there are perhaps more elegant equivalent ways of writing them down, my original explicit versions of the definitions still seem to be the most convenient, especially for concrete calculational purposes.

An operad $\mathscr{C}$ consists of objects $\mathscr{C}(j)$ with actions by the symmetric groups Σ_j, a unit map $\eta : k \longrightarrow \mathscr{C}(1)$ and product maps

$$\gamma : \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \to \mathscr{C}(j)$$

that are suitably associative, unital, and equivariant. The $\mathscr{C}(j)$ are thought of as parameter objects for j-ary operations that accept j inputs and produce one output.

A $\mathscr{C}$-algebra A is an object together with Σ_j-equivariant maps

$$\mathscr{C}(j) \otimes A^j \to A$$

that are suitably associative, unital, and equivariant, where A^j denotes the j-fold $\otimes$-power of A with $A^0 = \kappa$.

An A-module M is an object M together with Σ_{j-1}-equivariant maps

$$\mathscr{C}(j) \otimes A^{j-1} \otimes M \to M$$

that are suitably associative, unital, and equivariant.

These are the non-negotiable ingredients, but there are variants. If we drop all reference to symmetric groups, we obtain "non-Σ" versions of the concepts. If

[1]To quote MacLane (op cit, p. 134), "The frequent but unfortunate use of the word 'triple' in this sense has achieved a maximum of needless confusion, what with the conflict with ordered triple, plus the use of associated terms such as 'triple derived functors' for functors which are not three times derived from anything in the world. Hence the term *monad*."

[2]Reproduced in expanded form earlier in this volume.

we insist that $\mathscr{C}(0) = \kappa$, then we call $\mathscr{C}$ a "unital operad". The resulting map $\kappa = \mathscr{C}(0) \longrightarrow A$ encodes the unit elements of algebras over such operads (and is not to be confused with the unit map $\eta : \kappa \longrightarrow \mathscr{C}(1)$, which encodes the identity operation present on any kind of algebra). Unital operads come with "augmentation maps"

$$\varepsilon : \mathscr{C}(j) \longrightarrow \kappa$$

and "degeneracy maps"

$$\mathscr{C}(j) \longrightarrow \mathscr{C}(j-1), \quad 1 \leq i \leq j.$$

Thinking of elements of the $\mathscr{C}(j)$ as operations, we think of $\gamma(c \otimes d_1 \otimes \cdots \otimes d_k)$ as the composite of the operation c with the tensor product of the operations d_s. When $\mathscr{S}$ has an internal Hom functor, so that

$$\mathscr{S}(X \otimes Y, Z) \cong \mathscr{S}(X, \mathrm{Hom}(Y, Z)),$$

this is made precise by the endomorphism operad $\mathrm{End}(X)$ of an object X. We set

$$\mathrm{End}(X)(j) = \mathrm{Hom}(X^j, X).$$

The unit is given by the identity map $X \to X$, the right actions by symmetric groups are given by their left actions on tensor powers, and the maps γ are given by composition of tensor products of maps. The identities that define an operad are then forced by direct calculation. An action of $\mathscr{C}$ on A can be redefined in adjoint form as a morphism of operads $\mathscr{C} \to \mathrm{End}(A)$. The identities that define a $\mathscr{C}$-algebra are then also forced by direct calculation.

There is a notion of a monoid and of a commutative monoid in any symmetric monoidal category $\mathscr{S}$. These are objects A with unit maps $\kappa \longrightarrow A$ and product maps $A \otimes A \longrightarrow A$ such that the obvious diagrams commute. There are also notions of left A-modules, which are defined in terms of maps $A \otimes M \longrightarrow M$, and of A-bimodules. These classical notions are encoded in operad actions. Assume that $\mathscr{S}$ has finite coproducts; for a finite set S, let $\kappa[S]$ be the coproduct of a copy of κ for each element of S.

Example 1.1. The unital operad $\mathscr{M}$ in $\mathscr{S}$ has $\mathscr{M}(j) = \kappa[\Sigma_j]$. The unit map η is the identity and the product maps γ are dictated by the equivariance formulas. An $\mathscr{M}$-algebra A is the same thing as a monoid in $\mathscr{S}$. The operad action encodes all of the iterates and permutations of the product on the monoid. In terms of elements,

$$\theta(\sigma \otimes a_1 \otimes \cdots \otimes a_j) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(j)}.$$

An A-module M in the operadic sense is an A-bimodule in the classical sense. In terms of elements, given an operad action λ, we define

$$am = \lambda(e \otimes a \otimes m) \quad \text{and} \quad ma = \lambda(\tau \otimes a \otimes m),$$

where e and τ are the identity and transposition in Σ_2. Conversely, given an A-bimodule M, we define

$$\lambda(\sigma \otimes a_1 \otimes \cdots \otimes a_j) = a_{\sigma(1)} \cdots a_{\sigma(j)},$$

where $\sigma \in \Sigma_j$, $a_i \in A$ for $1 \leq i < j$ and $a_j \in M$.

Example 1.2. The unital operad $\mathscr{N}$ has $\mathscr{N}(j) = \kappa$ for all j. The Σ_j-actions are trivial, the unit map η is the identity, and the maps γ are the evident identifications. An $\mathscr{N}$-algebra A is the same thing as a commutative monoid in $\mathscr{S}$ and an A-module M in the operadic sense is the same thing as a left A-module. We may regard $\mathscr{N}$

as a non-Σ operad. In the non-Σ sense, an $\mathscr{N}$-algebra A is a monoid in $\mathscr{S}$, and an A-module is a left A-module in the classical sense. For any unital operad $\mathscr{C}$, the augmentations give a map $\epsilon : \mathscr{C} \to \mathscr{N}$ of operads. Therefore, by pullback along ϵ, an $\mathscr{N}$-algebra may be viewed as a $\mathscr{C}$-algebra.

2. Monadic reinterpretation of algebras

We recall some standard categorical definitions.

Definition 2.1. Let $\mathscr{S}$ be any category. A monad in $\mathscr{S}$ is a functor $C : \mathscr{S} \to \mathscr{S}$ together with natural transformations $\mu : CC \to C$ and $\eta : \mathrm{Id} \to C$ such that the following diagrams commute:

$$\begin{array}{ccccc} C & \xrightarrow{\eta C} & CC & \xleftarrow{C\eta} & C \\ & {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}} & \\ & & C & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} CCC & \xrightarrow{C\mu} & CC \\ {\scriptstyle \mu C}\downarrow & & \downarrow{\scriptstyle \mu} \\ CC & \xrightarrow{\mu} & C. \end{array}$$

A C-algebra is an object A of $\mathscr{S}$ together with a map $\xi : CA \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta} & CA \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \xi} \\ & & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} CCA & \xrightarrow{C\xi} & CA \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \xi} \\ CA & \xrightarrow{\xi} & A. \end{array}$$

Taking $\xi = \mu$, we see that CX is a C-algebra for any $X \in \mathscr{S}$. It is the free C-algebra generated by X. That is, for C-algebras A, restriction along $\eta : X \to CX$ gives an adjunction isomorphism

$$C[\mathscr{S}](CX, A) \cong \mathscr{S}(X, A),$$

where $C[\mathscr{S}]$ is the category of C-algebras. Formally, we are viewing C as a functor $\mathscr{S} \to C[\mathscr{S}]$ whose composite with the forgetful functor is our original monad. Thus the monad C is determined by its algebras. Quite generally, every pair $L : \mathscr{S} \to \mathscr{T}$ and $R : \mathscr{T} \to \mathscr{S}$ of left and right adjoints determines a monad RL on $\mathscr{S}$, but many different pairs of adjoint functors can define the same monad.

Now return to our symmetric monoidal category $\mathscr{S}$ and assume that it is cocomplete. We have the following construction of the monad of free algebras over an operad $\mathscr{C}$.

Definition 2.2. Define the monad C associated to an operad $\mathscr{C}$ by letting

$$CX = \coprod_{j \geq 0} \mathscr{C}(j) \otimes_{\kappa[\Sigma_j]} X^j.$$

The unit $\eta : X \to CX$ is $\eta \otimes \mathrm{id} : X = \kappa \otimes X \to \mathscr{C}(1) \otimes X$. The product $\mu : CCX \to CX$ is induced by the following maps, where $j = \sum j_s$:

$$\begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_k} \\ \downarrow \text{shuffle} \\ \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^j \\ \downarrow \gamma\otimes\mathrm{id} \\ \mathscr{C}(j) \otimes X^j. \end{array}$$

Proposition 2.3. *A $\mathscr{C}$-algebra structure on an object A determines and is determined by a C-algebra structure on A. Formally, the identity functor on $\mathscr{S}$ restricts to give an isomorphism between the categories of $\mathscr{C}$-algebras and of C-algebras.*

Proof. Maps $\theta_j : \mathscr{C}(j) \otimes_{\kappa[\Sigma_j]} A^j \to A$ that together specify an action of $\mathscr{C}$ on A are the same as a map $\xi : CA \to A$ that specifies an action of C on A. □

Not all monads come from operads. Rather, operads single out a particularly convenient, algebraically manageable, collection of monads.

For the operad $\mathscr{M}$, the free algebra MX is the free monoid in $\mathscr{S}$ generated by X. For the operad $\mathscr{N}$, the free algebra NX is the free associative and commutative algebra generated by X.

Now suppose that $\mathscr{C}$ is a unital operad. In this case, there is a monad that is different from that defined above but that nevertheless has essentially the same algebras. Since $\mathscr{C}$ is unital, a $\mathscr{C}$-algebra A comes with a unit $\eta \equiv \theta_0 : \kappa \to A$. Thinking of η as preassigned, it is natural to change ground categories to the category of objects under κ. Working in this ground category, we obtain a reduced monad $\tilde{C}$. This monad is so defined that the units of algebras that are built in by the θ_0 component of operad actions coincide with the preassigned units η. Formally, for an object X with unit $\eta : \kappa \longrightarrow X$, the reduced monad $\tilde{C}X$ is obtained from CX as an appropriate coequalizer. Informally, elementwise, we identify $\sigma_i(c) \otimes y$ with $c \otimes s_i y$, where $c \in \mathscr{C}(j)$, $y \in X^j$, and

$$s_i = \mathrm{id}^{i-1} \otimes \eta \otimes \mathrm{id}^{j-i} : X^{j-1} \longrightarrow X^j.$$

Proposition 2.4. *Let $\mathscr{C}$ be a unital operad. Then a $\mathscr{C}$-algebra structure satisfying $\eta = \theta_0$ on an object of $\mathscr{S}$ under κ determines and is determined by a $\tilde{C}$-algebra structure on A.*

The reduced construction is more general than the unreduced one.

Lemma 2.5. *For an object X, $CX \cong \tilde{C}(X \coprod \kappa)$ as $\mathscr{C}$-algebras.*

In algebraic contexts, there is not much difference between the two constructions. There, $\mathscr{S}$ will be the category of modules over a commutative ground ring k. If X has an augmentation $\varepsilon : X \longrightarrow k$ such that $\varepsilon\eta = \mathrm{id}$, then $X \cong \mathrm{Ker}\,\varepsilon \oplus k$ and therefore $\tilde{C}X \cong C(\mathrm{Ker}\,\varepsilon)$. In topological contexts, nothing like this works and there is a huge difference between the two constructions, with the reduced construction being by far the more important one.

Modules also admit a monadic reinterpretation: there is a monad $C[1]$ in the category $\mathscr{S}^2$ such that a $C[1]$-algebra (A, M) is a C-algebra A together with an

A-module M. There is a free A-module functor F_A for any $\mathscr{C}$-algebra A, and $C[1](X;Y)$ is the pair $(CX, F_{CX}(Y))$. This construction also has a reduced variant. See [14, I§4].

3. The specialization to algebraic operads

Let k be a commutative ring and write $\otimes$ and Hom for $\otimes_k$ and Hom_k. We shall work in the tensor category $\mathscr{M}_k$ of $\mathbb{Z}$-graded differential graded k-modules, with differential decreasing degrees by one. We implicitly use the standard convention that a sign $(-1)^{pq}$ is to be inserted whenever an element of degree p is permuted past an element of degree q. If one prefers the opposite grading convention, one can reindex chain complexes C_* by setting $C^n = C_{-n}$.

I will refer to $\mathbb{Z}$-graded chain complexes over k simply as "k-modules". As usual, we consider graded k-modules without differential to be k-modules with differential zero, and we view ungraded k-modules as graded k-modules concentrated in degree 0. These conventions allow us to view the theory of generalized algebras as a special case of the theory of differential graded generalized algebras.

When the differentials on the $\mathscr{C}(j)$ are zero, we think of $\mathscr{C}$ as purely algebraic, and it then determines an appropriate class of (differential) algebras. When the differentials on the $\mathscr{C}(j)$ are non-zero, $\mathscr{C}$ determines a class of (differential) algebras "up to homotopy", where the homotopies are determined by the homological properties of the $\mathscr{C}(j)$. Recall that a map of k-modules is said to be a quasi-isomorphism if it induces an isomorphism of homology groups.

Definition 3.1. Let $\mathscr{C}$ be a unital operad with each $\mathscr{C}(j)_n = 0$ for $n < 0$. We say that $\mathscr{C}$ is acyclic if its augmentations are quasi-isomorphisms. We say that $\mathscr{C}$ is Σ-free if $\mathscr{C}(j)$ is $k[\Sigma_j]$-free for each j. We say that $\mathscr{C}$ is an E_∞ operad if it is both acyclic and Σ-free; $\mathscr{C}(j)$ is then a $k[\Sigma_j]$-free resolution of k.

By an E_∞ algebra, we mean a $\mathscr{C}$-algebra for any E_∞ operad $\mathscr{C}$. These were called "May algebras" when they were introduced by Hinich and Schechtman [10]. If we ignore symmetric groups, we obtain the notion of an A_∞ algebra. These are commutative and non-commutative differential graded algebras up to homotopy. Similarly, there is a class of operads that is related to Lie algebras as E_∞ operads are related to commutative algebras, and there is a concomitant notion of a differential graded Lie algebra "up to homotopy," or L_∞ algebra. Hinich and Schechtman [11] called these "Lie May algebras".

An E_∞ algebra A has a product for each degree zero element, necessarily a cycle, of $\mathscr{C}(2)$. Each such product is unital, associative, and commutative up to all possible coherence homotopies, and all such products are homotopic. There is a long history in topology and category theory that makes precise what these "coherence homotopies" are. However, since the homotopies are all encoded in the operad action, there is no need to be explicit. In the last lecture of the conference, I will explain a beautiful new way of thinking about A_∞ and E_∞ algebras which hides the operads in an "operadic tensor product". We will there focus on one particular E_∞ operad, but all such operads give equivalent classes of A_∞ and E_∞ algebras.

One can treat operads as algebraic systems to which one can apply versions of classical algebraic constructions. An ideal $\mathscr{I}$ in an operad $\mathscr{C}$ consists of a sequence of sub $\kappa[\Sigma_j]$-modules $\mathscr{I}(j)$ of $\mathscr{C}(j)$ such that $\gamma(c \otimes d_1 \otimes \cdots \otimes d_k)$ is in $\mathscr{I}$ if either c or any of the d_s is in $\mathscr{I}$. There is then a quotient operad $\mathscr{C}/\mathscr{I}$ with j^{th} k-module

$\mathscr{C}(j)/\mathscr{I}(j)$. As observed by Ginzburg and Kapranov [9], one can construct the free operad $\mathscr{F}\mathscr{G}$ generated by any sequence $\mathscr{G} = \{\mathscr{G}(j)\}$ of $\kappa[\Sigma_j]$-modules, and one can then construct an operad that describes a particular type of algebra by quotienting out by the ideal generated by an appropriate sequence $\mathscr{R} = \{\mathscr{R}(j)\}$ of defining relations, where $\mathscr{R}(j)$ is a sub $\kappa[\Sigma_j]$-module of $(\mathscr{F}\mathscr{G})(j)$. Actually, there are two variants of the construction, one unital and one non-unital.

In many familiar examples, called quadratic operads by Ginzburg and Kapranov [9], $\mathscr{G}(j) = 0$ for $j \neq 2$ and $\mathscr{R}(j) = 0$ for $j \neq 3$. Here, if $\mathscr{G}(2)$ is $k[\Sigma_2]$ and $\mathscr{R}(3) = 0$, this reconstructs $\mathscr{M}$. If $\mathscr{G}(2) = k$ with trivial Σ_2-action and $\mathscr{R}(3) = 0$, this reconstructs $\mathscr{N}$. In these cases, we use the unital variant. If k is a field of characteristic other than 2 or 3, we can use the non-unital variant to construct an operad $\mathscr{L}$ whose algebras are the Lie algebras over k. To do this, we take $\mathscr{G}(2) = k$, with the transposition in Σ_2 acting as -1, and take $\mathscr{R}(3)$ to be the space $(\mathscr{F}\mathscr{G})(3)^{\Sigma_3}$ of invariants, which is one dimensional. Basis elements of $\mathscr{G}(2)$ and $\mathscr{R}(3)$ correspond to the bracket operation and the Jacobi identity. We shall see shortly that $\mathscr{L}$ can be realized homologically by the topological little n-cubes operads for any $n > 1$. Note that, in these "purely algebraic" examples, all $\mathscr{C}(j)$ are concentrated in degree zero, with zero differential.

The definition of a Lie algebra over a field k requires the additional relations $[x,x] = 0$ if $\mathrm{char}(k) = 2$ and $[x,[x,x]] = 0$ if $\mathrm{char}(k) = 3$. Purely algebraic operads are not well adapted to codify such relations with repeated variables, still less such nonlinear operations as the restriction (or p^{th} power operation) of restricted Lie algebras in characteristic p. The point is simply that the elements of an operad specify operations, and operations by their nature cannot know about special properties (such as repetition) of the variables to which they are applied.

As an aside, since in the absence of diagonals it is unclear that there is a workable algebraic analog, we note that a topological theory of E_∞ ring spaces has been developed [20, 21]. The sum and product, with the appropriate version of the distributive law, are codified in actions by two suitably interrelated operads. I may say a bit more about this in my second talk.

Fix an operad $\mathscr{C}$ and a $\mathscr{C}$-algebra A. It is clear that the category of A-modules is abelian. In fact, it is equivalent to the category of modules over the universal enveloping algebra $U(A)$ of A, where $U(A)$ is a certain differential graded algebra.

Definition 3.2. Let A be a $\mathscr{C}$-algebra. The action maps

$$\lambda : \mathscr{C}(j) \otimes A^{j-1} \otimes M \to M$$

of an A-module M together define an action map

$$\lambda : C(A;k) \otimes M = C(A;M) \to M.$$

Thus $C(A;k)$ may be viewed as a k-module of operators on A-modules. The free DGA $M(C(A;k))$ generated by $C(A;k)$ therefore acts iteratively on all A-modules. Define the universal enveloping algebra $U(A)$ to be the quotient of $M(C(A;k))$ by the ideal of universal relations.

Actually, $U(A)$ can be described more economically as a quotient of $C(A;k)$.

Proposition 3.3. *The category of A-modules is isomorphic to the category of $U(A)$-modules.*

It follows that the free $U(A)$-module functor gives us the free A-module functor. This implies the following identifications of $U(A)$ in special cases.

Examples 3.4. (i) For an $\mathscr{M}$-algebra A, $U(A)$ is isomorphic to $A \otimes A^{\mathrm{op}}$.

(ii) For an $\mathscr{N}$-algebra A, $U(A)$ is isomorphic to A.

(iii) For an $\mathscr{L}$-algebra L, an L-module in our sense is the same as a Lie algebra module in the classical sense, hence $U(L)$ is isomorphic to the classical universal enveloping algebra of L.

In my second talk, I shall construct a tensor product on the category of modules over an E_∞ algebra A. From the universal enveloping algebra point of view, this should look wholly implausible: a $U(A)$-module is just a left module, and, since $U(A)$ is far from being commutative, there is no obvious way to define a tensor product of A-modules, let alone a tensor product that is again an A-module. Moreover, I will give a precise description of $U(A)$ and show that it is quasi-isomorphic to A, as one would hope. Such calculations are often hard to come by, even rationally, as the work of Hinich and Schechtman [11] on L_∞ algebras shows.

4. Algebraic operads associated to topological operads

An operad $\mathscr{E}$ of topological spaces consists of spaces $\mathscr{E}(j)$ with right actions of Σ_j, a unit element $1 \in \mathscr{E}(1)$, and maps

$$\gamma : \mathscr{E}(k) \times \mathscr{E}(j_1) \times \cdots \times \mathscr{E}(j_k) \to \mathscr{E}(j)$$

such that the appropriate associativity, unity, and equivariance diagrams commute. For definiteness, we assume that $\mathscr{E}(0)$ is a point.

Via the singular chain complex functor, an operad $\mathscr{E}$ of spaces gives rise to an operad $C_*(\mathscr{E})$ of (differential graded) k-modules for any commutative ring k of coefficients. The operad $\mathscr{E}$ is said to an E_∞ operad if each space $\mathscr{E}(j)$ is Σ_j-free and contractible (a universal Σ_j-bundle), and $C_*(\mathscr{E})$ is then an E_∞ operad in the algebraic sense. Similarly, the chain functor C_* carries $\mathscr{E}$-algebras (= $\mathscr{E}$-spaces) to $C_*(\mathscr{E})$-algebras and carries modules over an $\mathscr{E}$-algebra to modules over the associated $C_*(\mathscr{E})$-algebra.

Taking coefficients in a field k, so that we have a Künneth isomorphism, we can go further and define homology operads.

Definition 4.1. Let $\mathscr{E}$ be an operad of spaces. Define $H_*(\mathscr{E})$ to be the unital operad whose jth k-module is the graded k-module $H_*(\mathscr{E}(j))$, with algebraic structure maps γ induced by the topological structure maps. For $n \geq 0$, define $H_n(\mathscr{E})$ to be the suboperad of $H_*(\mathscr{E}(j))$ whose jth k-module is $H_{n(j-1)}(\mathscr{E}(j))$ for $j \geq 0$; in particular, the 0th k-module is zero unless $n = 0$. The degrees are so arranged that the definition makes sense. We retain the grading that comes naturally, so that the jth term of $H_n(\mathscr{E})$ is concentrated in degree $n(j-1)$, but these operads also have evident "degree zero translates".

If the spaces $\mathscr{E}(j)$ are all connected, then $H_0(\mathscr{E}) = \mathscr{N}$ and $H_*(X)$ is a commutative algebra for any $\mathscr{E}$-space X. If the spaces $\mathscr{E}(j)$ are all contractible, for example if $\mathscr{E}$ is an E_∞ operad, then $H_*(\mathscr{E}) = \mathscr{N}$. Thus, on passage to homology, E_∞ operads record only the algebra structure on the homology of $\mathscr{E}$-spaces, although the chain level operad action gives rise to homology operations, including the Dyer-Lashof and Steenrod operations. It is for this reason that topologists did not formally introduce homology operads decades ago.

In fact, there is a sharp dichotomy between the calculational behavior of operads in characteristic zero and in positive characteristic. The depth of the original topological theory lies in positive characteristic, where passage to homology operads jettisons most of the interesting structure. In characteristic zero, in contrast, the homology operads completely determine the homology of the monads E and $\tilde{E}$ associated to an operad $\mathscr{E}$. Here, for a space X,

$$EX = \coprod \mathscr{E}(j) \times_{\Sigma_j} X^j.$$

For a based space X, $\tilde{E}X$ is the quotient of EX obtained by the appropriate basepoint identifications, and $\tilde{E}X$ has a natural filtration with successive quotients

$$\mathscr{E}(j)_+ \wedge_{\Sigma_j} X^{(j)},$$

where $X^{(j)}$ denotes the j-fold smash power of X. Here the smash product $X \wedge Y$ is the quotient of the product $X \times Y$ obtained by identifying the wedge $X \vee Y$ to a point.

The calculational difference comes from a simple general fact: if a finite group π acts on a space X, then, with coefficients in a field k of characteristic zero, $H_*(X/\pi)$ is naturally isomorphic to $H_*(X)/k[\pi]$. This leads to the following result.

Theorem 4.2. *Let $\mathscr{E}$ be an operad of spaces. Let E_H denote the monad in the category of k-modules associated to $H_*(\mathscr{E})$ and let $\tilde{E}_H$ denote the monad in the category of k-modules under k associated to $H_*(\mathscr{E})$. If char$(k) = 0$, then*

$$H_*(EX) \cong E_H(H_*(X)) \quad \text{and} \quad H_*(\tilde{E}X) \cong \tilde{E}_H(H_*(X))$$

as $H_*(\mathscr{E})$-algebras for all spaces X, or for all based spaces in the reduced case.

This allows us to realize free algebras topologically. Recall that we have the topological (actually, discrete) versions of the operads $\mathscr{M}$ and $\mathscr{N}$: $\mathscr{M}(j) = \Sigma_j$ and $\mathscr{N}(j) = \{*\}$. For a based space X, $\tilde{M}X$ is the James construction, or free topological monoid, on X, and it is homotopy equivalent to $\Omega\Sigma X$ if X is connected. Similarly, $\tilde{N}X$ is the infinite symmetric product, or free commutative topological monoid, on X, and it is homotopy equivalent to the product over $n \geq 1$ of the Eilenberg-MacLane spaces $K(H_n(X), n)$ if X is connected. Note that the unreduced constructions MX and NX are just disjoint unions of Cartesian powers and symmetric Cartesian powers and are thus much less interesting. At least in characteristic zero, we conclude that

$$H_*(\tilde{M}X) \cong \tilde{M}_H(H_*(X)) \quad \text{and} \quad H_*(\tilde{N}X) \cong \tilde{N}_H(H_*(X)).$$

The functors $\tilde{M}_H$ and $\tilde{N}_H$ are the free and free commutative algebra functors on unital k-modules that we previously denoted by $\tilde{M}$ and $\tilde{N}$. Thus the right sides are the free and free commutative algebras generated by $\tilde{H}_*(X)$.

5. Operads, loop spaces, n-Lie algebras, and n-braid algebras

We obtain deeper examples by considering the operads that come from the study of iterated loop spaces. Their homology operads turn out to describe n-Lie algebras and n-braid algebras. Implicitly or explicitly, the case $n = 2$ has received a good deal of attention in the recent literature of string theory. Although the theorems I'm about to state were proven in the early 1970's, their statements came much later, in work of Ginzberg and Kapranov [9] and Getzler and Jones [8].

For each $n > 0$, there is a little n-cubes operad $\mathscr{C}_n$. It was invented, before the introduction of operads, by Boardman and Vogt [3]. Its jth space $\mathscr{C}_n(j)$ consists of j-tuples of little n-cubes embedded with parallel axes and disjoint interiors in the standard n-cube. There is an analogous little n-disks operad defined in terms of embeddings of little disks in the unit disk via radial contraction and translation. These are better suited to considerations of group actions and of geometry, but they do not stabilize over n. There is a more sophisticated variant, due to Steiner [24], that enjoys the good properties of both the little n-cubes and the little n-disks operads. Each of these operads comes with a canonical equivalence from its jth space to the configuration space $F(\mathbb{R}^n, j)$ of j-tuples of distinct points of $\mathbb{R}^n$. The little n-cubes operad, and any of its variants, acts naturally on all n-fold loop spaces $\Omega^n Y$.

Since $\mathscr{C}_1$ maps by a homotopy equivalence to $\mathscr{M}$, we concentrate on the case $n > 1$. When k is a field of characteristic $p > 0$, the homology of a $\mathscr{C}_n$-space, such as $\Omega^n Y$, has an extremely rich and complicated algebraic structure, carrying Browder operations and some of the Dyer-Lashof operations that are present in the homology of E_∞ algebras. I will describe the characteristic zero information and a portion of the mod p information in Cohen's exhaustive mod p calculations [4, 5]. (There were earlier partial calculations by Arnol'd [1] in the case $n = 2$.) Again, we take k to be a field.

Cohen's calculations have two starting points. One is his complete and explicit calculation of the integral homology of $F(\mathbb{R}^n, j)$, with its action by Σ_j, for all n and j. He used this to define homology operations. The other is my "approximation theorem" [19]. It asserts that, for a based space X, there is a natural map of $\mathscr{C}_n$-spaces $\tilde{C}_n X \longrightarrow \Omega^n \Sigma^n X$ that is an equivalence when X is connected. This allowed Cohen to combine the homology operations with the Serre spectral sequence to compute simultaneously both $H_*(\tilde{C}_n X)$ and $H_*(\Omega^n \Sigma^n X)$ for any X.

In characteristic zero, the calculations simplify drastically since calculation of the homology operads $H_*(\mathscr{C}_n)$ determines $H_*(\tilde{C}_n X)$. Cohen showed that each space $F(\mathbb{R}^n, j)$ has the same integral homology as a certain product of wedges of $(n-1)$-spheres. Therefore, the operad $H_*(\mathscr{C}_n)$ can be written additively as the reduced sum $\mathscr{N} \tilde{\oplus} H_{n-1}(\mathscr{C}_n)$ of its suboperads $\mathscr{N}$ and $H_{n-1}(\mathscr{C}_n)$, where the reduced sum is obtained from the direct sum by identifying the unit elements in $\mathscr{N}(1)$ and $H_0(\mathscr{C}_n(1))$. For $n \geq 1$, the algebras over $H_n(\mathscr{C}_{n+1})$ turn out to be the n-Lie algebras and the algebras over $H_*(\mathscr{C}_{n+1})$ turn out to be the n-braid algebras.

Definition 5.1. An n-Lie algebra is a k-module L together with a map of k-modules $[\ ,\ ]_n : L \otimes L \to L$ that raises degrees by n and satisfies the following identities, where $\deg(x) = q - n$, $\deg(y) = r - n$, and $\deg(z) = s - n$.

(i) (Anti-symmetry)

$$[x, y]_n = -(-1)^{qr}[y, x]_n.$$

(ii) (Jacobi identity)

$$(-1)^{qs}[x, [y, z]_n]_n + (-1)^{qr}[y, [z, x]_n]_n + (-1)^{rs}[z, [x, y]_n]_n = 0.$$

(iii) $[x, x]_n = 0$ if $\mathrm{char}(k) = 2$ and $[x, [x, x]_n]_n = 0$ if $\mathrm{char}(k) = 3$.

Of course, a 0-Lie algebra is just a Lie algebra.

For a k-module Y and an integer n, define the n-fold suspension $\Sigma^n Y$ by $(\Sigma^n Y)_q = Y_{q-n}$.

Proposition 5.2. *The category of n-Lie algebras is isomorphic to the category of Lie algebras. There is an operad $\mathscr{L}_n$ whose algebras are the n-Lie algebras, and its "degree zero translate" is isomorphic to $\mathscr{L}$.*

Proof. For an n-Lie algebra L, $\Sigma^n L$ is a Lie algebra with bracket

$$[\Sigma^n x, \Sigma^n y] = \Sigma^n [x, y]_n.$$

Similarly, for a Lie algebra L, $\Sigma^{-n} L$ is an n-Lie algebra. □

Theorem 5.3. *If char(k) $\neq 2$ or 3, then, for all $n \geq 1$, the operad $H_n(\mathscr{C}_{n+1})$ defines n-Lie algebras. That is, $H_n(\mathscr{C}_{n+1}) \cong \mathscr{L}_n$.*

For a k-module V, let $L_n V = \Sigma^{-n} L \Sigma^n V$ be the free n-Lie algebra generated by V.

Theorem 5.4. *Assume that char(k) $= 0$. For any based space X,*

$$(\tilde{C}_{n+1})_H(H_*(X)) \cong H_*(\tilde{C}_{n+1}X) \cong NL_n\tilde{H}_*(X)$$

is the free commutative algebra generated by the free n-Lie algebra generated by $\tilde{H}_*(X)$.

In fact, this can be interpreted as a theorem about n-braid algebras.

Definition 5.5. An n-braid algebra is a k-module A that is an n-Lie algebra and a commutative DGA such that the bracket and product satisfy the following identity, where $\deg(x) = q - n$ and $\deg(y) = r - n$.

(Poisson formula)

$$[x, yz]_n = [x, y]_n z + (-1)^{q(r-n)} y[x, z]_n.$$

The Poisson formula asserts that the map $d_x = [x, ?]_n$ is a graded derivation, in the sense that

$$d_x(yz) = d_x(y)z + (-1)^{\deg(y)\deg(d_x)} y d_x(z).$$

Batalin-Vilkovisky algebras are examples of 1-braid algebras [7], hence the general case, with non-zero differentials, is relevant to string theory. However, our concern here is with structures that have zero differential.

Theorem 5.6. *The homology $H_*(X)$ is an n-braid algebra for any $\mathscr{C}_{n+1}$-space X and any field of coefficients. If char(k) $\neq 2$ or 3, then, for all $n \geq 1$, the algebras over the operad $H_*(\mathscr{C}_{n+1})$ are exactly the n-braid algebras. Moreover, the free n-braid algebra generated by a k-module V is isomorphic to NL_nV.*

The last statement could in principle be proven algebraically, but it is much easier to deduce it from the topology, even in characteristic zero. The n-bracket is denoted λ_n and called a Browder operation in the context of iterated loop spaces. The special characteristic 2 and 3 identities (iii) in the definition of an n-Lie algebra are of conceptual interest: they cannot be visible in the operad $H_n(\mathscr{C}_{n+1})$, but they follow directly from the chain level definition of λ_n.

6. Homology operations in characteristic p

When $\mathscr{C}$ is an E_∞ operad, an action of $\mathscr{C}$ on A builds in the kinds of higher homotopies for the multiplication of A that are the source, for example, of the Dyer-Lashof operations in the homology of infinite loop spaces and the Steenrod operations in the cohomology of general spaces. We describe the form that these operations take in the homology of general E_∞ algebras A in this section. Many other examples are known to topologists, such as the Steenrod operations in the

Ext groups of cocommutative Hopf algebras and in the cohomology of simplicial restricted Lie algebras.

We take $k = \mathbb{Z}$ and consider algebras A over an integral E_∞ operad $\mathscr{C}$. Let $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and consider the mod p homology $H_*(A;\mathbb{Z}_p)$.

Theorem 6.1. *For $s \geq 0$, there exist natural homology operations*

$$Q^s : H_q(A;\mathbb{Z}_2) \to H_{q+s}(A;\mathbb{Z}_2)$$

and

$$Q^s : H_q(A;\mathbb{Z}_p) \to H_{q+2s(p-1)}(A;\mathbb{Z}_p)$$

if $p > 2$. These operations satisfy the following properties

1. *$Q^s(x) = 0$ if $p = 2$ and $s < q$ or if $p > 2$ and $2s < q$.*
2. *$Q^s(x) = x^p$ if $p = 2$ and $s = q$ or if $p > 2$ and $2s = q$.*
3. *$Q^s(1) = 0$ if $s > 0$, where $1 \in H_0(A;\mathbb{Z}_p)$ is the identity element.*
4. *(Cartan formula) $Q^s(xy) = \sum Q^t(x)Q^{s-t}(y)$.*
5. *(Adem relations) If $p \geq 2$ and $t > ps$, then*

$$Q^tQ^s = \sum_i (-1)^{t+i}(pi - t, t - (p-1)s - i)Q^{s+t-i-1}Q^i;$$

if $p > 2$, $t \geq ps$, and β denotes the mod p Bockstein, then

$$Q^t\beta Q^s = \sum_i (-1)^{t+i}(pi - t, t - (p-1)s - i)\beta Q^{s+t-i}Q^i - \sum_i (-1)^{t+i}(pi - t - 1, t - (p-1)s - i)Q^{s+t-i}\beta Q^i;$$

here $(i,j) = \dfrac{(i+j)!}{i!j!}$ if $i \geq 0$ and $j \geq 0$ (where $0! = 1$), and $(i,j) = 0$ if i or j is negative; the sums run over $i \geq 0$.

For the proof, one simply checks that one is in the general algebraic framework of a 1970 paper of mine [18] that does the relevant homological algebra once and for all. That paper should have been about operad actions. As I mentioned, however, it was actually written shortly before I invented operads. The point is that $\mathscr{C}(p)$ is a Σ_p-free resolution of $\mathbb{Z}$, so that the homology of $\mathscr{C}(p) \otimes_{\Sigma_p} A^p$ is readily computed, and computation of $\theta_* : H_*(\mathscr{C}(p) \otimes_{\Sigma_p} A^p;\mathbb{Z}_p) \to H_*(A;\mathbb{Z}_p)$ allows one to read off the operations. The Cartan formula and the Adem relations are derived from special cases of the diagrams in the definition of an operad action via calculations in the homology of groups.

Notice the grading. The first non-zero operation is the pth power, and there can be infinitely many non-zero operations on a given element. This is in marked contrast with Steenrod operations in the cohomology of spaces, where the last non-zero operation is the pth power. In fact, Steenrod operations are defined on cohomologically graded E_∞ algebras that are concentrated in positive degrees, in which context the complexes $\mathscr{C}(j)$ of the relevant E_∞ operad must be regraded as cochain complexes concentrated in negative degrees. If we systematically regrade homologically, then Dyer-Lashof and Steenrod operations both fit into the general context of the theorem, except that the adjective "Dyer-Lashof" is to be used when the underlying chain complexes are positively graded and the adjective "Steenrod" is to be used when the underlying chain complexes are negatively graded.

7. A conversion theorem

In characteristic zero, E_∞ operads carry no more homological information than the operad $\mathscr{N}$, and similarly for more general types of algebras.

Lemma 7.1. *Let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of operads over a field k of characteristic zero, such as the augmentation $\epsilon : \mathscr{C} \to \mathscr{N}$ of an acyclic operad. Then the map $CX \to PX$ induced by ϵ is a quasi-isomorphism for all k-modules X.*

Taking $\mathscr{P} = \mathscr{N}$ and $\mathscr{P} = \mathscr{L}$, this leads to a proof that, when k is a field of characteristic zero, E_∞ algebras are quasi-isomorphic as E_∞ algebras to commutative DGA's and L_∞ algebras are quasi-isomorphic as L_∞ algebras to differential graded Lie algebras, and similarly for modules. That is, one can convert operadic algebras and modules to genuine algebras and modules without loss of information. The proof is a typical exercise in the use of the two-sided monadic bar construction that I introduced in the same paper [19] in which I first defined operads.

Theorem 7.2. *Let k be a field of characteristic zero and let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of operads of k-complexes. Then there is a functor W that assigns a quasi-isomorphic $\mathscr{P}$-algebra WA to a $\mathscr{C}$-algebra A. There is also a functor W that assigns a quasi-isomorphic WA-module WM to an A-module M.*

An acyclic operad augments by a quasi-isomorphism to the operad $\mathscr{N}$ that defines commutative DGA's, so the following consequence is immediate.

Corollary 7.3. *Let k be a field of characteristic zero and let $\mathscr{C}$ be an acyclic operad of k-complexes. Then there is a functor W that assigns a quasi-isomorphic commutative DGA WA to a $\mathscr{C}$-algebra A. There is also a functor W that assigns a quasi-isomorphic WA-module WM to an A-module M.*

The following consequence for differential graded Lie algebras is also immediate.

Corollary 7.4. *Let k be a field of characteristic zero and let $\mathscr{J}$ be an operad with a quasi-isomorphism to $\mathscr{L}$. Then there is a functor W that assigns a quasi-isomorphic differential graded Lie algebra WL to a $\mathscr{J}$-algebra L. There is also a functor W that assigns a quasi-isomorphic WL-module WM to an L-module M.*

The second corollary may be of interest in applications to string theory. The motivation for the first corollary originally came from algebraic geometry, where it answered a question of Deligne concerning mixed Tate motives.

The techniques of proof also apply to replace partial algebras over operads by genuine algebras over operads, in any characteristic. This has real force. Using it, Kriz and I were able to carry out a suggestion of Deligne for the construction of categories of both integral and rational mixed Tate motives [13, 14]. However, that is a subject for another talk.

I hope that this has given a bit of a feel for some of the ways that operads work, and I look forward to learning more from all of you during the conference.

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