ORTHOGONAL SPECTRA AND S-MODULES

M. A. MANDELL AND J. P. MAY

Abstract. There are two general approaches to the construction of symmetric monoidal categories of spectra, one based on an encoding of operadic structure in the definition of the smash product and the other based on the categorical observation that categories of diagrams with symmetric monoidal domain are symmetric monoidal. The first was worked out by Elmendorf, Kriz, and the authors in the theory of "S-modules". The second was worked out in the case of symmetric spectra by Hovey, Shipley, and Smith and, in a general topological setting, by Schwede, Shipley, and the authors. A comparison between symmetric spectra and S-modules was given by Schwede.

Orthogonal spectra are intermediate between symmetric spectra and S-modules: they are defined in the same diagrammatic fashion as symmetric spectra, but, as with S-modules, their stable weak equivalences are just the maps that induce isomorphisms on homotopy groups. We prove that the categories of orthogonal spectra and S-modules are Quillen equivalent and that this equivalence induces Quillen equivalences between the respective categories of ring spectra, of modules over ring spectra, and of commutative ring spectra. The equivalence is given by a functor that is closely related to an older and more intuitive functor from orthogonal spectra to S-modules, and a comparison between the two leads to a precise understanding of the relationship between the definitions of orthogonal spectra and of S-modules in terms of a category of Thom spaces.

Contents

There are several symmetric monoidal categories of "spectra" that are model categories with homotopy categories equivalent to the stable homotopy category. The most highly developed, and the first to be made rigorous, is the category $\mathscr{M} = \mathscr{M}_S$ of S-modules of [1]. Its objects are quite complicated, but the complication encodes computationally important information. A second such category is the category $\Sigma\mathscr{S}$ of symmetric spectra, which is due to Jeff Smith and whose rigorous development is given in [4]. That paper is written simplicially, but a logically independent topological treatment has since been given [7]. Symmetric spectra are far simpler objects than S-modules, but the simplicity comes at a price: the weak equivalences are not just the maps that induce isomorphisms on homotopy groups, and this makes the homotopy theory of symmetric spectra quite subtle. A third such category is the category $\mathscr{I}\mathscr{S}$ of orthogonal spectra, which was first defined by

Date: December 30, 1999.

the second author [8] and was fully developed in [7]. Orthogonal spectra are just as simple to define as symmetric spectra, yet their weak equivalences *are* just the maps that induce isomorphisms on homotopy groups. Thus, philosophically, orthogonal spectra are intermediate between S-modules and symmetric spectra, enjoying some of the best features of both.

It is proven in [7] that the categories of symmetric spectra and of orthogonal spectra are Quillen equivalent. It is proven by Schwede in [11] that the categories of symmetric spectra and S-modules are Quillen equivalent. However, this does not give a satisfactory Quillen equivalence between the categories of orthogonal spectra and S-modules since the resulting functor from orthogonal spectra to S-modules is the composite of a right adjoint (to symmetric spectra) and a left adjoint.

We shall give a Quillen equivalence between the categories of orthogonal spectra and S-modules such that the Quillen equivalence of [11] is the composite of the Quillen equivalence between symmetric spectra and orthogonal spectra of [7] and our new Quillen equivalence. Thus orthogonal spectra are mathematically as well as philosophically intermediate between symmetric spectra and S-modules. Our proofs will give a concrete Thom space level understanding of the relationship between orthogonal spectra and S-modules. To complete the picture, we also point out Quillen equivalences relating coordinatized prespectra, coordinate-free prespectra, and spectra to S-modules and orthogonal spectra (in §3).

To separate formalities from substance, we begin in §1 by establishing a formal framework for constructing symmetric monoidal left adjoint functors whose domain is a category of diagram spaces. In fact, this elementary category theory sheds new light on the basic constructions that are studied in all work on diagram spectra. In §2, we explain in outline how this formal theory combines with model theory to prove the following comparison theorems. We recall the relevant model structures and give the homotopical parts of the proofs in §3. We defer the basic construction that gives substance to the theory to §4.

Theorem 0.1. *There is a strong symmetric monoidal functor* $\mathbb{N} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$ *and a lax symmetric monoidal functor* $\mathbb{N}^{\#} : \mathscr{M} \longrightarrow \mathscr{I}\mathscr{S}$ *such that* $(\mathbb{N}, \mathbb{N}^{\#})$ *is a Quillen equivalence between* $\mathscr{I}\mathscr{S}$ *and* $\mathscr{M}$*. The induced equivalence of homotopy categories preserves smash products.*

Theorem 0.2. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of orthogonal ring spectra and of* S*-algebras.*

Theorem 0.3. *For a cofibrant orthogonal ring spectrum* R*, the pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of* R*-modules and of* $\mathbb{N}R$*-modules.*

By [7, 12.1(iv)], the assumption that R is cofibrant results in no loss of generality. As in [7, §13], this result implies the following one.

Corollary 0.4. *For an* S*-algebra* R*, the categories of* R*-modules and of* $\mathbb{N}^{\#}R$*-modules are Quillen equivalent.*

Theorem 0.5. *The pair* $(\mathbb{N}, \mathbb{N}^{\#})$ *induces a Quillen equivalence between the categories of commutative orthogonal ring spectra and of commutative* S*-algebras.*

Theorem 0.6. *Let* R *be a cofibrant commutative orthogonal ring spectrum. The categories of* R*-modules,* R*-algebras, and commutative* R*-algebras are Quillen equivalent to the categories of* $\mathbb{N}R$*-modules,* $\mathbb{N}R$*-algebras, and commutative* $\mathbb{N}R$*-algebras.*

By [7, 12.1(iv) and 15.2(ii)], the assumption that R is cofibrant results in no loss of generality. Again, as in [7, §16], this result implies the following one.

Corollary 0.7. *Let R be a commutative S-algebra. The categories of R-modules, R-algebras, and commutative R-algebras are Quillen equivalent to the categories of $\mathbb{N}^{\#}R$-modules, $\mathbb{N}^{\#}R$-algebras, and commutative $\mathbb{N}^{\#}R$-algebras.*

These last results are the crucial comparison theorems since most of the deepest applications of structured ring and module spectra concern E_∞-ring spectra or, equivalently by [1], commutative S-algebras. By [7, 22.4], commutative orthogonal ring spectra are the same objects as commutative orthogonal FSP's. Under the name "$\mathscr{I}_*$-prespectra", these were defined and shown to give rise to E_∞ ring spectra in [9]. Theorem 0.5 shows that, up to equivalence, all E_∞ ring spectra arise this way. The second author has wondered since 1973 whether or not that is so.

The analogues of the results above with orthogonal spectra and S-modules replaced by symmetric spectra and orthogonal spectra are proven in [7]. This has the following immediate consequence, which reproves all of the results of [11].

Theorem 0.8. *The analogues of the results above with orthogonal spectra replaced by symmetric spectra are also true.*

The functor $\mathbb{N}$ that occurs in the results above has all of the formal and homotopical properties that one might desire. However, a quite different and considerably more intuitive functor $\mathbb{M}$ from orthogonal spectra to S-modules is implicit in [9]. The functor $\mathbb{M}$ gives the most natural way to construct Thom spectra as commutative S-algebras, and its equivariant version was used in an essential way in the proof of the localization and completion theorem for complex cobordism given in [2]. We define $\mathbb{M}$ and compare it with $\mathbb{N}$ in §5.

Since orthogonal spectra, like S-modules, are intrinsically coordinate-free, they are well adapted to equivariant generalization. That is studied in [6].

It is a pleasure to thank our collaborators Brooke Shipley and Stefan Schwede. Like Schwede's paper [11], which gives a blueprint for some of §2 here, this paper is an outgrowth of our joint work in [7].

1. Right exact functors on categories of diagram spaces

To clarify our arguments, we first give the formal structure of our construction of the adjoint pair $(\mathbb{N}, \mathbb{N}^{\#})$ in a suitably general framework. We consider categories $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-shaped diagrams of based spaces for some domain category $\mathscr{D}$, and we show that, to construct left adjoint functors from $\mathscr{D}\mathscr{T}$ to suitable categories $\mathscr{C}$, we need only construct contravariant functors $\mathscr{D} \longrightarrow \mathscr{C}$. The proof is an exercise in the use of representable functors and must be standard category theory, but we do not know a convenient reference.

Let $\mathscr{T}$ be the category of based spaces and $\mathscr{D}$ be any based topological category with a small skeleton $sk\mathscr{D}$. A $\mathscr{D}$-space is a continuous based functor $\mathscr{D} \longrightarrow \mathscr{T}$. Let $\mathscr{D}\mathscr{T}$ be the category of $\mathscr{D}$-spaces. As observed in [7, §1], the evident levelwise constructions define limits, colimits, smash products with spaces, and function $\mathscr{D}$-spaces that give $\mathscr{D}\mathscr{T}$ a structure of complete and cocomplete, tensored and cotensored, topological category. We call such a category *topologically bicomplete*. We fix a topologically bicomplete category $\mathscr{C}$ for the rest of this section. We write $C \wedge A$ for the tensor of an object C of $\mathscr{C}$ and a based space A. All functors are assumed to be continuous.

Definition 1.1. A functor between topologically cocomplete categories is *right exact* if it commutes with colimits and tensors. For example, any functor that is a left adjoint is right exact.

For a contravariant functor $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$ and a $\mathscr{D}$-space X, we have the coend

$$\mathbb{E} \otimes_{\mathscr{D}} X = \int^{d} \mathbb{E}(d) \wedge X(d) \tag{1.2}$$

in $\mathscr{C}$. Explicitly, $\mathbb{E} \otimes_{\mathscr{D}} X$ is the coequalizer in $\mathscr{C}$ of the diagram

$$\bigvee_{d,e} \mathbb{E}(e) \wedge \mathscr{D}(d,e) \wedge X(d) \underset{\mathrm{id}\,\wedge\varepsilon}{\overset{\varepsilon\wedge\mathrm{id}}{\rightrightarrows}} \bigvee_{d} \mathbb{E}(d) \wedge X(d),$$

where the wedges run over pairs of objects and objects of $sk\mathscr{D}$ and the parallel arrows are wedges of smash products of identity and evaluation maps of $\mathbb{E}$ and X.

For an object $d \in \mathscr{D}$, we have a left adjoint $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ to the functor given by evaluation at d. If d^* is defined by $d^*(e) = \mathscr{D}(d,e)$, then $F_d A = d^* \wedge A$. In particular, $F_d S^0 = d^*$.

Definition 1.3. Let $\mathbb{D} = \mathbb{D}_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{D}\mathscr{T}$ be the evident contravariant functor that sends d to d^*.

The following observation is [7, 1.6].

Lemma 1.4. *The evaluation maps $\mathscr{D}(d,e) \wedge X(d) \longrightarrow X(e)$ of $\mathscr{D}$-spaces X induce a natural isomorphism of $\mathscr{D}$-spaces $\mathbb{D} \otimes_{\mathscr{D}} X \longrightarrow X$.*

Together with elementary categorical observations, this has the following immediate implication. It shows that (covariant) right exact functors $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ determine and are determined by contravariant functors $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$.

Theorem 1.5. *If $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ is a right exact functor, then $(\mathbb{F} \circ \mathbb{D}) \otimes_{\mathscr{D}} X \cong \mathbb{F}X$. Conversely, if $\mathbb{E} : \mathscr{D} \longrightarrow \mathscr{C}$ is a contravariant functor, then the functor $\mathbb{F} : \mathscr{D} \longrightarrow \mathscr{C}$ specified by $\mathbb{F}X = \mathbb{E} \otimes_{\mathscr{D}} X$ is right exact and $\mathbb{F} \circ \mathbb{D} \cong \mathbb{E}$.*

Notation 1.6. Write $\mathbb{F} \leftrightarrow \mathbb{F}^*$ for the correspondence between right exact functors $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ and contravariant functors $\mathbb{F}^* : \mathscr{D} \longrightarrow \mathscr{C}$. Thus, given $\mathbb{F}$, $\mathbb{F}^* = \mathbb{F} \circ \mathbb{D}$, and, given $\mathbb{F}^*$, $\mathbb{F} = \mathbb{F}^* \otimes_{\mathscr{D}} (-)$. In particular, on representable $\mathscr{D}$-spaces, $\mathbb{F}d^* \cong \mathbb{F}^*d$.

Corollary 1.7. *Natural transformations between right exact functors $\mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ determine and are determined by natural transformations between the corresponding contravariant functors $\mathscr{D} \longrightarrow \mathscr{C}$.*

Proposition 1.8. *Any right exact functor $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ has the right adjoint $\mathbb{F}^{\#}$ specified by*

$$(\mathbb{F}^{\#}C)(d) = \mathscr{C}(\mathbb{F}^*d, C)$$

for $C \in \mathscr{C}$ and $d \in \mathscr{D}$. The evaluation maps

$$\mathscr{D}(d,e) \wedge \mathscr{C}(\mathbb{F}^*d, C) \longrightarrow \mathscr{C}(\mathbb{F}^*e, C)$$

of the functor $\mathbb{F}^{\#}$ are the adjoints of the composites

$$\mathbb{F}^*e \wedge \mathscr{D}(d,e) \wedge \mathscr{C}(\mathbb{F}^*d, C) \xrightarrow{\varepsilon\wedge\mathrm{id}} \mathbb{F}^*d \wedge \mathscr{C}(\mathbb{F}^*d, C) \xrightarrow{\zeta} C,$$

where ε is an evaluation map of the functor $\mathbb{F}^*$ and ζ is an evaluation map of the category $\mathscr{C}$.

Proof. We must show that

$$\mathscr{C}(\mathbb{F}X, C) \cong \mathscr{D}\mathscr{T}(X, \mathbb{F}^{\#}C). \tag{1.9}$$

The description of $\mathbb{F}X$ as a coend implies a description of $\mathscr{C}(\mathbb{F}X, C)$ as an end constructed out of the spaces $\mathscr{C}(\mathbb{F}^*d \wedge X(d), C)$. Under the adjunction isomorphisms

$$\mathscr{C}(\mathbb{F}^*d \wedge X(d), C) \cong \mathscr{T}(X(d), \mathscr{C}(\mathbb{F}^*d, C)),$$

this end transforms to the end that specifies $\mathscr{D}\mathscr{T}(X, \mathbb{F}^{\#}C)$. □

As an illustration of the definitions, we show how the prolongation and forgetful functors studied in [7] fit into the present framework.

Example 1.10. A (covariant) functor $\iota : \mathscr{D} \longrightarrow \mathscr{D}'$ induces the forgetful functor $\mathbb{U} : \mathscr{D}'\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ that sends Y to $Y \circ \iota$. It also induces the contravariant functor $\mathbb{D}_{\mathscr{D}'} \circ \iota : \mathscr{D} \longrightarrow \mathscr{D}'\mathscr{T}$. Let $\mathbb{P}X = (\mathbb{D}_{\mathscr{D}'} \circ \iota) \otimes_{\mathscr{D}} X$. Then $\mathbb{P}$ is the prolongation functor left adjoint to $\mathbb{U}$.

Now let $\mathscr{D}$ be symmetric monoidal with product $\oplus$ and unit $u_{\mathscr{D}}$. By [7, §21], $\mathscr{D}\mathscr{T}$ is symmetric monoidal with unit $u^*_{\mathscr{D}}$. We denote the smash product of $\mathscr{D}\mathscr{T}$ by $\wedge_{\mathscr{D}}$. Actually, the construction of the smash product is another simple direct application of the present framework.

Example 1.11. We have the external smash product $\bar{\wedge} : \mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T} \longrightarrow (\mathscr{D} \times \mathscr{D})\mathscr{T}$ specified by $(X \bar{\wedge} Y)(d, e) = X(d) \wedge Y(e)$ [7, 21.1]. We also have the contravariant functor $\mathbb{D}_{\mathscr{D}} \circ \oplus : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}\mathscr{T}$. The internal smash product is given by

$$X \wedge_{\mathscr{D}} Y = (\mathbb{D}_{\mathscr{D}} \circ \oplus) \otimes_{\mathscr{D} \times \mathscr{D}} (X \bar{\wedge} Y). \tag{1.12}$$

It is an exercise to rederive the universal property

$$\mathscr{D}\mathscr{T}(X \wedge_{\mathscr{D}} Y, Z) \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(X \bar{\wedge} Y, Z \circ \oplus) \tag{1.13}$$

that characterizes $\wedge_{\mathscr{D}}$ from this definition.

Proposition 1.14. *Let $\mathbb{F}^* : \mathscr{D} \longrightarrow \mathscr{C}$ be a strong symmetric monoidal contravariant functor. Then $\mathbb{F} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}$ is a strong symmetric monoidal functor and $\mathbb{F}^{\#} : \mathscr{C} \longrightarrow \mathscr{D}\mathscr{T}$ is a lax symmetric monoidal functor.*

Proof. We are given an isomorphism $\lambda : \mathbb{F}^*u_{\mathscr{D}} \longrightarrow u_{\mathscr{C}}$ and a natural isomorphism

$$\phi : \mathbb{F}^*d \wedge_{\mathscr{C}} \mathbb{F}^*e \longrightarrow \mathbb{F}^*(d \oplus e).$$

Since $\mathbb{F}^*u_{\mathscr{D}} \cong \mathbb{F}u^*_{\mathscr{D}}$, we may view λ as an isomorphism $\mathbb{F}u^*_{\mathscr{D}} \longrightarrow u_{\mathscr{C}}$. By (1.9) and (1.13), we have

$$\mathscr{C}(\mathbb{F}(X \wedge_{\mathscr{D}} Y), C) \cong (\mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T})(X \bar{\wedge} Y, \mathbb{F}^{\#}C \circ \oplus).$$

Commuting coends past smash products and using isomorphisms

$$(\mathbb{F}^*d \wedge X(d)) \wedge (\mathbb{F}^*e \wedge Y(e)) \cong \mathbb{F}^*(d \oplus e) \wedge X(d) \wedge Y(e)$$

induced by ϕ, we obtain the first of the following two isomorphisms. We obtain the second by using the tensor adjunction of $\mathscr{C}$ and applying the defining universal property of coends.

$$\begin{aligned} \mathscr{C}(\mathbb{F}X \wedge_{\mathscr{C}} \mathbb{F}Y, C) &\cong \mathscr{C}\Big(\int^{(d,e)} \mathbb{F}^{\#}(d \oplus e) \wedge X(d) \wedge Y(e), C\Big) \\ &\cong (\mathscr{D}\mathscr{T} \times \mathscr{D}\mathscr{T})(X \bar{\wedge} Y, \mathbb{F}^{\#}C \circ \oplus). \end{aligned}$$

There results a natural isomorphism $\mathbb{F}X \wedge_{\mathscr{C}} \mathbb{F}Y \cong \mathbb{F}(X \wedge_{\mathscr{D}} Y)$, and coherence is easily checked.

For $\mathbb{F}^{\#}$, the adjoint $u^*_{\mathscr{D}} \longrightarrow \mathbb{F}^{\#}u_{\mathscr{C}}$ of λ gives the unit map. Taking the smash products of maps in $\mathscr{C}$ and applying isomorphisms ϕ, we obtain maps

$$\mathscr{C}(\mathbb{F}^*(d), C) \wedge \mathscr{C}(\mathbb{F}^*(e), C') \longrightarrow \mathscr{C}(\mathbb{F}^*(d \oplus e), C \wedge_{\mathscr{C}} C')$$

that together define a map

$$\mathbb{F}^{\#}C \,\bar{\wedge}\, \mathbb{F}^{\#}C' \longrightarrow \mathbb{F}^{\#}(C \wedge_{\mathscr{C}} C') \circ \oplus.$$

Using (1.13), there results a natural map

$$\mathbb{F}^{\#}C \wedge_{\mathscr{D}} \mathbb{F}^{\#}C' \longrightarrow \mathbb{F}^{\#}(C \wedge_{\mathscr{C}} C'),$$

and coherence is again easily checked. □

2. The proofs of the comparison theorems

We refer to [7] for details of the category $\mathscr{I}\mathscr{S}$ of orthogonal spectra and to [1] for details of the category $\mathscr{M} = \mathscr{M}_S$ of S-modules. Much of our work depends only on basic formal properties. Both of these categories are closed symmetric monoidal and topologically bicomplete. They are Quillen model categories, and their model structures are compatible with their smash products. Actually, in [7], the category of orthogonal spectra is given two model structures with the same (stable) weak equivalences. In one of them, the sphere spectrum is cofibrant, in the other, the "positive stable model structure", it is not. In [7], use of the positive stable model structure was essential to obtain an induced model structure on the category of commutative orthogonal ring spectra. It is also essential here, since the sphere S-module S is not cofibrant. We will review the model structures in §3.

We begin by giving a quick summary of definitions from [7], recalling how orthogonal spectra fit into the framework of the previous section. Let $\mathscr{I}$ be the symmetric monoidal category of finite dimensional real inner product spaces and linear isometric isomorphisms. We call an $\mathscr{I}$-space an *orthogonal space*. The category $\mathscr{I}\mathscr{T}$ of orthogonal spaces is closed symmetric monoidal under its smash products $X \wedge Y$ and function objects $F(X, Y)$.

The sphere orthogonal space $S_{\mathscr{I}}$ has Vth space the one-point compactification S^V of V; $S_{\mathscr{I}}$ is a commutative monoid in $\mathscr{I}\mathscr{T}$. An *orthogonal spectrum*, or *$\mathscr{I}$-spectrum*, is a (right) $S_{\mathscr{I}}$-module. The category $\mathscr{I}\mathscr{S}$ of orthogonal spectra is closed symmetric monoidal. We denote its smash products and function spectra by $X \wedge_{\mathscr{I}} Y$ and $F_{\mathscr{I}}(X, Y)$ (although this is not consistent with the previous section).

There is a symmetric monoidal category $\mathscr{I}_S$ with the same objects as $\mathscr{I}$ such that the category of *$\mathscr{I}_S$-spaces* is isomorphic to the category of *$\mathscr{I}$-spectra*; $\mathscr{I}_S$ contains $\mathscr{I}$ as a subcategory. The construction of $\mathscr{I}_S$ is given in [7, 2.1]. Its space of morphisms $\mathscr{I}_S(V, W)$ is $(V^* \wedge S_{\mathscr{I}})(W)$, where $V^*(W) = \mathscr{I}(V, W)_+$. In §4, we shall give a concrete alternative description of $\mathscr{I}_S$ in terms of Thom spaces, and we shall construct a coherent family of cofibrant $(-V)$-sphere S-modules $\mathbb{N}^*(V)$ that give us a contravariant "negative spheres" functor $\mathbb{N}^*$ to which we can apply the constructions of the previous section. Note that the unit of $\mathscr{I}$ is 0, the unit of $\mathscr{I}\mathscr{S}$ is $S_{\mathscr{I}}$, and, as required for consistency, $\mathscr{I}_S(0, W) = S^W$.

Theorem 2.1. *There is a strong symmetric monoidal contravariant functor $\mathbb{N}^* : \mathscr{I}_S \longrightarrow \mathscr{M}$. If $V \neq 0$, then $\mathbb{N}^*(V)$ is a cofibrant S-module and the evaluation map*

$$\varepsilon : \mathbb{N}^*(V) \wedge S^V = \mathbb{N}^*(V) \wedge \mathscr{I}_S(0, V) \longrightarrow \mathbb{N}^*(0) \cong S$$

of the functor is a weak equivalence.

Here $\mathbb{N}^*(0) \cong S$ since $\mathbb{N}^*$ is strong symmetric monoidal. Propositions 1.8 and 1.14 give the following immediate consequence.

Theorem 2.2. *Define functors* $\mathbb{N} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$ *and* $\mathbb{N}^\# : \mathscr{M} \longrightarrow \mathscr{I}\mathscr{S}$ *by letting* $\mathbb{N}(X) = \mathbb{N}^* \otimes_{\mathscr{I}_S} X$ *and* $(\mathbb{N}^\# M)(V) = \mathscr{M}(N^*(V), M)$*. Then* $(\mathbb{N}, \mathbb{N}^\#)$ *is an adjoint pair such that* $\mathbb{N}$ *is strong symmetric monoidal and* $\mathbb{N}^\#$ *is lax symmetric monoidal.*

This gives the formal properties of $\mathbb{N}$ and $\mathbb{N}^\#$, and we turn to their homotopical properties. According to [7, A.2], to show that these functors give a Quillen equivalence between $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$, it suffices to prove the following three results. Thus, since its last statement is formal [3, 4.3.3], these results will prove Theorem 0.1. We give the proofs in the next section. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between model categories is said to create the weak equivalences in $\mathscr{A}$ if the weak equivalences in $\mathscr{A}$ are exactly the maps f such that Ff is a weak equivalence in $\mathscr{B}$, and similarly for other classes of maps.

Lemma 2.3. *The functor* $\mathbb{N}^\#$ *creates the weak equivalences in* $\mathscr{M}$*.*

Lemma 2.4. *The functor* $\mathbb{N}^\#$ *preserves q-fibrations.*

Proposition 2.5. *The unit* $\eta : X \longrightarrow \mathbb{N}^\#\mathbb{N}X$ *of the adjunction is a weak equivalence for all cofibrant orthogonal spectra* X*.*

In Lemma 2.4, we are concerned with q-fibrations of orthogonal spectra in the positive stable model structure. To prove Theorem 0.1, we only need Proposition 2.5 for orthogonal spectra that are cofibrant in the positive stable model structure, but we shall prove it more generally for orthogonal spectra that are cofibrant in the stable model structure. We refer to *positive cofibrant* and *cofibrant* orthogonal spectra to distinguish these classes.

In the rest of this section, we show that these results imply their multiplicatively enriched versions needed to prove Theorems 0.2, 0.3, 0.5, and 0.6. That is, in all cases, we have an adjoint pair $(\mathbb{N}, \mathbb{N}^\#)$ such that $\mathbb{N}^\#$ creates weak equivalences and preserves q-fibrations and the unit of the adjunction is a weak equivalence on cofibrant objects. The one subtlety is that, to apply Proposition 2.5, we must relate cofibrancy of multiplicatively structured orthogonal spectra with cofibrancy of their underlying orthogonal spectra.

The proof of Theorem 0.2. The category of orthogonal ring spectra has two model structures. The respective weak equivalences and q-fibrations are created in the category of orthogonal spectra with its stable model structure or its positive stable model structure. The category of S-algebras is a model category with weak equivalences and q-fibrations created in the category of S-modules. Our claim is that $(\mathbb{N}, \mathbb{N}^\#)$ restricts to a Quillen equivalence relating the category of orthogonal ring spectra with its positive stable model structure to the category of S-algebras. It is clear from Lemmas 2.3 and 2.4 that $\mathbb{N}^\#$ creates weak equivalences and preserves q-fibrations. We must show that $\eta : R \longrightarrow \mathbb{N}^\#\mathbb{N}R$ is a weak equivalence when R is a positive cofibrant orthogonal ring spectrum. More generally, if R is a cofibrant orthogonal ring spectrum, then the underlying orthogonal spectrum of R is cofibrant (although not positive cofibrant) by [7, 12.1]. The conclusion follows from Proposition 2.5. □

The proof of Theorem 0.3. The category of R-modules is a model category with weak equivalences and q-fibrations created in the category of orthogonal spectra with its positive stable model structure. The category of $\mathbb{N}R$-modules is a model category with weak equivalences and q-fibrations created in the category of S-modules. Again, it is clear that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations. We must show that $\eta : Y \longrightarrow \mathbb{N}^{\#}\mathbb{N}Y$ is a weak equivalence when Y is a positive cofibrant R-module. We are assuming that R is positive cofibrant as an orthogonal ring spectrum, and it follows from [7, 12.1] that the underlying orthogonal spectrum of a cofibrant R-module is cofibrant (although not necessarily positive cofibrant). The conclusion follows from Proposition 2.5. □

The proof of Theorem 0.5. The category of commutative orthogonal ring spectra has a model structure with weak equivalences and q-fibrations created in the category of orthogonal spectra with its positive stable model structure [7, 15.1]. The category of commutative S-algebras has a model structure with weak equivalences and q-fibrations created in the category of S-modules [1, VII.4.8]. Again, it is clear that $\mathbb{N}^{\#}$ creates weak equivalences and preserves q-fibrations, and we must prove that $\eta : R \longrightarrow \mathbb{N}^{\#}\mathbb{N}R$ is a weak equivalence when R is a cofibrant commutative orthogonal ring spectrum. Since the underlying orthogonal spectrum of R is not cofibrant, we must work harder here. We use the notations and results of [7, §§15, 16], where the structure of cofibrant commutative orthogonal ring spectra is analyzed and the precisely analogous proof comparing commutative symmetric ring spectra and commutative orthogonal ring spectrum is given.

We may assume that R is a $\mathbb{C}F^{+}I$-cell complex (see [7, 15.1]), where $\mathbb{C}$ is the free commutative orthogonal ring spectrum functor, and we claim first that η is a weak equivalence when $R = \mathbb{C}X$ for a positive cofibrant orthogonal spectrum X. It suffices to prove that $\eta : X^{(i)}/\Sigma_i \longrightarrow \mathbb{N}^{\#}\mathbb{N}(X^{(i)}/\Sigma_i)$ is a weak equivalence for $i \geq 1$. On the right, $\mathbb{N}(X^{(i)}/\Sigma_i) \cong (\mathbb{N}X)^{(i)}/\Sigma_i$, and $\mathbb{N}X$ is a cofibrant S-module. Consider the commutative diagram

$$\begin{array}{ccccc}
E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} & \xrightarrow{\ \eta\ } & \mathbb{N}^{\#}\mathbb{N}(E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)}) & \cong & \mathbb{N}^{\#}(E\Sigma_{i+} \wedge_{\Sigma_i} (\mathbb{N}X)^{(i)}) \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle \mathbb{N}^{\#}\mathbb{N}q} & & \Big\downarrow{\scriptstyle \mathbb{N}^{\#}q} \\
X^{(i)}/\Sigma_i & \xrightarrow[\ \eta\ ]{} & \mathbb{N}^{\#}\mathbb{N}(X^{(i)}/\Sigma_i) & \cong & \mathbb{N}^{\#}((\mathbb{N}X)^{(i)}/\Sigma_i).
\end{array}$$

The q are the evident quotient maps, and the left and right arrows q are weak equivalences by [7, 15.5] and [1, III.5.1]. The top map η is a weak equivalence by Proposition 2.5 since an induction up the cellular filtration of $E\Sigma_i$, the successive subquotients of which are wedges of copies of $\Sigma_{i+} \wedge S^n$, shows that $E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)}$ is positive cofibrant since $X^{(i)}$ is positive cofibrant.

By passage to colimits, as in the analogous proof in [7, §16], the result for general R follows from the result for a $\mathbb{C}F^{+}I$-cell complex that is constructed in finitely many stages. We have proven the result when R requires only a single stage, and we assume the result when R is constructed in n stages. Thus suppose that R is constructed in $n+1$ stages. Then R is a pushout (in the category of commutative orthogonal ring spectra) of the form $R_n \wedge_{\mathbb{C}X} \mathbb{C}Y$, where R_n is constructed in n-stages and $X \longrightarrow Y$ is a wedge of maps in $F^{+}I$. As in the proof of [7, 15.9], $R \cong B(R_n, \mathbb{C}X, \mathbb{C}T)$, where T is a suitable wedge of orthogonal spectra F_rS^0. The bar construction here is the geometric realization of a proper simplicial orthogonal spectrum and $\mathbb{N}$ commutes with geometric realization. Tracing through the cofibration sequences used in the proof of the invariance of bar constructions in [1, X.4], we see that it suffices to show that η is a weak equivalence on the commutative orthogonal ring spectrum

$$R_n \wedge (\mathbb{C}X)^{(q)} \wedge \mathbb{C}T \cong R_n \wedge \mathbb{C}(X \vee \cdots \vee X \vee T)$$

of q-simplices for each q. By the definition of $\mathbb{C}F^+I$-cell complexes, we see that this smash product (= pushout) can be constructed in n-stages, hence the conclusion follows from the induction hypothesis. □

The proof of Theorem 0.6. For a cofibrant commutative orthogonal ring spectrum R, we must prove that the unit $\eta : X \longrightarrow \mathbb{N}^{\#}\mathbb{N}X$ of the adjunction is a weak equivalence when X is a cofibrant R-module, R-algebra, or commutative R-algebra. For R-modules, this reduces as in [7, §16] to an application of [1, III.3.8], which gives that the functor $\mathbb{N}R \wedge_S (-)$ preserves weak equivalences. The case of R-algebras follows since a cofibrant R-algebra is cofibrant as an R-module [7, 12.1]. The case of commutative R-algebras follows from the previous proof since a cofibrant commutative R-algebra is cofibrant as a commutative orthogonal ring spectrum. □

Remark 2.6. Consider the diagram

$$\Sigma\mathscr{I} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{I}\mathscr{S} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{M},$$

where $\Sigma\mathscr{S}$ is the category of symmetric spectra and $\mathbb{U}$ and $\mathbb{P}$ are the forgetful and prolongation functors of [7] (see Example 1.10). By (4.8) below, we have

$$(\mathbb{U} \circ \mathbb{N}^{\#})(M)(\mathbf{n}) \cong \mathscr{M}((S_S^{-1})^{(n)}, M)$$

as Σ_n-spaces, where S_S^{-1} is the canonical cofibrant (-1)-sphere in the category of S-modules. This is the right adjoint $\mathscr{M} \longrightarrow \Sigma\mathscr{S}$ used by Schwede [11], and $\mathbb{N} \circ \mathbb{P}$ is its left adjoint. Thus the adjunction studied in [11] is the composite of the adjunctions $(\mathbb{P}, \mathbb{U})$ and $(\mathbb{N}, \mathbb{N}^{\#})$.

3. Model structures and homotopical proofs

To complete the proofs and to place our results in context, we recall the relationship of $\mathscr{I}\mathscr{S}$ and $\mathscr{M}$ to various other model categories of prespectra and spectra. We have two categories of prespectra, coordinatized and coordinate-free. In [7], the former was described as the category of $\mathscr{N}$-spectra, where $\mathscr{N}$ is the discrete category with objects n, $n \geq 0$, and it was given a stable model structure and a positive stable model structure. We denote this category by $\mathscr{N}\mathscr{S}$. Then [7] gives the following result.

Proposition 3.1. *The forgetful functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{N}\mathscr{S}$ has a left adjoint prolongation functor $\mathbb{P} : \mathscr{N}\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$, and the pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence with respect to either the stable or the positive stable model structures.*

We shall focus on prespectra in the coordinate-free sense of [5, 1]. Thus a *prespectrum* X consists of based spaces $X(V)$ and based maps $\sigma : \Sigma^{W-V}X(V) \longrightarrow X(W)$, where V ranges over the finite dimensional sub inner product spaces of a countably infinite dimensional real inner product space U, which we may take to be $U = \mathbb{R}^\infty$. Let $\mathscr{P}$ denote the resulting category of prespectra. Exactly as in [7], $\mathscr{P}$ has stable and positive stable model structures.

Remark 3.2. We obtain a forgetful functor $\mathbb{U} : \mathscr{P} \longrightarrow \mathscr{N}\mathscr{S}$ by restricting to the subspaces $\mathbb{R}^n$ of U. We also have an underlying coordinate-free prespectrum functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}$. The composite of these two functors is the functor $\mathbb{U}$ of Proposition 3.1. All three functors $\mathbb{U}$ have left adjoints $\mathbb{P}$ given by Example 1.10, and Proposition 3.1 remains true with $\mathbb{U}$ replaced by either of our new functors $\mathbb{U}$.

A prespectrum X is an *Ω-spectrum* if its adjoint maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(W)$ are weak equivalences; it is a *positive* *Ω-spectrum* if these maps are weak equivalences for $V \neq 0$; it is a *spectrum* if these maps are homeomorphisms. Let $\mathscr{S}$ denote the category of spectra. It is given a stable model structure in [1, VII§§4, 5]. The following result is implicit in [1, 7]. We indicate the proof at the end of the section.

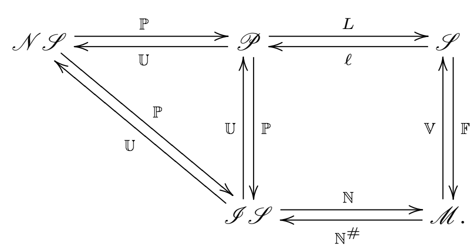
Proposition 3.3. *The forgetful functor $\ell : \mathscr{S} \longrightarrow \mathscr{P}$ has a left adjoint spectrification functor $L : \mathscr{P} \longrightarrow \mathscr{S}$, and the pair (L, ℓ) is a Quillen equivalence with respect to the stable model structures.*

Remark 3.4. This result applies to both the coordinatized and coordinate-free settings. The restriction of $\mathbb{U} : \mathscr{P} \longrightarrow \mathscr{N}\mathscr{S}$ to the respective subcategories of spectra is an equivalence of categories [5, I.2.4]; both $\mathbb{U}$ and its restriction to spectra are the right adjoints of Quillen equivalences of model categories.

There is an evident underlying spectrum functor $\mathscr{M} \longrightarrow \mathscr{S}$; that is, an S-module is a spectrum with additional structure. This functor is not the right adjoint of a Quillen equivalence, but it is weakly equivalent to such a right adjoint. The following result is implicit in [1], as we explain at the end of the section.

Proposition 3.5. *There is a "free functor" $\mathbb{F} : \mathscr{S} \longrightarrow \mathscr{M}$ that has a right adjoint $\mathbb{V} : \mathscr{M} \longrightarrow \mathscr{S}$, and there is a natural weak equivalence of spectra $M \longrightarrow \mathbb{V}M$ for S-modules M. The pair $(\mathbb{F}, \mathbb{V})$ is a Quillen equivalence of stable model categories.*

Thus, even before constructing $\mathbb{N}^*$ we have Quillen equivalences relating the categories $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, $\mathscr{P}$, $\mathscr{S}$, and $\mathscr{M}$, so that we know that all of our homotopy categories are equivalent. Of course, these equivalences are much less highly structured than the one we are after since $\mathscr{N}\mathscr{S}$, $\mathscr{P}$, and $\mathscr{S}$ are not symmetric monoidal under their smash products. To help orient the reader, we display our Quillen equivalences in the following (noncommutative) diagram:



To make our comparisons, we recall the definition of the model structures on all of our categories. The homotopy groups of a prespectrum, spectrum, orthogonal spectrum, or S-module are the homotopy groups of its underlying coordinatized prespectrum. A map in any of these categories is a weak equivalence if it induces an isomorphism of homotopy groups.

A map of spectra is a q-fibration if each of its component maps of spaces is a Serre fibration, and the functor $\mathbb{V}$ creates the q-fibrations of S-modules. The (positive) q-fibrations of prespectra or of orthogonal spectra are the (positive) level

Serre fibrations such that certain diagrams are homotopy pullbacks [7, 9.5]; all that we need to know about the latter condition is that it holds automatically for maps between (positive) Ω-spectra.

In all of our categories, the q-cofibrations are the maps that satisfy the LLP with respect to the acyclic q-fibrations. Equivalently, they are the retracts of relative cell complexes in the respective categories. These cell complexes are defined as usual in terms of attaching maps whose domains are appropriate "spheres". We have nth space or Vth space evaluation functors from the categories $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, $\mathscr{P}$, and $\mathscr{S}$ to the category $\mathscr{T}$. These have left adjoint shift desuspension functors, denoted

$$F_n : \mathscr{T} \longrightarrow \mathscr{N}\mathscr{S}, \quad F_V : \mathscr{T} \longrightarrow \mathscr{I}\mathscr{S}, \quad F_V : \mathscr{T} \longrightarrow \mathscr{P}, \quad \text{and} \quad \Sigma^\infty_V : \mathscr{T} \longrightarrow \mathscr{S}.$$

We write $F_n = F_{\mathbb{R}^n}$ in $\mathscr{P}$ and $\mathscr{I}\mathscr{S}$ and $\Sigma^\infty_n = \Sigma^\infty_{\mathbb{R}^n}$ in $\mathscr{S}$. Obvious isomorphisms between right adjoints imply isomorphisms between left adjoints

$$\mathbb{P}F_n \cong F_n \quad \text{and} \quad LF_V \cong \Sigma^\infty_V.$$

The domains of attaching maps are the F_nS^q in $\mathscr{N}\mathscr{S}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{P}$ where, for the positive stable model structures, we restrict to $n > 0$. The domains of attaching maps are the $\Sigma^\infty_n S^q$ in $\mathscr{S}$ and the $\mathbb{F}\Sigma^\infty_n S^q$ in $\mathscr{M}$.

Returning to $\mathbb{N}^*$, we will obtain the following description of its values on objects.

Lemma 3.6. *For an object $V \neq 0$ of $\mathscr{I}$, the S-module $\mathbb{N}^*(V)$ is non-canonically isomorphic to $\mathbb{F}\Sigma^\infty_V S^0$.*

The subtlety in the construction of $\mathbb{N}^*$ lies in its orthogonal functoriality. We cannot just define $\mathbb{N}^*(V)$ to be $\mathbb{F}\Sigma^\infty_V S^0$, since that would not give a functor of V. We begin our proofs with the following observation.

Lemma 3.7. *For S-modules M, $\mathbb{N}^\# M$ is a positive Ω-spectrum.*

Proof. We have $(\mathbb{N}^\# M)(V) = \mathscr{M}(\mathbb{N}^*(V), M)$. For $V \subset W$,

$$\Omega^{W-V}(\mathbb{N}^\# M)(W) \cong \mathscr{M}(\Sigma^{W-V}\mathbb{N}^*(W), M)$$

and the adjoint structure map $\tilde{\sigma} : \mathbb{N}^\#(V) \longrightarrow \Omega^{W-V}\mathbb{N}^\#(W)$ is induced from the evaluation map $\varepsilon : \Sigma^{W-V}\mathbb{N}^*(W) \longrightarrow \mathbb{N}^*(V)$. Let $V \neq 0$. Then ε is a weak equivalence between cofibrant S-modules and $\tilde{\sigma}$ is thus a weak equivalence. □

Proof of Lemma 2.3. By Lemma 3.6 and Proposition 3.5, for $V \subset U$ we have

$$(\mathbb{N}^\# M)(V) \cong \mathscr{M}(\mathbb{F}\Sigma^\infty_V S^0, M) \cong \mathscr{S}(\Sigma^\infty_V S^0, \mathbb{V}M) \cong \mathscr{T}(S^0, (\mathbb{V}M)(V)) = (\mathbb{V}M)(V),$$

which is weakly equivalent to $M(V)$. Since a map of orthogonal positive Ω-spectra or of S-modules is a weak equivalence if and only if its map on Vth spaces is a weak equivalence for $V \neq 0$ in U, this implies that a map f of S-modules is a weak equivalence if and only if $\mathbb{N}^\# f$ is a weak equivalence of orthogonal spectra. □

Proof of Lemma 2.4. Let $f : M \longrightarrow N$ be a q-fibration of S-modules. We must show that $\mathbb{N}^\# f$ is a positive q-fibration of orthogonal spectra. Since $\mathbb{N}^\# f$ is a map of positive Ω-spectra, we need only show that the Vth space map of $\mathbb{N}^\# f$ is a Serre fibration for $V \neq 0$, and it suffices to show this for $V = \mathbb{R}^n$, $n > 0$. By [1, VII.4.6], f is a q-fibration if and only if it satisfies the RLP with respect to all maps

$$i_0 : \mathbb{F}\Sigma^\infty_n CS^q \longrightarrow \mathbb{F}\Sigma^\infty_n CS^q \wedge I_+.$$

An easy adjunction argument from the isomorphism $\mathbb{N}^*(\mathbb{R}^n) \cong \mathbb{F}\Sigma_n^\infty S^0$ and the fact that $\mathbb{F}$ and the Σ_n^∞ are right exact shows that

$$f_* : \mathscr{M}(\mathbb{N}^*(\mathbb{R}^n), M) \longrightarrow \mathscr{M}(\mathbb{N}^*(\mathbb{R}^n), N)$$

satisfies the RLP with respect to the maps $i_0 : CS^q \longrightarrow CS^q \wedge I_+$ and is therefore a Serre fibration. □

Remark 3.8. In principle, the specified RLP property states that f_* is a based Serre fibration, whereas what we need is that f_* is a classical Serre fibration, that is, a based map that satisfies the RLP in $\mathscr{T}$ with respect to the maps $i_0 : D^q_+ \longrightarrow D^q_+ \wedge I$. However, when $n > 0$, f_* is isomorphic to the loop of a based Serre fibration, and the loop of a based Serre fibration is a classical Serre fibration.

Proof of Proposition 2.5. We first prove that $\eta : F_n A \to \mathbb{N}^{\#}\mathbb{N}F_n A$ is a weak equivalence for any based CW complex A; the only case we need is when A is a sphere. Here $F_n = \mathbb{P}F_n$ and it suffices to prove that the adjoint map of prespectra

$$\bar{\eta} : F_n A \to \mathbb{U}\mathbb{N}^{\#}\mathbb{N}F_n A$$

is a weak equivalence. By a check of definitions and use of Lemma 3.6,

$$\mathbb{N}F_n A \cong \mathbb{N}F_n S^0 \wedge A \cong \mathbb{N}^*(\mathbb{R}^n) \wedge A \cong \mathbb{F}\Sigma_n^\infty S^0 \wedge A \cong \mathbb{F}\Sigma_n^\infty A.$$

Therefore, using Lemma 3.6 and Proposition 3.5, we have weak equivalences

$$\begin{aligned}(\mathbb{N}^{\#}\mathbb{N}F_n A)(\mathbb{R}^q) \cong \mathscr{M}(\mathbb{F}\Sigma_q^\infty S^0, \mathbb{F}\Sigma_n^\infty A) \quad &\cong \quad \mathscr{S}(\Sigma_q^\infty S^0, \mathbb{V}\mathbb{F}\Sigma_n^\infty A)\\ &\simeq \quad \mathscr{S}(\Sigma_q^\infty S^0, \Sigma_n^\infty A) \cong (\Sigma_n^\infty A)(\mathbb{R}^q).\end{aligned}$$

Tracing through definitions, we find that, up to homotopy, the structural maps coincide under these weak equivalences with those of $\Sigma_n^\infty A \cong LF_n A$ and the map $\bar{\eta}$ induces the same map of homotopy groups as the unit $F_n A \longrightarrow \ell L F_n A$ of the adjunction of Lemma 3.3. Therefore $\bar{\eta}$ is a weak equivalence. By standard results on the homotopy groups of prespectra [7, §7] and their analogues for spectra, we see that the class of orthogonal spectra for which η is a weak equivalence is closed under wedges, pushouts along h-cofibrations, sequential colimits of h-cofibrations, and retracts. Therefore η is a weak equivalence for all cofibrant orthogonal spectra. □

The proofs of Propositions 3.1, 3.3, and 3.5. In each of these three propositions, it is immediate that the right adjoint creates weak equivalences and preserves q-fibrations. It remains to show that the units of the adjunctions are weak equivalences when evaluated on cofibrant objects. For Proposition 3.1, this is [7, 10.3]. For Proposition 3.3, Lemma 3.9 below gives the conclusion. For Proposition 3.5, Lemma 3.11 below gives the conclusion. □

A prespectrum X is an inclusion prespectrum if its adjoint structure maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V} X(W)$ are inclusions, and this holds when X is cofibrant. The following result is immediate from [5, I.2.2].

Lemma 3.9. *Let X be an inclusion prespectrum. Then*

$$LX(V) = \operatorname{colim}_{W \supset V} \Omega^{W-V} X(W).$$

The Vth map of the unit $\eta : X \longrightarrow \ell L X$ of the (L, ℓ)-adjunction is the map from the initial term $X(V)$ into the colimit, and η is a weak equivalence of prespectra.

Remark 3.10. For later use, we note a variant. We call X a positive inclusion prespectrum if $\tilde{\sigma}$ is an inclusion when $V \neq 0$. The description of $LX(V)$ is still valid and η is still a weak equivalence.

The notion of a *tame* spectrum is defined in [1, I.2.4]; cofibrant spectra are tame.

Lemma 3.11. *For tame spectra E, the unit $\eta : E \longrightarrow \mathbb{V}\mathbb{F}E$ is a weak equivalence. For S-modules M, there is a natural weak equivalence $\tilde{\lambda} : M \longrightarrow \mathbb{V}M$.*

Proof. With the notations of [1, I.4.1, I.5.1, I.7.1]

$$\mathbb{F}E = S \wedge_{\mathscr{L}} \mathbb{L}E \quad \text{and} \quad \mathbb{V}M = F_{\mathscr{L}}(S, M).$$

The unit η is the composite of the homotopy equivalence $\eta : E \longrightarrow \mathbb{L}E$ of [1, I.4.6], the weak equivalence $\tilde{\lambda} : \mathbb{L}E \longrightarrow F_{\mathscr{L}}(S, \mathbb{L}E)$ of [1, I.8.7], and the isomorphism $F_{\mathscr{L}}(S, \mathbb{L}E) \cong F_{\mathscr{L}}(S, S \wedge_{\mathscr{L}} \mathbb{L}E)$ of [1, II.2.5]. The result [1, I.8.7] also gives the weak equivalence $\tilde{\lambda} : M \longrightarrow \mathbb{V}M$. □

4. The construction of the functor $\mathbb{N}^*$

We prove Theorem 2.1 here. Implicitly, we shall give two constructions of the functor $\mathbb{N}^*$. The theory of S-modules is based on a functor called the twisted half-smash product, denoted $\ltimes$, the definitive construction of which is due to Cole [1, App]. The theory of orthogonal spectra is the theory of diagram spaces with domain category $\mathscr{I}_S$. Both $\ltimes$ and $\mathscr{I}_S$ are defined in terms of Thom spaces associated to spaces of linear isometries. We first define $\mathbb{N}^*$ in terms of twisted half-smash products. We then outline the definition of twisted half-smash products in terms of Thom spaces and redescribe $\mathbb{N}^*$ in those terms. That will make the connection with the category $\mathscr{I}_S$ transparent, since the morphism spaces of $\mathscr{I}_S$ are Thom spaces closely related to those used to define the relevant twisted half-smash products.

Here we allow the universe U on which we index our coordinate-free prespectra and spectra to vary. We write $\mathscr{P}^U$ and $\mathscr{S}^U$ for the categories of prespectra and spectra indexed on U. We have a forgetful functor $\ell : \mathscr{S}^U \longrightarrow \mathscr{P}^U$ with a left adjoint spectrification functor $L : \mathscr{P}^U \longrightarrow \mathscr{S}^U$. We have a suspension spectrum functor Σ^U that is left adjoint to the zeroth space functor Ω^U. Let $S^U = \Sigma^U(S^0)$. The functors Σ^U and Ω^U are usually denoted Σ^∞ and Ω^∞, but we wish to emphasize the choice of universe rather than its infinite dimensionality. We write Σ^∞ and Ω^∞ when $U = \mathbb{R}^\infty$, and we then write $S^U = S$. More generally, for a finite dimensional sub inner product space V of U, we have a shift desuspension functor $\Sigma^U_V : \mathscr{T} \longrightarrow \mathscr{S}^U$, denoted Σ^∞_V when $U = \mathbb{R}^\infty$. It is left adjoint to evaluation at V.

For inner product spaces U and U', let $\mathscr{I}(U, U')$ be the space of linear isometries $U \longrightarrow U'$, not necessarily isomorphisms. It is contractible when U' is infinite dimensional [9, 1.3]. We have a twisted half-smash functor

$$\mathscr{I}(U, U') \ltimes (-) : \mathscr{S}^U \longrightarrow \mathscr{S}^{U'},$$

whose definition we shall recall shortly. It is a "change of universe functor" that converts spectra indexed on U to spectra indexed on U' in a well-structured way.

Now fix $U = \mathbb{R}^\infty$ and consider the universes $V \otimes U$ for $V \in \mathscr{I}$. Identify V with $V \otimes \mathbb{R} \subset V \otimes U$. In the language of [1], we define

$$\mathbb{N}^*(V) = S \wedge_{\mathscr{L}} (\mathscr{I}(V \otimes U, U) \ltimes \Sigma^{V \otimes U}_V(S^0)). \tag{4.1}$$

To make sense of this, we must recall some of the definitional framework of [1]. We have the linear isometries operad $\mathscr{L}$ with nth space $\mathscr{L}(n) = \mathscr{I}(U^n, U)$.

The operad structure maps are given by compositions and direct sums of linear isometries, and they specialize to give a monoid structure on $\mathscr{L}(1)$, a left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$, and a right action of $\mathscr{L}(1)\times\mathscr{L}(1)$ on $\mathscr{L}(2)$. For a spectrum $E\in\mathscr{S}$, $\mathscr{L}(1)\ltimes E$ is denoted $\mathbb{L}E$. The monoid structure on $\mathscr{L}(1)$ induces a monad structure on the functor $\mathbb{L}:\mathscr{S}\longrightarrow\mathscr{S}$.

Definition 4.2. An $\mathbb{L}$-spectrum is an algebra over the monad $\mathbb{L}$. Let $\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra. The functor $\mathbb{L}$ takes values in $\mathbb{L}$-spectra and gives the free $\mathbb{L}$-spectrum functor $\mathbb{L}:\mathscr{S}\longrightarrow\mathscr{S}[\mathbb{L}]$.

By [1, I§5], we have an "operadic smash product"

$$E\wedge_{\mathscr{L}} E' = \mathscr{L}(2)\ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)} E\,\bar{\wedge}\,E' \tag{4.3}$$

between $\mathbb{L}$-spectra E and E', where $E\bar{\wedge}E'$ is the external smash product indexed on U^2 [1, I§2]. The sphere S is an $\mathbb{L}$-spectrum, and the action of $\mathscr{L}(1)$ by composition on $\mathscr{I}(V\otimes U,U)$ induces a structure of $\mathbb{L}$-spectrum on $\mathscr{I}(V\otimes U,U)\ltimes\Sigma^{V\otimes U}_V(S^0)$.

An $\mathbb{L}$-spectrum E has a unit map $\lambda: S\wedge_{\mathscr{L}} E\longrightarrow E$ that is always a weak equivalence and sometimes an isomorphism [1, I.§8 and II§1]. In particular, λ is an isomorphism when $E=S$, when $E=S\wedge_{\mathscr{L}}E'$ for any $\mathbb{L}$-spectrum E', and when E is the operadic smash product of two S-modules [1, I.8.2, II.1.2].

Definition 4.4. An S-module is an $\mathbb{L}$-spectrum E such that λ is an isomorphism. The smash product $\wedge_S$ in the category $\mathscr{M}$ of S-modules (denoted $\wedge$ earlier) is the restriction to S-modules of $\wedge_{\mathscr{L}}$. The functor $\mathbb{J}:\mathscr{S}[\mathbb{L}]\longrightarrow\mathscr{M}$ specified by

$$\mathbb{J}(E)=S\wedge_{\mathscr{L}}E$$

carries $\mathbb{L}$-spectra to weakly equivalent S-modules. The functor $\mathbb{F}:\mathscr{S}\longrightarrow\mathscr{M}$ of Proposition 3.5 is the composite $\mathbb{J}\circ\mathbb{L}$.

We can rewrite (4.1) as

$$\mathbb{N}^*(V)=\mathbb{J}(\mathscr{I}(V\otimes U,U)\ltimes\Sigma^{V\otimes U}_V(S^0)). \tag{4.5}$$

This makes sense of (4.1). It even makes sense when $V=\{0\}$. Here we interpret spectra indexed on the universe $\{0\}$ as based spaces. The space $\mathscr{I}(\{0\},U)$ is a point, namely the inclusion $i^U:\{0\}\longrightarrow U$. The functor $i^U_*=i^U\ltimes(-):\mathscr{T}\longrightarrow\mathscr{S}^U$ is left adjoint to the zeroth space functor, hence $i^U_*=\Sigma^U$. Thus (4.5) specializes to give $\mathbb{N}^*(0)=\mathbb{J}S$ and, as we have noted, $\lambda:\mathbb{J}S\longrightarrow S$ is an isomorphism.

The evident homeomorphisms

$$\Sigma^{V'-V}A\wedge\Sigma^{W'-W}B\cong\Sigma^{(V'-V)\oplus(W'-W)}(A\wedge B)$$

for $V\subset V'$ in $V\otimes U$ and $W\subset W'$ in $W\otimes U$, induce an isomorphism

$$\Sigma^{V\otimes U}_V(A)\,\bar{\wedge}\,\Sigma^{W\otimes U}_W(B)\cong\Sigma^{(V\oplus W)\otimes U}_{V\oplus W}(A\wedge B) \tag{4.6}$$

upon spectrification, where

$$\bar{\wedge}:\mathscr{S}^{V\otimes U}\times\mathscr{S}^{W\otimes U}\longrightarrow\mathscr{S}^{(V\oplus W)\otimes U}$$

is the external smash product. Using the formal properties [1, A.6.2 and A.6.3] of twisted half-smash products, the canonical homeomorphism

$$\mathscr{L}(2)\times_{\mathscr{L}(1)\times\mathscr{L}(1)}(\mathscr{I}(V\otimes U,U)\times\mathscr{I}(W\otimes U,U))\cong\mathscr{I}((V\oplus W)\otimes U,U)$$

given by Hopkins' lemma [1, I.5.4], and the associative and unital properties of $\wedge_{\mathscr{L}}$ of [1, I§§5,8], we see that the isomorphisms (4.6) induce isomorphisms

$$\phi : \mathbb{N}^*(V) \wedge_S \mathbb{N}^*(W) \longrightarrow \mathbb{N}^*(V \oplus W). \tag{4.7}$$

We may identify $\mathbb{R}^n \otimes U$ with U^n. With the notations of [1, II.1.7], the canonical cofibrant sphere S-modules are $S^n_S = \mathbb{F}S^n$, where S^n is the canonical sphere spectrum. For $n \geq 0$, $S^{-n} = \Sigma^\infty_n S^0$. Thus $\mathbb{N}^*(\mathbb{R}) = S^{-1}_S$ and, for $n \geq 1$,

$$\mathbb{N}^*(\mathbb{R}^n) \cong (S^{-1}_S)^{(n)} \cong S^{-n}_S = \mathbb{F}\Sigma^\infty_n S^0, \tag{4.8}$$

where the middle isomorphism is only canonical up to homotopy. The first isomorphism is Σ_n-equivariant, which is the essential point of Remark 2.6. If $\dim V = n$, $n > 0$, then $\mathbb{N}^*(V)$ is isomorphic to $\mathbb{N}^*(\mathbb{R}^n)$ and is thus cofibrant. Moreover, Σ^∞_V is isomorphic to Σ^∞_n, so that Lemma 3.6 holds.

Intuitively, (4.5) gives a coordinate-free generalization of the canonical cofibrant negative sphere S-modules used in [1]. We must still prove the contravariant functoriality in V of $\mathbb{N}^*(V)$, check the naturality of ϕ, and prove that the evaluation maps $\varepsilon : \mathbb{N}^*(V) \wedge S^V \longrightarrow \mathbb{N}^*(0)$ are weak equivalences. While this can be done directly in terms of the definitions on hand, it is more illuminating to review the definition of the half-smash product and relate it directly to the morphism spaces of the category $\mathscr{I}_S$. We introduce a category Θ of Thom spaces for this purpose. Its objects will be inclusions $V \subset U$, which we secretly think of as symbols $\frac{U}{V}$ since these objects are closely related to the functors Σ^U_V used in our definition of $\mathbb{N}^*$. We think of $T^{U,U'}_{V,V'}$ in the following definition as a slightly abbreviated notation for the morphism space $\Theta(\frac{U}{V}, \frac{U'}{V'})$.

Definition 4.9. Let U and U' be finite or countably infinite dimensional real inner product spaces. Let V and V' be finite dimensional sub inner product spaces of U and U'. Let $\mathscr{I}^{U,U'}_{V,V'}$ be the space of linear isometries $f : U \longrightarrow U'$ such that $f(V) \subset V'$. For $V \subset W$, let $W - V$ denote the orthogonal complement of V in W. Let $E^{U,U'}_{V,V'}$ be the subbundle of the product bundle $\mathscr{I}^{U,U'}_{V,V'} \times V'$ whose points are the pairs (f, x) such that $x \in V' - f(V)$. Let $T^{U,U'}_{V,V'}$ be the Thom space of $E^{U,U'}_{V,V'}$; it is obtained by applying fiberwise one-point compactification and identifying all of the points at ∞. The spaces $T^{U,U'}_{V,V'}$ are the morphism spaces of a based topological *Thom category* Θ whose objects are the inclusions $V \subset U$. Composition

$$\circ : T^{U',U''}_{V',V''} \wedge T^{U,U'}_{V,V'} \longrightarrow T^{U,U''}_{V,V''} \tag{4.10}$$

is defined by $(g, y) \circ (f, x) = (g \circ f, g(x) + y)$. Points $(\mathrm{id}_U, 0)$ give identity morphisms. If $\mathscr{I}^{U,U'}_{V,V'}$ is empty, $T^{U,U'}_{V,V'}$ is a point. For any U and any object $V' \subset U'$,

$$T^{U,U'}_{0,V'} = \mathscr{I}(U, U')_+ \wedge S^{V'}. \tag{4.11}$$

The category Θ is symmetric monoidal with respect to direct sums of inner product spaces. On morphism spaces, the map

$$\oplus : T^{U_1,U_1'}_{V_1,V_1'} \wedge T^{U_2,U_2'}_{V_2,V_2'} \longrightarrow T^{U_1\oplus U_2,U_1'\oplus U_2'}_{V_1\oplus V_2,V_1'\oplus V_2'} \tag{4.12}$$

sends $((f_1, x_1), (f_2, x_2))$ to $(f_1 \oplus f_2, x_1 + x_2)$. Note that we have a trivialization isomorphism of bundles

$$E^{U,U'}_{V,V'} \times V \cong \mathscr{I}^{U,U'}_{V,V'} \times V'$$

and thus an "untwisting isomorphism"

$$T^{U,U'}_{V,V'} \wedge S^V \cong \mathscr{I}^{U,U'}_{V,V'+} \wedge S^{V'}. \tag{4.13}$$

The theory of orthogonal spectra is based on the full sub-category of Θ whose objects are the identity inclusions $V \subset V$. If $V \subset V'$, then it is easily verified that

$$T^{V,V'}_{V,V'} \cong O(V')_+ \wedge_{O(V'-V)} S^{V'-V}.$$

Comparing with the definitions in [7, 2.1, 4.4], we obtain the following result.

Proposition 4.14. *The full subcategory of Θ whose objects are the identity inclusions $V \subset V$ is isomorphic as a based symmetric monoidal category to the category $\mathscr{I}_S$ such that an orthogonal spectrum is a continuous based functor $\mathscr{I}_S \longrightarrow \mathscr{T}$.*

We regard this isomorphism of categories as an identification.

In contrast, the twisted half-smash product is defined in terms of the full sub category of Θ whose objects are the inclusions $V \subset U$ in which U is infinite dimensional. The following definition and lemma are taken from [1, A.4.1–A.4.3].

Definition 4.15. Fix $V \subset U$ and U'. Define a prespectrum $T^{U,U'}_{V,-}$ indexed on U' by letting its V'th space be $T^{U,U'}_{V,V'}$ and letting its structure map for $V' \subset W'$ be induced by passage to Thom spaces from the evident bundle map

$$E^{U,U'}_{V,V'} \oplus (W' - V') \cong E^{U,U'}_{V,W'}|_{\mathscr{I}^{U,U'}_{V,V'}} \longrightarrow E^{U,U'}_{V,W'}.$$

For $V \subset W$, define a map $\tau : \Sigma^{W-V} T^{U,U'}_{W,-} \longrightarrow T^{U,U'}_{V,-}$ of prespectra indexed on U' by letting its V'th map be induced by passage to Thom spaces from the evident bundle map

$$E^{U,U'}_{W,V'} \oplus (W - V) \cong E^{U,U'}_{V,V'}|_{\mathscr{I}^{U,U'}_{W,V'}} \longrightarrow E^{U,U'}_{V,V'}.$$

Observe that $T^{U,U'}_{V,-}$ is an inclusion prespectrum and define $M^{U,U'}_{V,-} = LT^{U,U'}_{V,-}$. (That is, write M consistently for Thom spectra associated to Thom prespectra T.)

Lemma 4.16. *The spectrified map*

$$L\tau : \Sigma^{W-V} M^{U,U'}_{W,-} \cong L(\Sigma^{W-V} T^{U,U'}_{W,-}) \longrightarrow LT^{U,U'}_{V,-} = M^{U,U'}_{V,-}$$

is an isomorphism of spectra indexed on U'.

The following is a special case of the definition of the twisted half smash product given in [1, A.5.1].

Definition 4.17. Let E be a spectrum indexed on U. Define

$$\mathscr{I}(U, U') \ltimes E = \operatorname{colim}_V M^{U,U'}_{V,-} \wedge EV$$

where the colimit (in $\mathscr{S}^{U'}$) is taken over the maps

$$M^{U,U'}_{V,-} \wedge EV \cong \Sigma^{W-V} M^{U,U'}_{W,-} \wedge EV \cong M^{U,U'}_{W,-} \wedge \Sigma^{W-V} EV \longrightarrow M^{U,U'}_{W,-} \wedge EW$$

induced by the structure maps of E.

The following result of Cole [1, A.3.9] is pivotal.

Proposition 4.18. *For based spaces A, there is a natural isomorphism*

$$\mathscr{I}(U, U') \ltimes \Sigma^U_V A \cong M^{U,U'}_{V,-} \wedge A$$

of spectra indexed on U'.

The proof is simply the observation that, in this case, the defining colimit stabilizes at the Vth stage. Returning to the fixed choice of $U = \mathbb{R}^\infty$ and taking $A = S^0$, this gives the alternative description

$$\mathbb{N}^*(V) \cong \mathbb{J}M^{V\otimes U,U}_{V,-}. \tag{4.19}$$

We regard this isomorphism as an identification and use it to show the required functoriality of the $\mathbb{N}^*(V)$.

Definition 4.20. Tensoring linear isometries $V \longrightarrow W$ with id_U, we obtain a map $\mu : T^{V,W}_{V,W} \longrightarrow T^{V\otimes U,W\otimes U}_{V,W}$. The evaluation maps $\mathbb{N}^*(W) \wedge \mathscr{I}_S(W,V) \longrightarrow \mathbb{N}^*(V)$ of the contravariant functor $\mathbb{N}^*$ are defined to be the maps

$$\begin{aligned} \mathbb{J}M^{W\otimes U,U}_{W,-} \wedge T^{V,W}_{V,W} &\xrightarrow{\mathrm{id}\wedge\mu} \mathbb{J}M^{W\otimes U,U}_{W,-} \wedge T^{V\otimes U,W\otimes U}_{V,W} \\ &\cong \mathbb{J}L(T^{W\otimes U,U}_{W,-} \wedge T^{V\otimes U,W\otimes U}_{V,W}) \\ &\xrightarrow{\mathbb{J}L(\circ)} \mathbb{J}L(T^{V\otimes U,U}_{V,-}) = \mathbb{J}M^{V\otimes U,U}_{V,-} \end{aligned}$$

induced by composition in the category Θ.

The naturality of the maps ϕ of (4.7) is now checked by rewriting these maps in terms of Thom complexes, using (4.12). Finally, we have the following lemma.

Lemma 4.21. *The evaluation map* $\varepsilon : \mathbb{N}^*(V) \wedge S^V \longrightarrow \mathbb{N}^*(0) \cong S$ *of the functor* $\mathbb{N}^*$ *is a weak equivalence. When* $V = \mathbb{R}$, ε *factors as the composite of the canonical isomorphism* $\mathbb{N}^*(\mathbb{R}) \wedge S^1 \cong S_S$ *and the canonical cofibrant approximation* $S_S \longrightarrow S$.

Proof. Using the untwisting isomorphisms

$$T^{V\otimes U,U}_{V,V'} \wedge S^V \cong \mathscr{I}^{V\otimes U,U}_{V,V'} \wedge S^{V'}$$

and applying L, we obtain an isomorphism of $\mathbb{L}$-spectra

$$M^{V\otimes U,U}_{V,-} \wedge S^V \cong \mathscr{I}(V\otimes U, U)_+ \wedge S.$$

Applying $\mathbb{J}$ and using $\mathbb{J}S \cong S$, we find by (4.19) that

$$\mathbb{N}^*(V) \wedge S^V \cong \mathbb{J}((\mathscr{I}(V\otimes U,U)_+ \wedge S) \cong \mathscr{I}(V\otimes U,U)_+ \wedge S. \tag{4.22}$$

Under this isomorphism, the evaluation map corresponds to the homotopy equivalence induced by the evident homotopy equivalence $\mathscr{I}(V\otimes U,U)_+ \longrightarrow S^0$. When $V = \mathbb{R}$, $\mathbb{L}S \cong \mathscr{L}(1)_+ \wedge S$ and the isomorphism just given is the cited canonical isomorphism $\mathbb{N}^*(\mathbb{R}) \wedge S^1 \cong S_S$. □

5. The functor $\mathbb{M}$ and its comparison with $\mathbb{N}$

We begin with the underlying prespectrum and spectrification functors:

$$\mathscr{I}\mathscr{S} \xrightarrow{\mathbb{U}} \mathscr{P} \xrightarrow{L} \mathscr{S}. \tag{5.1}$$

The functor $\mathbb{M}$ is the composite of the following three functors:

$$\mathscr{I}\mathscr{S} \xrightarrow{\mathbb{U}} \mathscr{P}[\mathbb{L}] \xrightarrow{L} \mathscr{S}[\mathbb{L}] \xrightarrow{\mathbb{J}} \mathscr{M}. \tag{5.2}$$

The categories $\mathscr{P}[\mathbb{L}]$ and $\mathscr{S}[\mathbb{L}]$ are the categories of $\mathbb{L}$-prespectra and $\mathbb{L}$-spectra. We have already indicated what $\mathbb{L}$-spectra are, and we shall define $\mathbb{L}$-prespectra shortly. The functors $\mathbb{U}$ and L in (5.2) are restrictions of those of (5.1), and the functor $\mathbb{J}$ is specified in Definition 4.4. Thus, to construct $\mathbb{M}$, we must define $\mathbb{L}$-prespectra and show that the functors $\mathbb{U}$ and L induce functors from orthogonal spectra to $\mathbb{L}$-prespectra and from $\mathbb{L}$-prespectra to $\mathbb{L}$-spectra. The arguments are already implicit in [9].

Definition 5.3. For a prespectrum X and a linear isometry $f : U \longrightarrow U$, define a prespectrum f^*X by $(f^*X)(V) = X(fV)$, with structure maps

$$X(fV) \wedge S^{W-V} \xrightarrow{\text{id} \wedge S^f} X(fV) \wedge S^{f(W-V)} \xrightarrow{\sigma} X(fW).$$

Observe that f^*X is a spectrum if X is a spectrum.

Definition 5.4. An $\mathbb{L}$-prespectrum is a prespectrum X together with maps $\xi(f) : X \longrightarrow f^*X$ of prespectra for all linear isometries $f : U \longrightarrow U$ such that $\xi(\text{id}) = \text{id}$, $\xi(f') \circ \xi(f) = \xi(f' \circ f)$, and the function

$$\xi : T^{U,U}_{V,W} \wedge X(V) \longrightarrow X(W)$$

specified by

$$\xi((f, w), x)) = \sigma(\xi(f)(x), w)$$

is a continuous.

In Definition 4.2, we defined a $\mathbb{L}$-spectrum to be an algebra over the monad $\mathbb{L}$. Inspection of the construction of twisted half smash products in §4 (compare [10, XXII.5.3]) gives the following consistency statement. While this equivalence of definitions is not difficult, we emphasize that it is central to the mathematics: it converts structures that are defined one isometry at a time into structures that are defined globally in terms of spaces of isometries.

Lemma 5.5. *An $\mathbb{L}$-spectrum is an $\mathbb{L}$-prespectrum that is a spectrum.*

Lemma 5.6. *The functor $L : \mathscr{P} \longrightarrow \mathscr{S}$ induces a functor $\mathscr{P}[\mathbb{L}] \longrightarrow \mathscr{S}[\mathbb{L}]$.*

Proof. For a linear isometry $f : U \longrightarrow U$, the functor $f^* : \mathscr{P} \longrightarrow \mathscr{P}$ and its restriction $f^* : \mathscr{S} \longrightarrow \mathscr{S}$ have left adjoints f_*. The functor f_* on spectra is defined in terms of the functor f_* on prespectra by $f_* = Lf_*\ell$ [5, II§1]. Let X be an $\mathbb{L}$–prespectrum. The map $\xi(f)$ has an adjoint map $f_*X \longrightarrow X$; applying L, we obtain a map $f_*LX \longrightarrow LX$, and its adjoint gives an induced map $\xi(f) : LX \longrightarrow f^*LX$. The properties $\xi(\text{id}) = \text{id}$ and $\xi(f' \circ f) = \xi(f') \circ \xi(f)$ are inherited from their prespectrum level analogues. Since the functor L is continuous and commutes with smash products with spaces, the continuity and equivariance condition on ξ in Definition 5.4 are also inherited by LX. □

Lemma 5.7. *The functor $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}$ takes values in $\mathscr{P}[\mathbb{L}]$.*

Proof. We obtain $\xi(f) : X \longrightarrow f^*X$ by applying the functoriality of X and the naturality of σ to the restrictions of linear isometries $f : U \longrightarrow U$ to linear isometric isomorphisms $f : V \longrightarrow f(V)$ for indexing spaces V. It is clear by functoriality that $\xi(\text{id}) = \text{id}$ and $\xi(f' \circ f) = \xi(f') \circ \xi(f)$. The continuity and equivariance condition on ξ in Definition 5.4 follow from the continuity, naturality and equivariance of σ. □

Remark 5.8. For general $\mathbb{L}$-prespectra, the map $\xi(f) : X(V) \longrightarrow X(fV)$ may depend on the linear isometry $f : U \longrightarrow U$, not just on its restriction $V \longrightarrow f(V)$. For those $\mathbb{L}$-prespectra that come from orthogonal spectra, this map does depend solely on the restriction of f. For this reason, there is no obvious functor $\mathscr{P}[\mathbb{L}] \longrightarrow \mathscr{I}\mathscr{S}$.

The following lemmas give the basic formal properties of the functor $\mathbb{M}$.

Lemma 5.9. *The functor* $\mathbb{M}$ *is right exact.*

Proof. The functors $\mathbb{U}$, L, and $\mathbb{J}$ are each right exact. This is obvious for $\mathbb{U}$ from the spacewise specification of colimits and smash products with based spaces, and it holds for L and $\mathbb{J}$ since these functors are left adjoints. □

Lemma 5.10. *There is a canonical isomorphism* $\lambda : \mathbb{M}(S_{\mathscr{I}}) \longrightarrow S$.

Proof. Clearly $\mathbb{U}(S_{\mathscr{I}})$ is the usual sphere prespectrum and thus $S = L\mathbb{U}(S_{\mathscr{I}})$. As we have already used, $\mathbb{J}S \cong S$ by [1, I.8.2]. □

Lemma 5.11. *The functor* $\mathbb{M}$ *is lax symmetric monoidal.*

Proof. We have $\mathbb{M}S_{\mathscr{I}} \cong S$, and we must construct a natural map

$$\phi : \mathbb{M}(X) \wedge_S \mathbb{M}(X') \longrightarrow \mathbb{M}(X \wedge_{\mathscr{I}} X')$$

for orthogonal spectra X and X'. The functor $\mathbb{J}$ is strong symmetric monoidal, so

$$(\mathbb{J}E) \wedge_S (\mathbb{J}E') \cong \mathbb{J}(E \wedge_{\mathscr{L}} E')$$

for $\mathbb{L}$-spectra E and E'. Thus it suffices to construct a map of $\mathbb{L}$-spectra

$$\phi : L\mathbb{U}(X) \wedge_{\mathscr{L}} L\mathbb{U}(X') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X'),$$

and ϕ is obtained by passage to coequalizers from a map

$$\xi : \mathscr{L}(2) \ltimes L\mathbb{U}(X) \,\bar{\wedge}\, L\mathbb{U}(X') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X').$$

To construct ξ, it suffices to construct maps

$$\xi(f) : L\mathbb{U}(X)(V) \wedge L\mathbb{U}(X')(V') \longrightarrow L\mathbb{U}(X \wedge_{\mathscr{I}} X')(f(V \oplus V'))$$

for linear isometries $f \in \mathscr{L}(2)$ such that the $\xi(f)$ satisfy analogs of the conditions in Definition 5.4 [10, XXII.5.3]. The functoriality of X and X' gives maps

$$X(V) \wedge X'(V') \longrightarrow X(f(V)) \wedge X'(f(V')).$$

The universal property (1.13) that relates the external and internal smash product of orthogonal spectra gives a map of $(\mathscr{I}_S \times \mathscr{I}_S)$-spaces

$$X \,\bar{\wedge}\, X' \longrightarrow (X \wedge_{\mathscr{I}} X') \circ \oplus,$$

and this gives maps

$$X(f(V)) \wedge X'(f(V')) \longrightarrow (X \wedge_{\mathscr{I}} X')(f(V \oplus V')).$$

We obtain the required maps $\xi(f)$ from the composites

$$X(V) \wedge X'(V') \longrightarrow (X \wedge_{\mathscr{I}} X')(f(V \oplus V'))$$

by passing to prespectra and then to spectra, as in the proof of Lemma 5.6. The coherence properties of the maps ϕ obtained from these maps ξ are shown by formal verifications from the properties of the various smash products. □

Turning to homotopical properties, we have the following observation. Recall Remark 3.10.

Lemma 5.12. *If* X *is a positive inclusion orthogonal spectrum, then there are natural isomorphisms*

$$\pi_*(X) \cong \pi_*(\mathbb{M}(X)).$$

Proof. We have a natural weak equivalence $\lambda : \mathbb{M}(X) = \mathbb{J}L\mathbb{U}(X) \longrightarrow L\mathbb{U}(X)$ for any X, and the unit map $\eta : \mathbb{U}X \longrightarrow \ell L\mathbb{U}(X)$ is also a weak equivalence. □

Now the following theorem compares $\mathbb{M}$ and $\mathbb{N}$.

Theorem 5.13. *There is a symmetric monoidal natural transformation*

$$\alpha : \mathbb{N} \longrightarrow \mathbb{M}$$

such that $\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$ is a weak equivalence if X is cofibrant.

Proof. Recall the definition $\mathbb{M}^* = \mathbb{M} \circ \mathbb{D}_{\mathscr{I}_S} : \mathscr{I}_S \longrightarrow \mathscr{M}$ (see Definition 1.3 and Notation 1.6). By Corollary 1.7, to construct α, it suffices to construct a natural transformation $\alpha^* : \mathbb{N}^* \longrightarrow \mathbb{M}^*$. Thus consider the orthogonal spectra V^* specified by $V^*(W) = \mathscr{I}_S(V, W)$. By definition, $\mathbb{M}^*V = \mathbb{M}V^* = \mathbb{J}L\mathbb{U}V^*$. By Proposition 4.14, for $W \subset U$,

$$\mathbb{U}V^*(W) \cong T^{V,W}_{V,W}.$$

For $V \subset W \subset Z$, the structural map agrees under this isomorphism with

$$\oplus : T^{V,W}_{V,W} \wedge S^{Z-W} \cong T^{V,W}_{V,W} \wedge T^{0,Z-W}_{0,Z-W} \longrightarrow T^{V,Z}_{V,Z}.$$

We obtain a map of Thom spaces $T^{V\otimes U,U}_{V,W} \longrightarrow T^{V,W}_{V,W}$ by restricting to V the linear isometries $f : V \otimes U \longrightarrow U$ such that $f(V) \subset W$. These maps define a map of prespectra $T^{V\otimes U,U}_{V,-} \longrightarrow \mathbb{U}V^*$. Applying $\mathbb{J}L$ and using (4.19), there results a map of S-modules

$$\alpha^* : \mathbb{N}^*(V) = \mathbb{J}LT^{V\otimes U,U}_{V,-} \longrightarrow \mathbb{J}L\mathbb{U}V^* = \mathbb{M}^*(V).$$

It is an exercise to verify from Proposition 4.14 and the definitions that these maps specify a natural transformation that is compatible with smash products. Using Theorem 1.5, define

$$\alpha = \alpha^* \otimes_{\mathscr{I}_S} \mathrm{id} : \mathbb{N}X = \mathbb{N}^* \otimes_{\mathscr{I}_S} X \longrightarrow \mathbb{M}^* \otimes_{\mathscr{I}_S} X \cong \mathbb{M}X.$$

Then α is a symmetric monoidal natural transformation, and it remains to prove that $\alpha : \mathbb{N}X \longrightarrow \mathbb{M}X$ is a weak equivalence if X is cofibrant. It suffices to assume that X is an $\mathbb{F}I$-cell complex (see [7, §6]). Since $\mathbb{M}$ and $\mathbb{N}$ are right exact, it follows by the usual induction up the cellular filtration of X, using commutations with suspension, wedges, pushouts, and colimits, that it suffices to prove that α is a weak equivalence when $X = V^*$. In this case, α reduces to α^*. Again by suspension, it suffices to prove that

$$\Sigma^V \alpha^* : \Sigma^V \mathbb{N}^*(V) \longrightarrow \Sigma^V \mathbb{M}^*(V)$$

is a weak equivalence. We have an untwisting isomorphism (4.22) for the source of $\Sigma^V \alpha^*$ and an analogous isomorphism

$$\mathbb{M}(V^*) \wedge S^V \cong \mathscr{I}(V, U)_+ \wedge S$$

for its target. Under these isomorphisms, $\Sigma^V \alpha^*$ is the smash product with S of the map $\mathscr{I}(V \otimes U, U) \longrightarrow \mathscr{I}(V, U)$ induced by restriction of linear isometries, and this map is a homotopy equivalence since its source and target are contractible. □

Remark 5.14. By Proposition 1.8, the functor $\mathbb{M}$ has right adjoint $\mathbb{M}^{\#}$. However, $\mathbb{M}$ does not appear to preserve cofibrant objects and does not appear to be part of a Quillen equivalence.

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May (with an appendix by M. Cole). Rings, modules, and algebras in stable homotopy theory. Mathematical Surveys and Monographs Vol. 47. 1997. American Mathematical Society.

[2] J. P. C. Greenlees and J. P. May. Localization and completion theorems for MU-module spectra. Annals of Math. 146(1997), 509-544.

[3] M. Hovey. Model categories. Mathematical Surveys and Monographs Vol. 63. 1999. American Mathematical Society.

[4] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. Journal Amer. Math. Soc. 13(2000), 149-208.

[5] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Vol. 1213. 1986.

[6] M. A. Mandell and J. P. May. Equivariant orthogonal spectra. Preprint. 1999.

[7] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. Preprint, 1998; revised, 1999.

[8] J. P. May. Pairings of categories and spectra. J. Pure and Applied Algebra. 19(1980), 299-346.

[9] J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞-ring spaces and E_∞-ring spectra. Springer Lecture Notes in Mathematics Vol. 577. 1977.

[10] J. P. May, et al. Equivariant homotopy and cohomology theory. CBMS Regional Conference Series in Mathematics, Number 91. American Mathematical Society. 1996.

[11] S. Schwede. S-modules and symmetric spectra. Preprint, 1998; revised, 1999.

The University of Chicago, Chicago, IL 60637 USA

E-mail address: `mandell@math.uchicago.edu`

E-mail address: `may@math.uchicago.edu`