E_∞ RING THEORY

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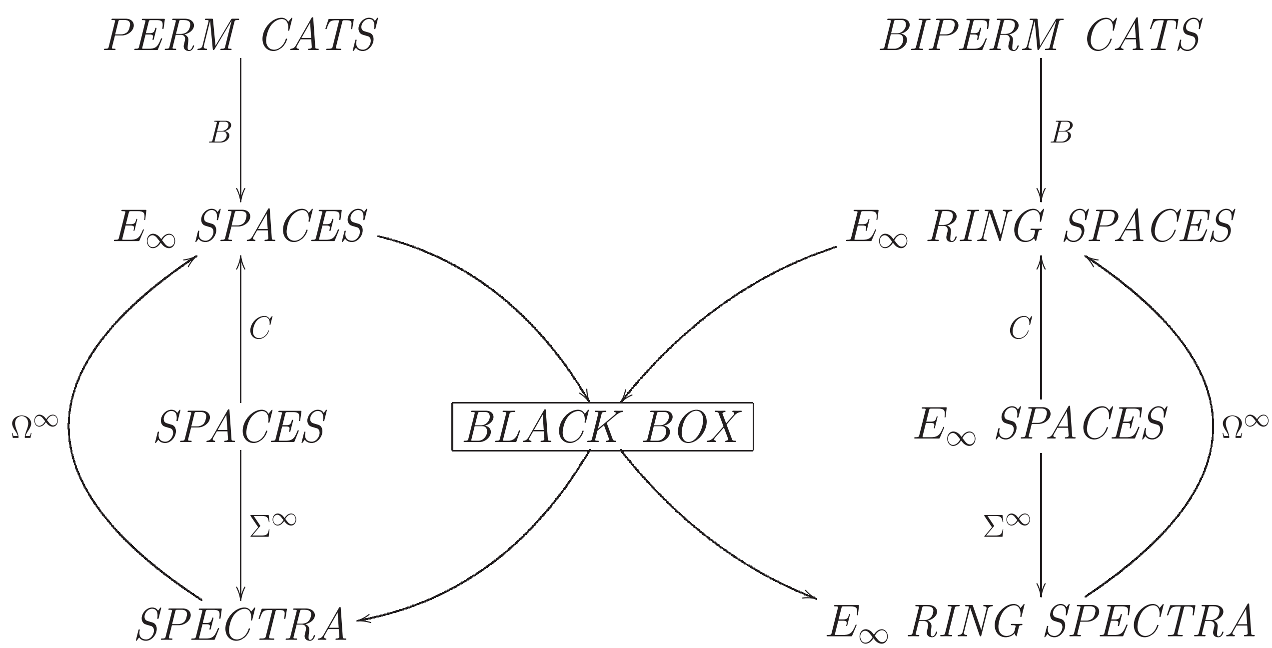

PERM CATS
B
E∞ SPACES
C
Ω∞
SPACES
Σ∞
SPECTRA
BLACK BOX
BIPERM CATS
B
E∞ RING SPACES
C
E∞ SPACES
Ω∞
Σ∞
E∞ RING SPECTRA

$$
\begin{array}{c}
\text{PERM CATS} \\
\Big\downarrow{\scriptstyle B} \\
\mathscr{F}-\text{SPACES} \\
\uparrow\downarrow \\
\hat{\mathscr{C}}-\text{SPACES} \\
\uparrow\downarrow \\
\mathscr{C}-\text{SPACES} \\
\Vert \\
E_\infty\ \text{SPACES}
\end{array}
\qquad\qquad
\begin{array}{c}
\text{BIPERM CATS} \\
\Big\downarrow{\scriptstyle B} \\
(\mathscr{F}\int\mathscr{F})-\text{SPACES} \\
\uparrow\downarrow \\
(\hat{\mathscr{G}}\int\hat{\mathscr{C}})-\text{SPACES} \\
\uparrow\downarrow \\
(\hat{\mathscr{C}},\hat{\mathscr{G}})-\text{SPACES} \\
\uparrow\downarrow \\
(\mathscr{C},\mathscr{G})-\text{SPACES} \\
\Vert \\
E_\infty\ \text{RING SPACES}
\end{array}
$$

E_∞ ring spaces

OPERAD: Σ_j-spaces $\mathscr{O}(j)$, $\mathscr{O}(0) = \{*\}$ (basepts), id $\in \mathscr{O}(1)$ (identity operation),

$$\gamma\colon \mathscr{O}(k)\times\mathscr{O}(j_1)\times\cdots\times\mathscr{O}(j_k) \longrightarrow \mathscr{O}(j_1+\cdots+j_k).$$

Associative, unital, equivariant.

$\mathscr{O}$-space X:

$$\theta\colon \mathscr{O}(j) \times X^j \longrightarrow X.$$

OPERAD PAIR: 'Additive', 'multiplicative' operads $\mathscr{C}$, $\mathscr{G}$,

$$\lambda\colon \mathscr{G}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1\cdots j_k).$$

Distributive, unital, equivariant, nullary.

$(\mathscr{C},\mathscr{G})$-space: $\mathscr{C}$-space and $\mathscr{G}$-space X

$$\begin{array}{ccc}
\mathscr{G}(k)\times\mathscr{C}(j_1)\times X^{j_1}\times\cdots\times\mathscr{C}(j_k)\times X^{j_k} & \xrightarrow{\mathrm{id}\times\theta^k} & \mathscr{G}(k)\times X^k \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
\mathscr{C}(j_1\cdots j_k)\times X^{j_1\cdots j_k} & \xrightarrow[\theta]{} & X
\end{array}$$

ξ on left induced from λ and ξ on X.

Monadic reinterpretation

Monad O in $\mathscr{T}$ = based spaces
$\mathscr{O}$-spaces $\cong$ O-algebras.

$$OX = \coprod \mathscr{O}(j) \times_{\Sigma_j} X^j/(\sim)$$

$\sim$= basepoint identifications

$(\mathscr{C}, \mathscr{G})$: G 'acts' on C;

C induces a monad on $G[\mathscr{T}]$,

CG becomes a monad on $\mathscr{T}$.

Isomorphic categories:

$$(\mathscr{C}, \mathscr{G}) - \text{spaces}$$

$$C - \text{algebras in } \; G[\mathscr{T}]$$

$$CG - \text{algebras in } \mathscr{T}.$$

Lewis May spectra; QX

$U = \mathbb{R}^\infty$; $V \subset W \subset U$ fin dim.

Prepectrum T: $TV \longrightarrow \Omega^{W-V} TW$

Spectrum E: $EV \xrightarrow{\cong} \Omega^{W-V} EW$

Spectrification L, forgetful ℓ:

$$\mathscr{S}(LT, E) \cong \mathscr{P}(T, \ell E)$$

$$\mathscr{P}(\{\Sigma^V X\}, T) \cong \mathscr{T}(X, T_0)$$

$$\Sigma^\infty X = L\{\Sigma^V X\} \quad \Omega^\infty E = E_0.$$

$$\mathscr{S}(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E)$$

$$QX = \operatorname{colim} \Omega^V \Sigma^V X = \Omega^\infty \Sigma^\infty X$$

Approximation Theorem

There is a map of monads

$$\alpha\colon C \longrightarrow Q$$

on $\mathscr{T}$ such that

$$\alpha\colon CX \longrightarrow QX$$

is a group completion for all X, hence a weak equivalence for all connected X.

Canonical operad pair

Linear isometries operad $\mathscr{L}$.

$$\mathscr{L}(j) = \mathscr{I}(U^j, U)$$

$\mathscr{L}$ acts on the Steiner operad $\mathscr{C}$, a variant of the infinite little cubes operad.

$\mathscr{C}$ acts on $\Omega^\infty E$; α is the composite

$$CX \xrightarrow{C\eta} CQX \xrightarrow{C\theta} QX.$$

$\mathscr{L}$ and $\mathscr{C}$ are E_∞ operads: their j^{th} spaces are Σ_j-free and contractible.

E_∞ ring spectra

External smash products $T \mathbin{\bar\wedge} T'$, $E \mathbin{\bar\wedge} E'$:

$$(T \mathbin{\bar\wedge} T')(V, V') = TV \wedge T'V'$$

$$(E \mathbin{\bar\wedge} E') = L(\ell E \mathbin{\bar\wedge} \ell E')$$

$T^{[j]}$: external j-fold smash power.

$f\colon U \longrightarrow U'$ induces $f^*\colon \mathscr{P}(U') \longrightarrow \mathscr{P}(U)$.

$$(f^*T')(V) = T'(fV)$$

Restricts to $f^*\colon \mathscr{S}(U') \longrightarrow \mathscr{S}(U)$.
Left adjoints f_* on $\mathscr{P}$, $f_* = Lf_*\ell$ on $\mathscr{S}$.

$\mathscr{L}$-prespectrum T: maps of prespectra

$$\xi_j(f)\colon f_*T^{[j]} \longrightarrow T$$

for all $f \in \mathscr{L}(j)$; suitably continuous and compatible with operad structure on $\mathscr{L}$.

$$\mathscr{L}(j) \ltimes T^{[j]} \longrightarrow T.$$

$$\mathscr{L}(j) \ltimes E^{[j]} \longrightarrow E.$$

Monadic reinterpretation; Ω^∞

Monad L_+ on $\mathscr{T}$ wrt $\wedge$ rather than $\times$:

$$L_+X = \vee_{j\geq 0}\, \mathscr{L}(j)_+ \wedge_{\Sigma_j} X^{(j)}.$$

Monad L_+ on $\mathscr{S}$ analogously:

$$L_+E = \vee_{j\geq 0}\, \mathscr{L}(j) \ltimes_{\Sigma_j} E^{[j]}.$$

$$L_+\Sigma^\infty X \cong \Sigma^\infty L_+X$$

$$\mathscr{S}(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E)$$

induces

$$L_+[\mathscr{S}](\Sigma^\infty Y, R) \cong L_+[\mathscr{T}](Y, \Omega^\infty R).$$

• The monad Q on $\mathscr{T}$ induces a monad Q on $L_+[\mathscr{T}]$; for an $\mathscr{L}$-spectrum R, $\Omega^\infty R$ is a Q-algebra, hence a C-algebra, in $L_+[\mathscr{T}]$.

• Therefore $\Omega^\infty R$ is an E_∞ ring space.

• The 1-component and unit components SL_1R and GL_1R of $\Omega^\infty R$ are $\mathscr{L}$-spaces.

• Therefore SL_1R and GL_1R are the 0^{th} spaces of spectra sl_1R and gl_1R.

Orientation theory

$F(X,Y)$: based maps $X \longrightarrow Y$

$$\circ\colon F(Y,Z)\times F(X,Y)\longrightarrow F(X,Z)$$

$E \in \mathscr{S}$, $E_0 \cong \Omega^V EV$ for $V \subset U = \mathbb{R}^\infty$.

$$\circ\colon F(S^V, EV)\times F(S^V, S^V)\longrightarrow F(S^V, EV)$$

$$E_0 \times \Omega^V\Sigma^V S^0 \longrightarrow E_0$$

R a ring spectrum. 1970's notations:

$$SF = SL_1 S \qquad F = GL_1 S$$

$$SFR = SL_1 R \qquad FR = GL_1 R$$

Think of F as a functor $\mathscr{I} \longrightarrow \mathit{monoids}$:

$$FV = hAut(S^V) \subset F(S^V, S^V)$$

G an $\mathscr{I}$-monoid (or group) mapping to F:

$$GV \times GV \longrightarrow GV \quad GV \longrightarrow FV$$

$$SGV \times SGV \longrightarrow SGV \quad SGV \longrightarrow SFV$$

$$B(FR, GV, S^V) \to B(FR, GV, *) \equiv B(GV; R)$$

(or with FR, GV replaced by SFR, SGV)

Classifying spaces for R-oriented V-sphere bundles, or with preassigned $H\mathbb{Z}$-orientation.

Pass to colimits. Get classifying spaces

$$B(G; R) \quad B(SG; R)$$

for stable R-oriented sphere bundles.

$$\begin{array}{ccccccc}
SG & \longrightarrow & SFR & \longrightarrow & B(SG;R) & \longrightarrow & BSG \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G & \longrightarrow & FR & \longrightarrow & B(G;R) & \longrightarrow & BG \\
\| & & \downarrow & & \downarrow & & \| \\
G & \longrightarrow & \pi_0(R)^\times & \longrightarrow & B(G;\pi_0 R) & \longrightarrow & BG
\end{array}$$

PUNCH LINE: If R is an E_∞ ring spectrum, this is a diagram of $\mathscr{L}$-spaces, hence has an associated diagram of connective spectra.

Universal obstruction to R-orientability:

$$w\colon BSG \longrightarrow BSFR = BSGL_1R$$

E.g: $G = O$ or $G = F$, $R = kO$:

$$w = w_2\colon BSF \longrightarrow BO_\otimes$$

Theorem 1. *At* $p > 2$,
$$BSTOP \simeq B(SF; kO)$$
as E_∞*-spaces: equivalent spectra.*

Theorem 2. *At* $p > 2$,
$$MSTOP \simeq M(SF; kO)$$
as E_∞*-ring spectra.*

$BCokerJ$, $MCokerJ$: Can compute!!!

Algebraic and topological K-theory

$$KR \equiv EB\mathscr{G}\mathscr{L}R$$

$$\mathscr{G}\mathscr{L}R = \coprod_{n\geq 0} GL(n,R)$$

E is the BLACK BOX functor

Fix a prime q.

Brauer lift (Quillen, plus May-Tornehave):

Theorem 3. *Completed away from q, $K\bar{\mathbb{F}}_q$ and kU are equivalent ring spectra. For $r = q^a$, Frobenius and Adams agree:*

$$\begin{array}{ccc} K\bar{\mathbb{F}}_q & \xrightarrow{\phi^r} & K\bar{\mathbb{F}}_q \\ \simeq\downarrow & & \downarrow\simeq \\ kU & \xrightarrow[\psi^r]{} & kU \end{array}$$

Multiplicative Brauer lift (May-Tornehave):

Theorem 4. *Completed away from q, $sl_1K\bar{\mathbb{F}}_q$ and sl_1KU are equivalent spectra.*

$$sl_1KU_p \simeq K(\mathbb{Z}_p, 2) \times bsu_p$$

Complete at $p > 2$, $p \neq q$.
Let $r = q^a$ be a unit mod p^2.

Write bR for the connected cover of KR.

$$\begin{array}{ccccc}
K\mathbb{F}_r & \longrightarrow & K\bar{\mathbb{F}}_q & \xrightarrow{\phi^r - 1} & bR \\
\simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
j & \longrightarrow & kU & \xrightarrow[\psi^r - 1]{} & bU
\end{array}$$

Theorem 5. *There is a composite "exponential equivalence"*

$$b\mathbb{F}_r \longrightarrow sl_1S \longrightarrow sl_1\mathbb{F}_r$$

So sl_1S splits. Infinite loop space splitting

$$SF \simeq J \times \operatorname{Coker} J.$$

As infinite loop spaces,

$$B \operatorname{Coker} J \simeq B(SF; K\mathbb{F}_r)$$

$$BO \times B \operatorname{Coker} J \simeq B(SF; kO) \simeq B(SF; K\bar{\mathbb{F}}_q).$$

Additive recognition principle

Theorem 6. *For a $\mathscr{C}$-space X, define*

$$EX = B(\Sigma^\infty, C, X).$$

EX is connective. There is a diagram of maps of C-spaces

$$X \xleftarrow{\varepsilon} B(C, C, X) \xrightarrow{B\alpha} B(Q, C, X) \xrightarrow{\zeta} \Omega^\infty EX.$$

ε is a homotopy equivalence, inverse η; ζ is a weak equivalence; $B\alpha$ is a group completion. Therefore the composite

$$\eta\colon X \longrightarrow \Omega^\infty EX$$

is a group completion. It is a weak equivalence if X is grouplike.

For a spectrum Y, there is a composite map of spectra

$$\varepsilon\colon E\Omega^\infty Y \xrightarrow{B\alpha} B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\varepsilon} Y.$$

Apply Ω^∞. The maps of $\mathscr{C}$-spaces

$$\Omega^\infty E\Omega^\infty Y \xrightarrow{\Omega^\infty B\alpha} \Omega^\infty B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\Omega^\infty \varepsilon} \Omega^\infty Y$$

are weak equivalences. Therefore ε is a weak equivalence if Y is connective.

Conclusion: E and Ω^∞ induce an equivalence between the homotopy category of grouplike E_∞ spaces and the homotopy category of connective spectra.

[Can replace by $\mathscr{C}$ by $\mathscr{C} \times \mathscr{L}$, $\mathscr{C} \times \mathscr{D}$, etc.]

Multiplicative recognition principle

Theorem 7. *For a* $(\mathscr{C},\mathscr{L})$*-space* X*,*

$$EX = B(\Sigma^\infty, C, X)$$

is a connective $\mathscr{L}$*-spectrum. All maps in the diagram*

$$X \xleftarrow{\varepsilon} B(C,C,X) \xrightarrow{B\alpha} B(Q,C,X) \xrightarrow{\zeta} \Omega^\infty EX$$

are maps of $(\mathscr{C},\mathscr{L})$*-spaces. Therefore the composite*

$$\eta\colon X \longrightarrow \Omega^\infty EX$$

is a ring completion.

For an $\mathscr{L}$-spectrum Y, the maps

$$\varepsilon\colon E\Omega^\infty Y \xrightarrow{B\alpha} B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\varepsilon} Y$$

are maps of $\mathscr{L}$-spectra and the maps

$$\Omega^\infty E\Omega^\infty Y \xrightarrow{\Omega^\infty B\alpha} \Omega^\infty B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\Omega^\infty \varepsilon} \Omega^\infty Y$$

are maps of $(\mathscr{C}, \mathscr{L})$-spaces.

<u>Conclusion</u>: E and Ω^∞ induce an equivalence between the homotopy category of ringlike E_∞ ring spaces and the homotopy category of connective E_∞ ring spectra.

Localizations of unit spectra

Let X be an E_∞ ring space, e.g. $\Omega^\infty R$.

Let $N = \{0, 1, 2, \cdots\} \subset \pi_0 X$.

Let $M \subset N$ be a multiplicative subset, for example $\{p^i\}$ or $\{n|(n,p) = 1\}$.

Let $X_M = \coprod X_m$, union of components.

Mild homological convergence condition.

Theorem 8. *As an E_∞ space, the localization of $SL_1 E = \Omega_1^\infty E(X, \theta)$ at M is the basepoint component $\Omega_1^\infty E(X_M, \xi)$.*

$E(X, \theta)$ is defined using X as a $\mathscr{C}$-space. The localizations of $SL_1 E(X)$ depend only on X as an $\mathscr{L}$-space.

Let $X = \Omega^\infty R$. $E(X, \theta)$ is the connective cover of R, $SL_1 E(X)$ is $SL_1 R$.

Corollary 9. *For an E_∞ ring spectrum R, $sl_1(R)[M^{-1}]$ is the connected cover of $E((\Omega^\infty R)_M, \xi)$.*

Lewis conundrum

Everything enriched over based spaces (or simplicial sets) in the following hypotheses.

Theorem 10 (Lewis). *Let $\mathscr{S}$ satisfy*

(i) $\mathscr{S}$ is closed symmetric monoidal. Have functors $\wedge$ and F such that

$$\mathscr{S}(E \wedge E', E'') \cong \mathscr{S}(E, F(E', E'')).$$

(ii) Have functors Σ^∞ and Ω^∞ such that

$$\mathscr{S}(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E).$$

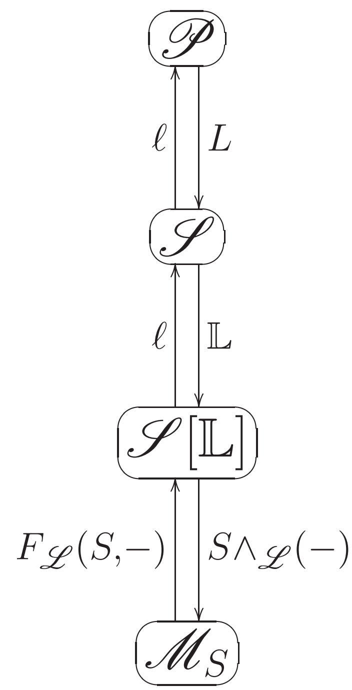
(iii) The unit for $\wedge$ is $S \equiv \Sigma^\infty S^0$.

Let E be a commutative monoid in $\mathscr{S}$ (e.g. S). Then $SL_1(E)$ is a product of Eilenberg-MacLane spaces.

Corollary 11. *No model structure on $\mathscr{S}$ with S cofibrant can give the right $Ho\mathscr{S}$.*

EKMM S-modules

Quillen equivalences:



Monad $\mathbb{L}$: $\mathbb{L}E = \mathscr{L}(1) \ltimes E$. $\eta : E \xrightarrow{\simeq} \mathbb{L}E$.

$\mathbb{L}$-spectrum: action $\mathbb{L}E \longrightarrow E$; e.g. $\mathbb{L}E$.

$$E \wedge_{\mathscr{L}} E' \equiv \mathscr{L}(2) \ltimes_{\mathscr{L}(1)\times\mathscr{L}(1)} E \mathbin{\overline{\wedge}} E'$$

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge_{\mathscr{L}} \Sigma^\infty Y.$$

Associative and commutative.

$$\tau\colon E \wedge_{\mathscr{L}} E' \cong E' \wedge_{\mathscr{L}} E.$$

Unit weak equivalence $\lambda\colon S \wedge_{\mathscr{L}} E \xrightarrow{\simeq} E$.

E is an S-module if λ is an isomorphism.

Examples: $\Sigma^\infty X$, $S \wedge_{\mathscr{L}} E$.

$\mathscr{S}[\mathbb{L}]$ is the category of $\mathbb{L}$-spectra.

$\mathscr{M}_S$ is the category of S-modules.

Definition 12. *A commutative 'monoid' in $\mathscr{S}[\mathbb{L}]$ is an $\mathbb{L}$-spectrum R with a unit map $\eta\colon S \longrightarrow R$ and a commutative and associative product $\phi\colon R \wedge_{\mathscr{L}} R \longrightarrow R$ such that the following diagram commutes.*

$$\begin{array}{ccccc} S \wedge_{\mathscr{L}} R & \xrightarrow{\eta \wedge \mathrm{id}} & R \wedge_{\mathscr{L}} R & \xleftarrow{\mathrm{id} \wedge \eta} & R \wedge_{\mathscr{L}} S \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \phi} & \swarrow_{\lambda\tau} & \\ & & R & & \end{array}$$

Theorem 13. *The category of commutative monoids in $\mathscr{S}[\mathbb{L}]$ is isomorphic to the category of E_∞ ring spectra.*

$$E \wedge_S E' = E \wedge_{\mathscr{L}} E'$$

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge_S \Sigma^\infty Y.$$

$\mathscr{M}_S$ is closed symmetric monoidal, unit S.

Definition 14. *A commutative S-algebra is a commutative monoid in $\mathscr{M}_S$. That is, it is an E_∞ ring spectrum which is an S-module.*

If R is an E_∞ ring spectrum, then $S \wedge_{\mathscr{L}} R$ is a weakly equivalent commutative S-algebra.

$S = \Sigma^\infty S^0$ is cofibrant in $\mathscr{S}$; $\mathbb{L}S$, $S \wedge_{\mathscr{L}} \mathbb{L}S$ are cofibrant approximations in $\mathscr{S}[\mathbb{L}]$, $\mathscr{M}_S$.

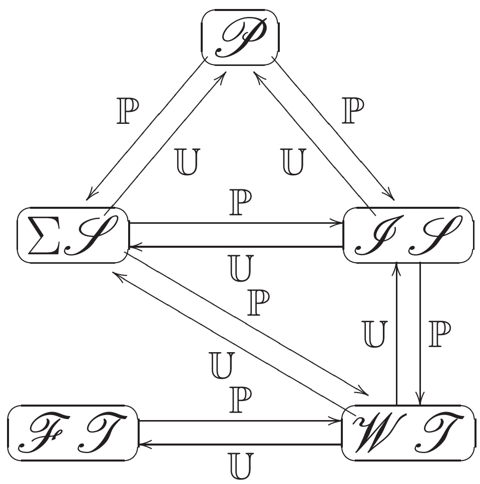
Adjunctions in $\mathscr{S}$, $\mathscr{S}[\mathbb{L}]$, $\mathscr{M}_S$:

$$(\Sigma^\infty, \Omega^\infty)$$

$$(\mathbb{L}\Sigma^\infty, \Omega^\infty \ell)$$

$$(S \wedge_{\mathscr{L}} \mathbb{L}\Sigma^\infty, F_{\mathscr{L}}(S, \Omega^\infty \ell))$$

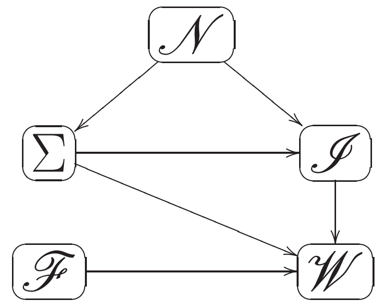
Diagram spectra



Lexicon:

- $\mathscr{P}$ is the category of $\mathscr{N}$-spectra, or (coordinatized) prespectra.
- $\Sigma\mathscr{S}$ is the category of Σ-spectra, or symmetric spectra.
- $\mathscr{I}\mathscr{S}$ is the category of $\mathscr{I}$-spectra, or orthogonal spectra.
- $\mathscr{F}\mathscr{T}$ is the category of $\mathscr{F}$-spaces, or Γ-spaces.
- $\mathscr{W}\mathscr{T}$ is the category of $\mathscr{W}$-spaces.

$\mathscr{D}$-spectra for diagram of domains $\mathscr{D}$:



Start with $\mathscr{D}$-spaces, $\mathscr{D}\mathscr{T}$. For a sphere functor $S\colon \mathscr{D} \longrightarrow \mathscr{T}$ with smash products, S-modules are $\mathscr{D}$-spectra. No distinction when $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$: $\mathscr{D}\mathscr{T} = \mathscr{D}\mathscr{S}$.

Lexicon:

- $\mathscr{N}$ is the category of natural numbers.
- Σ is the category of symmetric groups.
- $\mathscr{I}$ is the category of linear isometric iso's.
- $\mathscr{F}$ is the category of finite based sets.
- $\mathscr{W}$ is the category of based spaces that are homeomorphic to finite CW complexes.

Forgetful, prolongation functors $\mathbb{U}$, $\mathbb{P}$.
All $(\mathbb{P}, \mathbb{U})$ are Quillen equivalences.
(Connective spectra only for $\mathscr{F}\mathscr{T}$).

For $\mathscr{D}$-spaces T and T', have external smash product $T \,\bar{\wedge}\, T'$, a $\mathscr{D} \times \mathscr{D}$-space.

$$(T \,\bar{\wedge}\, T')(d,e) = Td \wedge T'e.$$

Given $\oplus\colon \mathscr{D}\times\mathscr{D} \longrightarrow \mathscr{D}$, left Kan extension gives a $\mathscr{D}$-space $T \wedge T'$.

$$((\mathscr{D}\times\mathscr{D})\mathscr{T})(T\bar{\wedge}T', V\circ\oplus) \cong \mathscr{D}\mathscr{T}(T\wedge T', V)$$

For S-modules T and T', get coequalizer

$$T \wedge_S T'.$$

All but $\mathscr{N}\mathscr{S}$ above are symmetric monoidal.

Quillen equivalence

$$\mathscr{I}\mathscr{S} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{M}_S$$

from *positive* model structure on $\mathscr{I}\mathscr{S}$.

Theorem 15. *The functors $\mathbb{P}$ and $\mathbb{N}$ induce Quillen equivalences from commutative symmetric ring spectra to commutative orthogonal ring spectra and from the latter to commutative S-algebras.*

We have comparison functors

$$\begin{array}{c}
\text{Commutative symmetric ring spectra} \\
\big\downarrow \mathbb{P} \\
\text{Commutative orthogonal ring spectra} \\
\big\downarrow \mathbb{N} \\
\text{Commutative } S\text{-algebras} \\
\big\uparrow S \wedge_{\mathscr{L}} (-) \\
E_\infty \text{ ring spectra} \\
\big\downarrow \Omega^\infty \\
E_\infty \text{ ring spaces.}
\end{array}$$

But: no 0^{th} space information in diagram commutative ring spectra. The E_∞ ring theory relating spaces and spectra is lost.

Naive E_∞ ring spectra

$$\mathscr{O}(j)_+ \wedge R^{(j)} \longrightarrow R.$$

Proposition 16. *For a positive cofibrant symmetric or orthogonal spectrum or for a cofibrant S-module E,*

$$\pi\colon (E\Sigma_j)_+ \wedge_{\Sigma_j} E^{(j)} \longrightarrow E^{(j)}/\Sigma_j$$

is a weak equivalence.

Proposition 17. *The homotopy categories of naive $\mathscr{O}$-spectra and commutative ring spectra (in any of $\Sigma\mathscr{S}$, $\mathscr{I}\mathscr{S}$, or $\mathscr{M}_S$) are equivalent.*