Categories, posets, Alexandrov spaces, simplicial complexes, with emphasis on finite spaces

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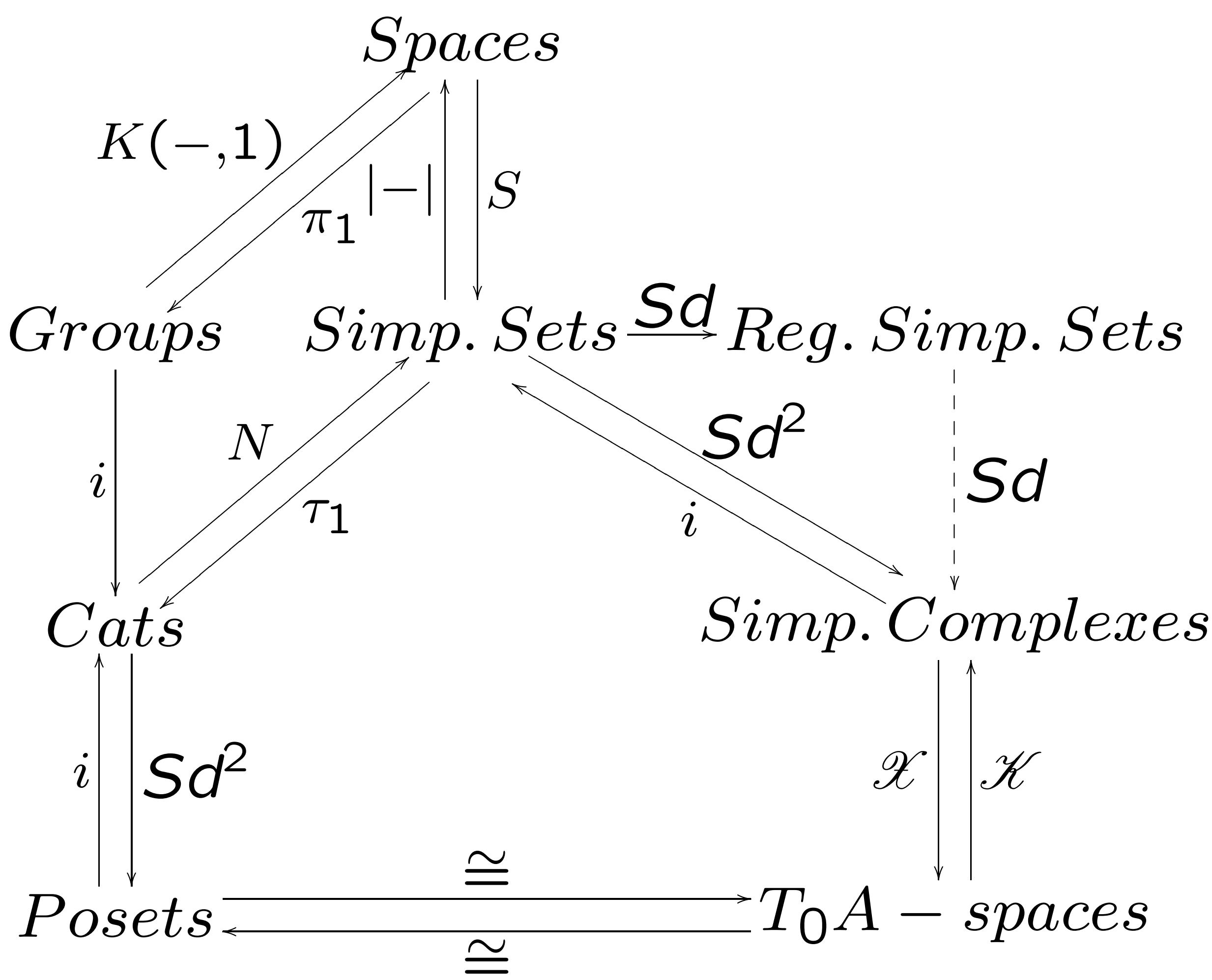
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Simplicial sets and subdivision

(Any new results are due to Rina Foygel)

$\mathbf{\Delta} \equiv$ standard simplicial category.

$\Delta[n]$ is represented on $\mathbf{\Delta}$ by $\mathbf{n}$.

It is $N\underline{\mathbf{n}}$, where $\underline{\mathbf{n}}$ is the poset $\{0, 1, \cdots, n\}$.

$Sd\Delta[n] \equiv \Delta[n]' \equiv N sd\,\underline{\mathbf{n}}$, where

$sd\,\underline{\mathbf{n}} \equiv \underline{\mathbf{n}}' \equiv monos/\mathbf{n}$.

$SdK \equiv K \otimes_{\mathbf{\Delta}} \Delta'$.

Lemma 1 *$SdK \cong SdL$ does not imply $K \cong L$ but does imply $K_n \cong L_n$ as sets, with corresponding simplices having corresponding faces.*

Regular simplicial complexes

A nondegenerate $x \in K_n$ is regular if the subcomplex $[x]$ it generates is the pushout of

$$\Delta[n] \xleftarrow{\delta^n} \Delta[n-1] \xrightarrow{d_n x} [d_n x].$$

K is regular if all x are so.

Theorem 1 *For any K, Sd K is regular.*

Theorem 2 *If K is regular, then $|K|$ is a regular CW complex: $(e^n, \partial e^n) \cong (D^n, S^{n-1})$ for all closed n-cells e.*

Theorem 3 *If X is a regular CW complex, then X is triangulable; that is X is homeomorphic to some $|i(K)|$.*

Properties of simplicial sets K

Let $x \in K_n$ be a nondegenerate simplex of K.

A: For all x, all faces of x are nondegenerate.

B: For all x, x has $n+1$ distinct vertices.

C: Any $n+1$ distinct vertices are the vertices of at most one x.

Lemma 2 *K has B iff for all x and all monos $\alpha, \beta\colon \mathbf{m} \longrightarrow \mathbf{n}$, $\alpha^* x = \beta^* x$ implies $\alpha = \beta$.*

Lemma 3 *If K has B, then K has A.*

No other general implications among A, B, C.

Properties A, B, C and subdivision

Lemma 4 *K has A iff SdK has A.*

Lemma 5 *K has A iff SdK has B.*

Lemma 6 *K has B iff SdK has C.*

Characterization of simplicial complexes

Lemma 7 *K has A iff Sd^2K has C,*

and then Sd^2K also has B.

Lemma 8 *K has B and C iff $K \in$ Im(i).*

Theorem 4 *K has A iff Sd$^2K \in Im(i)$.*

Subdivision and horn-filling

Lemma 9 *If SdK is a Kan complex, then K is discrete.*

Lemma 10 *If K does not have A, then SdK cannot be a quasicategory.*

Relationship of the properties to categories

Theorem 5 *If K has A, then $Sd\,K \in Im(N)$.*

Proof: Check the Segal maps criterion.

Definition 1 *A category $\mathscr{C}$ satisfies A, B, or C if $N\mathscr{C}$ satisfies A, B, or C.*

Lemma 11 *$\mathscr{C}$ has A iff for any $i\colon C \longrightarrow D$ and $r\colon D \longrightarrow C$ such that $r \circ i =$ id, $C = D$ and $i = r =$ id. (Retracts are identities.)*

Lemma 12 *$\mathscr{C}$ has B iff for any $i\colon C \longrightarrow D$ and $r\colon D \longrightarrow C$, $C = D$ and $i = r =$ id.*

Lemma 13 *$\mathscr{C}$ has B and C iff $\mathscr{C}$ is a poset.*

Definition 2 *Define a category $T\mathscr{C}$:*

Objects: nondegenerate simplices of $N\mathscr{C}$. e.g.

$$\underline{C} = C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_q$$

$$\underline{D} = D_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow D_r$$

Morphisms: maps $\underline{C} \longrightarrow \underline{D}$ are maps $\alpha\colon \mathbf{q} \longrightarrow \mathbf{r}$ in $\mathbf{\Delta}$ such that $\alpha^*\mathbf{D} = \mathbf{C}$ (implying α is mono).

Quotient category $sd\mathscr{C}$ with the same objects:

$$\alpha \circ \beta_1 \sim \alpha \circ \beta_2 \colon \underline{C} \longrightarrow \underline{D}$$

if $\sigma \circ \beta_1 = \sigma \circ \beta_2$ for a surjection $\sigma\colon \mathbf{p} \longrightarrow \mathbf{q}$ such that $\alpha^*\mathbf{D} = \sigma^*\mathbf{C}$ ($\alpha\colon \mathbf{p} \longrightarrow \mathbf{r}$, $\beta_i\colon \mathbf{q} \longrightarrow \mathbf{p}$).

$$(\beta_i^*\alpha^*\underline{D} = \beta_i^*\sigma^*\underline{C} = \underline{C}, \quad i = 1, 2)$$

(Anderson, Thomason, Fritsch-Latch, del Hoyo)

Lemma 14 *For any* $\mathscr{C}$, $T\mathscr{C}$ *has B.*

Corollary 1 *For any* $\mathscr{C}$, $sd\mathscr{C}$ *has B.*

Lemma 15 $\mathscr{C}$ *has B iff* $sd\mathscr{C}$ *is a poset.*

Theorem 6 *For any* $\mathscr{C}$, $sd^2\mathscr{C}$ *is a poset.*

Compare with K has A iff $Sd^2K \in Im(i)$.

Del Hoyo: Equivalence $\varepsilon\colon sd\mathscr{C} \longrightarrow \mathscr{C}$.

(Relate to equivalence $\varepsilon\colon Sd\,K \longrightarrow K$?)

Left adjoint τ_1 to N (Gabriel–Zisman).

Objects of $\tau_1 K$ are the vertices.

Think of 1-simplices y as maps

$$d_1 y \longrightarrow d_0 y,$$

form the free category they generate, and impose the relations

$$s_0 x = id_x \quad \textit{for} \quad x \in K_0$$

$$d_1 z = d_0 z \circ d_2 z \quad \textit{for} \quad z \in K_2.$$

The counit $\varepsilon\colon \tau_1 N\mathscr{A} \longrightarrow \mathscr{A}$ is an isomorphism.

$\tau_1 K$ depends only on the 2-skeleton of K. When

$K = \partial\Delta[n]$ for $n > 2$, the unit $\eta\colon K \longrightarrow N\tau_1 K$

is the inclusion $\partial\Delta[n] \longrightarrow \Delta[n]$.

Direct combinatorial proof:

Theorem 7 *For any* $\mathscr{C}$*,* $\mathsf{sd}\,\mathscr{C} \cong \tau_1\,\mathsf{Sd}\,N\mathscr{C}$*.*

Corollary 2 $\varepsilon = \tau_1\varepsilon\colon \mathsf{sd}\,\mathscr{C} \longrightarrow \tau_1 N\mathscr{C} \cong \mathscr{C}$*.*

Corollary 3 $\mathscr{C}$ *has A iff* $\mathsf{Sd}\,N\mathscr{C} \cong N\,\mathsf{sd}\mathscr{C}$*.*

Remark 1 *Even for posets* P *and* Q*,*
$sdP \cong sdQ$ *does not imply* $P \cong Q$*.*

In the development above, there is a counterexample to the converse of each implication that is not stated to be iff.

Sheds light on Thomason model structure.

Alexandrov and finite spaces

Alexandrov space, abbreviated A-space:

ANY intersection of open sets is open.

Finite spaces are A-spaces.

T_0-space: topology distinguishes points.

Kolmogorov quotient $K(A)$. McCord:

$A \longrightarrow K(A)$ is a homotopy equivalence.

Space = T_0-A-space from now on

T_1 finite spaces are discrete,

but any finite X has a closed point.

Define

$$U_x \equiv \cap\{U | x \in U\}$$

$\{U_x\}$ is unique minimal basis for the topology.

$$x \leq y \equiv x \in U_y; \quad \textit{that is}, \quad U_x \subset U_y$$

Transitive and reflexive; $T_0 \Longrightarrow$ antisymmetric.

For a poset X, define $U_x \equiv \{y | x \leq y\}$: basis for a T_0-A-space topology on the set X.

$f: X \longrightarrow Y$ is continuous $\Longleftrightarrow$ f preserves order.

Theorem 8 *The category $\mathscr{P}$ of posets is isomorphic to the category $\mathscr{A}$ of T_0-A-spaces.*

Finite spaces: $f\colon X \longrightarrow X$ is a homeomorphism iff f is one-to-one or onto.

Can describe n-point topologies by restricted kind of $n \times n$-matrix and enumerate them.

Combinatorics: count the isomorphism classes of posets with n points; equivalently count the homeomorphism classes of spaces with n points. HARD! For $n = 4$, $X = \{a, b, c, d\}$, 33 topologies, with bases as follows:

1	all
2	a, b, c, (a,b), (a,c), (b,c), (a,b,c)
3	a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d)
4	a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
5	a, b, (a,b)
6	a, b, (a,b), (a,b,c)
7	a, b, (a,b), (a,c,d)
8	a, b, (a,b), (a,b,c), (a,b,d)
9	a, b, (a,b), (a,c), (a,b,c)
10	a, b, (a,b), (a,c), (a,b,c), (a,c,d)
11	a, b, (a,b), (a,c), (a,b,c), (a,b,d)
12	a, b, (a,b), (c,d), (a,c,d), (b,c,d)
13	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d)
14	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
15	a
16	a, (a,b)
17	a, (a,b), (a,b,c)
18	a, (b,c), (a,b,c)
19	a, (a,b), (a,c,d)
20	a, (a,b), (a,b,c), (a,b,d)
21	a, (b,c), (a,b,c), (b,c,d)
22	a, (a,b), (a,c), (a,b,c)
23	a, (a,b), (a,c), (a,b,c), (a,b,d)
24	a, (c,d), (a,b), (a,c,d)
25	a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
26	a, (a,b,c)
27	a, (b,c,d)
28	(a,b)
29	(a,b), (c,d)
30	(a,b), (a,b,c)
31	(a,b), (a,b,c), (a,b,d)
32	(a,b,c)
33	none

Homotopies and homotopy equivalence

$f, g\colon X \longrightarrow Y$: $f \leq g$ if $f(x) \leq g(x)\ \forall\ x \in X$.

Proposition 1 *X, Y finite. $f \leq g$ implies $f \simeq g$.*

Proposition 2 *If $y \in U \subset X$ with U open (or closed) implies $U = X$, then X is contractible.*

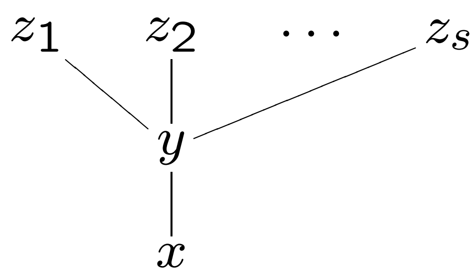
If X has a unique maximum or minimal point, X is contractible. Each U_x is contractible.

Definition 3 *Let X be finite.*

(a) $x \in X$ is upbeat *if there is a $y > x$ such that $z > x$ implies $z \geq y$.*

(b) $x \in X$ is downbeat *if there is a $y < x$ such that $z < x$ implies $z \leq y$.*

Upbeat:



Downbeat: upside down.

X is minimal if it has no upbeat or downbeat points. A *core* of X is a subspace Y that is minimal and a deformation retract of X.

Stong:

Theorem 9 *Any finite X has a core.*

Theorem 10 *If $f \simeq id: X \longrightarrow X$, then $f = id$.*

Corollary 4 *Minimal homotopy equivalent finite spaces are homeomorphic.*

REU results of Alex Fix and Stephen Patrias

Can now count homotopy types with n points.

Hasse diagram $Gr(X)$ of a poset X: directed graph with vertices $x \in X$ and an edge $x \to y$ if $y < x$ but there is no other z with $x \leq z \leq y$.

Translate minimality of X to a property of $Gr(X)$ and count the number of such graphs.

Find a fast enumeration algorithm.

Run it on a computer.

Get number of homotopy types with n points.

Compare with number of homeomorphism types.

n	$\simeq$	$\cong$
1	1	1
2	2	2
3	3	5
4	5	16
5	9	63
6	20	318
7	56	2,045
8	216	16,999
9	1,170	183,231
10	9,099	2,567,284
11	101,191	46,749,427
12	1,594,293	1,104,891,746

Exploit known results from combinatorics.

Astonishing conclusion:

Theorem 11 *(Fix and Patrias) The number of homotopy types of finite T_0-spaces is asymptotically equivalent to the number of homeomorphism types of finite T_0-spaces.*

T_0-A-spaces and simplicial complexes

Category $\mathscr{A}$ of T_0-A-spaces (= posets);

Category $\mathscr{B}$ of simplicial complexes.

McCord:

Theorem 12 *There is a functor* $\mathscr{K}: \mathscr{A} \longrightarrow \mathscr{B}$ *and a natural weak equivalence*

$$\psi: |\mathscr{K}(X)| \longrightarrow X.$$

The n*-simplices of* $\mathscr{K}(X)$ *are*

$$\{x_0, \cdots, x_n | x_0 < \cdots < x_n\},$$

and $\psi(u) = x_0$ *if* u *is an interior point of the simplex spanned by* $\{x_0, \cdots, x_n\}$*.*

Let $Sd\,K$ be the barycentric subdivision of a simplicial complex K; let b_σ be the barycenter of a simplex σ.

Theorem 13 *There is a functor* $\mathscr{X}: \mathscr{B} \longrightarrow \mathscr{A}$ *and a natural weak equivalence*

$$\phi: |K| \longrightarrow \mathscr{X}(K).$$

The points of $\mathscr{X}(K)$ *are the barycenters* b_σ *of*

simplices of K *and* $b_\sigma < b_\tau$ *if* $\sigma \subset \tau$*.*

$\mathscr{K}(\mathscr{X}(K)) = Sd\,K$ *and*

$$\phi_K = \psi_{\mathscr{X}(K)}: |K| \cong |Sd\,K| \longrightarrow \mathscr{X}(K).$$

Problem: not many maps between finite spaces!

Solution: subdivision: $Sd\,X \equiv \mathscr{X}(\mathscr{K}(X))$.

Theorem 14 *There is a natural weak equiv.*

$$\xi\colon Sd\,X \longrightarrow X.$$

Classical result and an implied analogue:

Theorem 15 *Let $f\colon |K| \longrightarrow |L|$ be continuous, where K and L are simplicial complexes, K finite. For some large n, there is a simplicial map $g\colon K^{(n)} \longrightarrow L$ such that $f \simeq |g|$.*

Theorem 16 *Let $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$ be continuous, where X and Y are T_0-A-spaces, X finite. For some large n there is a continuous map $g\colon X^{(n)} \longrightarrow Y$ such that $f \simeq |\mathscr{K}(g)|$.*

Definition 4 *Let X be a space. Define the* non-Hausdorff cone $\mathbb{C}X$ *by adjoining a new point $+$ and letting the proper open subsets of $\mathbb{C}X$ be the non-empty open subsets of X.*

Define the non-Hausdorff suspension $\mathbb{S}X$ *by adjoining two points $+$ and $-$ such that $\mathbb{S}X$ is the union under X of two copies of $\mathbb{C}X$.*

Let SX be the unreduced suspension of X.

Definition 5 *Define a natural map*

$$\gamma = \gamma_X \colon SX \longrightarrow \mathbb{S}X$$

by $\gamma(x,t) = x$ if $-1 < t < 1$ and $\gamma(\pm 1) = \pm$.

Theorem 17 *γ is a weak equivalence.*

Corollary 5 *$\mathbb{S}^n S^0$ is a minimal finite space with $2n+2$ points, and it is weak equivalent to S^n.*

The height $h(X)$ of a poset X is the maximal length h of a chain $x_1 < \cdots < x_h$ in X.

$$h(X) = \mathit{dim}\,|\mathscr{K}(X)| + 1.$$

Barmak and Minian:

Proposition 3 *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$.*

Corollary 6 *If $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere S^n, then X has at least $2n+2$ points, and if it has exactly $2n+2$ points it is homeomorphic to $\mathbb{S}^n S^0$.*

Remark 2 *If X has six elements, then $h(X)$ is 2 or 3. There is a six point finite space that is weak homotopy equivalent to S^1 but is not homotopy equivalent to $\mathbb{S}S^0$.*

Really finite H-spaces

Let X be a finite space and an H-space with unit e: $x \to ex$ and $x \to xe$ are each homotopic to the identity. Stong:

Theorem 18 *If X is minimal, these maps are homeomorphisms and e is both a maximal and a minimal point of X, so $\{e\}$ is a component.*

Theorem 19 *X is an H-space with unit e iff e is a deformation retract of its component in X. Therefore X is an H-space iff a component of X is contractible. If X is a connected H-space, X is contractible.*

Hardie, Vermeulen, Witbooi:

Let $\mathbb{T} = \mathbb{S}S^0$, $\mathbb{T}' = Sd\,\mathbb{T}$.

Brute force write it down proof (8×8 matrix)

Example 1 *There is product $\mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$ that realizes the product on S^1 after realization.*

Finite groups and finite spaces

X, Y finite T_0-spaces and G-spaces. Stong:

Theorem 20 *X has an equivariant core, namely a sub G-space that is a core and a G-deformation retract of X.*

Corollary 7 *Let X be contractible. Then X is G-contractible and has a point fixed by every self-homeomorphism.*

Corollary 8 *If $f\colon X \longrightarrow Y$ is a G-map and a homotopy equivalence, then it is a G-homotopy equivalence.*

Quillen's conjecture

G finite, p prime.

$\mathscr{S}_p(G)$: poset of non-trivial p-subgroups of G, ordered by inclusion.

G acts on $\mathscr{S}_p(G)$ by conjugation.

$\mathscr{A}_p(G)$: Sub G-poset of p-tori.

p-torus $\equiv$ elementary Abelian p-group.

$r_p(G)$ is the rank of a maximal p-torus in G.

$$\begin{array}{ccc}
|\mathscr{K}\mathscr{A}_p(G)| & \xrightarrow{|\mathscr{K}(i)|} & |\mathscr{K}\mathscr{S}_p(G)| \\
\downarrow \psi & & \downarrow \psi \\
\mathscr{A}_p(G) & \xrightarrow[i]{} & \mathscr{S}_p(G)
\end{array}$$

Vertical maps ψ are weak equivalences.

Proposition 4 *If G is a p-group, $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$ are contractible.*

Note: genuinely contractible, not just weakly.

Proposition 5 $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ *is a weak equivalence.*

Example 2 *If $G = \Sigma_5$, $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$ are not homotopy equivalent.*

$P \in \mathscr{S}_p(G)$ is normal iff P is a G-fixed point.

Theorem 21 *If $\mathscr{S}_p(G)$ or $\mathscr{A}_p(G)$ is contractible, then G has a non-trivial normal p-subgroup. Conversely, if G has a non-trivial normal p-subgroup, then $\mathscr{S}_p(G)$ is contractible, hence $\mathscr{A}_p(G)$ is weakly contractible.*

Conjecture 1 *(Quillen) If $\mathscr{A}_p(G)$ is weakly contractible, then G contains a non-trivial normal p-subgroup.*

Easy: True if $r_P(G) \leq 2$.

Quillen: True if G is solvable.

Aschbacker and Smith: True if $p > 5$ and G has no component $U_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd.

(Component of G: normal subgroup that is simple modulo its center).

Horrors: proof from the classification theorem.

Their 1993 article summarizes earlier results.

And as far as Jon Alperin and I know, that is where the problem stands. Finite space version may not help with the proof, but is intriquing.