Equivariant cohomology

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May 3, 2012
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May 5, 2012
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(collated slides)

OUTLINE

- Two classical definitions: Borel and Bredon
- P.A. Smith theory on fixed point spaces
- The Conner conjecture on orbit spaces
- The Oliver transfer and $RO(G)$-graded cohomology
- Mackey functors for finite and compact Lie groups
- Extending Bredon cohomology to $RO(G)$-grading
- A glimpse of the modern world of spectra and G-spectra

Borel's definition (1958):

G a topological group, X a (left) G-space, action $G \times X \longrightarrow X$

$$g(hx) = (gh)x, ex = x$$

EG a contractible (right) G-space with free action

$$yg = y \text{ implies } g = e$$

$$EG \times_G X = EG \times X/\sim \qquad (yg, x) \sim (y, gx)$$

"homotopy orbit space of X"

A an abelian group

$$H^*_{Bor}(X; A) = H^*(EG \times_G X; A)$$

Characteristic classes in Borel cohomology

$B(G,\Pi)$: classifying G-space for principal (G,Π)-bundles, principal Π-bundles with G-acting through bundle maps.

Theorem

$EG \times_G B(G,\Pi) \simeq BG \times B\Pi$ over BG.

Therefore, with field coefficients,

$$H^*_{Bor}(B(G,\Pi)) = H^*(EG \times_G B(G,\Pi)) \cong H^*(BG) \otimes H^*(B\Pi)$$

as an $H^*(BG)$-module.

Not very interesting theory of equivariant characteristic classes.

Bredon's definition (1967):

Slogan: "orbits are equivariant points" since $(G/H)/G = *$.

A coefficient system $\mathscr{A}$ is a contravariant functor

$$\mathscr{A} : h\mathscr{O}_G \longrightarrow \mathscr{A}b$$

$\mathscr{O}_G$ is the category of orbits G/H and G-maps,
$h\mathscr{O}_G$ is its homotopy category ($= \mathscr{O}_G$ if G is discrete)

$$H^*_G(X; \mathscr{A})$$

satisfies the Eilenberg-Steenrod axioms plus

"the equivariant dimension axiom":

$$H^0_G(G/H; \mathscr{A}) = \mathscr{A}(G/H), \quad H^n_G(G/H; \mathscr{A}) = 0 \;\text{ if }\; n \neq 0$$

Axioms for reduced cohomology theories

Cohomology theory $\widetilde{E}^*$ on based G-spaces (G-CW $\simeq$ types):

Contravariant homotopy functors $\widetilde{E}^n$ to Abelian groups, $n \in \mathbb{Z}$.

Natural suspension isomorphisms

$$\widetilde{E}^n(X) \longrightarrow \widetilde{E}^{n+1}(\Sigma X)$$

For $A \subset X$, the following sequence is exact:

$$\widetilde{E}^n(X/A) \longrightarrow \widetilde{E}^n(X) \longrightarrow \widetilde{E}^n(A)$$

The following natural map is an isomorphism:

$$\widetilde{E}^n(\bigvee_{i \in I} X_i) \longrightarrow \prod_{i \in I} \widetilde{E}^n(X_i)$$

$$E^n(X) = \widetilde{E}^n(X_+), \quad X_+ = X \amalg \{*\}; \quad E^n(X, A) = \widetilde{E}^n(X/A)$$

Borel vs Bredon:

$\underline{A}$ = the constant coefficient system, $\underline{A}(G/H) = A$

$$H^*(X/G; A) \cong H^*_G(X; \underline{A})$$

since both satisfy the dimension axiom and Bredon is unique.
Therefore

$$H^*_{Bor}(X; A) \equiv H^*(EG \times_G X; A) \cong H^*_G(EG \times X; \underline{A})$$

On "equivariant points", $EG \times_G (G/H) \cong EG/H = BH$, hence

$$H^*(EG \times_G (G/H); A) = H^*(BH; A).$$

Cellular (or singular) cochain construction:

G-CW complex X, cells of the form $G/H \times D^n$:

$X = \cup X^n$, $X^0 =$ disjoint union of orbits, pushouts

$$\begin{array}{ccc} \coprod_i G/H_i \times S^n & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ \coprod_i G/H_i \times D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

$$X^\bullet \colon \mathscr{O}_G^{op} \longrightarrow \mathit{Spaces}, \quad X^\bullet(G/H) = X^H$$

Chain complex $C_*(X)$ of coefficient systems:

$$C_n(X)(G/H) = C_n((X^n/X^{n-1})^H; \mathbb{Z})$$

Cochain complex of abelian groups:

$$C^*(X; \mathscr{A}) = \mathrm{Hom}_{\mathit{Coeff}}(C_*(X), \mathscr{A})$$

P.A. Smith theory (1938):

G a finite p-group, X a finite dimensional G-CW complex.

Consider mod p cohomology. Assume that $H^*(X)$ is finite.

Theorem

If $H^*(X) \cong H^*(S^n)$, then X^G is empty or $H^*(X^G) \cong H^*(S^m)$ for some $m \leq n$.

If $p > 2$, then $n - m$ is even and $X^G \neq \emptyset$ if n is even.

If H is a normal subgroup of G, then $X^G = (X^H)^{G/H}$.

Finite p-groups are nilpotent.

By induction on the order of G,

we may assume that G is cyclic of order p.

The Bockstein exact sequence

A short exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$$

of coefficient systems implies a short exact sequence

$$0 \longrightarrow C^*(X;\mathscr{A}) \longrightarrow C^*(X;\mathscr{B}) \longrightarrow C^*(X;\mathscr{C}) \longrightarrow 0$$

of cochain complexes, which implies a long exact sequence

$$\cdots \longrightarrow H^q_G(X;\mathscr{A}) \longrightarrow H^q_G(X;\mathscr{B}) \longrightarrow H^q_G(X;\mathscr{C}) \longrightarrow \cdots$$

Connecting homomorphism

$$\beta\colon H^q_G(X;\mathscr{C}) \longrightarrow H^{q+1}_G(X;\mathscr{A})$$

is called a “Bockstein operation”.

Smith theory

Let $FX = X/X^G$. Define $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ so that

$$H^*_G(X;\mathscr{A}) \cong \widetilde{H}^*(FX/G),$$

$$H^*_G(X;\mathscr{B}) \cong H^*(X),$$

$$H^*_G(X;\mathscr{C}) \cong H^*(X^G)$$

On orbits $G = G/e$ and $* = G/G$,

$$\mathscr{A}(G) = \mathbb{F}_p, \quad \mathscr{A}(*) = 0$$

$$\mathscr{B}(G) = \mathbb{F}_p[G], \quad \mathscr{B}(*) = \mathbb{F}_p$$

$$\mathscr{C}(G) = 0, \quad \mathscr{C}(*) = \mathbb{F}_p$$

Let

$$a_q = \dim \widetilde{H}^q(FX/G), \quad b_q = \dim H^q(X), \quad c_q = \dim H^q(X^G)$$

Beginning of proof of Smith theorem for $p = 2$

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \oplus \mathscr{C} \longrightarrow 0$$

On G, $\quad 0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2[G] \longrightarrow \mathbb{F}_2 \oplus 0 \longrightarrow 0.$

On $*$, $\quad 0 \longrightarrow 0 \longrightarrow \mathbb{F}_2 \longrightarrow 0 \oplus \mathbb{F}_2 \longrightarrow 0.$

$$H^*(X; \mathscr{A} \oplus \mathscr{C}) \cong H^*(X; \mathscr{A}) \oplus H^*(X; \mathscr{C})$$

Bockstein LES implies

$$\chi(X) = \chi(X^G) + 2\widetilde{\chi}(FX/G)$$

and

$$a_q + c_q \leq b_q + a_{q+1}$$

Beginning of proof of Smith theorem for $p > 2$

Let $I = \mathrm{Ker}(\varepsilon)$, $\varepsilon \colon \mathbb{F}_p[G] \longrightarrow \mathbb{F}_p$, where $\varepsilon(g) = 1$.

Define $\mathscr{I}^n$ by $\mathscr{I}^n(G) = I^n$ and $\mathscr{I}^n(*) = 0$. Then $\mathscr{I}^{p-1} = \mathscr{A}$.

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \oplus \mathscr{C} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{I} \oplus \mathscr{C} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I}^{n+1} \longrightarrow \mathscr{I}^n \longrightarrow \mathscr{A} \longrightarrow 0, \quad 1 \le n < p$$

Let

$$\bar{a}_q = \dim H^q_G(X; \mathscr{I})$$

Bockstein LES implies

$$\chi(X) = \chi(X^G) + p\widetilde{\chi}(FX/G)$$

and

$$a_q + c_q \le b_q + \bar{a}_{q+1}, \quad \bar{a}_q + c_q \le b_q + a_{q+1}$$

Completion of proof for any p

Inductively, for $q \geq 0$ and $r \geq 0$, with r odd if $p > 2$,

$$a_q + c_q + \cdots c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$

Let $n = \dim(X)$. With $q = n + 1$ and $r > n$, get $c_i = 0$ for $i > n$. With $q = 0$ and $r > n$, get

$$\sum c_q \leq \sum b_q.$$

So far, all has been general. If $H^*(X) \cong H^*(S^n)$, then $\sum b_q = 2$.

$\chi(X) \equiv \chi(X^G) \bmod p$ implies $\sum c_q = 0$ ($X^G = \emptyset$) or $\sum c_q = 2$.

If $p > 2$, it also implies $n - m$ is even and, if n is even, $X^G \neq \emptyset$.

The Conner conjecture (1960); first proven by Oliver (1976)

G a compact Lie group, X a finite dimensional G-CW complex with finitely many orbit types, A an abelian group.

Theorem

If $\widetilde{H}^*(X;A) = 0$, then $\widetilde{H}^*(X/G;A) = 0$.

Conner (implicitly): True if G is a finite extension of a torus.

If H is a normal subgroup of G, then $X/G = (X/H)\,/\,(G/H)$.

Reduces to $G = S^1$ and G finite. Standard methods apply.

General case: let N be the normalizer of a maximal torus T in G.

Then $\chi(G/N) = 1$ and $\widetilde{H}^n(X/N;A) = 0$.

The Oliver transfer

Theorem

Let $H \subset G$, $\pi\colon X/H \longrightarrow X/G$. For $n \geq 0$, there is a transfer map

$$\tau\colon \widetilde{H}^n(X/H; A) \longrightarrow \widetilde{H}^n(X/G; A)$$

such that $\tau \circ \pi^*$ is multiplication by $\chi(G/H)$.

Proof of the Conner conjecture.

Take $H = N$. The composite

$$\widetilde{H}^n(X/G; A) \xrightarrow{\pi^*} \widetilde{H}^n(X/N; A) \xrightarrow{\tau} \widetilde{H}^n(X/G; A)$$

is the identity and $\widetilde{H}^n(X/N; A) = 0$. $\square$

How do we get the Oliver transfer?

$RO(G)$-graded cohomology

$$X \wedge Y = X \times Y / X \vee Y$$

V a representation of G, S^V its 1-point compactification.

$$\Sigma^V X = X \wedge S^V, \quad \Omega^V X = \mathrm{Map}_*(S^V, X)$$

Suspension axiom on an "$RO(G)$-graded cohomology theory E^*":

$$\widetilde{E}^{\alpha}(X) \cong \widetilde{E}^{\alpha+V}(\Sigma^V X)$$

for all $\alpha \in RO(G)$ and all representations V.

Theorem
If $\mathscr{A} = \underline{\mathbb{Z}}$ (hence if $\mathscr{A} = \underline{A} = \underline{\mathbb{Z}} \otimes A$), then $H^*_G(-;\mathscr{A})$ extends to an $RO(G)$-graded cohomology theory.

Construction of the Oliver transfer

Let $X_+ = X \amalg \{*\}$. Consider $\varepsilon\colon (G/H)_+ \longrightarrow S^0$.

Theorem

For large enough V, there is a map

$$t\colon S^V = \Sigma^V S^0 \longrightarrow \Sigma^V G/H_+$$

such that $\Sigma^V \varepsilon \circ t$ has (nonequivariant) degree $\chi(G/H)$.

The definition of $\tau\colon \widetilde{H}^n(X/H; A) \longrightarrow \widetilde{H}^n(X/G; A)$.

$$\widetilde{H}^n(X/H; A) \cong \widetilde{H}^n_H(X; \underline{A}) \cong \widetilde{H}^n_G(X \wedge G/H_+; \underline{A}) \cong \widetilde{H}^{n+V}_G(X \wedge \Sigma^V G/H_+; \underline{A})$$

$$\widetilde{H}^n(X/G; A) \cong \widetilde{H}^n_G(X; \underline{A}) \cong \widetilde{H}^{n+V}_G(\Sigma^V X; \underline{A}) = \widetilde{H}^{n+V}_G(X \wedge S^V; \underline{A})$$

Smashing with X, t induces τ. □

How do we get the map t?

Generalizing, let M be a smooth G-manifold.

Embed M in a large V. The embedding has a normal bundle ν.

The embedding extends to an embedding of the total space of ν as a tubular neighborhood in V.

The Pontryagin Thom construction gives a map $S^V \longrightarrow T\nu$, where $T\nu$ is the Thom space of the normal bundle.

Compose with $T\nu \longrightarrow T(\tau \oplus \nu) \cong T\varepsilon = M_+ \wedge S^V$.

The composite is the transfer $t\colon S^V \longrightarrow \Sigma^V M_+$.

Atiyah duality: M_+ and $T\nu$ are Spanier-Whitehead dual.
This is the starting point for equivariant Poincaré duality,
for which $RO(G)$-grading is essential.

$RO(G)$-graded Bredon cohomology

Theorem

$H^*_G(-;\mathscr{A})$ extends to an $RO(G)$-graded theory if and only if the coefficient system $\mathscr{A}$ extends to a Mackey functor.

Theorem

$\underline{\mathbb{Z}}$, hence $\underline{A}$, extends to a Mackey functor.

What is a Mackey functor?

First definition, for finite G

Let $G\mathscr{S}$ be the category of finite G-sets. A Mackey functor $\mathscr{M}$ consists of covariant and contravariant functors

$$\mathscr{M}^*, \mathscr{M}_*\colon G\mathscr{S} \longrightarrow \mathscr{A}b,$$

which are the same on objects (written M) and satisfy:

$$M(A \amalg B) \cong M(A) \oplus M(B)$$

and a pullback of finite sets gives a commutative diagram:

$$\begin{CD} P @>{g}>> T \\ @V{i}VV @VV{j}V \\ S @>>{f}> B \end{CD} \qquad \begin{CD} M(P) @>{g_*}>> M(T) \\ @A{i^*}AA @AA{j^*}A \\ M(S) @>>{f_*}> M(B) \end{CD}$$

Suffices to define on orbits.

Pullback condition gives the "double coset formula".

Example: $\mathscr{M}(G/H) = R(H)$ (representation ring of H).

Restriction and induction give $\mathscr{M}^*$ and $\mathscr{M}_*$.

Second definition, G finite

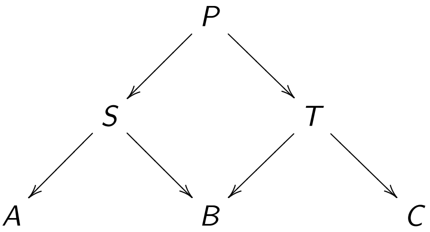
Category *G-Span* of "spans" of finite G-sets.

Objects are finite G-sets. Morphisms $A \longrightarrow B$ are diagrams

$$A \longleftarrow S \longrightarrow B$$

Really equivalence classes: $S \sim S'$ if $S \cong S'$ over A and B.

Composition by pullbacks:



A Mackey functor $\mathscr{M}$ is a (contravariant) functor

$$\mathscr{M}: G\text{-}Span \longrightarrow \mathscr{A}b,$$

written M on objects and satisfying $M(A \amalg B) \cong M(A) \oplus M(B)$.

Lemma

A Mackey functor is a Mackey functor.

Given $\mathscr{M}$,

$$A \xleftarrow{=} A \longrightarrow B, \quad A \longleftarrow B \xrightarrow{=} B$$

give $\mathscr{M}^*$ and $\mathscr{M}_*$. Given $\mathscr{M}^*$ and $\mathscr{M}_*$, composites give $\mathscr{M}$.

Topological reinterpretation

For based G-spaces X and Y with X a finite G-CW complex,

$$\{X, Y\}_G \equiv \mathrm{colim}_V [\Sigma^V X, \Sigma^V Y]_G$$

"Stable orbit category" or "Burnside category" $\mathscr{B}_G$:
objects G/H, abelian groups of morphisms

$$\mathscr{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G$$

Theorem

If G is finite, $\mathscr{B}_G$ is isomorphic to the full subcategory of orbits G/H in G-Span.

Mackey functors are contravariant additive functors $\mathscr{B}_G \longrightarrow \mathscr{A}b$.

Theorem if G is finite. Definition if G is a compact Lie group.

The Mackey functor $\underline{\mathbb{Z}}$

Define

$$\mathscr{A}_G(G/H) = \mathscr{B}_G(G/H, *) \cong \{S^0, S^0\}_H = A(H).$$

This gives the Burnside ring Mackey functor $\mathscr{A}_G$.

Augmentation ideal sub Mackey functor $\mathscr{I}_G(G/H) = IA(H)$.

The quotient Mackey functor $\mathscr{A}_G/\mathscr{I}_G$ is $\underline{\mathbb{Z}}$.

How can we extend $\mathbb{Z}$-grading to $RO(G)$-grading?

Represent ordinary $\mathbb{Z}$-graded theories on G-spectra by Eilenberg-MacLane G-spectra, which then represent $RO(G)$-graded theories!

What are spectra?

- Prespectra (naively, spectra): sequences of spaces T_n and maps $\Sigma T_n \longrightarrow T_{n+1}$
- Ω-(pre)spectra: Adjoints are equivalences $T_n \xrightarrow{\simeq} \Omega T_{n+1}$
- Spectra: Spaces E_n and homeomorphisms $E_n \longrightarrow \Omega E_{n+1}$
- Spaces to prespectra: $\{\Sigma^n X\}$ and $\Sigma(\Sigma^n X) \xrightarrow{\cong} \Sigma^{n+1} X$
- Prespectra to spectra, when $T_n \xrightarrow{\subset} \Omega T_{n+1}$:

$$(LT)_n = \operatorname{colim} \Omega^q T_{n+q}$$

- Spaces to spectra: $\Sigma^\infty X = L\{\Sigma^n X\}$
- Spectra to spaces: $\Omega^\infty E = E_0$
- Coordinate-free: spaces T_V and maps $\Sigma^W T_V \longrightarrow T_{V \oplus W}$

What are spectra good for?

- First use: Spanier-Whitehead duality [1958]
- Cobordism theory [1959] (Milnor; MSO has no odd torsion)
- Stable homotopy theory [1959] (Adams; ASS for spectra)
- Generalized cohomology theories [1960] (Atiyah-Hirzebruch; K-theory, AHSS)
- Generalized homology theories [1962] (G.W. Whitehead)
- Stable homotopy category [1964] (Boardman's thesis)

Representing cohomology theories

Fix Y. If $Y \simeq \Omega^2 Z$, then $[X, Y]$ is an abelian group.

For $A \subset X$, the following sequence is exact:

$$[X/A, Y] \longrightarrow [X, Y] \longrightarrow [A, Y]$$

The following natural map is an isomorphism:

$$[\bigvee_{i \in I} X_i, Y] \longrightarrow \prod_{i \in I} [X_i, Y]$$

For an Ω-spectrum $E = \{E_n\}$,

$$\widetilde{E}^n(X) = \begin{cases} [X, E_n] & \text{if } n \geq 0 \\ [X, \Omega^{-n} E_0] & \text{if } n < 0 \end{cases}$$

Suspension:

$$\widetilde{E}^n(X) = [X, E_n] \cong [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \widetilde{E}^{n+1}(\Sigma X)$$

What are naive G-spectra (any G)?

- Naive G-spectra: spectra with G-action
- G-spaces T_n and G-maps $\Sigma T_n \longrightarrow T_{n+1}$
- Naive Ω-G-spectra: $T_n \xrightarrow{\simeq} \Omega T_{n+1}$

Naive Ω-G-spectra $E = \{E_n\}$ represent $\mathbb{Z}$-graded cohomology.

$$\widetilde{E}_G^n(X) = \begin{cases} [X, E_n]_G & \text{if } n \geq 0 \\ [X, \Omega^{-n} E_0]_G & \text{if } n < 0 \end{cases}$$

Ordinary theories

Eilenberg-Mac Lane spaces:

$$\pi_n K(A,n) = A, \quad \pi_q K(A,n) = 0 \text{ if } q \neq n.$$

$$\widetilde{H}^n(X;A) = [X, K(A,n)]$$

For based G-spaces X,

$$\underline{\pi}_n(X) = \pi_n(X^\bullet); \quad \underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Eilenberg-Mac Lane G-spaces:

$$\underline{\pi}_n K(\mathscr{A},n) = \mathscr{A}, \quad \underline{\pi}_q K(\mathscr{A},n) = 0 \text{ if } q \neq n.$$

$$\widetilde{H}^n_G(X;\mathscr{A}) = [X, K(\mathscr{A},n)]_G$$

What are genuine G-spectra (G compact Lie)?

- G-spaces T_V, G-maps $\Sigma^W T_V \longrightarrow T_{V\oplus W}$ where V, W are real representations of G
- Ω-G-spectra: G-equivalences $T_V \xrightarrow{\simeq} \Omega^W T_{V\oplus W}$

Genuine Ω-G-spectra E represent $RO(G)$-graded theories.

Imprecisely,

$$E_G^{V-W}(X) = [\Sigma^W X, E_V].$$

Ordinary? Need genuine Eilenberg-Mac Lane G-spectra.

A quick and dirty construction (1981)

Build a good "equivariant stable homotopy category" of G-spectra.

Use sphere G-spectra $G/H_+ \wedge S^n$ to get a theory of G-CW spectra.

Mimic Bredon's construction of ordinary $\mathbb{Z}$-graded cohomology, but in the category of G-spectra, using Mackey functors instead of coefficient systems.

Apply Brown's representability theorem to represent the 0th term by a G-spectrum $H\mathscr{M}$: for G-spectra X,

$$H^0_G(X; \mathscr{M}) \cong \{X, H\mathscr{M}\}_G.$$

Then $H\mathscr{M}$ is the required Eilenberg-Mac Lane G-spectrum.

What are G-spectra good for?

- Equivariant K-theory [1968] (Atiyah, Segal)
- Equivariant cobordism [1964] (Conner and Floyd)
- $RO(G)$-graded homology and cohomology theories
- Equivariant Spanier-Whitehead and Poincaré duality
- Equivariant stable homotopy category (Lewis-May)
- Completion theorems (KU_G, π_G^*, MU_G-modules): (Atiyah-Segal, Segal conjecture, Greenlees-May)
- Nonequivariant applications!!!

Kervaire invariant one problem

Framed manifold M: trivialization of its (stable) normal bundle.

Ω_n^{fr}: Cobordism classes of (smooth closed) framed n-manifolds.

Is every framed n-manifold M, $n = 4k+2$, framed cobordant to a homotopy sphere (a topological sphere by Poincaré conjecture)?

$$\kappa\colon \Omega_{4k+2}^{fr} \longrightarrow \mathbb{F}_2$$

$\kappa[M]$ is the Kervaire invariant, the Arf invariant of a quadratic refinement of the cup product form on $H^{2k+1}(M;\mathbb{F}_2)$ that is determined by the given framing.

$\kappa[M] = 0$ if and only if $[M] = [\Sigma]$ for some homotopy sphere Σ.

History

$n = 2, 6, 14$: $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ have $\kappa = 1$ framings.

Kervaire (1960): PL, non-smoothable, 10-manifold M with $\kappa = 1$.

Kervaire and Milnor (1963): maybe $\kappa = 0$ for $n \neq 2$, 6, 14?

Browder (1969): $\kappa = 0$ unless $n = 2^{j+1} - 2$ for some j, and then $\kappa = 0$ if and only if h_j^2 does not survive in the ASS, $h_j \leftrightarrow Sq^{2^j}$.

Calculation/construction (Barratt, Jones, Mahowald, Tangora (using May SS)):

h_4^2 and h_5^2 survive the ASS. (h_6^2 doable??)

Hill, Hopkins, Ravenel

Theorem (2009)

$\kappa = 0$ unless n is 2, 6, 14, 30, 62, or maybe 126:
h_j^2 has a non-zero differential in the ASS, $j \geq 7$.

Calculations of $RO(G)$-graded groups $H^*_G(*; \underline{\mathbb{Z}})$ are critical!

Haynes Miller quote (Bourbaki Séminaire survey):

Hill, Hopkins, and Ravenel marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

- The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers.
- The theory of structured ring spectra, implemented by May and many others; and
- Equivariant stable homotopy theory, as developed by May and collaborators.

Structured ring spectra and structured ring G-spectra

E_∞ ring spectra (May-Quinn-Ray [1972])

E_∞ ring G-spectra (Lewis-May [1986])

Recent paradigm shift in stable homotopy theory.

Symmetric monoidal category of spectra $\mathscr{S}$ under $\wedge$;

E_∞ ring spectra are just commutative monoids in $\mathscr{S}$.

Elmendorf-Kriz-Mandell-May [1997]: S-modules, operadic $\wedge$
Hovey-Shipley-Smith [2000]: Symmetric spectra, categorical $\wedge$
Mandell-May-Shipley-Schwede [2001]: Orthogonal, comparisons
Mandell-May [2002]: Orthogonal G-spectra and S_G-modules

New subjects:

"Brave new algebra" (Waldhausen's name, now apt)

"Derived algebraic geometry" (Toen-Vezzosi, Lurie)

Revitalized areas

Equivariant ∞ loop space theory

Equivariant algebraic K-theory

(Guillou-Merling-May, [2011-2012]).

Prospective applications to algebraic K-theory of number rings?

Theorem

Let L be a Galois extension of a field F with Galois group G.

There is an E_∞ ring G-spectrum $K_G(L)$ such that

$$(K_G(L))^H = K(L^H) \quad \textit{for} \quad H \subset G$$

where $\pi_* K(R) =$ Quillen's algebraic K-groups of R.