Equivariant cohomology

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May 3, 2012
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May 5, 2012
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(collapsed slides)
OUTLINE

- Two classical definitions: Borel and Bredon
- P.A. Smith theory on fixed point spaces
- The Conner conjecture on orbit spaces
- The Oliver transfer and $RO(G)$-graded cohomology
- Mackey functors for finite and compact Lie groups
- Extending Bredon cohomology to $RO(G)$-grading
- A glimpse of the modern world of spectra and $G$-spectra
Borel’s definition (1958):

$G$ a topological group, $X$ a (left) $G$-space, action $G \times X \rightarrow X$

$$g(hx) = (gh)x, \ ex = x$$

$EG$ a contractible (right) $G$-space with free action

$$yg = y \text{ implies } g = e$$

$$EG \times_G X = EG \times X / \sim \quad (yg, x) \sim (y, gx)$$

“homotopy orbit space of $X$”

$A$ an abelian group

$$H^*_{Bor}(X; A) = H^*(EG \times_G X; A)$$
Characteristic classes in Borel cohomology

\( B(G, \Pi) \): classifying \( G \)-space for principal \((G, \Pi)\)-bundles, principal \( \Pi \)-bundles with \( G \)-acting through bundle maps.

**Theorem**

\[ EG \times_G B(G, \Pi) \simeq BG \times B\Pi \text{ over } BG. \]

*Therefore, with field coefficients,*

\[ H^*_{Bor}(B(G, \Pi)) = H^*(EG \times_G B(G, \Pi)) \cong H^*(BG) \otimes H^*(B\Pi) \]

*as an \( H^*(BG) \)-module.*

Not very interesting theory of equivariant characteristic classes.
Bredon's definition (1967):
Slogan: “orbits are equivariant points” since \((G/H)/G = \ast\).

A coefficient system \(\mathcal{A}\) is a contravariant functor

\[
\mathcal{A} : h\mathcal{O}_G \longrightarrow \text{Ab}
\]

\(\mathcal{O}_G\) is the category of orbits \(G/H\) and \(G\)-maps,
\(h\mathcal{O}_G\) is its homotopy category \((= \mathcal{O}_G \text{ if } G \text{ is discrete})\)

\[
H^*_G(X; \mathcal{A})
\]

satisfies the Eilenberg-Steenrod axioms plus

“the equivariant dimension axiom”:

\[
H^0_G(G/H; \mathcal{A}) = \mathcal{A}(G/H), \quad H^n_G(G/H; \mathcal{A}) = 0 \text{ if } n \neq 0
\]
Axioms for reduced cohomology theories

Cohomology theory $\tilde{E}^*$ on based $G$-spaces ($G$-CW $\simeq$ types):

Contravariant homotopy functors $\tilde{E}^n$ to Abelian groups, $n \in \mathbb{Z}$.

Natural suspension isomorphisms

$$\tilde{E}^n(X) \rightarrow \tilde{E}^{n+1}(\Sigma X)$$

For $A \subset X$, the following sequence is exact:

$$\tilde{E}^n(X/A) \rightarrow \tilde{E}^n(X) \rightarrow \tilde{E}^n(A)$$

The following natural map is an isomorphism:

$$\tilde{E}^n(\bigvee_{i \in I} X_i) \rightarrow \prod_{i \in I} \tilde{E}^n(X_i)$$

$$E^n(X) = \tilde{E}^n(X_+), \quad X_+ = X \amalg \{\ast\}; \quad E^n(X, A) = \tilde{E}^n(X/A)$$
Borel vs Bredon:

\( A = \) the constant coefficient system, \( A(G/H) = A \)

\[ H^*(X/G; A) \cong H^*_G(X; A) \]

since both satisfy the dimension axiom and Bredon is unique. Therefore

\[ H^*_{Bor}(X; A) \equiv H^*(EG \times_G X; A) \cong H^*_G(EG \times X; A) \]

On “equivariant points”, \( EG \times_G (G/H) \cong EG/H = BH \), hence

\[ H^*(EG \times_G (G/H); A) = H^*(BH; A). \]
Cellular (or singular) cochain construction:

$G$-CW complex $X$, cells of the form $G/H \times D^n$:

$X = \bigcup X^n$, $X^0 =$ disjoint union of orbits, pushouts

\[
\bigcup_i G/H_i \times S^n \rightarrow X^n \rightarrow \bigcup_i G/H_i \times D^{n+1} \rightarrow X^{n+1}
\]

$X^\bullet: O_G^{op} \rightarrow \text{Spaces}, \quad X^\bullet(G/H) = X^H$

Chain complex $C_*(X)$ of coefficient systems:

$C_n(X)(G/H) = C_n((X^n/X^{n-1})^H; \mathbb{Z})$

Cochain complex of abelian groups:

$C^*(X; \mathcal{A}) = \text{Hom}_{\text{Coeff}}(C_*(X), \mathcal{A})$
P.A. Smith theory (1938):

$G$ a finite $p$-group, $X$ a finite dimensional $G$-CW complex. Consider mod $p$ cohomology. Assume that $H^*(X)$ is finite.

**Theorem**

If $H^*(X) \cong H^*(S^n)$, then $X^G$ is empty or $H^*(X^G) \cong H^*(S^m)$ for some $m \leq n$.

If $p > 2$, then $n - m$ is even and $X^G \neq \emptyset$ if $n$ is even.

If $H$ is a normal subgroup of $G$, then $X^G = (X^H)^{G/H}$.

Finite $p$-groups are nilpotent.

By induction on the order of $G$,
we may assume that $G$ is cyclic of order $p$. 
The Bockstein exact sequence

A short exact sequence

\[ 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \]

of coefficient systems implies a short exact sequence

\[ 0 \longrightarrow C^*(X; \mathcal{A}) \longrightarrow C^*(X; \mathcal{B}) \longrightarrow C^*(X; \mathcal{C}) \longrightarrow 0 \]

of cochain complexes, which implies a long exact sequence

\[ \cdots \longrightarrow H^q_G(X; \mathcal{A}) \longrightarrow H^q_G(X; \mathcal{B}) \longrightarrow H^q_G(X; \mathcal{C}) \longrightarrow \cdots \]

Connecting homomorphism

\[ \beta: H^q_G(X; \mathcal{C}) \longrightarrow H^{q+1}_G(X; \mathcal{A}) \]

is called a “Bockstein operation”.
Smith theory
Let \( FX = X / X^G \). Define \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) so that

\[
H^*_G(X; \mathcal{A}) \cong \tilde{H}^*(FX / G),
\]

\[
H^*_G(X; \mathcal{B}) \cong H^*(X),
\]

\[
H^*_G(X; \mathcal{C}) \cong H^*(X^G)
\]

On orbits \( G = G / e \) and \( * = G / G \),

\[
\mathcal{A}(G) = \mathbb{F}_p, \quad \mathcal{A}(*) = 0
\]

\[
\mathcal{B}(G) = \mathbb{F}_p[G], \quad \mathcal{B}(*) = \mathbb{F}_p
\]

\[
\mathcal{C}(G) = 0, \quad \mathcal{C}(*) = \mathbb{F}_p
\]

Let

\[
a_q = \dim \tilde{H}^q(FX / G), \quad b_q = \dim H^q(X), \quad c_q = \dim H^q(X^G)
\]
Beginning of proof of Smith theorem for \( p = 2 \)

\[
0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{A} \oplus \mathcal{C} \rightarrow 0
\]

On \( G \), \( 0 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_2[G] \rightarrow \mathbb{F}_2 \oplus 0 \rightarrow 0 \).

On \( \ast \), \( 0 \rightarrow \mathbb{F}_2 \rightarrow 0 \oplus \mathbb{F}_2 \rightarrow 0 \).

\[
H^\ast(X; \mathcal{A} \oplus \mathcal{C}) \cong H^\ast(X; \mathcal{A}) \oplus H^\ast(X; \mathcal{C})
\]

Bockstein LES implies

\[
\chi(X) = \chi(X^G) + 2\tilde{\chi}(FX/G)
\]

and

\[
a_q + c_q \leq b_q + a_{q+1}
\]
Beginning of proof of Smith theorem for \( p > 2 \)

Let \( I = \text{Ker}(\varepsilon) \), \( \varepsilon : \mathbb{F}_p[G] \to \mathbb{F}_p \), where \( \varepsilon(g) = 1 \).

Define \( \mathcal{I}^n \) by \( \mathcal{I}^n(G) = I^n \) and \( \mathcal{I}^n(*) = 0 \). Then \( \mathcal{I}^{p-1} = \mathcal{A} \).

\[
0 \to \mathcal{I} \to \mathcal{B} \to \mathcal{A} \oplus \mathcal{C} \to 0
\]

\[
0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{I} \oplus \mathcal{C} \to 0
\]

\[
0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \mathcal{A} \to 0, \quad 1 \leq n < p
\]

Let

\[
\bar{a}_q = \text{dim}H^q_G(X; \mathcal{I})
\]

Bockstein LES implies

\[
\chi(X) = \chi(X^G) + p\bar{\chi}(FX/G)
\]

and

\[
a_q + c_q \leq b_q + \bar{a}_{q+1}, \quad \bar{a}_q + c_q \leq b_q + a_{q+1}
\]
Completion of proof for any $p$

Inductively, for $q \geq 0$ and $r \geq 0$, with $r$ odd if $p > 2$,

$$a_q + c_q + \cdots c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$  

Let $n = \dim(X)$. With $q = n + 1$ and $r > n$, get $c_i = 0$ for $i > n$. With $q = 0$ and $r > n$, get

$$\sum c_q \leq \sum b_q.$$  

So far, all has been general. If $H^*(X) \cong H^*(S^n)$, then $\sum b_q = 2$. $\chi(X) \equiv \chi(X^G) \mod p$ implies $\sum c_q = 0 (X^G = \emptyset)$ or $\sum c_q = 2$. If $p > 2$, it also implies $n - m$ is even and, if $n$ is even, $X^G \neq \emptyset$. 
The Conner conjecture (1960); first proven by Oliver (1976)

$G$ a compact Lie group, $X$ a finite dimensional $G$-CW complex with finitely many orbit types, $A$ an abelian group.

**Theorem**

*If $\tilde{H}^*(X; A) = 0$, then $\tilde{H}^*(X/G; A) = 0$.*

Conner (implicitly): True if $G$ is a finite extension of a torus.

If $H$ is a normal subgroup of $G$, then $X/G = (X/H) / (G/H)$.

Reduces to $G = S^1$ and $G$ finite. Standard methods apply.

General case: let $N$ be the normalizer of a maximal torus $T$ in $G$.

Then $\chi(G/N) = 1$ and $\tilde{H}^n(X/N; A) = 0$. 
The Oliver transfer

Theorem
Let \( H \subset G \), \( \pi: X/H \longrightarrow X/G \). For \( n \geq 0 \), there is a transfer map

\[
\tau: \tilde{H}^n(X/H; A) \longrightarrow \tilde{H}^n(X/G; A)
\]

such that \( \tau \circ \pi^* \) is multiplication by \( \chi(G/H) \).

Proof of the Conner conjecture.
Take \( H = N \). The composite

\[
\tilde{H}^n(X/G; A) \xrightarrow{\pi^*} \tilde{H}^n(X/N; A) \xrightarrow{\tau} \tilde{H}^n(X/G; A)
\]

is the identity and \( \tilde{H}^n(X/N; A) = 0 \).

How do we get the Oliver transfer?
$RO(G)$-graded cohomology

\[ X \wedge Y = X \times Y / X \lor Y \]

$V$ a representation of $G$, $S^V$ its 1-point compactification.

\[ \Sigma^V X = X \wedge S^V, \quad \Omega^V X = \text{Map}_*(S^V, X) \]

Suspension axiom on an “$RO(G)$-graded cohomology theory $E^*$”:

\[ \tilde{E}^\alpha(X) \cong \tilde{E}^{\alpha+V}(\Sigma^V X) \]

for all $\alpha \in RO(G)$ and all representations $V$.

*Theorem*

If $\mathcal{A} = \mathbb{Z}$ (hence if $\mathcal{A} = A = \mathbb{Z} \otimes A$), then $H^*_G(-; \mathcal{A})$ extends to an $RO(G)$-graded cohomology theory.
Construction of the Oliver transfer

Let \( X_+ = X \amalg \{\ast\} \). Consider \( \varepsilon: (G/H)_+ \to S^0 \).

**Theorem**

*For large enough \( V \), there is a map*

\[
t: S^V = \Sigma^V S^0 \to \Sigma^V G/H_+
\]

*such that \( \Sigma^V \varepsilon \circ t \) has (nonequivariant) degree \( \chi(G/H) \).*

The definition of \( \tau: \tilde{H}^n(X/H; A) \to \tilde{H}^n(X/G; A) \).

\[
\tilde{H}^n(X/H; A) \cong \tilde{H}^n_H(X; A) \cong \tilde{H}^n_G(X \land G/H_+; A) \cong \tilde{H}^{n+V}_G(X \land \Sigma^V G/H_+; A)
\]

\[
\tilde{H}^n(X/G; A) \cong \tilde{H}^n_G(X; A) \cong \tilde{H}^{n+V}_G(\Sigma^V X; A) = \tilde{H}^{n+V}_G(X \land S^V; A)
\]

Smashing with \( X \), \( t \) induces \( \tau \).
How do we get the map $t$?

Generalizing, let $M$ be a smooth $G$-manifold.

Embed $M$ in a large $V$. The embedding has a normal bundle $\nu$.

The embedding extends to an embedding of the total space of $\nu$ as a tubular neighborhood in $V$.

The Pontryagin Thom construction gives a map $S^V \longrightarrow T\nu$, where $T\nu$ is the Thom space of the normal bundle.

Compose with $T\nu \longrightarrow T(\tau \oplus \nu) \cong T\varepsilon = M_+ \wedge S^V$.

The composite is the transfer $t: S^V \longrightarrow \Sigma^V M_+$.

Atiyah duality: $M_+$ and $T\nu$ are Spanier-Whitehead dual. This is the starting point for equivariant Poincaré duality, for which $RO(G)$-grading is essential.
**RO(G)-graded Bredon cohomology**

**Theorem**

$H^*_G(-; \mathcal{A})$ extends to an RO(G)-graded theory if and only if the coefficient system $\mathcal{A}$ extends to a Mackey functor.

**Theorem**

$\mathbb{Z}$, hence $A$, extends to a Mackey functor.

**What is a Mackey functor?**

**First definition, for finite $G$**

Let $G\mathcal{I}$ be the category of finite $G$-sets. A Mackey functor $\mathcal{M}$ consists of covariant and contravariant functors

$$\mathcal{M}^*, \mathcal{M}^*: G\mathcal{I} \longrightarrow Ab,$$

which are the same on objects (written $M$) and satisfy:
\[ M(A \amalg B) \cong M(A) \oplus M(B) \]

and a pullback of finite sets gives a commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{g} & T \\
\downarrow{i} & & \downarrow{j} \\
S & \xrightarrow{f} & B
\end{array}
\quad
\begin{array}{ccc}
M(P) & \xrightarrow{g^*} & M(T) \\
\downarrow{i^*} & & \downarrow{j^*} \\
M(S) & \xrightarrow{f^*} & M(B)
\end{array}
\]

Suffices to define on orbits.

Pullback condition gives the “double coset formula”.

Example: \( M(G/H) = R(H) \) (representation ring of \( H \)).

Restriction and induction give \( M^* \) and \( M_* \).
Second definition, $G$ finite

Category $G$-$Span$ of “spans” of finite $G$-sets.

Objects are finite $G$-sets. Morphisms $A \rightarrow B$ are diagrams

$$A \leftarrow S \rightarrow B$$

Really equivalence classes: $S \sim S'$ if $S \cong S'$ over $A$ and $B$.

Composition by pullbacks:

```
\begin{tikzcd}
& P \\
S & & T \\
& A & B & C
\end{tikzcd}
```
A Mackey functor $\mathcal{M}$ is a (contravariant) functor

$$\mathcal{M} : G\text{-Span} \longrightarrow Ab,$$

written $M$ on objects and satisfying $M(A \amalg B) \cong M(A) \oplus M(B)$.

**Lemma**

A Mackey functor is a Mackey functor.

Given $\mathcal{M}$, 

$$A \leftarrow \cong A \rightarrow B, \quad A \leftarrow \cong B \rightarrow B$$

give $\mathcal{M}^*$ and $\mathcal{M}_*$. Given $\mathcal{M}^*$ and $\mathcal{M}_*$, composites give $\mathcal{M}$. 
Topological reinterpretation

For based $G$-spaces $X$ and $Y$ with $X$ a finite $G$-CW complex,

$$\{X, Y\}_G \equiv \colim_V [\Sigma^V X, \Sigma^V Y]_G$$

“Stable orbit category” or “Burnside category” $\mathcal{B}_G$: objects $G/H$, abelian groups of morphisms

$$\mathcal{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G$$

**Theorem**

If $G$ is finite, $\mathcal{B}_G$ is isomorphic to the full subcategory of orbits $G/H$ in $G$-Span.

Mackey functors are contravariant additive functors $\mathcal{B}_G \to \text{Ab}$.

**Theorem** if $G$ is finite. Definition if $G$ is a compact Lie group.
The Mackey functor $\mathbb{Z}$

Define

\[ A_G(G/H) = B_G(G/H, \ast) \cong \{S^0, S^0\}_H = A(H). \]

This gives the Burnside ring Mackey functor $A_G$.

Augmentation ideal sub Mackey functor $I_G(G/H) = IA(H)$.

The quotient Mackey functor $A_G/I_G$ is $\mathbb{Z}$.

How can we extend $\mathbb{Z}$-grading to $RO(G)$-grading?

Represent ordinary $\mathbb{Z}$-graded theories on $G$-spectra by Eilenberg-MacLane $G$-spectra, which then represent $RO(G)$-graded theories!
What are spectra?

- Prespectra (naively, spectra): sequences of spaces $T_n$ and maps $\Sigma T_n \to T_{n+1}$

- $\Omega$-(pre)spectra: Adjoints are equivalences $T_n \xrightarrow{\sim} \Omega T_{n+1}$

- Spectra: Spaces $E_n$ and homeomorphisms $E_n \to \Omega E_{n+1}$

- Spaces to prespectra: $\{\Sigma^n X\}$ and $\Sigma(\Sigma^n X) \xrightarrow{\sim} \Sigma^{n+1} X$

- Prespectra to spectra, when $T_n \xrightarrow{\subset} \Omega T_{n+1}$: 
  \[
  (LT)_n = \text{colim} \Omega^q T_{n+q}
  \]

- Spaces to spectra: $\Sigma^\infty X = L\{\Sigma^n X\}$

- Spectra to spaces: $\Omega^\infty E = E_0$

- Coordinate-free: spaces $T_V$ and maps $\Sigma^W T_V \to T_{V \oplus W}$
What are spectra good for?

- First use: Spanier-Whitehead duality [1958]
- Cobordism theory [1959] (Milnor; MSO has no odd torsion)
- Stable homotopy theory [1959] (Adams; ASS for spectra)
- Generalized homology theories [1962] (G.W. Whitehead)
- Stable homotopy category [1964] (Boardman’s thesis)
Representing cohomology theories

Fix $Y$. If $Y \simeq \Omega^2 \mathbb{Z}$, then $[X, Y]$ is an abelian group.

For $A \subset X$, the following sequence is exact:

$$[X/A, Y] \longrightarrow [X, Y] \longrightarrow [A, Y]$$

The following natural map is an isomorphism:

$$\bigvee_{i \in I} [X_i, Y] \longrightarrow \prod_{i \in I} [X_i, Y]$$

For an $\Omega$-spectrum $E = \{E_n\}$,

$$\tilde{E}^n(X) = \begin{cases} [X, E_n] & \text{if } n \geq 0 \\ [X, \Omega^{-n}E_0] & \text{if } n < 0 \end{cases}$$

Suspension:

$$\tilde{E}^n(X) = [X, E_n] \cong [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \tilde{E}^{n+1}(\Sigma X)$$
What are naive $G$-spectra (any $G$)?

- Naive $G$-spectra: spectra with $G$-action
- $G$-spaces $T_n$ and $G$-maps $\Sigma T_n \longrightarrow T_{n+1}$
- Naive $\Omega$-$G$-spectra: $T_n \xrightarrow{\sim} \Omega T_{n+1}$

Naive $\Omega$-$G$-spectra $E = \{E_n\}$ represent $\mathbb{Z}$-graded cohomology.

$$\tilde{E}^n_G(X) = \begin{cases} 
[X, E_n]_G & \text{if } n \geq 0 \\
[X, \Omega^{-n}E_0]_G & \text{if } n < 0
\end{cases}$$
Ordinary theories

Eilenberg-Mac Lane spaces:

\[ \pi_n K(A, n) = A, \quad \pi_q K(A, n) = 0 \text{ if } q \neq n. \]

\[ \tilde{H}^n(X; A) = [X, K(A, n)] \]

For based \(G\)-spaces \(X\),

\[ \underline{\pi}_n(X) = \pi_n(X^\bullet); \quad \underline{\pi}_n(X)(G/H) = \pi_n(X^H). \]

Eilenberg-Mac Lane \(G\)-spaces:

\[ \underline{\pi}_n K(\mathcal{A}, n) = \mathcal{A}, \quad \underline{\pi}_q K(\mathcal{A}, n) = 0 \text{ if } q \neq n. \]

\[ \tilde{H}_G^n(X; \mathcal{A}) = [X, K(\mathcal{A}, n)]_G \]
What are genuine $G$-spectra ($G$ compact Lie)?

- $G$-spaces $T_V$, $G$-maps $\Sigma^W T_V \longrightarrow T_V \oplus W$
  where $V, W$ are real representations of $G$

- $\Omega$-$G$-spectra: $G$-equivalences $T_V \xrightarrow{\sim} \Omega^W T_V \oplus W$

Genuine $\Omega$-$G$-spectra $E$ represent $RO(G)$-graded theories.

Imprecisely,

$$E_G^{V-W}(X) = [\Sigma^W X, E_V].$$

Ordinary? Need genuine Eilenberg-Mac Lane $G$-spectra.
A quick and dirty construction (1981)

Build a good “equivariant stable homotopy category” of $G$-spectra.

Use sphere $G$-spectra $G/H_+ \wedge S^n$ to get a theory of $G$-CW spectra.

Mimic Bredon’s construction of ordinary $\mathbb{Z}$-graded cohomology, but in the category of $G$-spectra, using Mackey functors instead of coefficient systems.

Apply Brown’s representability theorem to represent the 0th term by a $G$-spectrum $H\mathcal{M}$: for $G$-spectra $X$,

$$H^0_G(X; \mathcal{M}) \cong \{X, H\mathcal{M}\}_G.$$

Then $H\mathcal{M}$ is the required Eilenberg-Mac Lane $G$-spectrum.
What are $G$-spectra good for?

- Equivariant $K$-theory [1968] (Atiyah, Segal)
- Equivariant cobordism [1964] (Conner and Floyd)
- $RO(G)$-graded homology and cohomology theories
- Equivariant Spanier-Whitehead and Poincaré duality
- Equivariant stable homotopy category (Lewis-May)
- Completion theorems ($KU_G$, $\pi_G^*$, $MU_G$-modules): (Atiyah-Segal, Segal conjecture, Greenlees-May)
- Nonequivariant applications!!!
Kervaire invariant one problem

Framed manifold $M$: trivialization of its (stable) normal bundle.

$\Omega_{n}^{fr}$: Cobordism classes of (smooth closed) framed $n$-manifolds.

Is every framed $n$-manifold $M$, $n = 4k + 2$, framed cobordant to a homotopy sphere (a topological sphere by Poincaré conjecture)?

$$\kappa: \Omega_{4k+2}^{fr} \rightarrow \mathbb{F}_2$$

$\kappa[M]$ is the Kervaire invariant, the Arf invariant of a quadratic refinement of the cup product form on $H^{2k+1}(M; \mathbb{F}_2)$ that is determined by the given framing.

$\kappa[M] = 0$ if and only if $[M] = [\Sigma]$ for some homotopy sphere $\Sigma$. 
History

\( n = 2, 6, 14: \ S^1 \times S^1, \ S^3 \times S^3, \ S^7 \times S^7 \) have \( \kappa = 1 \) framings.

Kervaire (1960): PL, non-smoothable, 10-manifold \( M \) with \( \kappa = 1 \).

Kervaire and Milnor (1963): maybe \( \kappa = 0 \) for \( n \neq 2, 6, 14 \)?

Browder (1969): \( \kappa = 0 \) unless \( n = 2^{j+1} - 2 \) for some \( j \), and then \( \kappa = 0 \) if and only if \( h^2_j \) does not survive in the ASS, \( h_j \leftrightarrow Sq^{2j} \).

Calculation/construction (Barratt, Jones, Mahowald, Tangora (using May SS)):

\( h^2_4 \) and \( h^2_5 \) survive the ASS. (\( h^2_6 \) doable??)
Theorem (2009)
\[ \kappa = 0 \text{ unless } n \text{ is } 2, 6, 14, 30, 62, \text{ or maybe } 126: \]
\[ h_j^2 \text{ has a non-zero differential in the ASS, } j \geq 7. \]

Calculations of \( RO(G) \)-graded groups \( H_G^*(\ast; \mathbb{Z}) \) are critical!

Haynes Miller quote (Bourbaki Séminaire survey):
Hill, Hopkins, and Ravenel marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

- The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers.
- The theory of structured ring spectra, implemented by May and many others; and
- Equivariant stable homotopy theory, as developed by May and collaborators.
Structured ring spectra and structured ring \( G \)-spectra

\( E_\infty \) ring spectra (May-Quinn-Ray [1972])

\( E_\infty \) ring \( G \)-spectra (Lewis-May [1986])

Recent paradigm shift in stable homotopy theory.

Symmetric monoidal category of spectra \( \mathcal{S} \) under \( \wedge \);
\( E_\infty \) ring spectra are just commutative monoids in \( \mathcal{S} \).

Elmendorf-Kriz-Mandell-May [1997]: \( S \)-modules, operadic \( \wedge \)

Hovey-Shipley-Smith [2000]: Symmetric spectra, categorical \( \wedge \)

Mandell-May-Shipley-Schwede [2001]: Orthogonal, comparisons

Mandell-May [2002]: Orthogonal \( G \)-spectra and \( S_G \)-modules

New subjects:

“Brave new algebra” (Waldhausen’s name, now apt)

“Derived algebraic geometry” (Toen-Vezzosi, Lurie)
Revitalized areas
Equivariant $\infty$ loop space theory
Equivariant algebraic $K$-theory
(Guillou-Merling-May, [2011-2012]).

Prospective applications to algebraic $K$-theory of number rings?

Theorem
Let $L$ be a Galois extension of a field $F$ with Galois group $G$. There is an $E_\infty$ ring $G$-spectrum $K_G(L)$ such that

$$(K_G(L))^H = K(L^H) \text{ for } H \subset G$$

where $\pi_* K(R) =$ Quillen’s algebraic $K$-groups of $R$. 