# Equivariant infinite loop space theory

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# OUTLINE

- Nonequivariant background (1968-1974)
- Equivariant background (circa 1990)
- Operadic theory
- Segalic theory
- Comparison
- Multiplicative theory
- Categorical input
- Multiplicative input and output
- Prospects

## Background: Higher homotopies, $X \times X \longrightarrow X$

$X = \Omega Y_1 \Longleftrightarrow \infty$ of associativity homotopies

$X = \Omega^2 Y_2 \Longleftrightarrow \infty$ of commutativity homotopies

$X = \Omega^n Y_n$, $n \geq 1$, $\Longleftrightarrow \cdots$? How to organize the mess?

Analogy with higher category theory. $\mathscr{I} = \bullet \longrightarrow \bullet$

A natural transformation is: $H \colon \mathscr{C} \times \mathscr{I} \longrightarrow \mathscr{D}$

A "modification" is: $J \colon \mathscr{C} \times \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{D}$

Quasicategories; Segal categories; operads; $\cdots$

All can be blamed on infinite loop space theory.

## Background: Group actions

What groups? $G$ finite (for now)

$V$ a representation, $S^V = V \cup \infty$

$X$ a $G$-space.

$$\Sigma^V X = X \wedge S^V, \;\; \Omega^V X = \mathrm{Map}_*(S^V, X)$$

($G$ acting diagonally and by conjugation.)

$X = \Omega^V Y_V$ for all $V \iff$ what structure on $X$?

Operadic and Segalic methods adapt, nontrivially

## Operads

Spaces $\mathscr{C}(k)$ parametrize products of $k$-variables

Structure maps control composites of products

$\mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 + \cdots + j_k)$

$\mathscr{C}$-space $X$: $\mathscr{C}(k) \times X^k \longrightarrow X$, compatibility diagrams

$E_n$ and $E_\infty$ operads:

$E_n$-spaces $\Longrightarrow$ $n$-fold loop spaces

$E_\infty$-spaces $\Longrightarrow$ infinite loop spaces myriads of applications

myriads of calculations

Invariants: Dyer-Lashof operations.

$H_*(\Omega^n \Sigma^n X)$ calculated as functor of $H_*(X)$

Characteristic classes: $BF$, $BTop$, etc

Cobordism of topological manifolds

$\coprod_n BGL(n, R)$: algebraic $K$-theory $K(R)$

## Operadic machine

Monad $C$, $CX = \coprod_k (\mathscr{C}(k) \times_{\Sigma_k} X^k)/(\text{basept identifications})$

$C_n X \longrightarrow \Omega^n \Sigma^n X$, ‘group completion’ (on $\pi_0$, homology)

$C_n \longrightarrow \Omega^n \Sigma^n$ a map of monads (adjoint functors give monads)

$X$ an $E_n$-space, $\mathscr{C}_n$ acts on $X$. Get group completion

$$
\begin{array}{c}
X \\
{\scriptstyle \varepsilon}\uparrow \;\; \downarrow{\scriptstyle \eta} \\
B(C_n, C_n, X) \\
\downarrow \\
B(\Omega^n \Sigma^n, C_n, X) \\
\downarrow \\
\Omega^n B(\Sigma^n, C_n, X)
\end{array}
$$

# Equivariant operadic machine

Little cubes $\mathscr{C}_n$ (too square, useless equivariantly)

little discs $\mathscr{D}_n$ (too round, don't suspend, useless as $n$ varies)

Steiner operads: $R_V \subset \mathrm{Emb}(V)$ is the space of embeddings $f\colon V \longrightarrow V$ such that $|f(v) - f(w)| \leq |v - w|$ for $v, w \in V$.

Steiner paths: maps $c\colon I \longrightarrow R_V$ such that $c(1) = \mathrm{id}$.

$$\mathscr{K}_V(k) = \{\langle c_1, \cdots, c_k\rangle | c_i(0) \cap c_j(0) = \emptyset\}$$

$E_\infty$ operad: free $\Sigma_k$-action on $\mathscr{C}(k)$ and

nonequivariant (operad of spaces): $\mathscr{C}(k)$ is contractible

equivariant (operad of $G$-spaces): $\mathscr{C}(k)^\Pi$ is contractible

if $\Pi \subset G \times \Sigma_k$ and $\Pi \cap \Sigma_k = \{e\}$ (universal $(G, \Sigma_k)$-bundle)

## Equivariant adaptation

Let $\mathscr{C}$ be an $E_\infty$ operad, $\mathscr{C}_V = \mathscr{C} \times \mathscr{K}_V$ ($V$ varying).

Get a group completion $\alpha\colon C_V X \longrightarrow \Omega^V \Sigma^V X$, so that

$\alpha^H$ is a nonequivariant group completion for $H \subset G$.

$\alpha\colon \mathscr{C}_V \longrightarrow \Omega^V \Sigma^V$ is a map of monads.

$X$ an $E_\infty$-space, $\mathscr{C}$ acts on $X$. Get "Genuine $G$-spectrum"

$$\mathbb{E}_G X = \{B(\Sigma^V, C_V, X)\}.$$

Get group completions ($V^G \neq 0$)

$$\begin{array}{c} X \\ {\scriptstyle \varepsilon}\uparrow \;\; \downarrow{\scriptstyle \eta} \\ B(C_V, C_V, X) \\ \downarrow \\ B(\Omega^V \Sigma^V, C_V, X) \\ \downarrow \\ \Omega^V B(\Sigma^V, C_V, X) \end{array}$$

## The Segal machine

$\mathcal{F}$ the category of finite based sets $\mathbf{n} = \{0, 1, \cdots, n\}$,

basepoint 0.

$\mathcal{F}$-space: functor $X \colon \mathcal{F} \longrightarrow \mathrm{Spaces}$, notation $\mathbf{n} \mapsto X_n$;

reduced: $X_0 = *$.

$\phi \colon \mathbf{n} \to \mathbf{1}$, $\phi(i) = 1$, $1 \leq i \leq n$: "product" $\phi \colon X_n \longrightarrow X_1$

Coordinates $\delta_j \colon \mathbf{n} \to \mathbf{1}$, $\delta_j(i) = \delta_{i,j}$: maps $\delta \colon X_n \longrightarrow X_1^n$

$X$ is special if $\delta \colon X_n \longrightarrow X_1^n$ is a homotopy equivalence.

$$X_1^n \xleftarrow[\simeq]{\delta} X_n \xrightarrow{\phi} X_1$$

$K$ a space. Get $K^\bullet \colon \mathcal{F}^{op} \longrightarrow \mathrm{Spaces}$, $K^m = \mathrm{Map}_*(\mathbf{m}, K)$.

$X$ a special $\mathcal{F}$-space $X$. Have spaces $B(K^\bullet, \mathcal{F}, X)$.

Let $K$ run through $\{\mathbf{n}\}$. Get a special $\mathcal{F}$-space $B(\mathcal{F}, \mathcal{F}, X)$.

Let $K$ run through $\{S^n\}$. Get a spectrum $\{B((S^n)^\bullet, \mathcal{F}, X)\}$.

$\mathbf{1} = S^0$. Get group completions

$$\begin{array}{c}
X_1 \\
{\scriptstyle\varepsilon}\uparrow \;\; \downarrow{\scriptstyle\eta} \\
B(\mathcal{F}, \mathcal{F}, X)_1 \\
\Big\downarrow{\scriptstyle =} \\
B((S^0)^\bullet, \mathcal{F}, X) \\
\Big\downarrow \\
\Omega^n B((S^n)^\bullet, \mathcal{F}, X)
\end{array}$$

# Equivariant Segal machine

$\mathcal{F}$-$G$-space: functor $X\colon \mathcal{F} \longrightarrow G\text{-Spaces}$

$X$ is special if $\delta\colon (X_n)^{\Pi} \longrightarrow (X_1^n)^{\Pi}$ is a homotopy equivalence

if $\Pi \subset G \times \Sigma_n$ and $\Pi \cap \Sigma_n = \{e\}$. (In particular, $\Pi = H \subset G$.)

For a $G$-space $K$, get $B(K^{\bullet}, \mathcal{F}, X)$.

Let $K$ run through $\mathbf{n}$. Get an $\mathcal{F}$-$G$-space $B(\mathcal{F}, \mathcal{F}, X)$.

It is NOT SPECIAL!

Fixed point equivalences only for $H \subset G$, not all relevant $\Pi$.

New categorical and topological ideas fix this!

Construct variant bar construction $B_\Sigma(K^\bullet, \mathcal{F}, X)$

$B_\Sigma(\mathcal{F}, \mathcal{F}, X)$ is special. Genuine $G$-spectrum

$$\mathbb{S}_G X = \{B_\Sigma((S^V)^\bullet, \mathcal{F}, X)\}.$$

Get group completions ($V^G \neq 0$)

$$\begin{array}{c} X_1 \\ \varepsilon \uparrow \; \downarrow \eta \\ B_\Sigma(\mathcal{F}, \mathcal{F}, X)_1 \\ \downarrow = \\ B_\Sigma((S^0)^\bullet, \mathcal{F}, X) \\ \downarrow \\ \Omega^V B_\Sigma((S^V)^\bullet, \mathcal{F}, X) \end{array}$$

## Comparison

Input: Operad action vs $\mathcal{F}$-$G$-space

Common generalization (May-Thomason)

Category of operators $\mathscr{D} = \mathscr{D}(\mathscr{C})$, functor $\xi\colon \mathscr{D} \longrightarrow \mathcal{F}$.

$$\mathscr{D}(\mathbf{m}, \mathbf{n}) = \coprod_{\phi\colon \mathbf{m} \longrightarrow \mathbf{n}} \prod_{\mathbf{j}=\mathbf{1}}^{\mathbf{j}=\mathbf{n}} \mathscr{C}(|\phi^{-1}(\mathbf{j})|)$$

$\mathscr{D}$-$G$-space: functor $Y\colon \mathscr{D} \longrightarrow G$-Spaces, special as for $\mathcal{F}$.

A $\mathscr{C}$-$G$-space $X$ gives a $\mathscr{D}$-$G$-space $RX\colon \mathscr{D} \longrightarrow G$-Spaces

$$(RX)_n = X_1^n$$

A $\mathscr{D}$-$G$-space $Y\colon \mathscr{D} \longrightarrow G$-Spaces gives a $\mathscr{C}$-$G$-space

$$LY = Y_1.$$

An $\mathcal{F}$-$G$-space $X$ gives a $\mathscr{D}$-$G$-space $\xi^* X\colon \mathscr{D} \longrightarrow G\text{-Spaces}$.

A $\mathscr{D}$-$G$-space $Y$ gives an $\mathscr{F}$-$G$-space $\xi_* Y = B(\mathcal{F}, \mathscr{D}, Y)$.

Equivalences of homotopy categories

$$\mathscr{C}\text{-}G\text{-Spaces} \simeq \mathscr{D}(\mathscr{C})\text{-}G\text{-Spaces} \simeq \mathcal{F}\text{-}G\text{-Spaces}$$

Output: monadic vs categorical bar constructions.

MIRACLE: Steiner paths create a homotopy relating the machines

Let $Y$ be a $\mathscr{D}(\mathscr{C})$-$G$-space.

For a $G$-space $K$, define

• $K\colon \mathscr{D}(\mathscr{K}_V) \longrightarrow G\text{-Spaces}$, $\mathbf{n} \mapsto K \vee \cdots \vee K$, $n$ copies.

$$
\begin{array}{ccc}
(\mathbb{S}_G Y)(V) & = & B_\Sigma((S^V)^\bullet, \mathscr{D}(\mathscr{C}), Y) & & \\
& & \Big\uparrow \pi & & \\
& & B_\Sigma((S^V)^\bullet, \mathscr{D}(\mathscr{C} \times \mathscr{K}_V), Y) & & \\
& & \Big\downarrow i_1 & & \\
& & B_\Sigma(I \times (S^V)^\bullet, \mathscr{D}(\mathscr{C} \times \mathscr{K}_V), Y) & & \\
& & \Big\uparrow i_0 & & \\
& & B_\Sigma((S^V)^\bullet_{t=0}, \mathscr{D}(\mathscr{C} \times \mathscr{K}_V), Y) & & \\
& & \Big\uparrow \iota & & \\
& & B_\Sigma({}^\bullet(S^V), \mathscr{D}(\mathscr{C} \times \mathscr{K}_V), Y) & & \\
& & \Big\downarrow \omega & & \\
& & B(\Sigma^V L, \mathbb{D}(\mathscr{C} \times \mathscr{K}_V), Y) & = & \mathbb{E}_G Y.
\end{array}
$$

Multiplicative theory: the real work begins

Rings, modules, and algebras in equivariant stable homotopy theory

Categorical input is VITAL. Only space level additive theory so far.

Permutativity Operad $\mathscr{P} = \{\mathscr{E}\Sigma_j\}$

Permutativity $G$-Operad $\mathscr{P}_G = \{\mathbf{Cat}(\mathscr{E}\mathbf{G}, \mathscr{E}\mathbf{\Sigma_j}\}$

Permutative categories $\mathscr{A}$: action of $\mathscr{P}$

given by functors $\mathscr{P}(k) \times \mathscr{A}^k \longrightarrow \mathscr{A}$

permutative $G$-categories $\mathscr{A}$: action of $\mathscr{P}_G$

Symmetric monoidal categories: pseudoaction of $\mathscr{P}$

given by pseudofunctors $\mathscr{P}(k) \times \mathscr{A}^k \longrightarrow \mathscr{A}$

Symmetric monoidal $G$-categories: pseudoaction of $\mathscr{P}_G$

"Unbiased" structure: sees all $\mathscr{A}^k$, not just the first few.

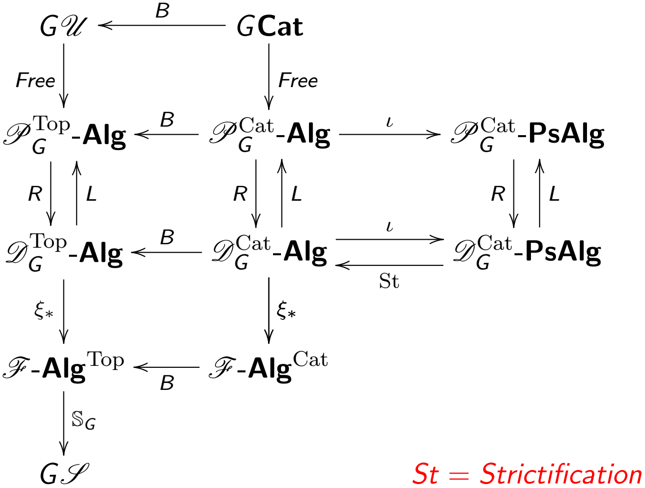
Operadic input vital:

no "biased" definitions are known equivariantly.

These give categorical input to the additive theory

## Additive Segal machine on $G$-categories and $G$-spaces

$$\mathscr{P}_G^{\mathrm{Cat}} = \mathscr{P}_G \qquad \mathscr{D}_G^{\mathrm{Cat}} = \mathscr{D}(\mathscr{P}_G^{\mathrm{Cat}})$$
$$\mathscr{P}_G^{\mathrm{Top}} = B\mathscr{P}_G \qquad \mathscr{D}_G^{\mathrm{Top}} = \mathscr{D}(\mathscr{P}_G^{\mathrm{Top}})$$



# Multicategories (many object operads)

Sets, or more structure, $\mathbf{Mult}(\mathscr{A}_{\mathbf{1}}, \cdots, \mathscr{A}_{\mathbf{k}}; \mathscr{B})$

of $k$-morphisms $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_k \longrightarrow \mathscr{B}$ for objects

$\mathscr{A}_i$ and $\mathscr{B}$. If only one object $\mathscr{A}$, just an operad:

$$\mathscr{C}(k) = \mathbf{Mult}(\mathscr{A}, \cdots, \mathscr{A}; \mathscr{A})$$

$k = 1$ gives underlying category $\mathscr{V}$; write $\mathbf{Mult}(\mathscr{V})$.

Let $(\mathscr{V}, \otimes)$ be a symmetric monoidal category.

$\mathbf{Mult}(\mathscr{V})$ is the multicategory whose $k$-morphisms

are all maps $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_k \longrightarrow \mathscr{B}$ in $\mathscr{V}$.

Algebraic examples: $R$-modules under $\otimes$, etc.

Cartesian monoidal examples: $G$-**Cat**, $G$-**Space**

Based $G$-spaces under smash product $\wedge$

$$X \wedge Y = X \times Y / X \vee Y$$

$G$-**Spectra** under smash product $\wedge$

Little multicategories $\mathscr{Q}$ parametrize algebraic structures

One object = operads: **Ass**, **Com**: monoids, comm. monoids

Two objects: multicategory for monoids acting on objects.

(Think of rings and modules). Many others.

Big multicategories $\mathscr{M}$, like $\mathrm{Mult}(\mathscr{V})$, are the home for

multiplicative structures given by morphisms of multicategories

$$X\colon \mathscr{Q} \longrightarrow \mathscr{M}.$$

Objects $\mathscr{X}(q)$ of $\mathscr{V}$; $k$-morphisms $\mathscr{Q}(q_1, \cdots, q_k; r)$ induce

$$\mathscr{X}(q_1) \otimes \cdots \otimes \mathscr{X}(q_k) \longrightarrow \mathscr{X}(r).$$

## Multiplicative infinite loop space theory

Transport a $\mathscr{Q}$-structure on $\mathscr{P}$-categories $\mathscr{A}(q)$

to a $\mathscr{Q}$-structure on the spectra $\mathbb{S}(B\mathscr{A}(q))$.

May 1977–82; Elmendorf-Mandell 2006.

Equivariant generalization needs new ideas.

Hard category theory, not an oxymoron!

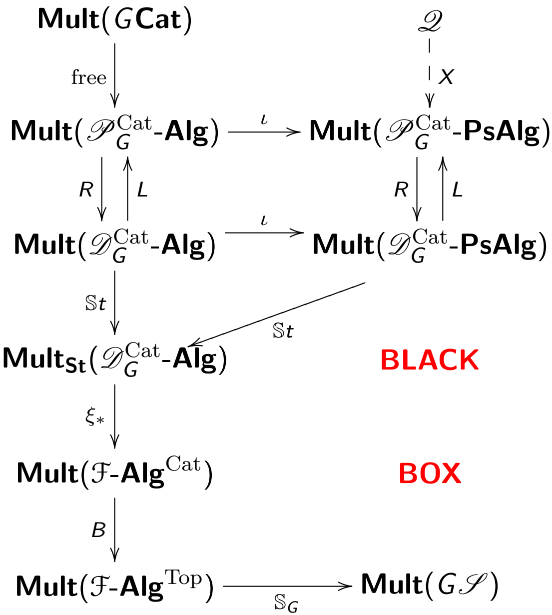

Mult(GCat)
free
Q
X
Mult(P_G^Cat-Alg)
ι
Mult(P_G^Cat-PsAlg)
R
L
R
L
Mult(D_G^Cat-Alg)
ι
Mult(D_G^Cat-PsAlg)
𝕊t
𝕊t
Mult_St(D_G^Cat-Alg)
BLACK
ξ_*
Mult(F-Alg^Cat)
BOX
B
Mult(F-Alg^Top)
𝕊_G
Mult(G𝒮)

Diagram and talk suppress categorical subleties.

$\mathbf{Mult}_{\mathbf{St}}(\mathscr{D}_G^{\mathrm{Cat}}\text{-}\mathbf{Alg})$

is a multicategory with strict $k$-morphisms,

given by distributivity natural isomorphisms.

From the operadic starting point,

it is surprising such things exist.

Classifying space construction $B$ wants such strict input.

A key point:

$\mathbb{S}_G \colon \mathcal{F}\text{-}\mathbf{Alg}^{\mathrm{Top}} \longrightarrow G\mathscr{S}$ is a symmetric monoidal functor

## Conclusion

Black box converts $G$-categorical input to $G$-spectrum output.

Free functors give an important class of examples

Nonequivariant structures $X$ extend equivariantly: $\mathbf{Cat}(\mathscr{E}G, X)$

New mathematical structures

Ring, module, and algebra spectra admit variants

Different kinds of commutative rings (Kervaire, Blumberg-Hill)

exist and can now be found on the level of categorical structure.

They are there. Let's find them!

I'll end at this beginning.