DUALITY IN BICATEGORIES AND TOPOLOGICAL APPLICATIONS

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In October, 1999, I organized a conference here in honor of MacLane's 90th birthday. I'll repeat how I started my talk then. "A great deal of modern mathematics would quite literally be unthinkable without the language of categories, functors, and natural transformations introduced by Eilenberg and MacLane in 1945. It was perhaps inevitable that some such language would have appeared eventually. It was certainly not inevitable that such an early systematization would have proven so remarkably durable and appropriate.

I talked then about another piece of language that Saunders introduced, that of symmetric monoidal category, which comes from a 1963 paper entitled "Natural associativity and commutativity". It gave the first clear articulation of the general categorical problem of coherence. The necessary isomorphisms are called "constraints" in the later literature of algebraic geometry. To quote from Mathematical Reviews, "A very pleasant feature of the paper is the skillful, economical and rigorous manner in which the problem is formulated".

More precisely, what I talked about then was duality theory in symmetric monoidal categories. As Mac Lane so well understood, category theory turns analogies into mathematical theory that turns the things being compared into examples. The category theory then allows one to understand new things about the examples, and to compare them rigorously. Duality theory gives a strikingly beautiful and important illustration of this kind of mathematics. The purpose of this talk is to explain duality theory in symmetric bicategories. This is a new theory whose basic definitions are less than a year old. It is joint work with Johann Sigurdsson, but its starting point was a key insight in work of Steven Costenoble and Stefan Waner.

Jean Benabou, who I think was here at the time, already introduced bicategories in 1967. There is a notion of a symmetric monoidal bicategory, but I only mention that for the well-informed category theorists. The notion of symmetric bicategories is different. The new theory is not difficult, and it could well have been developed immediately after Benabou's work. However, the need for it only became apparent with the development of parametrized stable homotopy theory. Niles Johnson is beginning to verify my belief that it also gives the definitively right framework for understanding Morita duality in derived categories. Surprisingly, Kate Ponto discovered that it also gives the definitively right framework for understanding topological fixed point theory. Here even the analogies are not immediately obvious.

OUTLINE

(1) Duality in symmetric monoidal categories
(2) Closed symmetric bicategories
(3) Duality in closed symmetric bicategories
(4) Composites of dualities
(5) Parametrized spectra
(6) Dictionaries
(7) Parametrized duality theory
(8) Parametrized homology & cohomology

1. Duality in symmetric monoidal categories

$$(\mathscr{C}, \otimes, \mathrm{Hom}, I, \gamma) \quad \otimes \colon \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

Unital, associative, and commutative up to coherent natural isomorphism; associativity and unit isomorphisms are implicit, and γ is the commutativity isomorphism.

$$I \otimes X \cong X \quad X \cong \mathrm{Hom}(I, X)$$

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

$$\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

$$\begin{array}{c} \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(X', Y') \\ \big\downarrow \otimes \\ \mathrm{Hom}(X \otimes X', Y \otimes Y') \end{array}$$

$$\nu : \mathrm{Hom}(X, Y) \otimes Z \longrightarrow \mathrm{Hom}(X, Y \otimes Z)$$

$$DX \equiv \mathrm{Hom}(X, I)$$

Evaluation map: $\varepsilon : DX \otimes X \longrightarrow I$.

Definition 1. X is *dualizable* if

$$\nu : DX \otimes X \longrightarrow \mathrm{Hom}(X, X)$$

is an isomorphism. Coevalutation map:

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[d, "\iota"'] & X \otimes DX \\ \mathrm{Hom}(X,X) & DX \otimes X \arrow[l, "\nu"] \arrow[u, "\gamma"'] \end{tikzcd}$$

Here ι is adjoint to $\mathrm{id}\colon X \longrightarrow X$.

$$\mathscr{M}_R, \quad \mathscr{D}_R, \quad \mathrm{Ho}\mathscr{S} \quad sh(X)$$

Finitely projective R-module

Perfect chain complex

Finite CW-spectrum

Theorem 2. *Fix X and Y. TFAE.*

(i) *X is dualizable and $Y \cong DX$.*

(ii) *Triangle identities: There are*

$$\eta : I \longrightarrow X \otimes Y \quad \textit{and} \quad \varepsilon : Y \otimes X \longrightarrow I \quad \textit{s.t.}$$

$$X \cong I \otimes X \xrightarrow{\eta \otimes \mathrm{id}} X \otimes Y \otimes X \xrightarrow{\mathrm{id} \otimes \varepsilon} X \otimes I \cong X$$

$$Y \cong Y \otimes I \xrightarrow{\mathrm{id} \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\varepsilon \otimes \mathrm{id}} I \otimes Y \cong Y$$

are identity maps.

(iii) *There is a map* $\eta : I \longrightarrow X \otimes Y$ *such that the composite*

$$\begin{tikzcd}
\mathscr{C}(W \otimes X, Z) \arrow[d, "(-)\otimes Y"] \\
\mathscr{C}(W \otimes X \otimes Y, Z \otimes Y) \arrow[d, "\mathscr{C}(\mathrm{id} \otimes \eta, \mathrm{id})"] \\
\mathscr{C}(W, Z \otimes Y)
\end{tikzcd}$$

is a bijection for all W *and* Z.

(iv) *There is a map* $\varepsilon : Y \otimes X \longrightarrow I$ *such that the composite*

$$\begin{tikzcd}
\mathscr{C}(W, Z \otimes Y) \arrow[d, "(-)\otimes X"] \\
\mathscr{C}(W \otimes X, Z \otimes Y \otimes X) \arrow[d, "\mathscr{C}(\mathrm{id}, \mathrm{id} \otimes \varepsilon)"] \\
\mathscr{C}(W \otimes X, Z)
\end{tikzcd}$$

is a bijection for all W *and* Z.

The adjoint $\tilde{\varepsilon} : Y \longrightarrow DX$ of ε satisfying (ii) or (iv) is an isomorphism under which the given map ε corresponds to the canonical evaluation map $\varepsilon : DX \otimes X \longrightarrow I$.

Terminology: (X, Y) is a dual pair.

Proposition 3. *If X and Y are dualizable, then DX and $X \otimes Y$ are dualizable and the canonical map $\rho : X \longrightarrow DDX$ is an isomorphism.*

Proposition 4. *If X or Z is dualizable, then the map ν above is an isomorphism.*

Proposition 5. *If X and X' are dualizable or if X is dualizable and $Y = I$, then the map $\otimes$ above is an isomorphism.*

Definition 6. X is invertible if there is a Y and an isomorphism $X \otimes Y \cong I$. Then (X, Y) is a dual pair. $\mathrm{Pic}(\mathscr{C})$ = group of isomorphism classes of invertible X.

2. Closed symmetric bicategories

A bicategory $\mathscr{C}$ is a monoidal category with many objects, called 0-cells. For each pair of 0-cells A, B, it has a category $\mathscr{C}(A, B)$ of object 1-cells $A \to B$ and morphism 2-cells $X \to Y$ between 1-cells $X, Y : A \to B$. There are "horizontal" compositions

$$\odot : \mathscr{C}(B,C) \times \mathscr{C}(A,B) \longrightarrow \mathscr{C}(A,C)$$

and unit 1-cells I_A [abbreviated A by abuse] such that $\otimes$ is associative and unital up to coherent natural isomorphism.

If $\mathscr{C}$ has one object $*$, then $\mathscr{C}(*,*)$ is a monoidal category.

$\mathscr{C}at$: categories, functors, nat. trans.

$\mathscr{B}_R$: R-algebras, (B, A)-bimodules, maps

$\mathscr{E}x$: spaces, spectra over $B \times A$, maps

Symmetric version? Directionality of 1-cells? Turns out that symmetric bicategories with one object are not just symmetric monoidal categories.

$\mathscr{C}^{\mathrm{op}}$: same 0-cells,

$$\mathscr{C}^{op}(A,B) = \mathscr{C}(B,A).$$

Definition 7. An *involution* on $\mathscr{C}$ consists of the following data subject to coherence diagrams.

(1) A bijection t on 0-cells with $ttA = A$.
(2) Equivalences of categories

$$t\colon \mathscr{C}(A,B) \longrightarrow \mathscr{C}(tB,tA) = \mathscr{C}^{\mathrm{op}}(tA,tB),$$

with isomorphism 2-cells $\xi\colon\ \mathrm{id} \cong tt$.
(3) Isomorphism 2-cells

$$\iota\colon I_{tA} \longrightarrow tI_A \quad \text{and}$$

$$\gamma\colon tY \odot^{\mathrm{op}} tX \equiv tX \odot tY \longrightarrow t(Y \odot X).$$

A *symmetric bicategory* $\mathscr{C}$ is a bicategory with an involution. A 0-cell A of $\mathscr{C}$ is said to be *commutative* if $tA = A$.

$t\colon \mathscr{C}(A,A) \longrightarrow \mathscr{C}(A,A)$ need not be the identity functor, so a one object symmetric bicategory need not be a symmetric monoidal category.

Example 8. $\mathscr{B}_R$. tA is the opposite R-algebra of A. The commutative 0-cells are the commutative R-algebras.

The 1-cells $X\colon A \to B$ are the (B,A)-bimodules, and $tX\colon tB \to tA$ is the same R-module X viewed as a (tA,tB)-bimodule.

The 2-cells $\alpha\colon X \to Y$ are the maps of (B,A)-bimodules, and $t(\alpha)$ is the same R-map viewed as a map of (tA,tB)-bimodules.

$$\begin{array}{c} \mathscr{B}_R(B,C) \times \mathscr{B}_R(A,B) \\ \big\downarrow{\scriptstyle \odot = \otimes_B} \\ \mathscr{B}_R(A,C). \end{array}$$

$$I_A \text{ is } A \text{ regarded as an } (A,A)\text{-bimodule,}$$

$$\gamma\colon tX \otimes_{tB} tY \longrightarrow t(Y \otimes_B X)$$

is given by $\gamma(x \otimes y) = y \otimes x$.

$$\mathscr{B}_R(A,R) = \text{right } A\text{-modules}$$

$$\mathscr{B}_R(R,A) = \text{left } A\text{-modules}$$

$tA = A \Rightarrow \mathscr{B}_R(A,A) = (A,A)$-bimodules.

Here $t \neq \mathrm{id}\colon \mathscr{B}_R(A,A) \longrightarrow \mathscr{B}_R(A,A)$. The symmetric monoidal full subcategory

$$\mathscr{M}_A \subset \mathscr{B}_R(A,A)$$

consists of the *central* (A,A)-bimodules X, those for which $ax = xa$ for all a and x.

Example 9. Graded version of $\mathscr{B}_R$ has γ defined with a sign. Leads to differential graded version. Topological version in brave new algebra.

$\mathscr{B}_R$ and other examples arise from anchored bicategories by neglect of structure. Maps $R \longrightarrow S$ of commutative rings and maps $A \longrightarrow B$ of algebras have been ignored.

LOGIC: $\mathscr{B}_R$ is EASY. Use it as a toy model to understand deeper examples, such as those above and parametrized spectra.

Definition 10. A bicategory $\mathscr{C}$ is right and left closed if there are right and left internal hom functors

$$\triangleright\colon \mathscr{C}(A,B)^{\mathrm{op}} \times \mathscr{C}(A,C) \longrightarrow \mathscr{C}(B,C)$$

and

$$\triangleleft\colon \mathscr{C}(A,C) \times \mathscr{C}(B,C)^{\mathrm{op}} \longrightarrow \mathscr{C}(A,B)$$

such that

$$\mathscr{C}(Y \odot X, Z) \cong \mathscr{C}(Y, X \triangleright Z)$$
$$\mathscr{C}(Y \odot X, Z) \cong \mathscr{C}(X, Z \triangleleft Y)$$

for

$X\colon A \longrightarrow B,\ Y\colon B \longrightarrow C,\ Z\colon A \longrightarrow C.$

Can't write $X \odot Y$, not defined!

Unit and counit 2-cells

$\varepsilon\colon (X\triangleright Z)\odot X \longrightarrow Z, \quad \eta\colon Y \longrightarrow X\triangleright(Y\odot X)$

$\varepsilon\colon Y\odot(Z\triangleleft Y) \longrightarrow Z, \quad \eta\colon X \longrightarrow (Y\odot X)\triangleleft Y.$

Definition 11. A symmetric bicategory is *closed* if it is left closed. It is then also right closed with

$$X \triangleright Z \cong t(tZ \triangleleft tX).$$

In symmetric monoidal categories, use of γ collapses $\lhd$ and $\rhd$ into Hom; replaced by involutive relation in symmetric bicategories.

Remark 12. $X \rhd Z$ is the 1-cell of "maps pointing right from X to Z"; $Z \lhd Y$ is the 1-cell of "maps pointing left from Y to Z". $X \rhd Z$ is a 1-cell from the *target* of X to the *target* of Z (with X and Z having the same source) and $Z \lhd Y$ is a 1-cell from the *source* of Z to the *source* of Y (with Y and Z having the same target).

Example 13. $\mathscr{B}_R$ is closed. For a

$$(B, A)\text{-bimodule} \;\; X,$$

$$(C, B)\text{-bimodule} \;\; Y,$$

$$(C, A)\text{-bimodule} \;\; Z,$$

$Z \lhd Y = \mathrm{Hom}_C(Y, Z)$, a (B, A)-bimodule and

$X \rhd Z = \mathrm{Hom}_A(X, Z)$, a (C, B)-bimodule.

$$\mathscr{B}_R(Y \otimes_B X, Z) \cong \mathscr{B}_R(Y, \mathrm{Hom}_A(X, Z))$$

$$\mathscr{B}_R(Y \otimes_B X, Z) \cong \mathscr{B}_R(X, \mathrm{Hom}_C(Y, Z))$$

Example 14. Base change bimodules. For an R-algebra map $f\colon A' \longrightarrow A$

$$A_f\colon A' \longrightarrow A, \quad \text{an} \quad (A, A')\text{-bimodule}$$

$${}_fA = tA_f\colon A \longrightarrow A' \quad \text{an}(A', A)\text{-bimodule.}$$

For an R-algebra B, pullback action functor

$$f^*\colon \mathscr{B}_R(A, B) \longrightarrow \mathscr{B}_R(A', B)$$

has a left adjoint $f_!$ and a right adjoint f_*, extension and coextension of scalars.

$$f^*M \cong M \odot A_f$$

$$f_!M' \cong M' \odot {}_fA$$

$$f_*M' \cong A_f \rhd M'.$$

For $g\colon B' \longrightarrow B$ and any A,

$$g^*\colon \mathscr{B}_R(A, B) \longrightarrow \mathscr{B}_R(A, B')$$

has a left adjoint $g_!$ and a right adjoint g_*,

$$g^*N \cong {}_gB \odot N,$$

$$g_!N' \cong B_g \odot N',$$

$$g_*N' \cong {}_gB \lhd N'.$$

Formal isomorphisms:

$$(Y \odot X) \triangleright Z \cong Y \triangleright (X \triangleright Z)$$

$$Z \triangleleft (Y \odot X) \cong (Z \triangleleft Y) \triangleleft X$$

$$(X \triangleright Z) \triangleleft Y \cong X \triangleright (Z \triangleleft Y)$$

Formal maps:

$$\mu\colon Z \odot (X \triangleright Y) \longrightarrow X \triangleright (Z \odot Y)$$

$$\nu\colon (Z \triangleleft Y) \odot W \longrightarrow (Z \odot W) \triangleleft Y$$

$$\omega\colon (Z \triangleleft Y) \odot (X \triangleright W) \longrightarrow X \triangleright (Z \odot W) \triangleleft Y$$

(Two ways of parenthesizing are isomorphic)

DUALITY THEORY:

WHEN ARE THESE ISOMORPHISMS?

3. Duality in closed symmetric bicategories

Write B for I_B [So B is a 0-cell and 1-cell].

Definition 15. (X,Y), $X\colon B \longrightarrow A$ and $Y\colon A \longrightarrow B$, is a *dual pair* of 1-cells if there are 2-cells

$$\eta\colon A \longrightarrow X \odot Y \quad \text{and} \quad \varepsilon\colon Y \odot X \longrightarrow B,$$

coevaluation and evaluation maps, such that the following diagrams commute in $\mathscr{C}(B,A)$ and $\mathscr{C}(A,B)$, respectively.

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}"'] & A \odot X \arrow[l, "\cong"'] \arrow[r, "\eta\odot\mathrm{id}"] & (X \odot Y) \odot X \arrow[d, "\cong"] \\
X & X \odot B \arrow[l, "\cong"] & X \odot (Y \odot X) \arrow[l, "\mathrm{id}\odot\epsilon"]
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[d, "\mathrm{id}"'] & Y \odot A \arrow[l, "\cong"'] \arrow[r, "\mathrm{id}\odot\eta"] & Y \odot (X \odot Y) \arrow[d, "\cong"] \\
Y & B \odot Y \arrow[l, "\cong"] & (Y \odot X) \odot Y \arrow[l, "\epsilon\odot\mathrm{id}"]
\end{tikzcd}$$

X is *right dualizable* with *right dual* Y.
Y is *left dualizable* with *left dual* X.

Example 16. Let $f\colon B \longrightarrow A$ be a map of R-algebras. We have

$$A_f\colon B \longrightarrow A \quad \text{and} \quad {}_fA\colon A \longrightarrow B.$$

${}_fA \odot A_f\colon B \longrightarrow B$ is A, regarded as a (B,B)-bimodule. Let

$$\eta = f\colon B \longrightarrow {}_fA \odot A_f.$$

$A_f \odot {}_fA = A \otimes_B A$. Let

$$\varepsilon\colon A_f \odot {}_fA \longrightarrow A$$

be given by the product on A. Then (η, ε) display $({}_fA, A_f)$ as a dual pair; the left and right unit laws for A imply the diagrams. $(A_f, {}_fA)$ is *not* a dual pair in general.

Example 17. Specialize. Take f to be the unit $\iota\colon R \longrightarrow A$ of an R-algebra with product $\phi\colon A \otimes_R A \longrightarrow A$. (ι, ϕ) display $({}_\iota A, A_\iota)$ as a dual pair. For $(A_\iota, {}_\iota A)$ to be a dual pair, we need a coproduct and counit

$$\eta\colon A \longrightarrow A \otimes_R A$$

$$\varepsilon\colon A \longrightarrow R$$

such that the left and right counit laws hold.

In topology, the situation is reversed: spaces have a coproduct and a counit, but not a product and a unit.

Proposition 18. *If $\mathscr{C}$ is symmetric, then η and ε exhibit (X, Y) as a dual pair iff*

$$\gamma^{-1}t(\eta) : tA \longrightarrow tY \odot tX$$

and

$$t(\varepsilon)\gamma \colon tX \odot tY \longrightarrow tB$$

exhibit (tY, tX) as a dual pair.

Analogous to symmetric monoidal duality, given 1-cells $X \colon B \longrightarrow A$ and $Y \colon A \longrightarrow B$ and a 2-cell $\varepsilon \colon Y \odot X \longrightarrow B$, we have

$$\varepsilon_{\#} \colon \mathscr{C}(W, Z \odot Y) \longrightarrow \mathscr{C}(W \odot X, Z).$$

For a 2-cell $\eta \colon A \longrightarrow X \odot Y$, we have

$$\eta_{\#} \colon \mathscr{C}(W \odot X, Z) \longrightarrow \mathscr{C}(W, Z \odot Y)$$

(Both for $W \colon A \longrightarrow C$, $Z \colon B \longrightarrow C$.) Duality (η, ε): these are inverse isomorphisms.

Proposition 19. *Let*

$$X\colon B \longrightarrow A \quad \text{and} \quad Y\colon A \longrightarrow B$$

be 1*-cells. TFAE for a given* 2*-cell*

$$\varepsilon\colon Y \odot X \longrightarrow B.$$

(1) (X, Y) *is a dual pair with evaluation map* ε.
(2) $\varepsilon_\#$ *is a bijection for all* W *and* Z.
(3) $\varepsilon_\#$ *is a bijection when* $W = A$ *and* $Z = X$ *and when* $W = Y$ *and* $Z = B$.

Dually, TFAE for a given 2*-cell*

$$\eta\colon A \longrightarrow X \odot Y.$$

(1) (X, Y) *is a dual pair with coevaluation map* η.
(2) $\eta_\#$ *is a bijection for all* W *and* Z.
(3) $\eta_\#$ *is a bijection when* $W = A$ *and* $Z = X$ *and when* $W = Y$ *and* $Z = B$.

Proposition 20. *Let $X, X' \colon B \longrightarrow A$ and $Y, Y' \colon A \longrightarrow B$ be 1-cells such that (X, Y) and (X', Y') are dual pairs and let $\alpha \colon X \longrightarrow X'$ and $\beta \colon Y \longrightarrow Y'$ be 2-cells.*

(1) *There is a unique 2-cell $\alpha^* \colon Y' \longrightarrow Y$ that makes either of the following diagrams commute, and then the other diagram also commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d, "\eta"'] & X' \odot Y' \arrow[d, "\mathrm{id} \odot \alpha^*"] \\
X \odot Y \arrow[r, "\alpha \odot \mathrm{id}"'] & X' \odot Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \odot X \arrow[r, "\mathrm{id} \odot \alpha"] \arrow[d, "\alpha^* \odot \mathrm{id}"'] & Y' \odot X' \arrow[d, "\varepsilon"] \\
Y \odot X \arrow[r, "\varepsilon"'] & B
\end{tikzcd}$$

(2) *There is a unique 2-cell $\beta^* \colon X' \to X$ that makes either of the following diagrams commute, and then the other diagram also commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d, "\eta"'] & X' \odot Y' \arrow[d, "\beta^* \odot \mathrm{id}"] \\
X \odot Y \arrow[r, "\mathrm{id} \odot \beta"'] & X \odot Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \odot X' \arrow[r, "\beta \odot \mathrm{id}"] \arrow[d, "\mathrm{id} \odot \beta^*"'] & Y' \odot X' \arrow[d, "\varepsilon"] \\
Y \odot X \arrow[r, "\varepsilon"'] & B
\end{tikzcd}$$

Now assume that $\mathscr{C}$ is closed.

Definition 21. For a 1-cell $X\colon B \longrightarrow A$, define

$$D_r X = X \triangleright B\colon A \longrightarrow B.$$

For a 1-cell $Y\colon A \longrightarrow B$, define

$$D_\ell Y = B \triangleleft Y\colon B \longrightarrow A.$$

A 2-cell $\varepsilon\colon Y \odot X \longrightarrow B$ has adjoints

$$\tilde{\varepsilon}\colon X \longrightarrow D_\ell Y \quad \text{and} \quad \tilde{\varepsilon}\colon Y \longrightarrow D_r X.$$

$$\begin{tikzcd}
Y \odot D_\ell Y \arrow[dr, "\varepsilon"'] & Y \odot X \arrow[l, "\tilde{\varepsilon}\odot \mathrm{id}"'] \arrow[r, "\mathrm{id}\odot\tilde{\varepsilon}"] \arrow[d, "\varepsilon"] & D_r X \odot X \arrow[dl, "\varepsilon"] \\
& B &
\end{tikzcd}$$

Proposition 22. *If $\varepsilon\colon Y \odot X \longrightarrow B$ is the evaluation map of a dual pair (X, Y), then the adjoint 2-cells*

$$\tilde{\varepsilon}\colon X \longrightarrow D_\ell Y \quad \textit{and} \quad \tilde{\varepsilon}\colon Y \longrightarrow D_r X$$

are isomorphisms.

The maps μ and ν specialize to

$$\mu\colon Z \odot D_r X \longrightarrow X \rhd Z \tag{1}$$

and

$$\nu\colon D_\ell Y \odot W \longrightarrow W \lhd Y. \tag{2}$$

Proposition 23. *The following are equivalent for a 1-cell $X\colon B \longrightarrow A$, and then $(X, D_r X)$ is canonically a dual pair.*

(1) *X is right dualizable.*
(2) *μ is an isomorphism when $Z = X$.*
(3) *μ is an isomorphism for all Z.*

When these hold, the adjoint of ε is an isomorphism $X \longrightarrow D_\ell D_r X$.

Dually, the following are equivalent for a 1-cell $Y\colon A \longrightarrow B$, and then $(D_\ell Y, Y)$ is canonically a dual pair.

(1) *Y is left dualizable.*
(2) *ν is an isomorphism when $W = Y$.*
(3) *ν is an isomorphism for all W.*

When these hold, the adjoint of ε is an isomorphism $Y \longrightarrow D_r D_\ell Y$.

Left and right analogues of $X \cong DDX$.
Other analogues.

Proposition 24. *Consider μ, ν, and ω.*

(1) *If X or Z is right dualizable, then*
$$\mu\colon Z \odot (X \rhd Y) \longrightarrow X \rhd (Z \odot Y)$$
is an isomorphism.

(2) *If W or Y is left dualizable, then*
$$\nu\colon (Z \lhd Y) \odot W \longrightarrow (Z \odot W) \lhd Y$$
is an isomorphism.

(3) *If X is right and Y is left dualizable, then*
$$\omega\colon (Z \lhd Y) \odot (X \rhd W) \longrightarrow X \rhd (Z \odot W) \lhd Y$$
is an isomorphism.

4. Composites of dualities

Theorem 25. *Consider* 1*-cells* W, X, Y, Z *as in the diagram*

$$\begin{tikzcd} C \arrow[r, shift left, "W"] & B \arrow[l, shift left, "Z"] \arrow[r, shift left, "X"] & A. \arrow[l, shift left, "Y"] \end{tikzcd}$$

If (η, ε) *exhibits* (X, Y) *as a dual pair and* (ζ, σ) *exhibits* (W, Z) *as a dual pair, then*

$$((\mathrm{id} \odot \zeta \odot \mathrm{id}) \circ \eta, \sigma \circ (\mathrm{id} \odot \epsilon \odot \mathrm{id}))$$

exhibits $(X \odot W, Z \odot Y)$ *as a dual pair.*

Theorem 26. *Let* $F\colon \mathscr{B} \longrightarrow \mathscr{C}$ *be a lax functor between symmetric bicategories. Let* (X, Y) *be a dual pair in* $\mathscr{B}$*,*

$$X\colon B \longrightarrow A \quad \text{and} \quad Y\colon A \longrightarrow B.$$

Assume that the unit and composition coherence 2*-cells*

$$I_{FB} \longrightarrow FI_B$$

and

$$FX \odot FY \longrightarrow F(X \odot Y)$$

are isomorphisms. Then (FX, FY) *is a dual pair in* $\mathscr{C}$*. There is a dual result for oplax functors.*

5. Parametrized spectra

$\mathrm{Ho}\mathscr{S}_B$: The homotopy category of spectra X parametrized over B. For each $b \in B$, there is a "fiber spectrum" X_b, and these are nicely glued together.

Base change functors for $f\colon A \longrightarrow B$:

$$f_!\colon \mathrm{Ho}\mathscr{S}_A \longrightarrow \mathrm{Ho}\mathscr{S}_B,$$
$$f^*\colon \mathrm{Ho}\mathscr{S}_B \longrightarrow \mathrm{Ho}\mathscr{S}_A,$$
$$f_*\colon \mathrm{Ho}\mathscr{S}_A \longrightarrow \mathrm{Ho}\mathscr{S}_B.$$

The closed symmetric bicategory $\mathscr{E}x$ has as 0-cells spaces B, with $tB = B$, and it has

$$\mathscr{E}x(A, B) = \mathrm{Ho}\mathscr{S}_{B\times A}.$$

$t\colon B\times A \cong A\times B$ induces involution $t = t^*$. For X over $B \times A$ and Y over $C \times B$, we have a fiberwise smash product

$$Y \mathbin{\bar\wedge} X \quad \text{over} \quad C \times B \times B \times A.$$

$$Y \odot X = (\mathrm{id}_C \times r \times \mathrm{id}_A)_!(\mathrm{id}_C \times \Delta \times \mathrm{id}_A)^*(Y \mathbin{\bar\wedge} X)$$

where $\Delta\colon B \longrightarrow B \times B$ is the diagonal and $r\colon B \longrightarrow *$ is the unique map to a point. The unit I_B over $B \times B$ is $\Delta_! S_B$, where S_B is the sphere spectrum over B.

$\mathrm{Ho}\mathscr{S}_B$ is closed symmetric monoidal with unit S_B under the fiberwise smash product

$$X \wedge_B Y = \Delta^*(X \mathbin{\bar\wedge} Y).$$

DUALITY IN $\mathrm{Ho}\mathscr{S}_B$
VERSUS
DUALITY IN $\mathscr{E}x$

Embed $\mathrm{Ho}\mathscr{S}_B$ in $\mathscr{E}x$ as

$$\mathrm{Ho}\mathscr{S}_{*\times B} = \mathscr{E}x(B,*)$$

and

$$\mathrm{Ho}\mathscr{S}_{B\times *} = \mathscr{E}x(*,B)$$

1-cells $X\colon B \longrightarrow *$ and $tX\colon * \longrightarrow B$.
$\mathscr{E}x$ informs on $\mathrm{Ho}\mathscr{S}_B$ by relating it with

$$\mathrm{Ho}\mathscr{S} = \mathscr{E}(*,*)$$

and

$$\mathrm{Ho}\mathscr{S}_{B\times B} = \mathscr{E}x(B,B).$$

For $f\colon B \longrightarrow A$, have base change spectra

$$S_f\colon B \longrightarrow A \quad \text{and} \quad tS_f\colon A \longrightarrow B$$

Theorem 27. *(tS_f, S_f) is a dual pair.*

Ignoring source and target, for $r\colon B \longrightarrow *$,

$$S_r \simeq S_B \simeq tS_r.$$

$$(f \times \mathrm{id})_! B \simeq S_f \simeq (\mathrm{id} \times f)^* A$$

$$(\mathrm{id} \times f)_! B \simeq tS_f \simeq (f \times \mathrm{id})^* A.$$

Change of source: For $f\colon A \longrightarrow A'$

$$Y \odot S_f \simeq (\mathrm{id} \times f)^* Y \simeq tS_f \rhd Y\colon A \longrightarrow B,$$
$$X \odot tS_f \simeq (\mathrm{id} \times f)_! X\colon A' \longrightarrow B,$$
$$S_f \rhd X \simeq (\mathrm{id} \times f)_* X\colon A' \longrightarrow B$$

for 1-cells $X\colon A \longrightarrow B$ and $Y\colon A' \longrightarrow B$.

Change of target: For $g\colon B \longrightarrow B'$

$$tS_g \odot Z \simeq (g \times \mathrm{id})^* Z \simeq Z \lhd S_g\colon A \longrightarrow B,$$
$$S_g \odot X \simeq (g \times \mathrm{id})_! X\colon A \longrightarrow B',$$
$$X \lhd tS_g \simeq (g \times \mathrm{id})_* X\colon A \longrightarrow B'$$

for 1-cells $X\colon A \longrightarrow B$ and $Z\colon A \longrightarrow B'$.

6. Dictionaries

Let X, Y be spectra over B.
Let Z be a spectrum.
$X \odot tY$ is a spectrum.
$tY \odot X$ is a spectrum over $B \times B$.

Translate $\odot$, $\triangleleft$, $\triangleright$ into base change, $\wedge_B$, and the internal hom $F_B(X, Y)$ in $\mathrm{Ho}\mathscr{S}_B$:

As spectra,

$$Y \odot tX \simeq r_!(Y \wedge_B X)$$

$$tY \triangleleft tX \simeq r_* F_B(X, Y) \simeq X \triangleright Y.$$

As spectra over B,

$$tX \odot Z \simeq X \,\overline{\wedge}\, Z$$

$$Z \odot X \simeq Z \,\overline{\wedge}\, X$$

$$Z \triangleleft X \simeq F_B(X, r^* Z) \simeq tX \triangleright Z.$$

As spectra over $B \times B$,

$$tY \odot X \simeq Y \,\overline{\wedge}\, X.$$

Translate $\wedge_B$, F_B into $\odot$, $\triangleleft$, $\triangleright$.

$$\Delta_! X \odot tY \simeq X \wedge_B Y \simeq X \odot \Delta_! Y,$$

$$\Delta_! X \triangleright Y \simeq F_B(X, Y) \simeq \Delta^*(X \triangleright Y),$$

$$X \triangleleft \Delta_! Y \simeq F_B(Y, X) \simeq \Delta^*(X \triangleleft Y).$$

Translate base change into $\odot$, $\triangleleft$, $\triangleright$.
Let $f\colon B \longrightarrow A$ be a map, X be a spectrum over A, and Y be a spectrum over B. 1-cells

$$A \longrightarrow * \quad \text{and} \quad B \longrightarrow *.$$

$$f_! Y \simeq Y \odot tS_f \qquad tf_! Y \simeq S_f \odot tY$$

$$X \odot S_f \simeq f^* X \simeq tS_f \triangleright X$$

$$tS_f \odot tX \simeq tf^* X \simeq tX \triangleleft S_f$$

$$f_* Y \simeq S_f \triangleright Y \qquad tf_* Y \simeq tY \triangleleft tS_f.$$

7. Parametrized duality theory

Fiberwise duality:
symmetric monoidal duality in $\mathrm{Ho}\mathscr{S}_B$.
$S =$ unit in $\mathscr{E}(*,*)$ $\Delta_! S_B$=unit in $\mathscr{E}(B,B)$.
Duals wrt S:

$$D_\ell X = S \lhd X \colon * \longrightarrow B$$

$$D_r tX = tX \rhd S \colon B \longrightarrow *.$$

Duals wrt $\Delta_! S_B$:

$$D_\ell tX = \Delta_! S_B \lhd tX \colon B \longrightarrow *$$

$$D_r X = X \rhd \Delta_! S_B \colon * \longrightarrow B.$$

Duality wrt S: tX is right dualizable iff X is left dualizable, and $tD_\ell X \simeq D_r tX$. Viewed in $\mathrm{Ho}\mathscr{S}_B$, these are

$$D_B X = F_B(X, S_B) = \bar{F}(X, \Delta_* S_B).$$

($\bar{F}$ is external hom that goes with $\bar{\wedge}$).
(tX, Y) is a dual pair iff (tY, X) is a dual pair, and then $Y \simeq D_B X$.

$$\eta \colon \Delta_! S_B \longrightarrow tX \odot Y \qquad \varepsilon \colon Y \odot tX \longrightarrow S$$

Proposition 28. *(tX, Y) is a dual pair iff (X, Y) is a fiberwise dual pair.*

Duality wrt $\Delta_! S_B$: X is right dualizable iff tX is left dualizable, and $D_\ell tX \simeq tD_r X$. Viewed in $\mathrm{Ho}\mathscr{S}_B$, these are

$$D_B^{CW} X = \bar{F}(X, \Delta_! S_B).$$

(X, tY) is a dual pair iff (Y, tX) is a dual pair, and then $Y \simeq D_B^{CW} X$.

$$\eta\colon S \longrightarrow X \odot tY \qquad \varepsilon\colon tY \odot X \longrightarrow \Delta_! S_B$$

Right dualizable $\equiv$ Costenoble-Waner dualizable, abbreviated CW-dualizable.

Proposition 29. *X is CW-dualizable with dual Y iff Y is CW-dualizable with dual X, and then $X \simeq D_B^{CW} D_B^{CW} X$.*

Proposition 30. *If X is CW dualizable and J is any spectrum over B, then*

$$\begin{array}{c} r_!(J \wedge_B D_B^{CW} X) \simeq J \odot tD_B^{CW} X \\ \big\downarrow{\scriptstyle \mu} \\ r_* F_B(X, J) \simeq X \rhd J \end{array}$$

is an equivalence of spectra.

Definition 31. An ex-space K over B is CW-dualizable if $\Sigma_B^\infty K$ is CW dualizable.

Theorem 32. *If M is a smooth compact n-manifold, then $S_M^0 = M \amalg M$ over M is CW-dualizable. Its dual is $\Sigma_M^{-q} S^\nu$.*

S^ν is the fiberwise one-point compactification of the normal bundle ν of $M \subset \mathbb{R}^q$.

Proposition 33. *If S_K^0 is CW-dualizable and (K,p) is a space over B, then*

$$(K,p) \amalg B \cong p_! S_K^0$$

is CW-dualizable.

Theorem 34. *If X is a wedge summand of a "finite cell spectrum", then X is CW-dualizable.*

Proof. A cell is $(D^n,p) \amalg B$; induction. $\square$

Duality in $\mathscr{E}x$ when $A = B = *$ is SW-duality (Spanier-Whitehead).

Proposition 35. *If S_B is CW-dualizable, then $r_! S_B = \Sigma^\infty B_+$ is SW-dualizable, so B is equivalent to a finite CW-complex.*

If B infinite, then S_B is not CW-dualizable. It is invertible, hence fiberwise dualizable.

Let M be a smooth closed manifold. Then $(S_M, t\Sigma_M^{-q} S^\nu)$ is a dual pair. Also, $\Sigma_M^{-q} S^\nu$ is invertible in $\mathrm{Ho}\mathscr{S}_M$ with inverse $\Sigma_M^\infty S^\tau$:

$$\Sigma_M^\infty S^\tau \wedge_M \Sigma_M^{-q} S^\nu \simeq \Sigma^{-q}(S^\tau \wedge_M S^\nu) \simeq S_M$$

since $\tau \oplus \nu$ is trivial and $S^\tau \wedge_M S^\nu \cong S_M^V$. Combining these dualities leads to homotopical Poincaré duality.

Theorem 36. *For a spectrum k,*

$$k \wedge M_+ \simeq S_M \triangleright (k \wedge S^\tau).$$

8. Parametrized homology and cohomology

For spectra J and X over B, define

$$\begin{aligned} J_n(X) &= \pi_n(r_!(J \wedge_B X)) \\ J^n(X) &= \pi_{-n}(r_* F_B(X, J)) \\ &\cong [S_B^{-n}, F_B(X, J)]_B. \end{aligned}$$

For a spectrum k,

$$\begin{aligned} (r^*k)_*(X) &\cong k_*(r_!X) \equiv k_*^B(X) \\ (r^*k)^*(X) &\cong k^*(r_!X) \equiv k_B^*(X). \end{aligned}$$

$$\begin{aligned} r_!(J \wedge_B X) &\simeq J \odot tX \\ r_* F_B(X, J) &\simeq X \rhd J. \end{aligned}$$

$$\begin{aligned} J_*(X) &= \pi_*(J \odot tX) \\ J^*(X) &= \pi_{-*}(X \rhd J). \end{aligned}$$

If (X, Y) is a dual pair,

$$J \odot tY \simeq X \rhd J$$

Theorem 37 (Costenoble-Waner duality)**.**

$$J_*(Y) \cong J^{-*}(X).$$

Let M be a smooth closed n-manifold. Have

$$k \wedge M_+ \simeq S_M \triangleright (k \wedge S^\tau).$$

Take $X = S_M$ and $J = k \wedge S^\tau$.

Theorem 38 (Poincaré duality)**.**

$$k_*(M_+) \cong (k \wedge S^\tau)^{-*}(S_M).$$

Definition 39. The Thom complex of an n-plane bundle ξ over B is

$$T\xi = S^\xi / s(B) = r_! S^\xi.$$

Let k be a commutative ring spectrum. A k-orientation of ξ is a class $\mu \in k^n(T\xi)$ whose fiber restriction

$$\mu_b \in k^n(T\xi_b) \cong k^n(S^n) \cong k^0(S^0) = \pi_0(k)$$

is a unit in the ring $\pi_0(k)$ for each $b \in B$.

Proposition 40. *A k-orientation μ of ξ induces an equivalence of spectra over B*
$k \wedge S^\xi \simeq r^* k \wedge_B S^\xi \longrightarrow r^* k \wedge_B S^n_B \simeq k \wedge S^n$.

"ξ is trivial to the eyes of k"
$r_! S^0_B = B_+$ and thus

$$(k \wedge S^n_B)^{-*}(S_B) \cong (r^* \Sigma^n k)^{-*}(S^0_B) \cong k^{n-*}(B_+).$$

Take $B = M$. A k-orientation of M is a k-orientation of its tangent bundle.

Theorem 41 (Poincaré duality)**.** *Let k be a commutative ring spectrum and M be a k-oriented smooth closed n-manifold. Then*
$$k_*(M_+) \cong k^{n-*}(M_+).$$

New result by the same methods.

Theorem 42. *Let L^d be a smooth closed submanifold of a smooth closed manifold M^n, both k-oriented. Then*
$$k_{n-d+p}(T\nu_{M,L}) \cong k^{d-p}(L_+) \cong k_p(L_+).$$