What are parametrized spectra good for?

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Outline

- Background: history and naive basic definitions
- Parametrized duality
  - Categorical duality and transfer maps
  - Bicategorical and Costenoble-Waner duality
    (This is parametrized Spanier-Whitehead duality)
  - Homotopical Poincaré duality
  - The homotopical meaning of orientations
Homology and cohomology

- Parametrized homology and cohomology
- Relationship with nonparametrized theories
- Classical Poincaré duality
- Twisted homology and cohomology theories
What are parametrized spaces?

- Answer: Maps $p: E \rightarrow B$
- Better: Maps $p$ and sections $s: B \rightarrow E$
  (Stable algebraic topologists want basepoints)
  These are often called “ex-spaces”

One focus: fiberwise 1-point compactifications of vector bundles and their Thom spaces $E/sB$. 
What are spectra?

- Naively, sequences of spaces $T_n$ and maps $\Sigma T_n \rightarrow T_{n+1}$

- $\Omega$-spectra: $T_n \xrightarrow{\sim} \Omega T_{n+1}$

  We will be naive today: pretend everything is for the best in this best of all possible worlds, and ignore all technicalities

- Spaces to spectra: $\Sigma^{\infty} X$

  $$\Sigma (\Sigma^n X) \xrightarrow{\sim} \Sigma^{n+1} X$$

- $\Omega$-spectra to spaces: $\Omega^{\infty} T$

  $$T \mapsto T_0$$
What are spectra good for?

- Spanier-Whitehead duality [1958]
- Cobordism theory [1959] (Milnor; MSO)
- Stable homotopy theory [1959] (Adams; ASS)
- Generalized homology theories [1962] (G.W. Whitehead)
- Stable homotopy category [1964] (Boardman)
What are parametrized spectra?

- ex-spectrum $E$ over $B$: ex-spaces $E_n$ and maps of ex-spaces

  $$\Sigma_B E_n \longrightarrow E_{n+1}$$

- $\Omega$-ex-spectrum $E$:

  $$E_n \xrightarrow{\sim} \Omega_B E_{n+1}$$

  Sections give fibers basepoints; $\Sigma_B$ and $\Omega_B$ restrict to suspension and loops on fibers

- Ex-spaces to ex-spectra: $\Sigma^\infty_B X$

  $$\Sigma_B (\Sigma^n_B X) \xrightarrow{\sim} \Sigma^{n+1}_B X$$

- $\Omega$-ex-spectra to spaces: $\Omega^\infty_B E$

  $$E \leftrightarrow E_0$$

Basic constructions are done spacewise and fiberwise
Precursors and sources

- J.C. Becker and D.H. Gottlieb
  The transfer map and fiber bundles, 1975

- Monica Clapp
  Duality and transfer for parametrized spectra, 1981

- Monica Clapp and Dietrich Puppe
  The homotopy category of parametrized spectra, 1984

- Po Hu
  Duality for Smooth Families in Equivariant Stable Homotopy Theory, 2003

- S.R. Costenoble and S. Waner
  Equivariant ordinary homology and cohomology theory, 2003

- J.P. May and Johann Sigurdsson
  Parametrized homotopy theory, 2006
The category $\mathcal{I}_B$ of spectra over $B$

- Good homotopy category: Approximate ex-spectra $X$ by $\Omega$-ex-spectra such that all $p: X_n \to B$ are fibrations. Fibers are spectra. Weak equivalences are maps that give isomorphisms on the (stable) homotopy groups of fibers.

- $\text{Ho} \mathcal{I}_B$; invert weak equivalences; maps $[X, Y]_B$

- Base change functors for $f: A \to B$

\[ f^*: \mathcal{I}_B \to \mathcal{I}_A \]
\[ f!: \mathcal{I}_A \to \mathcal{I}_B \]
\[ f_*: \mathcal{I}_A \to \mathcal{I}_B \]

\[ [f! X, Y]_B \cong [X, f^* Y]_A \quad [f^* Y, X]_A \cong [Y, f_* X]_B \]
Comparing $\mathcal{S}_B$ to $\mathcal{S} = \mathcal{S}_{pt}$

$r: B \rightarrow \ast$; for based spaces $X$, $r^*X = B \times X$.

$$r^*: \mathcal{S} \rightarrow \mathcal{S}_B, \quad r!: \mathcal{S}_B \rightarrow \mathcal{S}, \quad r_*: \mathcal{S}_B \rightarrow \mathcal{S}$$

- $r!$ “quotients out sections spacewise”
- For a map $p: E \rightarrow B$, $(E, p)_+ = E \amalg B$:
  $$r!\Sigma^\infty_B (E, p)_+ = \Sigma^\infty E_+$$

- For a spherical fibration $p: E \rightarrow B$ with section $s$,
  $$r!\Sigma^\infty_B E = \Sigma^\infty Tp,$$
  $Tp = E/sB$. Thom spectra work similarly.

- $r_*$ is the “global sections functor”, $\text{Sec}(B, E)$ spacewise
Fiberwise external smash product $X \bar{\wedge} Y$ over $A \times B$ for ex-spectra $X$ over $A$ and $Y$ over $B$. External hom function ex-spectrum $\tilde{F}(X, Y)$ over $A$ for $X$ over $B$ and $Y$ over $A \times B$.

Internal smash product of ex-spectra over $B$

$$X \wedge_B Y = \Delta^*(X \bar{\wedge} Y)$$

Unit $S_B = \Sigma^\infty_B S^0_B$, where $S^0_B = B \times S^0$

Internal hom function ex-spectrum $F_B(X, Y)$ over $B$

$$F_B(X, Y) = \tilde{F}(X, \Delta^*_Y)$$

$$[X \wedge_B Y, Z]_B \cong [X, F_B(Y, Z)]_B$$
Let \((\mathcal{C}, \otimes, I)\) be a symmetric monoidal category. Objects \(X\) and \(Y\) are **dual** if there are maps

\[
\eta : I \longrightarrow X \otimes Y
\]

\[
\varepsilon : Y \otimes X \longrightarrow I
\]

such that the composites

\[
\begin{align*}
X &\cong I \otimes X \xrightarrow{\eta \otimes \text{id}} X \otimes Y \otimes X \xrightarrow{\text{id} \otimes \varepsilon} X \otimes I \cong X \\
Y &\cong Y \otimes I \xrightarrow{\text{id} \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\varepsilon \otimes \text{id}} I \otimes Y \cong Y
\end{align*}
\]

are identity maps.
Given $\Delta: X \to X \otimes X$, the trace of $f: X \to X$ is

$$
\begin{array}{cccccc}
I & \xrightarrow{\eta} & X \otimes Y & \xrightarrow{\gamma} & Y \otimes X & \xrightarrow{id \otimes f} \\
\downarrow \tau(f) & & \downarrow & & \downarrow \text{id} \otimes \Delta \\
X & \xleftarrow{\varepsilon} & I \otimes X & \xleftarrow{\varepsilon \otimes \text{id}} & Y \otimes X \otimes X
\end{array}
$$

The transfer $\tau = \tau_X$ is $\tau(\text{id}_X)$. 

Fiberwise duality theorem

$$\text{Ho}\mathcal{I}_B$$ is symmetric monoidal, unit $$S_B$$.

**Theorem**

$$X$$ is dualizable iff each fiber $$X_b$$ is dualizable in $$\text{Ho}\mathcal{I}$$.

Headed towards

- Global transfer map for fibrations
- Fiberwise transfer map for bundles
- These agree when both are defined
Transfer for fibrations

Let \( p : E \to B \) be a fibration with finite CW fibers.

\[
(E, p)_+ = E \amalg B \quad \Delta : E_+ \to E_+ \wedge_B E_+
\]

\[
\tau_{(E, p)}_+ : S_B = \Sigma^\infty_B (B, \text{id})_+ \to \Sigma^\infty_B (E, p)_+
\]

Apply \( r_! \), \( r : B \to * \)

\[
r_! S_B = \Sigma^\infty B_+, \quad r_! \Sigma^\infty_B (E, p)_+ = \Sigma^\infty E_+
\]

This gives the transfer map

\[
\tau_E = r_! \tau_{(E, p)}_+ : \Sigma^\infty B_+ \to \Sigma^\infty E_+
\]

It induces transfer in all homology and cohomology theories.
The bundle construction

- $H$ a locally compact group (structural group)
- $p: P \longrightarrow B$ a principal $H$-bundle

Apply $P \times_H (-)$ spacewise to $H$-spectra to get spectra over $B$:

$$P \equiv P \times_H (-): H\mathcal{S} \longrightarrow \mathcal{S}_B$$

(Specialization of a fiberwise equivariant bundle construction.)
Fiberwise transfer

- Let $M$ be a compact $H$-manifold
- Let $p: P \rightarrow B$ be a principal $H$-bundle

$\Sigma^\infty_H M$ is dualizable as an $H$-spectrum:

$$\tau_M: S_H \rightarrow \Sigma^\infty_H M_+$$

Think of $\tau_M$ as transfer on fibers
Fiberwise transfer

Apply $P$ to $\tau_M$ and then apply $r!$:

$$P_{\tau_M} : PS_H \longrightarrow P\Sigma_{H} M_+$$

$$r!PS_H = \Sigma^\infty B_+ \quad r!P\Sigma_{H} M_+ = \Sigma^\infty E_+$$

$$\tau_E = r!P_{\tau_M} : \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+$$

The functor $P$ is **monoidal**! Therefore

**Theorem**

$$\tau_E = r!\tau(E,p)_+ = r!P_{\tau_M} = \tau_E$$
Costenoble-Waner duality

- Spanier-Whitehead duality is different!
- Cannot be fiberwise duality
- Due to Costenoble and Waner
- Cannot be understood without bicategorical duality
Bicategory $\mathcal{E}$ of ex-spectra

- **0-cells:** spaces
- **1-cells and 2-cells:** $\mathcal{E}(A, B) = \text{Ho}\mathcal{S}_{B \times A}$
- **Unit 1-cells** $I_B = \Delta_! S_B \in \mathcal{E}(B, B)$
- **Composition** $\circ: \mathcal{E}(B, C) \times \mathcal{E}(A, B) \to \mathcal{E}(A, C)$
  
  \[
  Y \circ X = (\text{id}_A \times r_B \times \text{id}_C)^!(\text{id}_A \times \Delta_B \times \text{id}_C)^*(Y \wedge X)
  \]

- $t: A \times B \to B \times A$ induces **involution**

Analogous to rings, bimodules, maps of bimodules, with

\[
Y \circ X = Y \otimes_B X
\]
Objects $X \in \mathcal{E}(A, B)$ and $Y \in \mathcal{E}(B, A)$ are dual, $X$ right dualizable with right dual $Y$, if there are maps $\eta : I_A \to X \odot Y$ and $\varepsilon : Y \odot X \to I_B$ such that the composites

$$X \cong I_A \odot X \xrightarrow{\eta \odot \text{id}} X \odot Y \xrightarrow{\text{id} \odot \varepsilon} X \odot I_B \cong X$$

$$Y \cong Y \odot I_A \xrightarrow{\text{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \text{id}} I_B \odot Y \cong Y$$

are identity maps.
The role of $\mathcal{E}$

Embed $\text{Ho}\mathcal{I}_B$ in $\mathcal{E}$ as

$$\text{Ho}\mathcal{I}_{* \times B} = \mathcal{E}(B, \ast)$$

and

$$\text{Ho}\mathcal{I}_{B \times \ast} = \mathcal{E}(\ast, B)$$

1-cells $X : B \to \ast$ and $tX : \ast \to B$, respectively

$\mathcal{E}$ informs on $\text{Ho}\mathcal{I}_B$ by relating it to

$$\text{Ho}\mathcal{I} = \mathcal{E}(\ast, \ast)$$

and

$$\text{Ho}\mathcal{I}_{B \times B} = \mathcal{E}(B, B).$$
Duality in Ho$\mathcal{S}_B$ versus duality in $\mathcal{E}$

$S_B$ is invertible, hence dualizable in Ho$\mathcal{S}_B$.

This means that $S_B$ is left dualizable:

$(tS_B, S_B)$ is a dual pair in $\mathcal{E}$.

$S_B$ is usually not right dualizable.

For spectra $X$ and $Y$ over $B$,

$$Y \circ tX = r_!(Y \wedge_B X) \in \mathcal{I}$$

$$tY \circ X = Y \bar{\wedge} X \in \mathcal{I}_{B \times B}$$
Duality in $\mathcal{E}$ with respect to $S$

Unit $I_* = S \in \mathcal{E}(\ast, \ast)$.

Let $(tX, Y)$ be a dual pair.

In $\text{Ho}\mathcal{S}_B$,

$$Y \simeq D_BX \equiv F_B(X, S_B) = \tilde{F}(X, \Delta_* S_B)$$

$$\eta: \Delta ! S_B \longrightarrow tX \odot Y \quad \varepsilon: Y \odot tX \longrightarrow S$$

**Theorem**

$(tX, Y)$ is a dual pair iff $(X, Y)$ is a fiberwise dual pair.
Duality in $\mathcal{E}$ with respect to $\Delta \downarrow S_B$

Unit $I_B = \Delta \downarrow S_B \in \mathcal{E}(B, B)$.

Let $(X, tY)$ be a dual pair.

In $\text{Ho} \mathcal{S}_B$,

$$Y \cong D^c_w X \equiv \bar{F}(X, \Delta \downarrow S_B)$$

$$\eta: S \rightarrow X \odot tY \quad \varepsilon: tY \odot X \rightarrow \Delta \downarrow S_B$$

Right dualizable $\equiv$ Costenoble-Waner (CW) dualizable

$$(X, tY) \text{ is a dual pair iff } (Y, tX) \text{ is a dual pair.}$$
**Definition**

An ex-space $K$ over $B$ is CW-dualizable if $\Sigma_B^\infty K$ is CW-dualizable.

$S^0_K \equiv (K, \text{id})_+ = K \times S^0$ is the 0-sphere ex-space over $K$.

**Theorem (Parametrized Atiyah duality)**

Let $M$ be a smooth compact $n$-manifold embedded in $\mathbb{R}^q$ with normal bundle $\nu$. Let $S^\nu$ be the one-point fiberwise compactification of $\nu$.

Then $S^0_M$ is CW-dualizable with (right) dual $\Sigma_M^{-q} S^\nu$. 
Proposition

If $S^0_K$ is CW-dualizable and $(K, p)$ is a space over $B$, then

$$(K, p)_+ \cong p_! S^0_K$$

is CW-dualizable over $B$.

A “cell” is a spectrum $\Sigma^\infty_B (D^n, p)_+$ over $B$; induct over cells:

Theorem

If $X$ is a wedge summand of a finite cell spectrum over $B$, then $X$ is CW-dualizable.

Duality in $\mathcal{E}$ when $A = B = \ast$ is Spanier-Whitehead duality in $\mathcal{I}$. If $S_B$ is CW-dualizable, then $r_! S_B = \Sigma^\infty B_+$ is SW-dualizable, so $B$ is a finite complex. If $B$ is infinite, $S_B$ cannot be CW-dualizable.
Let $X$, $Y$, $J$ be in $\mathcal{E}(B, \ast)$, with $(X, tY)$ a dual pair. Then

$$J \odot tY \simeq X \triangleright J.$$ 

$X \triangleright J$ is notation for a bicategorical right hom functor:

$$\mathcal{E}(X, Y \triangleright Z) \cong \mathcal{E}(X \odot Y, Z) \cong \mathcal{E}(Y, Z \triangleleft X)$$

Dictionary:

$$J \odot tY = r_!(J \wedge_B Y)$$

$$X \triangleright J = r_* F_B(X, J)$$

Therefore

$$r_!(J \wedge_B Y) \simeq r_* F_B(X, J)$$
Let $M$ be a smooth closed manifold, $M \subset \mathbb{R}^q$.

Then $(X, tY) = (S_M, t\Sigma^{-q}_M S^\nu)$ is a dual pair.

Since $\tau \oplus \nu$ is trivial,

$$S^\tau \wedge_M S^\nu \cong S^q_M.$$ 

Thus $\Sigma^{-q}_M S^\nu$ is invertible in $\text{Ho}\mathcal{S}_M$ with inverse $\Sigma^\infty_M S^\tau$.

With $J = k \wedge S^\tau$ (makes sense),

$$r_!(J \wedge_M Y) \cong r_* F_M(X, J)$$

becomes

**Theorem (Homotopical Poincaré duality)**

*For a spectrum $k$,*

$$k \wedge M_+ \cong S_M \triangleright (k \wedge S^\tau) = r_* F_M(S_M, k \wedge S^\tau).$$
For spectra $J$ and $X$ over $B$, define

$$J_n(X) = \pi_n(r_!(J \wedge_B X))$$

$$J^n(X) = \pi_{-n}(r_* F_B(X, J))$$

$$\cong [S_B^{-n}, F_B(X, J)]_B.$$ 

$$J_\bullet(S_B) = \pi_\bullet(r_! J) \quad J^\bullet(S_B) = \pi_{-\bullet}(r_* J)$$

The coefficient groups are genuinely different (but related)!
The “classical” case

For a spectrum $k$,

$$k^B_\bullet(X) \equiv k_\bullet(r_! X) \cong (r^* k)_\bullet(X)$$

$$k_B^\bullet(X) \equiv k^\bullet(r_! X) \cong (r^* k)^\bullet(X)$$

Isomorphisms are special cases of base change isomorphisms.

$f : B \longrightarrow A$. For $X \in \mathcal{S}_B$ and $k \in \mathcal{S}_A$,

$$k_\bullet(f_! X) \cong (f^* k)_\bullet(X) \quad k^\bullet(f_! X) \cong (f^* k)^\bullet(X)$$

Also, for $X \in \mathcal{S}_A$ and $J \in \mathcal{S}_B$,

$$(f_! J)_\bullet(X) \cong J_\bullet(f^* X) \quad (f_! J)^\bullet(X) \cong J^\bullet(f^* X)$$
Twisted Poincaré duality

Let \((X, tY)\) be a dual pair.

**Theorem (Costenoble-Waner duality)**

\[
J_\bullet(Y) \cong J^{-\bullet}(X).
\]

Let \(M\) be a smooth closed \(n\)-manifold and take \(X = S_M\) and \(J = k \wedge S^\tau\).

**Theorem (Twisted Poincaré duality)**

\[
k_\bullet(M_+) \cong (k \wedge S^\tau)^{-\bullet}(S_M).
\]
Orientations of bundles

The Thom complex of an $n$-plane bundle $\xi$ over $B$ is

$$T\xi = S^\xi / s(B) = r_! S^\xi.$$

Let $k$ be a commutative ring spectrum. A $k$-orientation of $\xi$ is a class $\mu \in k^n(T\xi)$ whose fiber restriction

$$\mu_b \in k^n(T\xi_b) \cong k^n(S^n) \cong k^0(S^0) = \pi_0(k)$$

is a unit in the ring $\pi_0(k)$ for each $b \in B$.

**Proposition (Parametrized trivialization)**

A $k$-orientation $\mu$ of $\xi$ induces an equivalence of spectra over $B$

$$k \wedge S^\xi \cong r^* k \wedge_B S^\xi \longrightarrow r^* k \wedge_B S^n_B \cong k \wedge S^n_B,$$

$S^n_B = B \times S^n$, and conversely. Orientations untwist cohomology:

$$(k \wedge S^\xi)^{-\bullet}(S_B) \cong (k \wedge S^n_B)^{-\bullet}(S_B) \cong k^{n-\bullet}(B_+).$$
With $B = M$, a $k$-orientation of $M$ is a $k$-orientation of $\tau M$.

**Theorem (Poincaré duality)**

Let $k$ be a commutative ring spectrum and $M$ be a $k$-oriented smooth closed $n$-manifold. Then

$$k_\bullet(M_+) \cong k^{n-\bullet}(M_+).$$

Works similarly for manifolds with boundary. Similarly:

**Theorem (Relative Poincaré duality)**

Let $L^d$ be a smooth closed submanifold of a smooth closed manifold $M^n$, both $k$-oriented. Then

$$k_{n-d+\bullet}(T\nu_{M,L}) \cong k^{d-\bullet}(L_+) \cong k_\bullet(L_+).$$
Homology of fibers and local coefficient systems

$b \in B$ gives $b: * \to B$

A spectrum $Y$ over $B$ has fibers $Y_b = b^* Y$

Have a topological local system $\Pi B \to \text{Ho}\mathcal{I}$, $b \mapsto Y_b$

Local coefficient systems $\Pi B \to \mathcal{A}b_*$ for $J \in \mathcal{I}_B$:

$L_\bullet(Y, J) \quad b \mapsto (J_b)_\bullet(Y_b)$

$L^\bullet(Y, J) \quad b \mapsto (J_b)^\bullet(Y_b)$
$B$ a CW complex, $Y \in \mathcal{S}_B$, Serre spectral sequences

$$E_{2}^{p,q} = H_{p}(B; \mathcal{L}_{q}(Y, J)) \Rightarrow J_{p+q}(Y)$$

$$E_{2}^{p,q} = H_{p}(B; \mathcal{L}_{q}(Y, J)) \Rightarrow J_{p+q}(Y)$$

Other spectral sequences:

- Atiyah-Hirzebruch (specialize Serre)
- Čech (for open covers of $B$)
- Rothenberg-Steenrod
- Eilenberg-Moore (parametrized Künneth theorem)
Take $P = EH$ in the bundle construction

Let $k$ be a (naive) $H$-spectrum

Define $k_H = EH \times_H k$, a spectrum over $BH$

For a space $p: X \to BH$ over $BH$, define

$$k_\bullet(X, p) = (k_H)_\bullet((X, p)_+) \quad \text{and} \quad k^\bullet(X, p) = (k_H)^\bullet((X, p)_+)$$

Extrinsic: Given by theories represented by spectra over $BH$. 

What are parametrized spectra good for?
Twisted homology and cohomology theories

Form the pullback

\[ \begin{array}{ccc} P & \rightarrow & EH \\ \downarrow & & \downarrow \\ X & \rightarrow & BH \end{array} \]

\(P\) is a principal \(H\)-bundle over \(X\). Write

\[ k^P_\bullet(X) = k_\bullet(X, p) \quad \text{and} \quad k_P^\bullet(X) = k^\bullet(X, p) \]

Depends only on the homotopy class of \(p\).
Twisted homology and cohomology theories

Recall the space $S_X^0 = X \times S^0$ over $X$

- $p_! S_X^0 = (X, p)_+$ as a space over $BH$
- $P \times_H k = p^* k_H$ as a spectrum over $X$
- Intrinsic reinterpretation

$$k^P_\bullet(X) \cong (P \times_H k)_\bullet(S_X^0) \quad \text{and} \quad k_P^\bullet(X) \cong (P \times_H k)^\bullet(S_X^0)$$

- $k^0_P(X) =$ homotopy classes of sections of $P \times_H k_0 \to X$
Twistings

- An $H$-action on $k$ is a homomorphism $H \rightarrow Iso(k) \subset Aut(k)$
  - $Iso(k) =$ group of isomorphisms $k \rightarrow k$
  - $Aut(k) =$ monoid of equivalences $k \rightarrow k$
- Sensitive to precise model for $k$: technicalities

- Take $k$ to be a ring spectrum. Define $GL_1(k) \subset Aut(k)$ to be the monoid of unit components of $k_0$. Pullback

\[
\begin{array}{ccc}
GL_1(k) & \rightarrow & k_0 \\
\downarrow & & \downarrow \\
\pi_0(k)^\times & \rightarrow & \pi_0(k)
\end{array}
\]

- Take $H \rightarrow GL_1(k)$: “twisting”
Twisted $K$-theory

- $GL_1(K) \simeq [\mathbb{Z}/2 \times K(\mathbb{Z}, 2)] \times BSU\otimes$
  (Detail: term in $[-]$ splits as $H$-spaces, not $\infty$ loop spaces)
- Projective unitary group $H = PU(\mathcal{H})$ model for $K(\mathbb{Z}, 2)$
- $H \longrightarrow GL_1(K) \subset K_0$ in a good model for $K_0 \simeq BU \times \mathbb{Z}$.

Details: Atiyah-Segal
(Equivariant details: J.-L. Tu, P. Xu, C. Laurent-Gengoux)
Twisted $K$-theory

\[ [X, BH] \cong H^3(X; \mathbb{Z}) \]

- Fix $p : X \to BH$ with pullback $P$
- $K^P_\bullet(X)$ is $K$-cohomology twisted by $p \in H^3(X; \mathbb{Z})$
- $K_\bullet^P(X)$ is $K$-homology twisted by $p \in H^3(X; \mathbb{Z})$

Methodology of modern algebraic topology is available

Complements monumental Freed-Hopkins-Teleman work
There is an equivariant elaboration of everything I’ve said.
There is a fiberwise elaboration of nearly everything I’ve said, starting from a bicategory $\mathcal{E}_B$ whose 0-cells are spaces over $B$. (This is analogous to working with algebras over a ring rather than just with rings, or working over a general base scheme.)
Equivariant and fiberwise duality of all types works similarly.
The combination gives fiberwise equivariant Poincaré duality and generalizes the Wirthmüller and Adams isomorphisms in equivariant stable homotopy theory, as pioneered by Po Hu.
Equivariant parametrized spectra encode equivariant twisted homology and cohomology theories, in particular $K$-theory.
Parametrized spectra encode lots of interesting topology!