What are parametrized spectra good for?

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Outline

- Background: history and naive basic definitions
- Parametrized duality
    - Categorical duality and transfer maps
    - Bicategorical and Costenoble-Waner duality
      (This is parametrized Spanier-Whitehead duality)
    - Homotopical Poincaré duality
    - The homotopical meaning of orientations

Homology and cohomology

- Parametrized homology and cohomology
- Relationship with nonparametrized theories
- Classical Poincaré duality
- Twisted homology and cohomology theories

What are parametrized spaces?

- Answer: Maps $p\colon E \longrightarrow B$
- Better: Maps p and sections $s\colon B \longrightarrow E$
  (Stable algebraic topologists want basepoints)

  These are often called "ex-spaces"

One focus: fiberwise 1-point compactifications
of vector bundles and their Thom spaces E/sB.

What are spectra?

- Naively, sequences of spaces T_n and maps $\Sigma T_n \longrightarrow T_{n+1}$
- Ω-spectra: $T_n \xrightarrow{\cong} \Omega T_{n+1}$

  We will be naive today: pretend everything is for the best in this best of all possible worlds, and ignore all technicalities

- Spaces to spectra: $\Sigma^\infty X$

$$\Sigma(\Sigma^n X) \xrightarrow{\cong} \Sigma^{n+1} X$$

- Ω-spectra to spaces: $\Omega^\infty T$

$$T \mapsto T_0$$

What are spectra good for?

- Spanier-Whitehead duality [1958]
- Cobordism theory [1959] (Milnor; MSO)
- Stable homotopy theory [1959] (Adams; ASS)
- Generalized cohomology theories [1960]
  (Atiyah-Hirzebruch; K-theory, AHSS)
- Generalized homology theories [1962] (G.W. Whitehead)
- Stable homotopy category [1964] (Boardman)

What are parametrized spectra?

- ex-spectrum E over B: ex-spaces E_n and maps of ex-spaces

$$\Sigma_B E_n \longrightarrow E_{n+1}$$

- Ω-ex-spectrum E:

$$E_n \xrightarrow{\simeq} \Omega_B E_{n+1}$$

  Sections give fibers basepoints; Σ_B and Ω_B restrict to suspension and loops on fibers

- Ex-spaces to ex-spectra: $\Sigma_B^\infty X$

$$\Sigma_B(\Sigma_B^n X) \xrightarrow{\cong} \Sigma_B^{n+1} X$$

- Ω-ex-spectra to spaces: $\Omega_B^\infty E$

$$E \mapsto E_0$$

Basic constructions are done spacewise and fiberwise

Precursors and sources

- J.C. Becker and D.H. Gottlieb
  The transfer map and fiber bundles, 1975
- Monica Clapp
  Duality and transfer for parametrized spectra, 1981
- Monica Clapp and Dietrich Puppe
  The homotopy category of parametrized spectra, 1984
- Po Hu
  Duality for Smooth Families in Equivariant Stable Homotopy Theory, 2003
- S.R. Costenoble and S. Waner
  Equivariant ordinary homology and cohomology theory, 2003
- J.P. May and Johann Sigurdsson
  Parametrized homotopy theory, 2006

The category $\mathscr{S}_B$ of spectra over B

- Good homotopy category: Approximate ex-spectra X by Ω-ex-spectra such that all $p\colon X_n \longrightarrow B$ are fibrations. Fibers are spectra. Weak equivalences are maps that give isomorphisms on the (stable) homotopy groups of fibers.
- $\mathrm{Ho}\mathscr{S}_B$; invert weak equivalences; maps $[X, Y]_B$
- Base change functors for $f\colon A \longrightarrow B$

$$f^*\colon \mathscr{S}_B \longrightarrow \mathscr{S}_A$$

$$f_!\colon \mathscr{S}_A \longrightarrow \mathscr{S}_B$$

$$f_*\colon \mathscr{S}_A \longrightarrow \mathscr{S}_B$$

$$[f_!X, Y]_B \cong [X, f^*Y]_A \qquad [f^*Y, X]_A \cong [Y, f_*X]_B$$

Comparing $\mathscr{S}_B$ to $\mathscr{S} = \mathscr{S}_{pt}$

$r\colon B \longrightarrow *$; for based spaces X, $r^*X = B \times X$.

$$r^*\colon \mathscr{S} \longrightarrow \mathscr{S}_B, \quad r_!\colon \mathscr{S}_B \longrightarrow \mathscr{S}, \quad r_*\colon \mathscr{S}_B \longrightarrow \mathscr{S}$$

- $r_!$ "quotients out sections spacewise"
- For a map $p\colon E \longrightarrow B$, $(E,p)_+ = E \amalg B$:

$$r_!\Sigma^\infty_B(E,p)_+ = \Sigma^\infty E_+$$

- For a spherical fibration $p\colon E \longrightarrow B$ with section s,

$$r_!\Sigma^\infty_B E = \Sigma^\infty Tp,$$

  $Tp = E/sB$. Thom spectra work similarly.
- r_* is the "global sections functor", $Sec(B,E)$ spacewise

Symmetric monoidal category

- Fiberwise external smash product $X \barwedge Y$ over $A \times B$ for ex-spectra X over A and Y over B. External hom function ex-spectrum $\bar{F}(X, Y)$ over A for X over B and Y over $A \times B$.
- Internal smash product of ex-spectra over B

$$X \wedge_B Y = \Delta^*(X \barwedge Y)$$

- Unit $S_B = \Sigma_B^\infty S_B^0$, where $S_B^0 = B \times S^0$
- Internal hom function ex-spectrum $F_B(X, Y)$ over B

$$F_B(X, Y) = \bar{F}(X, \Delta_* Y)$$

$$[X \wedge_B Y, Z]_B \cong [X, F_B(Y, Z)]_B$$

Categorical Duality

Let $(\mathscr{S}, \otimes, I)$ be a symmetric monoidal category.
Objects X and Y are dual if there are maps

$$\eta : I \longrightarrow X \otimes Y$$

$$\varepsilon : Y \otimes X \longrightarrow I$$

such that the composites

$$X \cong I \otimes X \xrightarrow{\eta \otimes \mathrm{id}} X \otimes Y \otimes X \xrightarrow{\mathrm{id} \otimes \varepsilon} X \otimes I \cong X$$

$$Y \cong Y \otimes I \xrightarrow{\mathrm{id} \otimes \eta} Y \otimes X \otimes Y \xrightarrow{\varepsilon \otimes \mathrm{id}} I \otimes Y \cong Y$$

are identity maps.

Given $\Delta\colon X \longrightarrow X \otimes X$, the trace of $f\colon X \longrightarrow X$ is

$$
\begin{array}{ccccc}
I & \xrightarrow{\eta} & X \otimes Y & \xrightarrow{\gamma} & Y \otimes X \\
 & & & & \big\downarrow{\scriptstyle \mathrm{id}\otimes f} \\
\big\downarrow{\scriptstyle \tau(f)} & & & & Y \otimes X \\
 & & & & \big\downarrow{\scriptstyle \mathrm{id}\otimes \Delta} \\
X & \xleftarrow{\cong} & I \otimes X & \xleftarrow{\varepsilon\otimes \mathrm{id}} & Y \otimes X \otimes X
\end{array}
$$

The transfer $\tau = \tau_X$ is $\tau(\mathrm{id_X})$.

Fiberwise duality theorem

$\mathrm{Ho}\mathscr{S}_B$ is symmetric monoidal, unit S_B.

Theorem

X is dualizable iff each fiber X_b is dualizable in $\mathrm{Ho}\mathscr{S}$.

Headed towards

- Global transfer map for fibrations
- Fiberwise transfer map for bundles
- These agree when both are defined

Transfer for fibrations

Let $p\colon E \to B$ be a fibration with finite CW fibers.

$$(E,p)_+ = E \amalg B \qquad \Delta\colon E_+ \longrightarrow E_+ \wedge_B E_+$$

$$\tau_{(E,p)_+}\colon S_B = \Sigma^\infty_B (B,\mathrm{id})_+ \longrightarrow \Sigma^\infty_B (E,p)_+$$

Apply $r_!$, $r\colon B \to *$

$$r_! S_B = \Sigma^\infty B_+, \quad r_! \Sigma^\infty_B (E,p)_+ = \Sigma^\infty E_+$$

This gives the transfer map

$$\tau_E = r_! \tau_{(E,p)_+}\colon \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+$$

It induces transfer in all homology and cohomology theories.

The bundle construction

- H a locally compact group (structural group)
- $p\colon P \longrightarrow B$ a principal H-bundle

Apply $P \times_H (-)$ spacewise to H-spectra
to get spectra over B:

$$\mathbf{P} \equiv P \times_H (-)\colon H\mathscr{S} \longrightarrow \mathscr{S}_B$$

(Specialization of a fiberwise equivariant bundle construction.)

Fiberwise transfer

- Let M be a compact H-manifold
- Let $p\colon P \longrightarrow B$ be a principal H-bundle

$\Sigma_H^\infty M$ is dualizable as an H-spectrum:

$$\tau_M\colon S_H \longrightarrow \Sigma_H^\infty M_+$$

Think of τ_M as transfer on fibers

Fiberwise transfer

Apply **P** to τ_M and then apply $r_!$:

$$\mathbf{P}\tau_M \colon \mathbf{P}S_H \longrightarrow \mathbf{P}\Sigma^\infty_H M_+$$

$$r_!\mathbf{P}S_H = \Sigma^\infty B_+ \quad r_!\mathbf{P}\Sigma^\infty_H M_+ = \Sigma^\infty E_+$$

$$\tau_E = r_!\mathbf{P}\tau_M \colon \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+$$

The functor **P** is monoidal! Therefore

Theorem

$$\tau_E = r_!\tau_{(E,p)_+} = r_!\mathbf{P}\tau_M = \tau_E$$

Costenoble-Waner duality

- Spanier-Whitehead duality is different!
- Cannot be fiberwise duality
- Due to Costenoble and Waner
- Cannot be understood without bicategorical duality

The bicategory $\mathscr{E}$

Bicategory $\mathscr{E}$ of ex-spectra

- 0-cells: spaces
- 1-cells and 2-cells: $\mathscr{E}(A, B) = \mathrm{Ho}\mathscr{S}_{B\times A}$
- Unit 1-cells $I_B = \Delta_! S_B \in \mathscr{E}(B, B)$
- Composition $\odot\colon \mathscr{E}(B, C) \times \mathscr{E}(A, B) \longrightarrow \mathscr{E}(A, C)$

$$Y \odot X = (\mathrm{id}_A \times r_B \times \mathrm{id}_C)_!(\mathrm{id}_A \times \Delta_B \times \mathrm{id}_C)^*(Y \barwedge X)$$

- $t\colon A \times B \longrightarrow B \times A$ induces involution

Analogous to rings, bimodules, maps of bimodules, with

$$Y \odot X = Y \otimes_B X$$

Bicategorical duality

Objects $X \in \mathscr{E}(A,B)$ and $Y \in \mathscr{E}(B,A)$ are dual,
X right dualizable with right dual Y, if there are maps

$$\eta : I_A \longrightarrow X \odot Y$$

$$\varepsilon : Y \odot X \longrightarrow I_B$$

such that the composites

$$X \cong I_A \odot X \xrightarrow{\eta \odot \mathrm{id}} X \odot Y \odot X \xrightarrow{\mathrm{id} \odot \varepsilon} X \odot I_B \cong X$$

$$Y \cong Y \odot I_A \xrightarrow{\mathrm{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \mathrm{id}} I_B \odot Y \cong Y$$

are identity maps.

The role of $\mathscr{E}$

Embed $\mathrm{Ho}\mathscr{S}_B$ in $\mathscr{E}$ as

$$\mathrm{Ho}\mathscr{S}_{*\times B} = \mathscr{E}(B,*)$$

and

$$\mathrm{Ho}\mathscr{S}_{B\times *} = \mathscr{E}(*,B)$$

1-cells $X\colon B \longrightarrow *$ and $tX\colon * \longrightarrow B$, respectively

$\mathscr{E}$ informs on $\mathrm{Ho}\mathscr{S}_B$ by relating it to

$$\mathrm{Ho}\mathscr{S} = \mathscr{E}(*,*)$$

and

$$\mathrm{Ho}\mathscr{S}_{B\times B} = \mathscr{E}(B,B).$$

Duality in $\mathrm{Ho}\mathscr{S}_B$ versus duality in $\mathscr{E}$

S_B is invertible, hence dualizable in $\mathrm{Ho}\mathscr{S}_B$.

This means that S_B is left dualizable:

(tS_B, S_B) is a dual pair in $\mathscr{E}$.

S_B is usually not right dualizable.

For spectra X and Y over B,

$$Y \odot tX = r_!(Y \wedge_B X) \in \mathscr{S}$$

$$tY \odot X = Y \mathbin{\bar{\wedge}} X \in \mathscr{S}_{B\times B}$$

Duality in $\mathscr{E}$ with respect to S

Unit $I_* = S \in \mathscr{E}(*, *)$.

Let (tX, Y) be a dual pair.

In $\mathrm{Ho}\mathscr{S}_B$,

$$Y \simeq D_B X \equiv F_B(X, S_B) = \bar{F}(X, \Delta_* S_B)$$

$$\eta\colon \Delta_! S_B \longrightarrow tX \odot Y \quad \varepsilon\colon Y \odot tX \longrightarrow S$$

Theorem

(tX, Y) is a dual pair iff (X, Y) is a fiberwise dual pair.

Duality in $\mathscr{E}$ with respect to $\Delta_! S_B$

Unit $I_B = \Delta_! S_B \in \mathscr{E}(B, B)$.

Let (X, tY) be a dual pair.

In Ho$\mathscr{S}_B$,

$$Y \simeq D_B^{cw} X \equiv \bar{F}(X, \Delta_! S_B)$$

$$\eta \colon S \longrightarrow X \odot tY \quad \varepsilon \colon tY \odot X \longrightarrow \Delta_! S_B$$

Right dualizable $\equiv$ Costenoble-Waner (CW) dualizable

(X, tY) is a dual pair iff (Y, tX) is a dual pair.

Parametrized Atiyah duality

Definition

An ex-space K over B is CW-dualizable if $\Sigma_B^\infty K$ is CW-dualizable.

$S_K^0 \equiv (K, \mathrm{id})_+ = K \times S^0$ is the 0-sphere ex-space over K.

Theorem (Parametrized Atiyah duality)

Let M be a smooth compact n-manifold embedded in $\mathbb{R}^q$ with normal bundle ν. Let S^ν be the one-point fiberwise compactification of ν.

Then S_M^0 is CW-dualizable with (right) dual $\Sigma_M^{-q} S^\nu$.

Proposition

If S^0_K is CW-dualizable and (K, p) is a space over B, then

$$(K, p)_+ \cong p_! S^0_K$$

is CW-dualizable over B.

A "cell" is a spectrum $\Sigma^\infty_B (D^n, p)_+$ over B; induct over cells:

Theorem

If X is a wedge summand of a finite cell spectrum over B, then X is CW-dualizable.

Duality in $\mathscr{E}$ when $A = B = *$ is Spanier-Whitehead duality in $\mathscr{S}$. If S_B is CW-dualizable, then $r_! S_B = \Sigma^\infty B_+$ is SW-dualizable, so B is a finite complex. If B is infinite, S_B cannot be CW-dualizable.

Bicategorical interpretation

Let X, Y, J be in $\mathscr{E}(B,*)$, with (X,tY) a dual pair. Then

$$J \odot tY \simeq X \triangleright J.$$

$X \triangleright J$ is notation for a bicategorical right hom functor:

$$\mathscr{E}(X, Y \triangleright Z) \cong \mathscr{E}(X \odot Y, Z) \cong \mathscr{E}(Y, Z \triangleleft X)$$

Dictionary:

$$J \odot tY = r_!(J \wedge_B Y)$$

$$X \triangleright J = r_* F_B(X, J)$$

Therefore

$$r_!(J \wedge_B Y) \simeq r_* F_B(X, J)$$

Homotopical Poincaré duality

Let M be a smooth closed manifold, $M \subset \mathbb{R}^q$.

Then $(X, tY) = (S_M, t\Sigma_M^{-q} S^\nu)$ is a dual pair.

Since $\tau \oplus \nu$ is trivial,

$$S^\tau \wedge_M S^\nu \cong S_M^q.$$

Thus $\Sigma_M^{-q} S^\nu$ is invertible in $\mathrm{Ho}\mathscr{S}_M$ with inverse $\Sigma_M^\infty S^\tau$.

With $J = k \wedge S^\tau$ (makes sense),

$$r_!(J \wedge_M Y) \simeq r_* F_M(X, J)$$

becomes

Theorem (Homotopical Poincaré duality)

For a spectrum k,

$$k \wedge M_+ \simeq S_M \triangleright (k \wedge S^\tau) = r_* F_M(S_M, k \wedge S^\tau).$$

Parametrized homology and cohomology

For spectra J and X over B, define

$$J_n(X) = \pi_n(r_!(J \wedge_B X))$$

$$\begin{aligned} J^n(X) &= \pi_{-n}(r_* F_B(X, J)) \\ &\cong [S_B^{-n}, F_B(X, J)]_B. \end{aligned}$$

$$J_\bullet(S_B) = \pi_\bullet(r_! J) \qquad J^\bullet(S_B) = \pi_{-\bullet}(r_* J)$$

The coefficient groups are genuinely different (but related)!

The "classical" case

For a spectrum k,

$$k_\bullet^B(X) \equiv k_\bullet(r_! X) \cong (r^* k)_\bullet(X)$$

$$k_B^\bullet(X) \equiv k^\bullet(r_! X) \cong (r^* k)^\bullet(X)$$

Isomorphisms are special cases of base change isomorphisms.

$f\colon B \longrightarrow A$. For $X \in \mathscr{S}_B$ and $k \in \mathscr{S}_A$,

$$k_\bullet(f_! X) \cong (f^* k)_\bullet(X) \qquad k^\bullet(f_! X) \cong (f^* k)^\bullet(X)$$

Also, for $X \in \mathscr{S}_A$ and $J \in \mathscr{S}_B$,

$$(f_! J)_\bullet(X) \cong J_\bullet(f^* X) \qquad (f_* J)^\bullet(X) \cong J^\bullet(f^* X)$$

Twisted Poincaré duality

Let (X, tY) be a dual pair.

Theorem (Costenoble-Waner duality)

$$J_\bullet(Y) \cong J^{-\bullet}(X).$$

Let M be a smooth closed n-manifold
and take $X = S_M$ and $J = k \wedge S^\tau$.

Theorem (Twisted Poincaré duality)

$$k_\bullet(M_+) \cong (k \wedge S^\tau)^{-\bullet}(S_M).$$

Orientations of bundles

The Thom complex of an n-plane bundle ξ over B is

$$T\xi = S^\xi / s(B) = r_! S^\xi.$$

Let k be a commutative ring spectrum. A k-orientation of ξ is a class $\mu \in k^n(T\xi)$ whose fiber restriction

$$\mu_b \in k^n(T\xi_b) \cong k^n(S^n) \cong k^0(S^0) = \pi_0(k)$$

is a unit in the ring $\pi_0(k)$ for each $b \in B$.

Proposition (Parametrized trivialization)

A k-orientation μ of ξ induces an equivalence of spectra over B

$$k \wedge S^\xi \simeq r^* k \wedge_B S^\xi \longrightarrow r^* k \wedge_B S^n_B \simeq k \wedge S^n_B,$$

$S^n_B = B \times S^n$, and conversely. Orientations untwist cohomology:

$$(k \wedge S^\xi)^{-\bullet}(S_B) \cong (k \wedge S^n_B)^{-\bullet}(S_B) \cong k^{n-\bullet}(B_+).$$

Homological Poincaré duality

With $B = M$, a k-orientation of M is a k-orientation of τ_M.

Theorem (Poincaré duality)

Let k be a commutative ring spectrum and M be a k-oriented smooth closed n-manifold. Then

$$k_\bullet(M_+) \cong k^{n-\bullet}(M_+).$$

Works similarly for manifolds with boundary. Similarly:

Theorem (Relative Poincaré duality)

Let L^d be a smooth closed submanifold of a smooth closed manifold M^n, both k-oriented. Then

$$k_{n-d+\bullet}(T\nu_{M,L}) \cong k^{d-\bullet}(L_+) \cong k_\bullet(L_+).$$

Homology of fibers and local coefficient systems

$b \in B$ gives $b \colon * \longrightarrow B$

A spectrum Y over B has fibers $Y_b = b^* Y$

Have a topological local system $\Pi B \longrightarrow \mathrm{Ho}\mathscr{S}$, $b \mapsto Y_b$

Local coefficient systems $\Pi B \longrightarrow \mathscr{A}b_*$ for $J \in \mathscr{S}_B$:

$$\mathscr{L}_\bullet(Y, J) \qquad b \mapsto (J_b)_\bullet(Y_b)$$

$$\mathscr{L}^\bullet(Y, J) \qquad b \mapsto (J_b)^\bullet(Y_b)$$

B a CW complex, $Y \in \mathscr{S}_B$, Serre spectral sequences

$$E^2_{p,q} = H_p(B; \mathscr{L}_q(Y, J)) \Longrightarrow J_{p+q}(Y)$$

$$E_2^{p,q} = H^p(B; \mathscr{L}^q(Y, J)) \Longrightarrow J^{p+q}(Y)$$

Other spectral sequences:

- Atiyah-Hirzebruch (specialize Serre)
- Čech (for open covers of B)
- Rothenberg-Steenrod
- Eilenberg-Moore (parametrized Künneth theorem)

Twisted homology and cohomology theories

- Take $P = EH$ in the bundle construction
- Let k be a (naive) H-spectrum
- Define $k_H = EH \times_H k$, a spectrum over BH
- For a space $p\colon X \longrightarrow BH$ over BH, define

$$k_\bullet(X,p) = (k_H)_\bullet((X,p)_+) \quad \text{and} \quad k^\bullet(X,p) = (k_H)^\bullet((X,p)_+)$$

Extrinsic: Given by theories represented by spectra over BH.

Twisted homology and cohomology theories

Form the pullback

$$\begin{array}{ccc} P & \longrightarrow & EH \\ \downarrow & & \downarrow \\ X & \underset{p}{\longrightarrow} & BH \end{array}$$

P is a principal H-bundle over X. Write

$$k^P_\bullet(X) = k_\bullet(X, p) \quad \text{and} \quad k_P^\bullet(X) = k^\bullet(X, p)$$

Depends only on the homotopy class of p.

Twisted homology and cohomology theories

Recall the space $S^0_X = X \times S^0$ over X

- $p_! S^0_X = (X, p)_+$ as a space over BH
- $P \times_H k = p^* k_H$ as a spectrum over X
- Intrinsic reinterpretation

$$k^P_\bullet(X) \cong (P \times_H k)_\bullet(S^0_X) \quad \text{and} \quad k^\bullet_P(X) \cong (P \times_H k)^\bullet(S^0_X)$$

- $k^0_P(X) =$ homotopy classes of sections of $P \times_H k_0 \longrightarrow X$

Twistings

- An H-action on k is a homomorphism $H \longrightarrow Iso(k) \subset Aut(k)$
  $Iso(k) =$ group of isomorphisms $k \longrightarrow k$
  $Aut(k) =$ monoid of equivalences $k \longrightarrow k$
  Sensitive to precise model for k: technicalities
- Take k to be a ring spectrum. Define $GL_1(k) \subset Aut(k)$ to be the monoid of unit components of k_0. Pullback

$$\begin{array}{ccc} GL_1(k) & \longrightarrow & k_0 \\ \downarrow & & \downarrow \\ \pi_0(k)^\times & \longrightarrow & \pi_0(k) \end{array}$$

- Take $H \longrightarrow GL_1(k)$: "twisting"

Twisted K-theory

- $GL_1(K) \simeq [\mathbb{Z}/2 \times K(\mathbb{Z},2)] \times BSU_{\otimes}$
  (Detail: term in [–] splits as H-spaces, not ∞ loop spaces)
- Projective unitary group $H = PU(\mathscr{H})$ model for $K(\mathbb{Z},2)$
- $H \longrightarrow GL_1(K) \subset K_0$ in a good model for $K_0 \simeq BU \times \mathbb{Z}$.

Details: Atiyah-Segal

(Equivariant details: J.-L. Tu, P. Xu, C. Laurent-Gengoux)

Twisted K-theory

$$[X, BH] \cong H^3(X; \mathbb{Z})$$

- Fix $p \colon X \longrightarrow BH$ with pullback P
- $K_P^\bullet(X)$ is K-cohomology twisted by $p \in H^3(X; \mathbb{Z})$
- $K_\bullet^P(X)$ is K-homology twisted by $p \in H^3(X; \mathbb{Z})$

Methodology of modern algebraic topology is available

Complements monumental Freed-Hopkins-Teleman work

Closing comments

- There is an equivariant elaboration of everything I've said.
- There is a fiberwise elaboration of nearly everything I've said, starting from a bicategory $\mathscr{E}_B$ whose 0-cells are spaces over B. (This is analogous to working with algebras over a ring rather than just with rings, or working over a general base scheme.)
- Equivariant and fiberwise duality of all types works similarly.
- The combination gives fiberwise equivariant Poincaré duality and generalizes the Wirthmüller and Adams isomorphisms in equivariant stable homotopy theory, as pioneered by Po Hu.
- Equivariant parametrized spectra encode equivariant twisted homology and cohomology theories, in particular K-theory.
- Parametrized spectra encode lots of interesting topology!