FROM THE BEGINNING (ROSENDAL TALK)

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First let me thank John for the opportunity to give a jet-lagged first talk. He wrote “I think it would be very nice if you could start it off with a talk about where this all began, plus whatever else you prefer to cover”. I'll give a four part talk, starting with where it began, probably ending before getting to the interesting part.

The real beginning is the Freudenthal suspension theorem, but we can jump ahead to Boardman’s definitive definition of the stable homotopy category in 1964. All people doing foundational work back then thought in terms of CW complexes, especially finite ones. That precluded taking a point-set category too seriously.

I first became convinced that spectra should be well-behaved point-set level objects for very naive reasons connected with homology operations on infinite loop spaces. I wanted the infinite little cubes operad to act naturally on the zeroth spaces of spectra. To make this true, I defined a spectrum to be a sequence of spaces $E_i$ and homeomorphisms $\tilde{\sigma}: E_i \longrightarrow \Omega E_{i+1}$. That was in 1968. I did have in mind a redevelopment of the stable homotopy category from that starting point, but nothing serious was done for some years. I was into calculations and not foundations back then.

In September of 1972, Frank Quinn posed the question of defining operad actions on that kind of point-set level spectrum. I will come back to his original motivation. In November of 1972, Nigel Ray gave a talk about Thom spectra in Chicago’s topology seminar, and we talked about how to get a good construction of them. Something clicked, and by the end of the following week I had defined $E_\infty$ ring spectra. They didn’t look like they do now, because I only defined twisted half-smash products and extended powers a few years later, but the definition was precisely equivalent to the one that is now standard.

The crucial idea was to model the definition on Thom spectra as they come in nature, starting from the Grassmannians of inner product spaces. In particular, they are coordinate-free spectra, given by spaces $TV$ and structure maps $\sigma: \Sigma^W TV \longrightarrow T(V \oplus W)$ for inner products $V$ and $W$. Actually, in retrospect, Thom spectra as we understood them in 1972 were FSP’s, functors with smash product, in a form that is the external smash product version of an orthogonal ring spectra with its internalized smash product.

Spectrifying to make the adjoint structure maps $\tilde{\sigma}$ into homeomorphisms, the result is an $E_\infty$ ring spectra. Up to language, I asked myself then
whether or not every $E_{\infty}$ ring spectrum arose as the spectrification of an orthogonal ring spectrum. I thought that the answer was, surely not, but Mandell, Schwede, Shipley, and I proved in 1999 that the answer is yes. Technology had moved on, and we proved that the two definitions, and their symmetric ring spectrum variant, are equivalent via Quillen equivalences of model categories. That is, any good definition of highly structured ring spectra gives a Quillen model category that is Quillen equivalent to the original category of $E_{\infty}$ ring spectra.

There was a conference at Northwestern in 1974 at which I proposed 18 problems and 3 conjectures in this area. Let me list a few of them to give the flavor of where we were then, compared with where we are now. Many of them, such as the very first one, are still open.

Problem 1. Does the Brown-Peterson spectrum admit a model as an $E_{\infty}$ ring spectrum?

We will hear two talks about that still open question this afternoon.

Problem 2. Are localizations and completions of $E_{\infty}$ ring spectra again $E_{\infty}$ ring spectra?

EKMM proves that Bousfield localization of an $R$-algebra $A$ at an $R$-module $E$ is a again an $R$-algebra, and similarly in the commutative case. For example, with $R = ko$, this gives a one line proof that $KO$ is a $ko$-algebra, hence an $E_{\infty}$-ring spectrum by neglect of structure.

Conjecture 1. The complex Adams conjecture holds on the infinite loop space level.

There were two proofs announced in a joint paper of Seymour and Friedlander. Seymour’s is wrong and Friedlander’s is right.

Conjecture 2. The Atiyah-Bott-Shapiro orientation $g : BSpin \to B(Spin; ko)$ is an infinite loop map.

Conjecture 3. The Sullivan orientation $\bar{g} : BSTop \to B(STop; ko[1/2])$ is an infinite loop map.

It was questions like the last one that motivated Quinn.

Problem 4. Is $\psi^r : ko[1/r] \to ko[1/r]$ a map of $E_{\infty}$ ring spectra?

Problem 5. Is Brauer lift $\hat{\lambda} : ko^\delta \to \tilde{ko}[1/q]$ a map of $E_{\infty}$ ring spectra?

I will say more about these last two problems shortly. For this conference, the most relevant of these 1974 problems is

Problem 10. Develop a theory of $E_{\infty}$ pairings of $E_{\infty}$ modules over an $E_{\infty}$ ring spectrum.
Ancient history; a book; Thom Thom spectra

1968-73: ∞ LOOP SPACE THEORY
1968: Spectra: $E = \{E_i \mid E_i \cong \Omega E_{i+1}\}$.
1972-73: $E_\infty$ ring spaces and ring spectra.
Coordinate-free spectra:

$$\{EV \mid EV \cong \Omega^W E_{V\oplus W}\}$$

$I = \text{category of fin dim inner product spaces.}$

$\mathcal{T} = \text{category of based spaces}$

$I$-functor: continuous functor

$$T: \mathcal{I} \longrightarrow \mathcal{T}$$

with commutative and associative nat. trans.

$$\omega: T \times T \longrightarrow T \circ \oplus.$$  

Inclusions on restriction to basepoints.

$I$-functors $\Rightarrow \mathcal{L}$-spaces,

$\mathcal{L}$ the linear isometries operad.
$\mathcal{I}$-monoid $G$: values in monoids.

$G = O, U, \text{Spin}, \text{String}, \text{Top}, F$, special variants $SG$.

Orthogonal spectrum $T$:
($\mathcal{I}_*$-prespectrum in 1980)

$$\Sigma^W T(V) \longrightarrow T(V \oplus W)$$

$\mathcal{I}$-FSP ($\mathcal{I}_*$-prefunctors in 1973):

$$T(V) \oplus T(W) \longrightarrow T(V \oplus W)$$

Thom spectra the original examples.

$\mathcal{I}$-FSP’s $\implies E_\infty$-ring spectra

Modern route: Left Kan extension shows $\mathcal{I}$-FSP’s are the same as orthogonal ring spectra, and they are (Quillen) equivalent to $E_\infty$ ring spectra, EKMM $S$-algebras, and symmetric ring spectra. ALL NECESSARY.
Motivation: calculations, e.g. $\pi_*(MSTOP)$.

Formalization of work of Sullivan; localize at some prime $p \neq 2$.

\[
\begin{array}{cccccc}
SF & \longrightarrow & F/Top & \longrightarrow & BSTop & \longrightarrow & BSF \\
\uparrow & & \downarrow \cong & & \downarrow \cong & & \uparrow \\
SF & \longrightarrow & BO \otimes & \longrightarrow & B(SF; kO) & \longrightarrow & BSF.
\end{array}
\]

Here $B(SF; kO)$ is the classifying space for $kO$-oriented stable spherical fibrations, and

\[BO \otimes = SL_1(kO) = SF(kO).\]

**Theorem 1.** The diagram is a commutative diagram of infinite loop spaces and infinite loop maps.

\[\infty \text{ loop spaces } J \text{ and } J \otimes:\]

\[J \longrightarrow BO \xrightarrow{\psi r - \text{id}} BO \]

and

\[J \otimes \longrightarrow BO \otimes \xrightarrow{\psi r / \text{id}} BO \otimes,\]

$(r \mod p^2$ generates the group of units).
\infty \text{ loop space } BCoker J:

\[ BCoker J \to B(SF; kO)^{c(\psi^r)}BO. \]

\[ c(\psi^r) \text{ is the “universal cannibalistic class”}. \]

\[ Coker J = \Omega BCoker J. \]

**Theorem 2.** There is an equivalence of infinite loop spaces

\[ SF \simeq J \times Coker J. \]

**Theorem 3.** There is an equivalence of infinite loop spaces

\[ B(SF; kO) \simeq BCoker J \times BO. \]

These three theorems distill a huge amount of work by many people: Boardman and Vogt; Adams; Peterson; Adams and Priddy; Madsen, Snaith, and Tornehave; Friedlander; Hodgkin and Snaith.

This reduces calculation of \( H^*(BSTop) \) to calculation of \( H^*(BCoker J) \).

Why is \( BCoker J \) an \( \infty \) loop space?

Is \( \psi^r \) a map of \( E_\infty \) ring spectra?

Discrete models from algebraic \( K \)-theory.
MULTIPLICATIVE INFINITE LOOP SPACE THEORY

0th space of an $E_\infty$ ring spectrum is an $E_\infty$ ring space. Converse up to completion of semi-ring to ring. (1975, 1983).

$E_\infty$ ring space: $(\mathcal{K}, \mathcal{L})$ an operad pair. $\mathcal{K}$: Steiner’s version of infinite little cubes operad. Monad $\mathbb{K}$ associated to $\mathcal{K}$ restricts to a monad on the category of $\mathcal{L}$-spaces with zero. An $E_\infty$ ring space is an algebra over this monad. Has two operad actions.

A bipermutative category $\mathcal{C}$ is a category with products $\oplus$ and $\otimes$; strictest version of symmetric bimonoidal. $B\mathcal{C}$ is equivalent to an $E_\infty$ ring space and so gives rise to an $E_\infty$ ring spectrum $R\mathcal{C}$.

**Example 4.** $kO$ from $O(n; \mathbb{R})$, $n \geq 0$. 
Let \( r = q^a \), \( q \) prime, where \( r \) mod \( p^2 \) generates the group of units. Allow \( p = 2 \).

Let \( \mathbb{F}_r \) be the field with \( r \) elements, \( \overline{\mathbb{F}}_q \) its algebraic closure. Bipermutative category \( \mathcal{O}(R) \) of orthogonal groups \( O(n, R) \) of a commutative ring \( R \) gives \( E_\infty \) ring spectrum \( kO(R) \). \( \infty \) loop commutative diagram:

\[
\begin{array}{c}
SF \xrightarrow{e} BO(\overline{\mathbb{F}}_q) \otimes \xrightarrow{\lambda} B(SF; kO(\overline{\mathbb{F}}_q)) \xrightarrow{\pi} BSF
\end{array}
\]

\[
\begin{array}{c}
SF \xrightarrow{e} BO \otimes \xrightarrow{\omega} B(SF; kO) \xrightarrow{\pi} B SF.
\end{array}
\]

\( \lambda \otimes \) is Brauer lift. Why an \( \infty \) loop map? Key: machine built property of \( SL_1(\mathbb{C}) \).

\( R \) an \( E_\infty \) ring spectrum. Unit components and component of identity (1973 notations)

\[
GL_1(R) = FR \quad \text{and} \quad SL_1(R) = SFR.
\]

Infinite loop spaces; spectra now denoted

\[
gl_1(R) \quad \text{and} \quad sl_1(R).
\]

\( \infty \) loop maps \( X \longrightarrow GL_1 R \)

\[
\begin{array}{c}
E_\infty \text{ ring maps } \Sigma^\infty X_+ \longrightarrow R.
\end{array}
\]
$X$ an $E_\infty$ space with $\pi_0(X) = \{n|n \geq 0\}$, associated $E_\infty$ ring spectrum $R$. Given a multiplicative set $M \subset \{n|n \geq 0\}$, let $X_M = \amalg X_m$. It is an $E_\infty$ space.

**Theorem 5.** As an $\infty$ loop space, the localization of $SL_1(R)$ at $M$ is the basepoint component of the 0th space of the spectrum associated to $X_M$.

$SL_1(R)$ comes from the additive structure, and is then given a multiplicative structure, but its localizations can be computed just from space level multiplicative structure.

Example of how this is used.

**Example 6.** Regard elements of $O(n, \mathbb{F}_r)$ as permutations of the set of $r^n$ elements. Get a map of permutative categories

$$\mathcal{O}(\mathbb{F}_r, \oplus) \longrightarrow (\mathcal{E}, \times),$$

$\mathcal{E}$=finite sets, hence $\infty$ loop map

$$J \simeq kO(\mathbb{F}_r)_0 \longrightarrow SF[1/q].$$

$SF[1/q]$ computed from $\mathcal{E}$ under $\times$, not $\amalg$. 

Frobenius automorphisms

$$\phi^r : O(n, \overline{F}_q) \to O(n, \overline{F}_q),$$

with fixed points $O(n, F_r)$. These give a map of bipermutative categories and so give an $E_\infty$ ring map

$$\phi^r : kO(\overline{F}_q) \to kO(\overline{F}_q).$$

$\phi^r$ corresponds to $\psi^r$ under Brauer lift, and $c(\phi^r)$ corresponds to $c(\psi^r)$:

$$BC\text{Coker} J \to B(SF; kO(\overline{F}_q)c(\phi^r)) \to SL_1(kO(\overline{F}_q))$$

$c(\phi^r)$ is an $\infty$ loop map, and

$$BC\text{Coker} J \simeq B(SF; kO(F_r)).$$

**Question 7.** Does the chromatic level 2 picture tell us anything about $BC\text{Coker} J$?
Coming soon to your local book seller:
Parametrized Homotopy Theory
by J.P. May and J. Sigurdsson

Everything to follow works equivariantly for any compact Lie group $G$ of equivariance. Spaces and spectra can mean $G$-spaces and $G$-spectra. Will describe things model theoretically, although model structures are insufficient for the proofs.

**BASIC STRUCTURE**

**Theorem 8.** For any space $B$, there is a proper stable topological model category $\mathcal{S}_B$ of spectra over $B$. For $b \in B$ and $X \in \mathcal{S}_B$, the fiber $X_b$ is an orthogonal spectrum. $X$ is fibrant if each projection $X(V) \to B$ is a fibration and each adjoint structure map

$$\tilde{\sigma} : X_b(V) \to \Omega^W X_b(V \oplus W)$$

is a weak equivalence.
Theorem 9. The stable homotopy category $\text{Ho}\mathcal{S}_B$ is closed symmetric monoidal under the internal smash product and function spectrum functors $\wedge_B$ and $F_B$.

**BASE CHANGE FUNCTORS**

Theorem 10. For $f: A \to B$, there is a pullback functor $f^*: \mathcal{S}_B \to \mathcal{S}_A$ with left and right adjoints $f_!$ and $f^*$.

Theorem 11. $(f_!, f^*)$ is a Quillen pair, and it is a Quillen equivalence if $f$ is an equivalence.

Theorem 12. $(f^*, f_*)$ is a Quillen pair if $f$ is a bundle with CW fibers, but not in general otherwise.
Example 13. $f: A \to B$ an inclusion.

$$f^*Y = Y|_A$$

$$(f_!X)(V) = X(V) \cup_{sA} B$$

$$f_*(X) = “skyscraper” \text{ over } A.$$ 

Example 14. $r: B \to \ast$. $X$ a based space, 

$E = (E, p, s)$ an ex-space over $B$.

$$r^*X = B \times X$$

$$r_!E = E/s(B) = \text{Thom space}$$

$$r_*E = \text{Sec}(B, E).$$

Example 15. $H \subset G$. $H\mathcal{I} \cong G\mathcal{I}_{G/H}$. 
Thom Thom spectra

Let $G$ be an $\mathcal{I}$-monoid. Bar construction

$$EG(V) = B(\ast, G(V), S^V)$$

Bundle (or quasifibration) over

$$BG(V) = B(\ast, G(V), \ast)$$

with section $s$ at $\infty$. Classical Thom spectra

$$TG(V) = EG(V)/sBG(V) = r_!EG(V).$$

Reinterpretation. $BG = \text{colim} \: BG(V)$.

$$i(V) : BG(V) \longrightarrow BG$$

$$i(V)!EG(V) = EG(V) \cup_{sBG(V)} BG$$

These give a spectrum $UG$ over $BG$, and

$$TG = r_!UG.$$
Definition 16. For $f : B \rightarrow BG$,

$$Tf = r_! f^* UG.$$ 

Here $r : B \rightarrow \ast$ is $r \circ f$, so get

$$Tf = r_! f_! f^* UG \rightarrow r_! UG = TG.$$ 

Reinterpretation of Mahowald and Gaunce Lewis construction. Properties easily proven.

$UG = U(G; S)$, $S$ the sphere spectrum. Let $X$ be an $\mathscr{I}$-$G$ orthogonal spectrum: maps

$$\xi : G(V)_+ \wedge X(V) \rightarrow X(V)$$

compatible with $\sigma$ and $G(V) \rightarrow G(W)$.

Definition 17. For an $\mathscr{I}$-$G$ spectrum $X$,

$$U(G; X)(V) = i(V)_! B(\ast, G(V), X(V))$$

gives a spectrum over $BG$, and

$$T(G; X) = r_! U(G; X).$$

If $X$ is an $\mathscr{I}$-FSP with product compatible with its $G$-action, called an $\mathscr{I}$-$G$-FSP, then $T(G; X)$ is an $\mathscr{I}$-FSP.
**Example 18.** If $G$ is group-valued, then $TG$ is an $\mathcal{I}$-$G$-FSP. Action induced by

$$G(V) \times G(V)^q \times S^V \longrightarrow G(V)^q \times S^V$$

$$(h, g_1, \ldots, g_q, v) \mapsto (hg_1 h^{-1}, \ldots, hg_q h^{-1}, hv)$$

**Definition 19.** $T(G; TG)$ is the Thom Thom spectrum of $G$. Since it is an $\mathcal{I}$-FSP, it is an orthogonal ring spectrum.

Two-sided variant starting from $B(Y, G, X)$.

The classifying space $B(SF; R)$ for $R$-oriented spherical fibrations discussed earlier is

$$B(SF; R) = B(SL_1(R), SF, *) .$$

$SF$ acts on $SL_1(R)$ since $R$ is a coordinate-free spectrum, *not* an orthogonal spectrum. Need precise 0th space functor.
HOMOLOGY and COHOMOLOGY

Definition 20. Let $E$ and $X$ be spectra over $B$. For integers $n$, define the $n$th $E$-homology and $E$-cohomology groups of $X$ by

$$E_n(X) = \pi_n(r_!(E \wedge_B X))$$

and

$$E^n(X) = \pi_{-n}(r_*F_B(X, E)).$$

Then

$$E^n X \cong [S_B^{\sim n}, F_B(X, E)]_B.$$  

Applied to $\Sigma_B^{\infty}K$, gives theories on ex-spaces $K$ over $B$. Axiomatization, representability as usual, although Adams’ variant of Brown representability does not apply.

- Twisted theories
- Bundle construction
- Twisted $K$-theory
- Fundamental groupoids visible
- Twisted $K$-theory
- Eilenberg-Moore spectral sequence is a parametrized Künneth theorem
Proposition 21. Fix $f: A \to B$. For $E \in \mathcal{I}_A$ and $X \in \mathcal{I}_B$, 

$$(f\_! E)_n(X) \cong E_n(f^* X)$$

and

$$(f_* E)^n(X) \cong E^n(f^* X).$$

For $X \in \mathcal{I}_A$ and $E \in \mathcal{I}_B$, 

$$(f^* E)_n(X) \cong E_n(f\_! X)$$

and

$$(f^* E)^n(X) \cong E^n(f\_! X).$$

Corollary 22. Let $E = r^* k$, where $k$ is a spectrum. Let $X$ be a spectrum over $B$. Then 

$$E_n(X) \cong k_n(r\_! X)$$

and

$$E^n(X) \cong k^n(r\_! X).$$

Parametrized Spanier-Whitehead duality? New Costenoble-Waner duality needed!
Back to Basics: Compatibilities

Theorem 23. \( f^*: \mathcal{S}_B \to \mathcal{S}_A \) is “closed symmetric monoidal”: 
\( f^* S_B \cong S_A \) and

\[
\begin{align*}
(1) \quad f^*(Y \land_B Z) & \cong f^*Y \land_A f^*Z \\
(2) \quad f_!(f^*Y \land_A X) & \cong Y \land_B f_!X \\
(3) \quad F_B(Y, f_*X) & \cong f_*F_A(f^*Y, X) \\
(4) \quad f^*F_B(Y, Z) & \cong F_A(f^*Y, f^*Z) \\
(5) \quad F_B(f_!X, Y) & \cong f_*F_A(X, f^*Y)
\end{align*}
\]

Theorem 24. Given a pullback

\[
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\downarrow i & & \downarrow j \\
A & \xrightarrow{f} & B
\end{array}
\]

with \( f \) or \( j \) a fibration,

\[
\begin{align*}
& j^* f_! \cong g_! i^* \quad f^* j_* \cong i_* g^* \\
& f^* j_! \cong i_! g^* \quad j^* f_* \cong g_* i^*.
\end{align*}
\]

Not true without fibration hypothesis.
EXTERNAL OPERATIONS

$X, Y, Z$ spectra over $A, B, A \times B$.

Get $X \sqcap Y$ over $A \times B$, $\bar{F}(Y, Z)$ over $A$:

$$\mathcal{S}_{A \times B}(X \sqcap Y, Z) \cong \mathcal{S}_A(X, \bar{F}(Y, Z))$$

$$X \sqcap Y \cong \pi_A^* X \sqcap_{A \times B} \pi_B^* Y$$

$$\bar{F}(Y, Z) \cong \pi_A^* F_{A \times B}(\pi_B^* Y, Z)$$

$X, Y$ ex-spectra over $B$, $\Delta : B \to B \times B$

$$X \sqcap_B Y \cong \Delta^*(X \sqcap Y)$$

$$F_B(X, Y) \cong \bar{F}(X, \Delta_* Y)$$

Quillen pairs

$$(- \sqcap Y, \bar{F}(Y, -))$$

Compatibilities with base change, e.g.

$$(f^* Y \sqcap g^* Z) \cong (f \times g)^*(Y \sqcap Z)$$
$\mathcal{K}_B$ category of ex-spaces over $B$. Adjoint

$$\Sigma^\infty_B : \mathcal{K}_B \to \mathcal{I}_B \quad \text{and} \quad \Omega^\infty_B : \mathcal{I}_B \to \mathcal{K}_B$$

Case $V = 0$ of adjoints

$$F_V : \mathcal{K}_B \to \mathcal{I}_B \quad \text{and} \quad \text{ev}_V : \mathcal{I}_B \to \mathcal{K}_B$$

where $\text{ev}_V = X(V)$.

Generating cofibrations are $F_V i$, where

$$i : S^{n-1} \to D^n$$

is a fiberwise cofibration over $B$.

Nota bene: $S_B = \Sigma^\infty_B B \times S^0$ is not among the generating cofibrations.

**Theorem 25.** $\text{HoG}\mathcal{I}_B$ is triangulated with compatible symmetric monoidal structure.
Theorem 26 (Fiberwise duality theorem). \((X\text{ fibrant})\). \(X\) is dualizable or invertible if and only if each fiber \(X_b\) is dualizable or invertible.

Dualizable in symmetric monoidal category:
\[
\eta: S_B \longrightarrow X \wedge_B Y, \quad \varepsilon: Y \wedge_B X \longrightarrow S_B
\]
such that
\[
(id \wedge \varepsilon)(\eta \wedge id) = \text{Id}_X, \quad (\varepsilon \wedge id)(id \wedge \eta) = \text{Id}_Y.
\]

\[
D_B(X) \equiv F_B(X, S_B), \quad Y \simeq D_B(X)
\]
Alternatively, \(X\) is dualizable if
\[

\nu: D_B X \wedge_B X \rightarrow F_B(X, X)
\]
is an isomorphism. Fiberwise, the map is
\[

\nu: D(X_b) \wedge X_b \rightarrow F(X_b, X_b),
\]
which gives the proof of the theorem.
Identifies dualizable and invertible guys.

**Theorem 27** (Lewis–May, Greenlees). A $G$-spectrum is dualizable if and only if it is isomorphic in $\text{Ho}G\mathcal{I}$ to a homotopy retract of a finite $G$-CW spectrum.

**Theorem 28** (Fausk, Lewis, May). A $G$-spectrum is invertible if and only if it is isomorphic in $\text{Ho}G\mathcal{I}$ to a “stable homotopy representation” $\Sigma^{-V}\Sigma^{\infty}T$, where $V$ is a representation of $G$ and $T$ is a finitely dominated based $G$-CW complex such that $T^H \simeq S^{n(H)}$ for $H \subset G$. 
Definition 29. $\Delta_X : X \to X \land_B C_X$ for some $C_X$. The trace $\tau(f)$ of $f : X \to X$ is

$$
\begin{array}{c}
S_B \xrightarrow{\eta} X \land_B D_B X \xrightarrow{\gamma} D_B X \land X \\
\tau(f) \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
C_X \cong S_B \land C_X \xleftarrow{\epsilon \land 1} D_B X \land_B X \land_B C_X
\end{array}
$$

$C_X = S_B$, $\Delta_X = \text{id}$:

$\tau(f)$ is the Lefschetz constant $\chi(f)$; $f = \text{id}$: Euler characteristic $\chi(X)$.

$C_X = X$, $\Delta_X$ a diagonal map:

$\tau_X = \tau(\text{id})$ is the transfer map of $X$.

Compatible triangulation implies additivity:

$$
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \\
\chi(Y) = \chi(X) + \chi(Z).
\end{array}
$$
Nonparametrized transfers by applying $r_!$. 

**Definition 30 (TRANSFER MAPS).** Let $p: E \to B$ be a $G$-fibration over $B$ such that each fiber $E_b$ is a homotopy retract of a finite $G_b$-CW-complex. Then $\Sigma^\infty_B(E \amalg B)$ is dualizable, and we have a transfer

$$\tau_{E \amalg B}: S_B \to \Sigma^\infty_B(E \amalg B)$$

of $G$-spectra over $B$. The usual transfer is

$$\tau_E = r_! \tau_{E \amalg B}: \Sigma^\infty B_+ \to \Sigma^\infty E_+.$$ 

“Bundle construction” leads to a fiberwise variant: insert pretransfer maps of spectra fiberwise into bundles. The two agree where both are defined.
SYMMETRIC BICATEGORIES

Bicategory \( \mathcal{C} \):
monoidal category with many objects.
0-cells = objects
Category \( \mathcal{C}(A, B) \) for each pair of 0-cells.
1-cell \( X : A \to B \) is an object of \( \mathcal{C}(A, B) \).
2-cell \( X \to Y \) is a morphism of \( \mathcal{C}(A, B) \).

\[
\odot : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)
\]
for each triple of 0-cells, unit functor
\[
U_A : \ast \to \mathcal{C}(A, A)
\]
for each 0-cell, written \( A = U_A \) by abuse.

Associative and left and right unital up to coherent natural isomorphism. That’s all.
\( \mathcal{C} \) is closed if it has left and right internal hom functors:
\[
\triangleleft : \mathcal{C}(A, C) \times \mathcal{C}(B, C)^{\text{op}} \to \mathcal{C}(A, B)
\]

\[
\triangleright : \mathcal{C}(A, B)^{\text{op}} \times \mathcal{C}(A, C) \to \mathcal{C}(B, C)
\]

\[
\mathcal{C}(X, Z \triangleleft Y) \cong \mathcal{C}(Y \odot X, Z) \cong \mathcal{C}(Y, X \triangleright Z).
\]
Definition 31. \( \mathcal{C} \) is \emph{symmetric} if it has

- A bijection \( t \) on the 0-cells of \( \mathcal{C} \) such that

\[ ttA = A. \]

- Equivalences of categories

\[ t: \mathcal{C}(A, B) \rightarrow \mathcal{C}(tB, tA) \]

- Natural isomorphism 2-cells

\[ \iota: tU_A \rightarrow U_{tA} \]

\[ \gamma: tX \odot tY \rightarrow t(Y \odot X) \]

for 1-cells \( X: A \rightarrow B \) and \( Y: B \rightarrow C \) such that appropriate coherence laws hold.

If \( \mathcal{C} \) is closed and symmetric, then

\[ X \triangleright Z \simeq t(tZ \triangleleft tX): B \rightarrow C \]

for \( X: A \rightarrow B \) and \( Z: A \rightarrow C \).
Example 32. $\mathcal{B}_R$, $R$ a commutative ring:

- 0-cells are $R$-algebras.
- $\mathcal{B}_R(A, B)$ is the category of $(B, A)$-bimodules.
- $\otimes_B$ defines the composition

$\odot: \mathcal{B}_R(B, C) \times \mathcal{B}_R(A, B) \longrightarrow \mathcal{B}_R(A, C)$

- The left and right homs are

$Z \triangleleft Y = \text{Hom}_C(Y, Z)$

$X \triangleright Z = \text{Hom}_A(X, Z)$.

- $tA$ is the opposite $R$-algebra of $A$
- For a $(B, A)$-bimodule $M$, $tM$ is $M$ regarded as a $(tA, tB)$-bimodule.
- $\gamma$ is the usual interchange isomorphism.

Example 33. DGA variant, using derived categories of bimodules.

Example 34. Structured ring spectrum variant, using stable homotopy categories.

Right home for Morita theory and variants.
DUALITY IN SYMMETRIC BICATEGORIES

Definition 35. Let

\[ X: B \to A \text{ and } Y: A \to B \]

be 1-cells. \((X, Y)\) is a dual pair if there are 2-cells

\[ \eta: A \to X \circ Y \]
\[ \varepsilon: Y \circ X \to B \]

such that the following diagrams commute in \(C(B, A)\) and \(C(A, B)\).

\[
\begin{array}{ccc}
X & \cong & A \circ X \\
\downarrow \text{id} & & \downarrow \cong \\
X & \cong & X \circ B \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \cong & (X \circ Y) \circ X \\
\downarrow \cong & & \downarrow \cong \\
X & \cong & X \circ (Y \circ X) \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \cong & Y \circ A \\
\downarrow \text{id} & & \downarrow \cong \\
Y & \cong & B \circ Y \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \cong & Y \circ (X \circ Y) \\
\downarrow \cong & & \downarrow \cong \\
Y & \cong & (Y \circ X) \circ Y \\
\end{array}
\]

We say that \(X\) is left dual to \(Y\) and \(Y\) is right dual to \(X\). Right dualizability is not equivalent to left dualizability.
**Definition 36.** For a 1-cell $X : B \to A$,

$$D_rX = X \triangleright B : A \to B.$$ 

For a 1-cell $Y : A \to B$,

$$D_\ell Y = B \triangleleft Y : B \to A.$$ 

$\varepsilon : Y \odot X \to B$ has left and right adjoints

$$\tilde{\varepsilon} : X \to D_\ell Y \quad \text{and} \quad \check{\varepsilon} : Y \to D_rX$$

such that the following diagram commutes:

\[
\begin{array}{ccc}
Y \odot D_\ell Y & \xrightarrow{\tilde{\varepsilon} \odot \text{id}} & Y \odot X \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
B & \xrightarrow{\check{\varepsilon}} & D_rX \odot X
\end{array}
\]

**Proposition 37.** If $\varepsilon : Y \odot X \to B$ is the evaluation map of a dual pair $(X, Y)$,

$$\tilde{\varepsilon} : X \to D_\ell Y \quad \text{and} \quad \check{\varepsilon} : Y \to D_rX$$

are isomorphisms.
Example 38. Let $A$ be a commutative $R$-algebra: $tA = A$. Write

$$U_A: A \rightarrow A; \quad A \text{ as an } (A, A)\text{-bimodule}$$

$$S_A: A \rightarrow R; \quad A \text{ as a left } A\text{-module}$$

$$tS_A: R \rightarrow A; \quad A \text{ as a right } A\text{-module}$$

$$tS_A \circ S_A = A \otimes_A A \cong A$$

and

$$S_A \circ tS_A = A \otimes_R A.$$ 

Let

$$\eta: U_R = R \rightarrow A \cong tS_A \circ S_A$$

and

$$\varepsilon: tS_A \circ S_A \cong A \otimes_R A \rightarrow A = U_A$$

be the unit and product on $A$. Left and right unit laws say $(tS_A, S_A)$ is a dual pair.

For $(S_A, tS_A)$ to be a dual pair, need maps

$$\eta: A \rightarrow A \otimes_R A \text{ and } \varepsilon: A \rightarrow R$$

of $(A, A)$ and $(R, R)$-bimodules such that left and right counit laws hold.
We have natural maps

\[ \mu: Z \circ D_r X \longrightarrow X \triangleright Z \]

and

\[ \nu: D_\ell Y \circ W \longrightarrow W \triangleleft Y \]

**Proposition 39.** A 1-cell \( X: B \longrightarrow A \) is right dualizable iff \( \mu \) is an isomorphism when \( Z = X \), and then \( \mu \) is an isomorphism for all \( Z \) and \( X \cong D_\ell D_r X \).

**Proposition 40.** A 1-cell \( Y: A \longrightarrow B \) is left dualizable iff \( \nu \) is an isomorphism when \( W = Y \), and then \( \nu \) is an isomorphism for all \( W \) and \( Y \cong D_r D_\ell Y \).
THE BICATEGORY \(\mathcal{E}x\)

- 0-cells are spaces, with \(tB = B\).
- \(\mathcal{E}x(A, B) = \text{Ho}\mathcal{I}_{B \times A}\) with \(t = t^*\),
  \[
  t: B \times A \cong A \times B.
  \]

- Composition
  \[
  \odot: \text{Ho}\mathcal{I}_{C \times B} \times \text{Ho}\mathcal{I}_{B \times A} \rightarrow \text{Ho}\mathcal{I}_{C \times A}
  \]
  \[
  Y \odot X = \theta_B(Y \tilde{\wedge} X)
  \]
  \(Y \tilde{\wedge} X\) is a spectrum over \(C \times B \times B \times A\).

- \(\theta_B: \text{Ho}\mathcal{I}_{C \times B \times B \times A} \rightarrow \text{Ho}\mathcal{I}_{C \times A}\)
  is
  \[
  \pi_{C \times A!}(\text{id} \times \Delta_B \times \text{id})^*.
  \]

- The unit \(U_B\) is \(\Delta_B S_B\).
- \(\theta_B\) has right adjoint \(\theta_B^* = (\text{id} \times \Delta_B \times \text{id})_* \pi^*_{C \times A}\).

  \[
  Z \triangleleft Y = \tilde{F}(Y, \theta_B^* Z): A \rightarrow B
  \]

  \[
  X \triangleright Z = \tilde{F}(X, \theta_B^* Z): B \rightarrow C
  \]

\(X: A \rightarrow B, Y: B \rightarrow C, Z: A \rightarrow C\).
TILTING SPECTRA

**Theorem 41.** For \( f : B \rightarrow A \), there are \( S_f : B \rightarrow A \) and \( tS_f : A \rightarrow B \) such that \((tS_f, S_f)\) is a dual pair and:

\[
f_!Y \simeq Y \odot tS_f
\]

\[
f_!tY \simeq S_f \odot tY
\]

\[
X \odot S_f \simeq f^*X \simeq tS_f \triangleright X
\]

\[
tS_f \odot tX \simeq f^*tX \simeq X \triangleleft S_f
\]

\[
f_*Y \simeq S_f \triangleright tY
\]

\[
f_*tX \simeq X \triangleleft S_f.
\]

For \( r : B \rightarrow * \),

\[
S_r \simeq S_B \simeq tS_r.
\]
Theorem 42. Let \((X, Y)\) be a dual pair, \(X : B \to A\) and \(Y : A \to B\), and let \(f : B \to C\) be a map. Then

\[ ((\text{id} \times f)!X, (f \times \text{id})!Y) \]

is a dual pair. With \(C = \ast\) and \(f = r\),

\[ (r!X, r!Y) \]

is a dual pair. \((A = \ast: \text{duality in } \mathcal{S})\).

Proof. This says that

\[ (X \odot tS_f, S_f \odot Y) \]

is a dual pair. Since \((tS_f, S_f)\) is a dual pair, this is formal. \qed
Have two copies $E_x(\ast, B)$ and $E_x(B, \ast)$ of $\text{Ho}G\mathcal{I}_B$ in $E_x$ and the copy $E_x(\ast, \ast)$ of $\text{Ho}G\mathcal{I}$. Also have $E_x(B, B)$ in play.

**Notations 43.** View a spectrum $X$ over $B$ as a 1-cell $B \rightarrow \ast$. Write $tX$ for $X$ viewed as a 1-cell $\ast \rightarrow B$. For $X$ and $Y$ over $B$,

$$X \odot tY : \ast \rightarrow \ast$$

is a spectrum, but

$$tY \odot X : B \rightarrow B$$

is a spectrum over $B \times B$.

**Proposition 44.** Let $X$ and $E$ be spectra over $B$. As spectra,

$$E \odot tX \simeq r!(E \wedge_B X)$$

and

$$tE \triangleleft tX \simeq r_* F_B(X, E) \simeq X \triangleright E.$$ 

Therefore

$$E_*(X) \simeq \pi_*(E \odot tX)$$

and

$$E^*(X) \simeq \pi_{-*}(X \triangleright E).$$
Duals with respect to $\ast$:

$$D_\ell X = S \lhd X : \ast \longrightarrow B$$

$$D_r tX = tX \rhd S : B \longrightarrow \ast,$$

Both are the spectrum $D_B(X)$ over $B$.

**Proposition 45.** A spectrum $X$ over $B$ is fiberwise dualizable with dual $Y$ if and only if $(tX, Y)$ is a dual pair in $\mathcal{E}x$.

Duals with respect to $B$; write $B$ for $\Delta! \mathcal{S}_B$.

$$D_\ell tX = B \lhd tX : B \longrightarrow \ast$$

$$D_r X = X \rhd B : \ast \longrightarrow B.$$

They are the same spectrum over $B$.

**Definition 46.** The Costenoble-Waner dual of $X$ in $\text{HoG}\mathcal{I}_B$ is

$$D^{CW}_B X = \overline{F}(X, B).$$

Viewed as a 1-cell in $\mathcal{E}x$,

$$D^{CW}_B X \simeq B \lhd tX = D_\ell X : B \longrightarrow \ast$$

and

$$tD^{CW}_B X \simeq X \rhd B = D_r X : \ast \longrightarrow B.$$
Definition 47. $X$ is Costenoble-Waner [CW] dualizable with dual $Y$ if $(X, tY)$ is a dual pair in $\mathcal{E}x$. Then $Y \simeq D_B^{CW} X$.

Proposition 48. $X$ is CW dualizable with dual $Y$ if and only if $Y$ is CW dualizable with dual $X$, and then

$$X \simeq D_B^{CW} D_B^{CW} X.$$

Proposition 49. If $X$ is CW dualizable and $E$ is any spectrum over $B$, then

$$r_!(E \wedge_B D_B^{CW} X) \simeq E \odot tD_B^{CW} X$$

$$\mu$$

$$r_* F_B(X, E) \simeq X \triangleright E$$

is an equivalence of spectra. Therefore

$$E_*(D_B^{CW} X) \simeq E^{-*}(X).$$
When $A = B = \ast$, duality theory in $\mathcal{E}x$ reduces to Spanier-Whitehead [SW] duality theory in $\text{Ho}\mathcal{I}$.

**Corollary 50.** If $S_B$ is CW dualizable, then $\Sigma^\infty(B_+)\}$ is SW dualizable.

*Proof.* $r!S_B \simeq \Sigma^\infty(B_+)$.

In general $S_B$ is fiberwise but not CW dualizable, whereas finite cell spectra are CW but not fiberwise dualizable.

**Theorem 51.** If a spectrum $X$ over $B$ is a retract in $\text{HoG}\mathcal{I}_B$ of a finite cell spectrum, then $X$ is CW dualizable.

Will state steps of proof. Say that a space $T$ is CW dualizable if $S_T$ is CW dualizable.

**Theorem 52** (Parametrized Atiyah duality). Any smooth compact $G$-manifold $M$ is CW dualizable.

Applying $r!$, this reproves Atiyah duality.
Corollary 53. Any $G$-space $G/H \times S^n$ is CW dualizable.

Lemma 54. $(K, p)$ a space over $B$. Then $(K, p)_+ = K \amalg B$ is isomorphic to $p!S^0_K$.

Proposition 55. If $K$ is CW dualizable and $(K, p)$ is a space over $B$, then $\Sigma^\infty_B (K, p)_+$ is CW dualizable.

Proof. $p!S_K = \Sigma^\infty_B (K, p)_+$, so done. □

Implies one cell case. Induction:

Lemma 56. The cofiber of a map between CW dualizable spectra is CW dualizable.

Lemma 57. A retract of a CW dualizable spectrum is CW dualizable.

Converse? Don’t know.

Question 58. Is $S_M$ equivalent in $\text{Ho}G\mathcal{S}_M$ to a retract of a finite cell spectrum over $M$?
Let \((F, C_F)\) be a dual pair, so that \(F\) is CW
dualizable with right dual \(C_F\). Then
\[
\mu : X \circ C_F \longrightarrow S_F \Delta X
\]
is an equivalence for \(X\) over \(F\). This is
\[
r_!(X \wedge_F C_F) \simeq r_* X.
\]
\(F = G/H\): the Wirthmüller isomorphism.

Fiberwise generalization:
Let \(G = \Gamma / \Pi\), where \(\Pi \triangleleft \Gamma\). Let \(p : E \longrightarrow B\) be a \((\Pi; \Gamma)\)-bundle. Fiber a CW dualizable
\(\Gamma\)-space \(F\) with right dual \(C_F\). Associated
principal \((\Pi; \Gamma)\)-bundle \(P \longrightarrow B\).

**Theorem 59.** Set
\[
C_p = P \times_\Pi C_F.
\]
Then
\[
(S_p, \Delta ! C_p \circ tS_p)
\]
is a dual pair and, via a duality map \(\mu\),
\[
p_!(X \wedge_E C_p) \simeq p_* X
\]
for all spectra \(X\) over \(E\).

The Adams isomorphism is a special case.