FROM THE BEGINNING (ROSENDAL TALK)

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First let me thank John for the opportunity to give a jet-lagged first talk. He wrote "I think it would be very nice if you could start it off with a talk about where this all began, plus whatever else you prefer to cover". I'll give a four part talk, starting with where it began, probably ending before getting to the interesting part.

The real beginning is the Freudenthal suspension theorem, but we can jump ahead to Boardman's definitive definition of the stable homotopy category in 1964. All people doing foundational work back then thought in terms of CW complexes, especially finite ones. That precluded taking a point-set category too seriously.

I first became convinced that spectra should be well-behaved point-set level objects for very naive reasons connected with homology operations on infinite loop spaces. I wanted the infinite little cubes operad to act naturally on the zeroth spaces of spectra. To make this true, I defined a spectrum to be a sequence of spaces E_i and homeomorphisms $\tilde{\sigma}\colon E_i \longrightarrow \Omega E_{i+1}$. That was in 1968. I did have in mind a redevelopment of the stable homotopy category from that starting point, but nothing serious was done for some years. I was into calculations and not foundations back then.

In September of 1972, Frank Quinn posed the question of defining operad actions on that kind of point-set level spectrum. I will come back to his original motivation. In November of 1972, Nigel Ray gave a talk about Thom spectra in Chicago's topology seminar, and we talked about how to get a good construction of them. Something clicked, and by the end of the following week I had defined E_∞ ring spectra. They didn't look like they do now, because I only defined twisted half-smash products and extended powers a few years later, but the definition was precisely equivalent to the one that is now standard.

The crucial idea was to model the definition on Thom spectra as they come in nature, starting from the Grassmannians of inner product spaces. In particular, they are coordinate-free spectra, given by spaces TV and structure maps $\sigma\colon \Sigma^W TV \longrightarrow T(V \oplus W)$ for inner products V and W. Actually, in retrospect, Thom spectra as we understood them in 1972 were FSP's, functors with smash product, in a form that is the external smash product version of an orthogonal ring spectra with its internalized smash product.

Spectrifying to make the adjoint structure maps $\tilde{\sigma}$ into homeomorphisms, the result is an E_∞ ring spectra. Up to language, I asked myself then whether or not every E_∞ ring spectrum arose as the spectrification of an orthogonal ring spectrum. I thought that the answer was, surely not, but Mandell, Schwede, Shipley, and I proved in 1999 that the answer is yes. Technology had moved on, and we proved that the two definitions, and their symmetric ring spectrum variant, are equivalent via Quillen equivalences of model categories. That is, any good definition of highly structured ring spectra gives a Quillen model category that is Quillen equivalent to the original category of E_∞ ring spectra.

There was a conference at Northwestern in 1974 at which I proposed 18 problems and 3 conjectures in this area. Let me list a few of them to give the flavor of where we were then, compared with where we are now. Many of them, such as the very first one, are still open.

Problem 1. Does the Brown-Peterson spectrum admit a model as an E_∞ ring spectrum?

We will hear two talks about that still open question this afternoon.

Problem 2. Are localizations and completions of E_∞ ring spectra again E_∞ ring spectra?

EKMM proves that Bousfield localization of an R-algebra A at an R-module E is a again an R-algebra, and similarly in the commutative case. For example, with $R = ko$, this gives a one line proof that KO is a ko-algebra, hence an E_∞-ring spectrum by neglect of structure.

Conjecture 1. The complex Adams onjecture holds on the infinite loop space level.

There were two proofs announced in a joint paper of Seymour and Friedlander. Seymour's is wrong and Friedlander's is right.

Conjecture 2. The Atiyah-Bott-Shapiro orientation $g\colon BSpin \longrightarrow B(Spin; ko)$ is an infinite loop map.

Conjecture 3. The Sullivan orientation $\bar{g}\colon BSTop \longrightarrow B(STop; ko[1/2])$ is an infinite loop map.

It was questions like the last one that motivated Quinn.

Problem 4. Is $\psi^r\colon ko[1/r] \longrightarrow ko[1/r]$ a map of E_∞ ring spectra?

Problem 5. Is Brauer lift $\hat{\lambda} : ko^\delta \longrightarrow \widehat{ko}[1/q]$ a map of E_∞ ring spectra?

I will say more about these last two problems shortly. For this conference, the most relevant of these 1974 problems is

Problem 10. Develop a theory of E_∞ pairings of E_∞ modules over an E_∞ ring spectrum.

Ancient history; a book; Thom Thom spectra

1968-73: ∞ LOOP SPACE THEORY
1968: Spectra: $E = \{E_i \,|\, E_i \cong \Omega E_{i+1}\}$.
1972-73: E_∞ ring spaces and ring spectra.
Coordinate-free spectra:

$$\{EV \,|\, EV \cong \Omega^W E_{V\oplus W}\}$$

$\mathscr{I}$ = category of fin dim inner product spaces.

$\mathscr{T}$ = category of based spaces

$\mathscr{I}$-functor: continuous functor

$$T \colon \mathscr{I} \longrightarrow \mathscr{T}$$

with commutative and associative nat. trans.

$$\omega \colon T \times T \longrightarrow T \circ \oplus.$$

Inclusions on restriction to basepoints.

$$\mathscr{I}\text{-functors} \implies \mathscr{L}\text{-spaces},$$

$\mathscr{L}$ the linear isometries operad.

$\mathscr{I}$-monoid G: values in monoids.

$G = O$, U, $Spin$, $String$, Top, F, special variants SG.

Orthogonal spectrum T:
($\mathscr{I}_*$-prespectrum in 1980)

$$\Sigma^W T(V) \longrightarrow T(V \oplus W)$$

$\mathscr{I}$-FSP ($\mathscr{I}_*$-prefunctors in 1973):

$$T(V) \oplus T(W) \longrightarrow T(V \oplus W)$$

Thom spectra the original examples.

$$\mathscr{I}\text{-FSP's} \implies E_\infty\text{-ring spectra}$$

Modern route: Left Kan extension shows $\mathscr{I}$-FSP's are the same as orthogonal ring spectra, and they are (Quillen) equivalent to E_∞ ring spectra, EKMM S-algebras, and symmetric ring spectra. ALL NECESSARY.

Motivation: calculations, e.g. $\pi_*(MSTOP)$.

Formalization of work of Sullivan;
localize at some prime $p \neq 2$.

$$\begin{tikzcd}
SF \arrow[r] \arrow[d, equal] & F/Top \arrow[r] \arrow[d, "\simeq"] & BSTop \arrow[r] \arrow[d, "\simeq"] & BSF \arrow[d, equal] \\
SF \arrow[r] & BO_{\otimes} \arrow[r] & B(SF;kO) \arrow[r] & BSF.
\end{tikzcd}$$

Here $B(SF;kO)$ is the classifying space for kO-oriented stable spherical fibrations, and

$$BO_{\otimes} = SL_1(kO) = SF(kO).$$

Theorem 1. *The diagram is a commutative diagram of infinite loop spaces and infinite loop maps.*

∞ loop spaces J and $J_{\otimes}$:

$$J \longrightarrow BO \xrightarrow{\psi^r - \mathrm{id}} BO$$

and

$$J_{\otimes} \longrightarrow BO_{\otimes} \xrightarrow{\psi^r/\mathrm{id}} BO_{\otimes},$$

(r mod p^2 generates the group of units).

∞ loop space $BCokerJ$:

$$BCokerJ \longrightarrow B(SF;kO) \xrightarrow{c(\psi^r)} BO_{\otimes}.$$

$c(\psi^r)$ is the "universal cannibalistic class".

$$CokerJ = \Omega BCokerJ.$$

Theorem 2. *There is an equivalence of infinite loop spaces*

$$SF \simeq J \times CokerJ.$$

Theorem 3. *There is an equivalence of infinite loop spaces*

$$B(SF;kO) \simeq BCokerJ \times BO.$$

These three theorems distill a huge amount of work by many people: Boardman and Vogt; Adams; Peterson; Adams and Priddy; Madsen, Snaith, and Tornehave; Friedlander; Hodgkin and Snaith.

This reduces calculation of $H^*(BSTop)$ to calculation of $H^*(BCokerJ)$.

Why is $BCokerJ$ an ∞ loop space?
Is ψ^r a map of E_∞ ring spectra?
Discrete models from algebraic K-theory.

MULTIPLICATIVE INFINITE LOOP SPACE THEORY

0th space of an E_∞ ring spectrum is an E_∞ ring space. Converse up to completion of semi-ring to ring. (1975, 1983).

E_∞ ring space: $(\mathscr{K}, \mathscr{L})$ an operad pair. $\mathscr{K}$: Steiner's version of infinite little cubes operad. Monad $\mathbb{K}$ associated to $\mathscr{K}$ restricts to a monad on the category of $\mathscr{L}$-spaces with zero. An E_∞ ring space is an algebra over this monad. Has two operad actions.

A bipermutative category $\mathscr{C}$ is a category with products $\oplus$ and $\otimes$; strictest version of symmetric bimonoidal. $B\mathscr{C}$ is equivalent to an E_∞ ring space and so gives rise to an E_∞ ring spectrum $R\mathscr{C}$.

Example 4. kO from $O(n;\mathbb{R})$, $n \geq 0$.

Let $r = q^a$, q prime, where r mod p^2 generates the group of units. Allow $p = 2$.

Let $\mathbb{F}_r$ be the field with r elements, $\bar{\mathbb{F}}_q$ its algebraic closure. Bipermutative category $\mathscr{O}(R)$ of orthogonal groups $O(n,R)$ of a commutative ring R gives E_∞ ring spectrum $kO(R)$. ∞ loop commutative diagram:

$$\begin{tikzcd}
SF \arrow[r, "e"] \arrow[d, equal] & BO(\bar{\mathbb{F}}_q)_\otimes \arrow[r] \arrow[d, "\lambda_\otimes", "\simeq"'] & B(SF;kO(\bar{\mathbb{F}}_q)) \arrow[r, "\pi"] \arrow[d, "\omega", "\simeq"'] & BSF \arrow[d, equal] \\
SF \arrow[r, "e"'] & BO_\otimes \arrow[r] & B(SF;kO) \arrow[r, "\pi"'] & BSF.
\end{tikzcd}$$

$\lambda_\otimes$ is Brauer lift. Why an ∞ loop map? Key: machine built property of $SL_1(R\mathscr{C})$.

R an E_∞ ring spectrum. Unit components and component of identity (1973 notations)

$$GL_1(R) = FR \quad \text{and} \quad SL_1(R) = SFR.$$

Infinite loop spaces; spectra now denoted

$$gl_1(R) \quad \text{and} \quad sl_1(R).$$

$$\infty \text{ loop maps } X \longrightarrow GL_1 R$$

$$\longleftrightarrow$$

$$E_\infty \text{ ring maps } \Sigma^\infty X_+ \longrightarrow R.$$

X an E_∞ space with $\pi_0(X) = \{n | n \geq 0\}$, associated E_∞ ring spectrum R. Given a multiplicative set $M \subset \{n | n \geq 0\}$, let $X_M = \amalg X_m$. It is an E_∞ space.

Theorem 5. *As an ∞ loop space, the localization of $SL_1(R)$ at M is the base-point component of the 0th space of the spectrum associated to X_M.*

$SL_1(R)$ comes from the *additive* structure, and is then given a *multiplicative* structure, but its localizations can be computed just from space level multiplicative structure.

Example of how this is used.

Example 6. Regard elements of $O(n, \mathbb{F}_r)$ as permutations of the *set* of r^n elements. Get a map of permutative categories

$$\mathscr{O}(\mathbb{F}_r, \oplus) \longrightarrow (\mathscr{E}, \times),$$

$\mathscr{E}$=finite sets, hence ∞ loop map

$$J \simeq kO(\mathbb{F}_r)_0 \longrightarrow SF[1/q].$$

$SF[1/q]$ computed from $\mathscr{E}$ under $\times$, not $\amalg$.

Frobenius automorphisms

$$\phi^r : O(n, \bar{\mathbb{F}}_q) \longrightarrow O(n, \bar{\mathbb{F}}_q),$$

with fixed points $O(n, \mathbb{F}_r)$. These give a map of bipermutative categories and so give an E_∞ ring map

$$\phi^r \colon kO(\bar{\mathbb{F}}_q) \longrightarrow kO(\bar{\mathbb{F}}_q).$$

ϕ^r corresponds to ψ^r under Brauer lift, and $c(\phi^r)$ corresponds to $c(\psi^r)$:

$$BCoker J \longrightarrow B(SF; kO(\bar{\mathbb{F}}_q) \xrightarrow{c(\phi^r)} SL_1(kO(\bar{\mathbb{F}}_q))$$

$c(\phi^r)$ *is* an ∞ loop map, and

$$BCoker J \simeq B(SF; kO(\mathbb{F}_r)).$$

Question 7. Does the chromatic level 2 picture tell us anything about $BCoker J$?

Coming soon to your local book seller:
Parametrized Homotopy Theory
by J.P. May and J. Sigurdsson

Everything to follow works equivariantly for any compact Lie group G of equivariance. Spaces and spectra can mean G-spaces and G-spectra. Will describe things model theoretically, although model structures are insufficient for the proofs.

BASIC STRUCTURE

Theorem 8. *For any space B, there is a proper stable topological model category $\mathscr{S}_B$ of spectra over B. For $b \in B$ and $X \in \mathscr{S}_B$, the fiber X_b is an orthogonal spectrum. X is fibrant if each projection $X(V) \longrightarrow B$ is a fibration and each adjoint structure map*

$$\tilde{\sigma}\colon X_b(V) \longrightarrow \Omega^W X_b(V \oplus W)$$

is a weak equivalence.

Theorem 9. *The stable homotopy category $Ho\mathscr{S}_B$ is closed symmetric monoidal under the internal smash product and function spectrum functors $\wedge_B$ and F_B.*

BASE CHANGE FUNCTORS

Theorem 10. *For $f\colon A \longrightarrow B$, there is a pullback functor $f^*\colon \mathscr{S}_B \longrightarrow \mathscr{S}_A$ with left and right adjoints $f_!$ and f^*.*

Theorem 11. $(f_!, f^*)$ *is a Quillen pair, and it is a Quillen equivalence if f is an equivalence.*

Theorem 12. (f^*, f_*) *is a Quillen pair if f is a bundle with CW fibers, but not in general otherwise.*

Example 13. $f\colon A \to B$ an inclusion.

$$f^*Y = Y|_A$$

$$(f_!X)(V) = X(V) \cup_{sA} B$$

$$f_*(X) = \text{“skyscraper” over } A.$$

Example 14. $r\colon B \to *$. X a based space,

$E = (E, p, s)$ an ex-space over B.

$$r^*X = B \times X$$

$$r_!E = E/s(B) = \text{Thom space}$$

$$r_*E = \mathrm{Sec}(B, E).$$

Example 15. $H \subset G$. $H\mathscr{S} \cong G\mathscr{S}_{G/H}$.

Thom Thom spectra

Let G be an $\mathscr{I}$-monoid. Bar construction

$$EG(V) = B(*, G(V), S^V)$$

Bundle (or quasifibration) over

$$BG(V) = B(*, G(V), *)$$

with section s at ∞. Classical Thom spectra

$$TG(V) = EG(V)/sBG(V) = r_! EG(V).$$

Reinterpretation. $BG = \operatorname{colim} BG(V)$.

$$i(V)\colon BG(V) \longrightarrow BG$$

$$i(V)_! EG(V) = EG(V) \cup_{sBG(V)} BG$$

These give a spectrum UG over BG, and

$$TG = r_! UG.$$

$$B(*, G(V), S^{V\oplus W}) \longrightarrow B(*, G(V\oplus W), S^{V\oplus W})$$

induces required structural maps

$$\sigma\colon \Sigma^W_{BG} U(V) \longrightarrow U(V \oplus W)$$

Definition 16. For $f\colon B \longrightarrow BG$,

$$Tf = r_! f^* UG.$$

Here $r\colon B \longrightarrow *$ is $r \circ f$, so get

$$Tf = r_! f_! f^* UG \longrightarrow r_! UG = TG.$$

Reinterpretation of Mahowald and Gaunce Lewis construction. Properties easily proven.

$UG = U(G; S)$, S the sphere spectrum. Let X be an $\mathscr{I}$-G orthogonal spectrum: maps

$$\xi\colon G(V)_+ \wedge X(V) \longrightarrow X(V)$$

compatible with σ and $G(V) \to G(W)$.

Definition 17. For an $\mathscr{I}$-G spectrum X,

$$U(G; X)(V) = i(V)_! B(*, G(V), X(V))$$

gives a spectrum over BG, and

$$T(G; X) = r_! U(G; X).$$

If X is an $\mathscr{I}$-FSP with product compatible with its G-action, called an $\mathscr{I}$-G-FSP, then $T(G; X)$ is an $\mathscr{I}$-FSP.

Example 18. If G is group-valued, then TG is an $\mathscr{I}$-G-FSP. Action induced by

$$G(V) \times G(V)^q \times S^V \longrightarrow G(V)^q \times S^V$$

$$(h, g_1, \ldots, g_q, v) \mapsto (hg_1h^{-1}, \ldots, hg_qh^{-1}, hv)$$

Definition 19. $T(G; TG)$ is the <u>Thom</u> <u>Thom</u> spectrum of G. Since it is an $\mathscr{I}$-FSP, it is an orthogonal ring spectrum.

Two-sided variant starting from $B(Y, G, X)$.

The classifying space $B(SF; R)$ for R-oriented spherical fibrations discussed earlier is

$$B(SF; R) = B(SL_1(R), SF, *).$$

SF acts on $SL_1(R)$ since R is a coordinate-free spectrum, *not* an orthogonal spectrum. Need precise 0th space functor.

HOMOLOGY and COHOMOLOGY

Definition 20. Let E and X be spectra over B. For integers n, define the nth E-homology and E-cohomology groups of X by

$$E_n(X) = \pi_n(r_!(E \wedge_B X))$$

and

$$E^n(X) = \pi_{-n}(r_* F_B(X, E)).$$

Then

$$E^n X \cong [S_B^{-n}, F_B(X, E)]_B.$$

Applied to $\Sigma_B^\infty K$, gives theories on ex-spaces K over B. Axiomatization, representability as usual, although Adams' variant of Brown representability does not apply.

- Twisted theories
- Bundle construction
- Twisted K -theory
- Fundamental groupoids visible
- Twisted K -theory
- Eilenberg-Moore spectral sequence

is a parametrized Künneth theorem

Proposition 21. *Fix $f\colon A \longrightarrow B$. For $E \in \mathscr{S}_A$ and $X \in \mathscr{S}_B$,*

$$(f_!E)_n(X) \cong E_n(f^*X)$$

and

$$(f_*E)^n(X) \cong E^n(f^*X).$$

For $X \in \mathscr{S}_A$ and $E \in \mathscr{S}_B$,

$$(f^*E)_n(X) \cong E_n(f_!X)$$

and

$$(f^*E)^n(X) \cong E^n(f_!X).$$

Corollary 22. *Let $E = r^*k$, where k is a spectrum. Let X be a spectrum over B. Then*

$$E_n(X) \cong k_n(r_!X)$$

and

$$E^n(X) \cong k^n(r_!X).$$

Parametrized Spanier-Whitehead duality?
New Costenoble-Waner duality needed!

BACK TO BASICS: COMPATIBILITIES

Theorem 23. *$f^*\colon \mathscr{S}_B \to \mathscr{S}_A$ is "closed symmetric monoidal": $f^*S_B \cong S_A$ and*

$$f^*(Y \wedge_B Z) \cong f^*Y \wedge_A f^*Z \tag{1}$$

$$f_!(f^*Y \wedge_A X) \cong Y \wedge_B f_!X \tag{2}$$

$$F_B(Y, f_*X) \cong f_*F_A(f^*Y, X) \tag{3}$$

$$f^*F_B(Y, Z) \cong F_A(f^*Y, f^*Z) \tag{4}$$

$$F_B(f_!X, Y) \cong f_*F_A(X, f^*Y) \tag{5}$$

Theorem 24. *Given a pullback*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "i"'] & D \arrow[d, "j"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with f or j a fibration,

$$j^*f_! \cong g_!i^* \quad f^*j_* \cong i_*g^*$$

$$f^*j_! \cong i_!g^* \quad j^*f_* \cong g_*i^*.$$

Not true without fibration hypothesis.

EXTERNAL OPERATIONS

X, Y, Z spectra over A, B, $A \times B$.

Get $X \,\bar{\wedge}\, Y$ over $A \times B$, $\bar{F}(Y, Z)$ over A:

$$\mathscr{S}_{A\times B}(X \,\bar{\wedge}\, Y, Z) \cong \mathscr{S}_A(X, \bar{F}(Y, Z))$$

$$X \,\bar{\wedge}\, Y \cong \pi_A^* X \wedge_{A\times B} \pi_B^* Y$$

$$\bar{F}(Y, Z) \cong \pi_{A\,*} F_{A\times B}(\pi_B^* Y, Z)$$

X, Y ex-spectra over B, $\Delta : B \to B \times B$

$$X \wedge_B Y \cong \Delta^*(X \,\bar{\wedge}\, Y)$$

$$F_B(X, Y) \cong \bar{F}(X, \Delta_* Y)$$

Quillen pairs

$$(- \,\bar{\wedge}\, Y,\ \bar{F}(Y, -))$$

Compatibilities with base change, e.g.

$$(f^* Y \,\bar{\wedge}\, g^* Z) \cong (f \times g)^*(Y \,\bar{\wedge}\, Z)$$

$\mathscr{K}_B$ category of ex-spaces over B. Adjoints

$$\Sigma^\infty_B \colon \mathscr{K}_B \to \mathscr{S}_B \quad \text{and} \quad \Omega^\infty_B \colon \mathscr{S}_B \to \mathscr{K}_B$$

Case $V = 0$ of adjoints

$$F_V \colon \mathscr{K}_B \to \mathscr{S}_B \quad \text{and} \quad \mathrm{ev}_V \colon \mathscr{S}_B \to \mathscr{K}_B$$

where $\mathrm{ev}_V = X(V)$.

Generating cofibrations are $F_V i$, where

$$i \colon S^{n-1}_+ \longrightarrow D^n_+$$

is a fiberwise cofibration over B.

Nota bene: $S_B = \Sigma^\infty_B B \times S^0$ is *not* among the generating cofibrations.

Theorem 25. *$HoG\mathscr{S}_B$ is triangulated with compatible symmetric monoidal structure.*

FIBERWISE DUALITY

Theorem 26 (Fiberwise duality theorem)**.** *(X fibrant). X is dualizable or invertible if and only if each fiber X_b is dualizable or invertible.*

Dualizable in symmetric monoidal category:

$\eta\colon S_B \longrightarrow X \wedge_B Y, \quad \varepsilon\colon Y \wedge_B X \longrightarrow S_B$

such that

$(\mathrm{id} \wedge \varepsilon)(\eta \wedge \mathrm{id}) = \mathrm{Id}_X, \quad (\varepsilon \wedge \mathrm{id})(\mathrm{id} \wedge \eta) = \mathrm{Id}_Y\,.$

$$D_B(X) \equiv F_B(X, S_B), \quad Y \simeq D_B(X)$$

Alternatively, X is dualizable if

$$\nu\colon D_B X \wedge_B X \to F_B(X, X)$$

is an isomorphism. Fiberwise, the map is

$$\nu\colon D(X_b) \wedge X_b \to F(X_b, X_b),$$

which gives the proof of the theorem.

Identifies dualizable and invertible guys.

Theorem 27 (Lewis–May, Greenlees)**.** *A G-spectrum is dualizable if and only if it is isomorphic in $HoG\mathscr{S}$ to a homotopy retract of a finite G-CW spectrum.*

Theorem 28 (Fausk, Lewis, May)**.** *A G-spectrum is invertible if and only if it is isomorphic in $HoG\mathscr{S}$ to a "stable homotopy representation" $\Sigma^{-V}\Sigma^{\infty}T$, where V is a representation of G and T is a finitely dominated based G-CW complex such that $T^H \simeq S^{n(H)}$ for $H \subset G$.*

TRACE and TRANSFER

Definition 29. $\Delta_X \colon X \to X \wedge_B C_X$ for some C_X. The *trace* $\tau(f)$ of $f \colon X \to X$ is

$$\begin{tikzcd} S_B \arrow[r, "\eta"] \arrow[d, "\tau(f)"'] & X \wedge_B D_B X \arrow[rr, "\gamma"] & & D_B X \wedge X \arrow[d, "D_B f \wedge \Delta_X"] \\ C_X & S_B \wedge C_X \arrow[l, "\cong"'] & & D_B X \wedge_B X \wedge_B C_X \arrow[ll, "\epsilon \wedge 1"'] \end{tikzcd}$$

$C_X = S_B$, $\Delta_X = \mathrm{id}$:

$\tau(f)$ is the *Lefschetz constant* $\chi(f)$;
$f = \mathrm{id}$: *Euler characteristic* $\chi(X)$.

$C_X = X$, Δ_X a diagonal map:

$\tau_X = \tau(\mathrm{id})$ is the *transfer map* of X.

Compatible triangulation implies additivity:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

$$\chi(Y) = \chi(X) + \chi(Z).$$

Nonparametrized transfers by applying $r_!$.

Definition 30 (TRANSFER MAPS)**.** Let $p\colon E \longrightarrow B$ be a G-fibration over B such that each fiber E_b is a homotopy retract of a finite G_b-CW-complex. Then $\Sigma^\infty_B(E \amalg B)$ is dualizable, and we have a transfer

$$\tau_{E\amalg B}\colon S_B \to \Sigma^\infty_B(E \amalg B)$$

of G-spectra over B. The usual transfer is

$$\tau_E = r_!\tau_{E\amalg B}\colon \Sigma^\infty B_+ \to \Sigma^\infty E_+.$$

"Bundle construction" leads to a fiberwise variant: insert pretransfer maps of spectra fiberwise into bundles. The two agree where both are defined.

SYMMETRIC BICATEGORIES

Bicategory $\mathscr{C}$:
monoidal category with many objects.
0-cells = objects
Category $\mathscr{C}(A,B)$ for each pair of 0-cells.
1-cell $X\colon A \longrightarrow B$ is an object of $\mathscr{C}(A,B)$.
2-cell $X \longrightarrow Y$ is a morphism of $\mathscr{C}(A,B)$.

$$\odot\colon \mathscr{C}(B,C) \times \mathscr{C}(A,B) \longrightarrow \mathscr{C}(A,C)$$

for each triple of 0-cells, unit functor

$$U_A\colon * \longrightarrow \mathscr{C}(A,A)$$

for each 0-cell, written $A = U_A$ by abuse.

Associative and left and right unital up to coherent natural isomorphism. That's all.

$\mathscr{C}$ is *closed* if it has left and right internal hom functors:

$$\triangleleft\colon \mathscr{C}(A,C) \times \mathscr{C}(B,C)^{\mathrm{op}} \longrightarrow \mathscr{C}(A,B)$$

$$\triangleright\colon \mathscr{C}(A,B)^{\mathrm{op}} \times \mathscr{C}(A,C) \longrightarrow \mathscr{C}(B,C)$$

$$\mathscr{C}(X, Z \triangleleft Y) \cong \mathscr{C}(Y \odot X, Z) \cong \mathscr{C}(Y, X \triangleright Z).$$

Definition 31. $\mathscr{C}$ is *symmetric* if it has

- A bijection t on the 0-cells of $\mathscr{C}$ such that

$$ttA = A.$$

- Equivalences of categories

$$t\colon \mathscr{C}(A,B) \longrightarrow \mathscr{C}(tB,tA)$$

- Natural isomorphism 2-cells

$$\iota\colon tU_A \longrightarrow U_{tA}$$

$$\gamma\colon tX \odot tY \longrightarrow t(Y \odot X)$$

for 1-cells $X\colon A \longrightarrow B$ and $Y\colon B \longrightarrow C$ such that appropriate coherence laws hold.

If $\mathscr{C}$ is closed and symmetric, then

$$X \triangleright Z \simeq t(tZ \triangleleft tX)\colon B \longrightarrow C$$

for $X\colon A \longrightarrow B$ and $Z\colon A \longrightarrow C$

Example 32. $\mathscr{B}_R$, R a commutative ring:
• 0-cells are R-algebras.
• $\mathscr{B}_R(A,B)$ is the category of (B,A)-bimodules.
• $\otimes_B$ defines the composition

$$\odot\colon \mathscr{B}_R(B,C)\times\mathscr{B}_R(A,B)\longrightarrow\mathscr{B}_R(A,C)$$

• The left and right homs are

$$Z\triangleleft Y = \mathrm{Hom}_C(Y,Z)$$

$$X\triangleright Z = \mathrm{Hom}_A(X,Z).$$

• tA is the opposite R-algebra of A
• For a (B,A)-bimodule M, tM is M regarded as a (tA,tB)-bimodule.
• γ is the usual interchange isomorphism.

Example 33. DGA variant, using derived categories of bimodules.

Example 34. Structured ring spectrum variant, using stable homotopy categories.

Right home for Morita theory and variants.

DUALITY IN SYMMETRIC BICATEGORIES

Definition 35. Let

$$X\colon B \longrightarrow A \quad \text{and} \quad Y\colon A \longrightarrow B$$

be 1-cells. (X, Y) is a *dual pair* if there are 2-cells

$$\eta\colon A \longrightarrow X \odot Y$$

$$\varepsilon\colon Y \odot X \longrightarrow B$$

such that the following diagrams commute in $\mathscr{C}(B, A)$ and $\mathscr{C}(A, B)$.

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}"'] & A \odot X \arrow[l, "\cong"'] \arrow[r, "\eta \odot \mathrm{id}"] & (X \odot Y) \odot X \arrow[d, "\cong"] \\
X & X \odot B \arrow[l, "\cong"] & X \odot (Y \odot X) \arrow[l, "\mathrm{id} \odot \epsilon"]
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[d, "\mathrm{id}"'] & Y \odot A \arrow[l, "\cong"'] \arrow[r, "\mathrm{id} \odot \eta"] & Y \odot (X \odot Y) \arrow[d, "\cong"] \\
Y & B \odot Y \arrow[l, "\cong"] & (Y \odot X) \odot Y \arrow[l, "\epsilon \odot \mathrm{id}"]
\end{tikzcd}$$

We say that X is *left dual* to Y and Y is *right dual* to X. Right dualizability is *not* equivalent to left dualizability.

Definition 36. For a 1-cell $X\colon B \longrightarrow A$,

$$D_r X = X \rhd B\colon A \longrightarrow B.$$

For a 1-cell $Y\colon A \longrightarrow B$,

$$D_\ell Y = B \lhd Y\colon B \longrightarrow A.$$

$\varepsilon\colon Y \odot X \longrightarrow B$ has left and right adjoints

$$\tilde{\varepsilon}\colon X \longrightarrow D_\ell Y \quad \text{and} \quad \tilde{\varepsilon}\colon Y \longrightarrow D_r X$$

such that the following diagram commutes:

$$\begin{tikzcd}
Y \odot D_\ell Y \arrow[dr, "\varepsilon"'] & Y \odot X \arrow[l, "\tilde{\varepsilon}\odot \mathrm{id}"'] \arrow[r, "\mathrm{id}\odot\tilde{\varepsilon}"] \arrow[d, "\varepsilon"] & D_r X \odot X \arrow[dl, "\varepsilon"] \\
 & B &
\end{tikzcd}$$

Proposition 37. *If $\varepsilon\colon Y \odot X \longrightarrow B$ is the evaluation map of a dual pair (X, Y),*

$$\tilde{\varepsilon}\colon X \longrightarrow D_\ell Y \quad \textit{and} \quad \tilde{\varepsilon}\colon Y \longrightarrow D_r X$$

are isomorphisms.

Example 38. Let A be a commutative R-algebra: $tA = A$. Write

$$U_A \colon A \longrightarrow A; \quad A \text{ as an } (A,A)\text{-bimodule}$$

$$S_A \colon A \longrightarrow R; \quad A \text{ as a left } A\text{-module}$$

$$tS_A \colon R \longrightarrow A; \quad A \text{ as a right } A\text{-module}$$

$$tS_A \odot S_A = A \otimes_A A \cong A$$

and

$$S_A \odot tS_A = A \otimes_R A.$$

Let

$$\eta \colon U_R = R \longrightarrow A \cong tS_A \odot S_A$$

and

$$\varepsilon \colon tS_A \odot S_A \cong A \otimes_R A \longrightarrow A = U_A$$

be the unit and product on A. Left and right unit laws say (tS_A, S_A) is a dual pair.

For (S_A, tS_A) to be a dual pair, need maps

$$\eta \colon A \longrightarrow A \otimes_R A \ \text{ and } \ \varepsilon \colon A \longrightarrow R$$

of (A,A) and (R,R)-bimodules such that left and right counit laws hold.

We have natural maps

$$\mu\colon Z \odot D_r X \longrightarrow X \triangleright Z$$

and

$$\nu\colon D_\ell Y \odot W \longrightarrow W \triangleleft Y$$

Proposition 39. *A 1-cell $X\colon B \longrightarrow A$ is right dualizable iff μ is an isomorphism when $Z = X$, and then μ is an isomorphism for all Z and $X \cong D_\ell D_r X$.*

Proposition 40. *A 1-cell $Y\colon A \longrightarrow B$ is left dualizable iff ν is an isomorphism when $W = Y$, and then ν is an isomorphim for all W and $Y \cong D_r D_\ell Y$.*

THE BICATEGORY $\mathscr{E}x$

- 0-cells are spaces, with $tB = B$.
- $\mathscr{E}x(A, B) = \mathrm{Ho}\mathscr{S}_{B\times A}$ with $t = t^*$,

$$t\colon B \times A \cong A \times B.$$

- Composition

$$\odot\colon \mathrm{Ho}\mathscr{S}_{C\times B} \times \mathrm{Ho}\mathscr{S}_{B\times A} \longrightarrow \mathrm{Ho}\mathscr{S}_{C\times A}$$

$$Y \odot X = \theta_B(Y \mathbin{\bar{\wedge}} X)$$

$Y \mathbin{\bar{\wedge}} X$ is a spectrum over $C \times B \times B \times A$.

$$\theta_B\colon \mathrm{Ho}\mathscr{S}_{C\times B\times B\times A} \longrightarrow \mathrm{Ho}\mathscr{S}_{C\times A}$$

is

$$\pi_{C\times A!}(\mathrm{id} \times \Delta_B \times \mathrm{id})^*.$$

- The unit U_B is $\Delta_! S_B$.
- θ_B has right adjoint $\theta_B^* = (\mathrm{id} \times \Delta_B \times \mathrm{id})_* \pi_{C\times A}^*$.

$$Z \lhd Y = \bar{F}(Y, \theta_B^* Z)\colon A \longrightarrow B$$

$$X \rhd Z = \bar{F}(X, \theta_B^* Z)\colon B \longrightarrow C$$

$X\colon A \longrightarrow B$, $Y\colon B \longrightarrow C$, $Z\colon A \longrightarrow C$.

TILTING SPECTRA

Theorem 41. *For $f\colon B \longrightarrow A$, there are*

$$S_f\colon B \longrightarrow A \quad \textit{and} \quad tS_f\colon A \longrightarrow B$$

such that (tS_f, S_f) is a dual pair and:

$$f_!Y \simeq Y \odot tS_f$$

$$f_!tY \simeq S_f \odot tY$$

$$X \odot S_f \simeq f^*X \simeq tS_f \rhd X$$

$$tS_f \odot tX \simeq f^*tX \simeq X \lhd S_f$$

$$f_*Y \simeq S_f \rhd tY$$

$$f_*tX \simeq X \lhd S_f.$$

For $r\colon B \longrightarrow *$,

$$S_r \cong S_B \cong tS_r.$$

Theorem 42. *Let (X, Y) be a dual pair, $X\colon B \longrightarrow A$ and $Y\colon A \longrightarrow B$, and let $f\colon B \longrightarrow C$ be a map. Then*

$$((\mathrm{id} \times f)_! X, (f \times \mathrm{id})_! Y)$$

is a dual pair. With $C = *$ and $f = r$,

$$(r_! X, r_! Y)$$

is a dual pair. ($A = *$: duality in $\mathscr{S}$).

Proof. This says that

$$(X \odot tS_f, S_f \odot Y)$$

is a dual pair. Since (tS_f, S_f) is a dual pair, this is formal. □

Have two copies $\mathscr{E}x(*, B)$ and $\mathscr{E}x(B, *)$ of $\mathrm{Ho}G\mathscr{S}_B$ in $\mathscr{E}x$ and the copy $\mathscr{E}x(*, *)$ of $\mathrm{Ho}G\mathscr{S}$. Also have $\mathscr{E}x(B, B)$ in play.

Notations 43. View a spectrum X over B as a 1-cell $B \longrightarrow *$. Write tX for X viewed as a 1-cell $* \longrightarrow B$. For X and Y over B,

$$X \odot tY : * \longrightarrow *$$

is a spectrum, but

$$tY \odot X : B \longrightarrow B$$

is a spectrum over $B \times B$.

Proposition 44. *Let X and E be spectra over B. As spectra,*

$$E \odot tX \simeq r_!(E \wedge_B X)$$

and

$$tE \lhd tX \simeq r_* F_B(X, E) \simeq X \rhd E.$$

Therefore

$$E_*(X) \cong \pi_*(E \odot tX)$$

and

$$E^*(X) \cong \pi_{-*}(X \rhd E).$$

Duals with respect to $*$:

$$D_\ell X = S \triangleleft X \colon * \longrightarrow B$$

$$D_r tX = tX \triangleright S \colon B \longrightarrow *,$$

Both are the spectrum $D_B(X)$ over B.

Proposition 45. *A spectrum X over B is fiberwise dualizable with dual Y if and only if (tX, Y) is a dual pair in $\mathscr{E}x$.*

Duals with respect to B; write B for $\Delta_! S_B$.

$$D_\ell tX = B \triangleleft tX \colon B \longrightarrow *$$

$$D_r X = X \triangleright B \colon * \longrightarrow B.$$

They are the same spectrum over B.

Definition 46. The *Costenoble-Waner dual* of X in $\mathrm{Ho}G\mathscr{S}_B$ is

$$D_B^{CW} X = \bar{F}(X, B).$$

Viewed as a 1-cell in $\mathscr{E}x$,

$$D_B^{CW} X \simeq B \triangleleft tX = D_\ell X \colon B \longrightarrow *$$

and

$$tD_B^{CW} X \simeq X \triangleright B = D_r X \colon * \longrightarrow B.$$

Definition 47. X is *Costenoble-Waner [CW] dualizable* with dual Y if (X, tY) is a dual pair in $\mathscr{E}x$. Then $Y \simeq D_B^{CW} X$.

Proposition 48. *X is CW dualizable with dual Y if and only if Y is CW dualizable with dual X, and then*

$$X \simeq D_B^{CW} D_B^{CW} X.$$

Proposition 49. *If X is CW dualizable and E is any spectrum over B, then*

$$\begin{array}{c} r_!(E \wedge_B D_B^{CW} X) \simeq E \odot t D_B^{CW} X \\ \Big\downarrow \mu \\ r_* F_B(X, E) \simeq X \rhd E \end{array}$$

is an equivalence of spectra. Therefore

$$E_*(D_B^{CW} X) \cong E^{-*}(X).$$

When $A = B = *$, duality theory in $\mathscr{E}x$ reduces to Spanier-Whitehead [SW] duality theory in $\mathrm{Ho}\mathscr{S}$.

Corollary 50. *If S_B is CW dualizable, then $\Sigma^\infty(B_+)$ is SW dualizable.*

Proof. $r_! S_B \simeq \Sigma^\infty(B_+)$. □

In general S_B is fiberwise but not CW dualizable, whereas finite cell spectra are CW but not fiberwise dualizable.

Theorem 51. *If a spectrum X over B is a retract in $HoG\mathscr{S}_B$ of a finite cell spectrum, then X is CW dualizable.*

Will state steps of proof. Say that a space T is CW dualizable if S_T is CW dualizable.

Theorem 52 (Parametrized Atiyah duality)**.** *Any smooth compact G-manifold M is CW dualizable.*

Applying $r_!$, this reproves Atiyah duality.

Corollary 53. *Any G-space $G/H \times S^n$ is CW dualizable.*

Lemma 54. (K, p) *a space over B. Then $(K, p)_+ = K \amalg B$ is isomorphic to $p_! S^0_K$.*

Proposition 55. *If K is CW dualizable and (K, p) is a space over B, then $\Sigma^\infty_B (K, p)_+$ is CW dualizable.*

Proof. $p_! S_K = \Sigma^\infty_B (K, p)_+$, so done. □

Implies one cell case. Induction:

Lemma 56. *The cofiber of a map between CW dualizable spectra is CW dualizable.*

Lemma 57. *A retract of a CW dualizable spectrum is CW dualizable.*

Converse? Don't know.

Question 58. Is S_M equivalent in $\mathrm{Ho}G\mathscr{S}_M$ to a retract of a finite cell spectrum over M?

Let (F, C_F) be a dual pair, so that F is CW dualizable with right dual C_F. Then

$$\mu \colon X \odot C_F \longrightarrow S_F \rhd X$$

is an equivalence for X over F. This is

$$r_!(X \wedge_F C_F) \simeq r_* X.$$

$F = G/H$: the Wirthmüller isomorphism.

Fiberwise generalization:

Let $G = \Gamma/\Pi$, where $\Pi \lhd \Gamma$. Let $p \colon E \longrightarrow B$ be a $(\Pi;\Gamma)$-bundle. Fiber a CW dualizable Γ-space F with right dual C_F. Associated principal $(\Pi;\Gamma)$-bundle $P \longrightarrow B$.

Theorem 59. *Set*

$$C_p = P \times_\Pi C_F.$$

Then

$$(S_p, \Delta_! C_p \odot tS_p)$$

is a dual pair and, via a duality map μ,

$$p_!(X \wedge_E C_p) \simeq p_* X$$

for all spectra X over E.

The Adams isomorphism is a special case.