AN OLD-FASHIONED ELEMENTARY TALK (VILLARS)

I have some beautiful brand new mathematics to show you. It is about the construction of some brave new rings and some new infinite loop spaces that I do not yet understand.

However, that's not the talk that I will give. You see, the organizers "are afraid that it might not be quite appropriate for this workshop". According to them, "A good number of the participants are not topologists at all, so probably wouldn't benefit much from it."

I had offered the organizers an alternative talk about operads in algebraic geometry. However, I do know a lot of you, and I can safely say that "A good number of the participants are not algebraic geometers at all, so probably wouldn't benefit much from that."

The really interesting problem addressed in that work is the one of understanding Voevodsky's Steenrod operations in motivic cohomology. Addressed, but not yet solved, and it couldn't be crammed into an hour in any case. Most algebraic geometers don't understand the algebraic topology behind Voevodsky's definitions, and most algebraic topologists don't understand the relevant algebraic geometry.

So, what should I talk about? Maybe I'll just start from scratch and describe a single way that operad actions leading to Steenrod operations arise in algebraic topology, algebraic geometry, and homological algebra. I will give a general theorem that includes all three as special cases. The essential point of the talk is to explain its statement and proof, together with its relevant specializations.

This idea is actually the one that led me to operads in the first place. My 1970 paper "A general algebraic approach to Steenrod operations" would have been about operads if they had been defined, but it was in fact the immediate precursor of the definition of operads the next year. Diagrams that lead to the Cartan formula and Adem relations look as follows. It was because I knew how important they were that I wrote out the general diagrams that we are all sick of in the definition of an action of an operad on an object. In algebraic contexts, these special cases of the associativity diagram are [slide].

In a sense, from the algebraic geometry perspective, this is mathematics that Godement understood in 1958 and promised to publish in his never to appear volume 2.

Fix a commutative ring R and take all chains and cochains with coefficients in R. Let L be the cosimplicial chain complex $L_n = C_*(\Delta_n)$, where C_* denotes the normalized simplicial chain complex functor from simplicial sets to chain complexes. The Eilenberg-Zilber operad is the endomorphism operad $\mathscr{Z} = \mathrm{End}(L)$.

Theorem 1. *Let F be a cosimplicial commutative DGA. Then $Hom_\Delta(L, F)$ is naturally a $\mathscr{Z}$-algebra.*

An old-fashioned elementary talk

$\mathscr{C}$ an E_∞ operad over $\mathbb{F}_p$, structure maps γ.

X a $\mathscr{C}$-algebra, action maps θ.

$$\theta\colon \mathscr{C}(p) \otimes X^p \longrightarrow X$$

Pass to homology (cochains: cohomology). Get Steenrod operations P^i.

1970: Pre-operadic framework in "A general algebraic approach to Steenrod operations".

1971: Diagrammatic definition of operads.

1972: "The geometry of iterated loop spaces".

CARTAN FORMULA

$$P^k(xy) = \sum_{i+j=k} P^i(x)P^j(y)$$

$$\begin{tikzcd}
\mathscr{C}(p) \otimes (\mathscr{C}(2) \otimes X^2)^p \arrow[r, "\mathrm{id} \otimes \theta^p"] \arrow[d, "\mathrm{shuffle}"'] & \mathscr{C}(p) \otimes X^p \arrow[dd, "\theta"] \\
\mathscr{C}(p) \otimes \mathscr{C}(2)^p \otimes X^{2p} \arrow[d, "\gamma \otimes \mathrm{id}"'] & \\
\mathscr{C}(2p) \otimes X^{2p} \arrow[r, "\theta"] & X \\
\mathscr{C}(2) \otimes \mathscr{C}(p)^2 \otimes X^{2p} \arrow[u, "\gamma \otimes \mathrm{id}"] & \\
\mathscr{C}(2) \otimes (\mathscr{C}(p) \otimes X^p)^2 \arrow[r, "\mathrm{id} \otimes \theta^2"'] \arrow[u, "\mathrm{shuffle}"] & \mathscr{C}(2) \otimes X^2 \arrow[uu, "\theta"']
\end{tikzcd}$$

ADEM RELATIONS

(Cohomological version)

If $i < pj$, then

$$P^i P^j = \sum_k (-1)^{i+k} (\text{binom coeff}) P^{i+j-k} P^k$$

$$\begin{tikzcd}
\mathscr{C}(p) \otimes (\mathscr{C}(p) \otimes X^p)^p \arrow[r, "\mathrm{id} \otimes \theta^p"] \arrow[dd, "\text{shuffle}"'] & \mathscr{C}(p) \otimes X^p \arrow[d, "\theta"] \\
 & X \\
\mathscr{C}(p) \otimes \mathscr{C}(p)^p \otimes X^{p^2} \arrow[r, "\gamma \otimes \mathrm{id}"'] & \mathscr{C}(p^2) \otimes X^{p^2} \arrow[u, "\theta"']
\end{tikzcd}$$

General diagram a small step from there.

Equivariance crucial to Steenrod operations, non-symmetric operads a simpler notion.

The classical source of E_∞ algebras

Fix a commutative ring R of coefficients.

Let $\mathrm{Ch}(R)$ be the category of $\mathbb{Z}$-graded R-cochain complexes. (Grading: $X_q = X^{-q}$)

Let $L\colon \Delta \longrightarrow \mathrm{Ch}(R)$ be the cosimplicial chain complex given by the (normalized) simplicial chains of the standard simplices,

$$n \mapsto L_n = C_*(\Delta_n),$$

regraded cohomologically.

Definition 2. The *Eilenberg-Zilber operad* $\mathscr{Z}$ is the endomorphism operad of L.

Theorem 3. *Let F be a cosimplicial commutative DGA. Then the cochain complex $Hom_\Delta(L, F)$ is a $\mathscr{Z}$-algebra.*

Hom and the endomorphism operad of L

For cosimplicial objects L,M in $\mathrm{Ch}(R)$, $\mathrm{Hom}_\Delta(L, M)$ is the equalizer in $\mathrm{Ch}(R)$:

$$\begin{tikzcd} \mathrm{Hom}_\Delta(L, M) \arrow[d] \\ \prod_n \mathrm{Hom}(L_n, M_n) \arrow[d, shift left] \arrow[d, shift right] \\ \prod_{\alpha: m\to n} \mathrm{Hom}(L_m, M_n) \end{tikzcd}$$

Parallel arrow components on (f_n) are

$$(f_n \circ L(\alpha)) \quad \text{and} \quad (L(\alpha) \circ f_m).$$

$\mathrm{Hom}_\Delta(L, -)$ is often denoted “Tot”.

$L^j \colon \Delta \longrightarrow \mathrm{Ch(R)}$ is defined by $n \mapsto L_n^{\otimes j}$.

$$\mathscr{L}(j) = \mathrm{End}(L)(j) = \mathrm{Hom}_\Delta(L, L^j).$$

Σ_j acts by permutations on L^j.

$\eta\colon R \longrightarrow \mathscr{Z}(1)$ sends 1 to (id_n).

$$\gamma : \mathscr{Z}(k)\otimes\mathscr{Z}(j_1)\otimes\cdots\otimes\mathscr{Z}(j_k) \longrightarrow \mathscr{Z}(j),$$

$j = j_1 + \cdots + j_k$, is the composite:

$$\begin{array}{c}
\mathscr{Z}(k) \otimes \mathscr{Z}(j_1) \otimes \cdots \otimes \mathscr{Z}(j_k) \\
\big\downarrow{\scriptstyle \mathrm{id}\otimes k\text{-fold }\otimes\text{-product}} \\
\mathscr{Z}(k) \otimes \mathrm{Hom}_\Delta(L^k, L^j) \\
\big\downarrow{\scriptstyle \text{twist}} \\
\mathrm{Hom}_\Delta(L^k, L^j) \otimes \mathrm{Hom}_\Delta(L, L^k) \\
\big\downarrow{\scriptstyle \text{composition}} \\
\mathscr{Z}(j).
\end{array}$$

McClure-Smith: this is a ‘functor operad’.

Proof of Theorem 2

$$\theta\colon \mathscr{Z}(j) \otimes \mathrm{Hom}_\Delta(L, F)^j \longrightarrow \mathrm{Hom}_\Delta(L, F)$$

$$\begin{array}{c}
\mathrm{Hom}_\Delta(L, L^j) \otimes \mathrm{Hom}_\Delta(L, F)^j \\
\Big\downarrow \text{id} \otimes j\text{-fold } \otimes\text{-product} \\
\mathrm{Hom}_\Delta(L, L^j) \otimes \mathrm{Hom}_\Delta(L^j, F^j) \\
\Big\downarrow \text{twist} \\
\mathrm{Hom}_\Delta(L^j, F^j) \otimes \mathrm{Hom}_\Delta(L, L^j) \\
\Big\downarrow \text{composition} \\
\mathrm{Hom}_\Delta(L, F^j) \\
\Big\downarrow \mathrm{Hom}_\Delta(\text{id},\phi) \\
\mathrm{Hom}_\Delta(L, F),
\end{array}$$

where $\phi\colon F^j \longrightarrow F$ is the unit map if $j = 0$ ($F^0 = R$), the identity if $j = 1$, and the iterated product of the DGA's F_n if $j \geq 2$.

Corollary 4. *$Hom_\Delta(L, F)$ is an E_∞-algebra.*

The Eilenberg-Zilber theorem

Let $\mathscr{C}om$ be the operad $\mathscr{C}om(j) = R$.

$C_*(\Delta_0) \cong R$, and restriction to cosimplicial level 0 gives $\varepsilon\colon \mathscr{Z} \longrightarrow \mathscr{C}om$.

Theorem 5 (Eilenberg-Zilber)**.** *The map ε is a quasi-isomorphism of operads.*

E_∞: $\mathscr{E}(j)$ is an $R[\Sigma_j]$-free resolution of R.

$\mathscr{Z}(j)_q \neq 0$ for $q < 0$ and $\mathscr{Z}(j)$ not Σ_j-free.

Proposition 6. *There is a quasi-isomorphism $\mathscr{E} \longrightarrow \mathscr{Z}$, where $\mathscr{E}$ is an E_∞-operad.*

(1) Mandell: truncate, tensor with $\mathscr{E}$.

(2) McClure-Smith: beautiful example.

(3) Tutti: Cofibrant approximation.

Example: simplicial cochains

DGA's with terms concentrated in degree 0, so zero differential, are just commutative R-algebras.

Let X be a simplicial set, $R[X]$ the free simplicial R-module generated by X, and

$$R^{[X]} = \mathrm{Hom}_R(R[X], R),$$

the dual cosimplicial R-module. Its nth term R^{X_n} is an R-algebra via the product of R. Diagonal $X_n \longrightarrow X_n \times X_n$ used implicitly.

$$C^*(X; R) \cong \mathrm{Hom}_\Delta(L, R^{[X]})$$

Proof.

$$C_*(X; R) \cong L \otimes_{\Delta^{op}} R[X]$$

In degree n, the right side is R-free on $i_n \otimes X_n$, and the differentials agree.

$$\mathrm{Hom}_R(L \otimes_{\Delta^{\mathrm{op}}} R[X], R) \cong \mathrm{Hom}_\Delta(L, R^{[X]})$$

$\square$

Example: Čech cochains of sheaves

Let X be a space, $\mathscr{U}$ an open cover indexed on an ordered set I. Let $\mathscr{U}_n$ be the set of ordered $(n+1)$-tuples

$$S = \{U_{i_0}, \ldots, U_{i_n}\}$$

(allowing repeats) of sets in $\mathscr{U}$ whose intersection U_S is non-empty. $\mathscr{U}_\bullet$ is a simplicial set. The qth face deletes the qth set. The qth degeneracy repeats the qth set.

Let $\mathscr{F}$ be a presheaf of R-modules on X. Define a cosimplicial R-module $\mathscr{F}^\bullet_{\mathscr{U}}$ by

$$\mathscr{F}^n_{\mathscr{U}} = \prod_{S \in \mathscr{U}_n} \mathscr{F}(U_S).$$

The cofaces and codegeneracies are induced by restriction maps associated to the faces and degeneracies of $\mathscr{U}_\bullet$. Čech cochains:

$$\check{\mathrm{C}}^*(\mathscr{U}, \mathscr{F}) = \mathrm{Hom}_\Delta(L, \mathscr{F}^\bullet_{\mathscr{U}}).$$

Since $C_*(\Delta_n)$ is normalized, the products implicit on the right have coordinates 0 when S contains repeats. This agrees with the usual definition of Čech cochains. Pass to colimits over refinements of covers to obain a cosimplicial R-module $\mathscr{F}^\bullet$. Then

$$\check{\mathrm{C}}^*(X, \mathscr{F}) = \mathrm{Hom}_\Delta(L, \mathscr{F}^\bullet).$$

Proposition 7. *If $\mathscr{F}$ is a presheaf of commutative R-algebras, $\mathscr{F}^\bullet_{\mathscr{U}}$ and $\mathscr{F}^\bullet$ are cosimplicial commutative R-algebras.*

Proof. The product on $\mathscr{F}^n_{\mathscr{U}}$ is

$$\begin{array}{c} \left(\prod_{S\in\mathscr{U}_n} \mathscr{F}(U_S)\right) \otimes \left(\prod_{T\in\mathscr{U}_n} \mathscr{F}(U_T)\right) \\ \downarrow \\ \prod_{S\in\mathscr{U}_n}(\mathscr{F}(U_S) \otimes \mathscr{F}(U_S)) \\ \downarrow \\ \mathscr{F}(U_S) \end{array}$$

First arrow is projection on diagonal factors. Pass to colimits for $\mathscr{F}^\bullet$. □

Get Steenrod operations when $R = \mathbb{F}_p$. Usual properties?
Cartan formula, Adem relations, but:

Remark 8. $C_{pq-i}(\Delta^q) = 0$ if $pq - i > q$ implies $P^s = 0$ for $s < 0$ in $\check{\mathrm{H}}^*(X, \mathscr{F})$. Not true in hypercohomology. $P^0 \neq \mathrm{Id}$ in $\check{\mathrm{H}}^*(X, \mathscr{F})$; rather P^0 is the Frobenius operator obtained by applying the pth power in the algebras $\mathscr{F}(U)$ to the coordinates of representative cocycles of cohomology classes.

Example: Hypercohomology

Generalize. For a presheaf $\mathscr{F}$ of cochain complexes on X, get cosimplicial cochain complexes $\mathscr{F}^\bullet_{\mathscr{U}}$ and Čech hypercochains

$$\check{\mathrm{C}}^*(\mathscr{U}, \mathscr{F}) = \mathrm{Hom}_\Delta(L, \mathscr{F}^\bullet_{\mathscr{U}}).$$

Passing to colimits over covers, get $\mathscr{F}^\bullet$ and

$$\check{\mathrm{C}}^*(X, \mathscr{F}) = \mathrm{Hom}_\Delta(L, \mathscr{F}^\bullet).$$

Proposition 9. *If $\mathscr{F}$ is a presheaf of commutative DGA's, then $\mathscr{F}^\bullet_{\mathscr{U}}$ and $\mathscr{F}^\bullet$ are cosimplicial commutative DGA's.*

Operadic generalization

We may encounter E_∞ algebras rather than commutative DGA's. For operads $\mathscr{O}$ and $\mathscr{P}$, $(\mathscr{O}\otimes\mathscr{P})(j) = \mathscr{O}(j)\otimes\mathscr{P}(j)$, with structure maps determined by those of $\mathscr{O}$ and $\mathscr{P}$.

Theorem 10. *If F is a cosimplicial $\mathscr{O}$-algebra with structure maps θ, then* $\mathrm{Hom}_\Delta(L,F)$ *is an algebra over $\mathscr{O}\otimes\mathscr{Z}$ with action maps*

$$\begin{array}{c}
\mathscr{O}(j)\otimes\mathscr{Z}(j)\otimes \mathrm{Hom}_\Delta(L,F)^j \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\xi} \\
\mathscr{O}(j)\otimes \mathrm{Hom}_\Delta(L,F^j) \\
\Big\downarrow{\scriptstyle \zeta} \\
\mathrm{Hom}_\Delta(L,\mathscr{O}(j)\otimes F^j) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_\Delta(\mathrm{id},\theta)} \\
\mathrm{Hom}_\Delta(L,F).
\end{array}$$

ξ: tensor, twist, and compose, as before.
ζ: induced from $\zeta(x\otimes f)(y) = x\otimes f(y)$,

$$\zeta\colon X\otimes \mathrm{Hom}(Y,Z)\longrightarrow \mathrm{Hom}(Y,X\otimes Z).$$

Theorem 11. *If $\mathscr{F}$ is a presheaf of $\mathscr{O}$-algebras, $\check{C}^*(\mathscr{U}, \mathscr{F}_{\mathscr{U}})$ and $\check{C}^*(X, \mathscr{F})$ are $\mathscr{O} \otimes \mathscr{Z}$-algebras.*

Proposition 12. *If $\mathscr{O}$ is acyclic, there is an E_∞ operad $\mathscr{E}$ and a quasi-isomorphism $\mathscr{E} \longrightarrow \mathscr{O} \otimes \mathscr{Z}$.*

Remark 13. Let $\mathscr{S}$ be a site. Modifying the Čech construction to deal with covers $\mathscr{U}$ of objects X in the site, everything adapts to the Čech cochain complexes of X with coefficients in sheaves on $\mathscr{S}$ of the specified algebraic types. Just replace intersections with finite limits and observe, e.g., that finite limits of $\mathscr{O}$-algebras are $\mathscr{O}$-algebras.

Example: Cocommutative Hopf algebras A

$B_n(A) = A^n$ gives n-th term of simplicial bar construction $B_\bullet(A) = B_\bullet(R, A, R)$.

$\psi \colon A \longrightarrow A \otimes A$ induces

$$B_\bullet(A) \longrightarrow B_\bullet(A \otimes A).$$

Shuffling tensor factors gives

$$B_\bullet(A \otimes A') \longrightarrow B_\bullet(A) \otimes B_\bullet(A').$$

Composing, $B_\bullet(A)$ is a simplicial coalgebra, cocommutative since A is cocommutative.

Cosimplicial cobar construction

$$C^\bullet(A) = \mathrm{Hom}_R(B_\bullet(A), R)$$

is a cosimplicial commutative R-algebra.

$$B(A) = L \otimes_{\Delta^{op}} B_\bullet(A)$$

$$C(A) = \mathrm{Hom}_\Delta(L, C^\bullet(A))$$

Steenrod operations in $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$. Used to study classical Adams spectral sequence and homotopical $\cup_i$ products.

More generally, if A is a cocommutative DG-Hopf algebra, $C^\bullet(A)$ is a cosimplicial commutative DGA. Hyperext $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ also has Steenrod operations.

Characteristic zero

Theorem 14. *Let R be a field of characteristic 0. Then E_∞ algebras are quasi-isomorphic to commutative DGA's.*

$$A \xleftarrow{\simeq} B(E,E,A) \xrightarrow{\simeq} B(Com,E,A)$$

This uses the passage from operads $\mathscr{E}$ to monads E that led to the name "operad". It is a portmanteau of "operations" and "monad".

Relationship with topological spaces

Consider connected E_∞ algebras A;
$A^q = 0$ if $q < 0$, $A^0 = R$.

Quillen-Sullivan and Mandell:

$R = \mathbb{Q}$: Homotopy category of nilpotent rational spaces embeds as a full subcategory of the homotopy category of DGA's.

Can apply rational homotopy theory to algebraic geometry (mixed Hodge structures Morgan, Hain, Navarro Aznar).

$R = \bar{\mathbb{F}}_p$: Homotopy category of nilpotent p-complete spaces of finite type embeds as a full subcategory of the homotopy category of $\mathscr{E}$-algebras, $\mathscr{E}$ an E_∞ operad.

Applications of p-adic homotopy theory to algebraic geometry?

Symmetric sequences

A permutative category $\mathscr{P}$ is a category with an associative and unital product and a natural commutativity isomorphism τ.

Definition 15. Let Σ be the category with objects $\mathbf{q} = \{1, \ldots, q\}$ and morphisms the symmetric groups. It is permutative under concatenation of sets, $(\mathbf{q}, \mathbf{r}) \mapsto \mathbf{q} + \mathbf{r}$, and induced homomorphisms $\Sigma_q \times \Sigma_r \longrightarrow \Sigma_{q+r}$; $\mathbf{0}$ is the unit and τ is given by the block transpositions $\tau_{q,r} \in \Sigma_{q+r}$.

Fix a symmetric monoidal category $(\mathscr{C}, \otimes, \kappa)$.

Symmetric sequence in $\mathscr{C}$: $F \colon \Sigma \longrightarrow \mathscr{C}$.

Example: Symmetric spectra

$\mathscr{C}$ = based spaces or simplicial sets under the smash product $\wedge$.

$$F(\mathbf{q}) \wedge S^r \longrightarrow F(\mathbf{q}+\mathbf{r})$$

Natural: $F \wedge S \longrightarrow F \circ \oplus$.

Example: Symmetric monoids in $\mathscr{C}$

$$\phi\colon F(\mathbf{q}) \otimes F(\mathbf{r}) \longrightarrow F(\mathbf{q}+\mathbf{r}),$$
$$\lambda\colon \kappa \longrightarrow T(0)$$

Associative, unital, and

$$\begin{tikzcd} F(q) \otimes F(r) \arrow[r, "\phi"] \arrow[d, "\tau"'] & F(q+r) \arrow[d, "F(\tau_{q,r})"] \\ F(r) \otimes F(q) \arrow[r, "\phi"'] & F(q+r). \end{tikzcd}$$

Symmetric ring spectra are examples.

Caterads versus PROP's

Definition 16. A *caterad in* $\mathscr{C}$ is an enriched permutative category $\mathscr{A}$ over $\mathscr{C}$ with a permutative functor $\iota\colon \Sigma \longrightarrow \mathscr{A}_0$ that is a bijection on objects.

$\mathscr{A}_0$ is the underlying category. Morphism objects $\mathscr{A}(\mathbf{p}, \mathbf{q})$ and morphism sets $\mathscr{A}_0(\mathbf{p}, \mathbf{q})$.

Portmanteau of categories and operads.

PROP: Take $\mathscr{C}$ to be sets.

Topological PROP: Take $\mathscr{C}$ to be spaces.

PACT: Take $\mathscr{C}$ to be $Ch(R)$.

Non-example: presheaf singular chains

Let $\mathscr{S} = \mathrm{Sm}/k$ be the category of smooth separated schemes of finite type over a field k. Let $\mathrm{Pre}(\mathscr{S})$ be the category of presheaves on $\mathscr{S}$. Let $\Delta^\bullet$ be the standard cosimplicial object in $\mathscr{S}$. Its nth scheme is

$$\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(\Sigma t_i - 1)),$$

with faces and degeneracies like those of the standard simplices Δ^n.

Definition 17. For a presheaf $\mathscr{F}$ on $\mathscr{S}$, define a simplicial presheaf $\mathscr{F}_\bullet$ by

$$\mathscr{F}_n(X) = \mathscr{F}(X \times \Delta^n)$$

for $X \in \mathscr{S}$, with faces and degeneracies induced by those of $\Delta^\bullet$. If $\mathscr{F}$ is Abelian, $\mathscr{F}_\bullet$ is a simplicial Abelian presheaf. Applying the simplicial chain functor to $\mathscr{F}_\bullet$ then gives the "singular chains" $C_*(\mathscr{F})$.

Now let $\mathscr{F} = \{\mathscr{F}(q)\}$ be a sequence of Abelian presheaves with 'external' pairings

$$\mathscr{F}(q)(X) \otimes \mathscr{F}(r)(Y) \xrightarrow{\phi} \mathscr{F}(q+r)(X \times Y)$$

for $X, Y \in \mathscr{S}$. These give products

$$\begin{array}{c} \mathscr{F}(q)(X \times \Delta^n) \otimes \mathscr{F}(r)(X \times \Delta^n) \\ \downarrow \\ \mathscr{F}(q+r)(X \times \Delta^n \times X \times \Delta^n). \end{array}$$

Pull back along the diagonal of $X \times \Delta^n$ to get an internal product

$$\phi\colon \mathscr{F}(q)_\bullet \otimes \mathscr{F}(r)_\bullet \longrightarrow \mathscr{F}(q+r)_\bullet$$

of simplicial Abelian presheaves.

Symmetric monoid $\mathscr{F}_\bullet$ in $\Delta^{\mathrm{op}}\mathrm{AbPre}(\mathscr{S})$.

Pass to chains and compose with the shuffle map g to get a product map of presheaves of chain complexes

$$\begin{array}{c} C_*(\mathscr{F}(q)) \otimes C_*(\mathscr{F}(r)) \\ \downarrow g \\ C_*(\mathscr{F}(q) \otimes \mathscr{F}(r)) \\ \downarrow \phi \\ C_*(\mathscr{F}(q+r)). \end{array}$$

Choosing $\mathscr{F}$ appropriately and reindexing cohomologically with a shift of grading, this is how products are defined formally on motivic cochains. The chain level product is not commutative because the pairing ϕ is not commutative. *Symmetric monoids are not commutative.* This has nothing to do with the Eilenberg-Zilber operad.

Voevodsky's Steenrod operations are like Steenrod operations: Eilenberg-MacLane objects central. But also like Dyer-Lashof operations: the shuffle chain map rather than the Eilenberg-Zilber map is used.

Operad action? Yes and no. One yes answer (Kriz-May) gives Steenrod operations. The no answer says they can't be Voevodsky's operations for dimensional reasons.

Theorem 18. *Partial commutative DG-algebras have associated quasi-isomorphic E_∞-algebras.*

Theorem 19. *Bloch's higher Chow complexes give a partial commutative DG-algebra under the intersection product.*

By a deep theorem of Suslin, the resulting E_∞-algebras usually compute Voevodsky's motivic cohomology.

There is a caterad action on Voevodsky's cochains, but the caterad known to work is not acyclic (Guillou-May).