# TOPOLOGICAL HOCHSCHILD AND CYCLIC HOMOLOGY AND ALGEBRAIC $K$-THEORY

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I would like to try to explain very roughly what topological cyclic homology is and what it is good for. This is not work of my own. It is mainly work of Ib Madsen, who was a student of mine once upon a time, and his collaborators Marcel Bökstedt and Lars Hesselholt. I will focus on work that is largely internal to TC. However, it gains interest from beautiful theorems of Randy McCarthy and Bjorn Dundas that relate TC to algebraic $K$-theory. When I suggested this title a year ago, I hoped both to learn some of this material and to carry out an idea I had for a new construction of topological cyclic homology. That idea may or may not work, but I am certainly not ready to talk about it. What I want to focus on is the role of equivariant stable homotopy theory in the study of TC. It is powerful equivariant techniques that make TC so computable. The relevant equivariant foundations were set out in work of Gaunce Lewis and myself and of John Greenlees and myself, which is my only excuse for talking on a subject about which I really know very little.

## 1. The relative isomorphism between $K(R)$ and TC(R)

Let me begin by stating the basic theorem that relates $K$-theory to TC, leaving all indications of what TC is to later. Let $R$ be a ring. There is a natural map of spectra (in the topological sense) $KR \longrightarrow TC(R)$. It is called the "cyclotomic trace" and was first defined by Bökstedt, Hsiang, and Madsen, following up ideas of Goodwillie. The homotopy groups of $KR$ are Quillen's algebraic $K$-groups of $R$. A rational version of the following theorem was proven by Goodwillie. The $p$-adic version is considerably more difficult.

**Theorem 1.1** (McCarthy)**.** *Let $f : R \longrightarrow S$ be a surjective homomorphism of rings whose kernel $I$ is a nilpotent ideal. Then the following square becomes homotopy cartesian after completion at $p$ for any prime $p$:*

$$\begin{array}{ccc} K(R) & \longrightarrow & TC(R) \\ \downarrow & & \downarrow \\ K(S) & \longrightarrow & TC(S). \end{array}$$

*Therefore the induced map of homotopy fibers $K(f) \longrightarrow TC(f)$ becomes an equivalence after $p$-adic completion.*

Write $K_i(R) = \pi_i(KR)$ and $TC_i(R) = \pi_i(TC(R))$. Then $K_i = 0$ for $i < 0$ and $TC_i = 0$ for $i < -1$. For any spectrum $T$, let $T[0, \infty)$ be the connective cover of $T$; it is obtained from $T$ by killing its homotopy groups in negative degrees.

The theorem implies that if $K(S)^{\wedge}_p \longrightarrow TC(S)[0, \infty)^{\wedge}_p$ is an equivalence, then $K(R)^{\wedge}_p \longrightarrow TC(R)[0, \infty)^{\wedge}_p$ is an equivalence. The theorem refers to an integral construction of $TC(R)$. The usual construction is equivalent to the product of its $p$-adic completions, although a somewhat different construction carries more integral information. In contrast with $K$-theory, if $R$ is finitely generated over $\mathbb{Z}$, then

$$TC(R)\hat{_p} \simeq TC(R\hat{_p})\hat{_p}.$$

Thus the fiber of $K(\mathbb{Z}_{(p)}) \longrightarrow K(\mathbb{Z}_p)$ is invisible to TC, and passage to TC loses number theoretic information. That may be a high price to pay for computability, but that is the nature of this approach to $K$-theory. We will work one prime at a time and consider only $p$-adically completed spectra, omitting notation for the completion.

## 2. The absolute isomorphism for finite $W(k)$-algebras

Now let me state the basic results of Madsen, Bökstedt, and Hesselholt. There is a general theorem on when $K(R) \cong TC(R)[0,\infty)$, and there are explicit calculations in favorable cases. The starting point is consideration of the field $\mathbb{F}_p$ or, more generally, a perfect field $k$ of characteristic $p$. We shall focus on an explanation of the proof of the following theorem.

**Theorem 2.1** (Hesselholt-Madsen)**.** *For any perfect field $k$ of characteristic $p$,*

$$K(k) \simeq TC(k)[0,\infty) \simeq H\mathbb{Z}_p.$$

The ring $W(k)$ of Witt vectors plays a fundamental role in the study of TC, and

$$TC(k) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H(\mathrm{coker}(F-1)),$$

where $F : W(k) \longrightarrow W(k)$ is the Frobenius homomorphism. The cited cokernel is zero when $k$ is algebraically closed. Recall that $W(\mathbb{F}_p) \cong \mathbb{Z}_p$. I will give you the definition of $W(A)$ for a general commutative ring later on.

**Theorem 2.2** (Hesselholt-Madsen)**.** *Let $A$ be a finitely generated $W(k)$-algebra.*

1. $K(A/p^i) \simeq TC(A/p^i)[0,\infty)$ *for* $i \geq 1$.
2. $K(A) \simeq \operatorname{holim} K(A/p^i)$.
3. $TC(A) \simeq \operatorname{holim} TC(A/p^i)$.
4. $K(A) \simeq TC(A)[0,\infty)$.

By induction and McCarthy's theorem, (i) holds in general if it holds for $i = 1$. Thus assume $A = A/p$. The radical $J$ of $A$ is nilpotent, so by McCarthy's theorem again we may assume that $J = 0$ and $A$ is central simple. If $A = M_n(k)$, (i) holds because both $K$ and $TC$ are Morita invariant and the result holds for $A = k$. The general case reduces to this case by use of a Galois splitting field for $A$. Part (ii) is a direct consequence of work of Suslin and others. Part (iii) is a direct homotopy theoretical argument from the definition of TC, the starting point being the analogue

$$HA \simeq \operatorname{holim} H(A/p^i)$$

for Eilenberg-Mac Lane spectra $HA = K(A,0)$. Part (iv) is an immediate consequence of parts (i) through (iii).

## 3. Explicit calculations

To give an explicit computation, let $\tilde{K}(k[\varepsilon])$ be the homotopy fiber of the map

$$K(k[\varepsilon]) \longrightarrow K(k)$$

that sends $\varepsilon$ to 0; here $k[\varepsilon]$ is the ring of dual numbers, $\varepsilon^2 = 0$. Define $\widetilde{TC}(k[\varepsilon])$ similarly. Thinking of these as integral for the moment, they are both rationally trivial, the first by a theorem of Goodwillie. Thinking of them $p$-adically, McCarthy's theorem gives

$$\tilde{K}(k[\varepsilon]) \simeq \widetilde{TC}(k[\varepsilon]).$$

Let $W_n(k)$ be the ring of Witt vectors of length $n$. In particular, $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$. I will recall the general definition later.

**Theorem 3.1** (Hesselholt-Madsen). *The group $\widetilde{TC}_i(k[\varepsilon])$ is zero if $i$ is even and*

$$\widetilde{TC}_i(k[\varepsilon]) = \bigoplus_{(d,2p)=1,\ 1\le d\le i} W_{n(i,d)}(k)$$

*if $i$ is odd, where $n = n(i,d)$ is characterized by $p^{n-1}d \le i < p^n d$.*

More deeply, Hesselholt and Madsen have recently generalized this to a calculation of $\widetilde{TC}_*(k[x]/(x^q))$, the result just stated being the special case $q = 1$.

Some other deep explicit computations are given in work of Bökstedt and Madsen, Tsalidis, and, at $p = 2$, Rognes. Unfortunately, the relevant theorems are written up in pieces in a total of nine separate papers. Nevertheless, the statements of the final results are remarkably simple and explicit. Let $A_s$ be the Witt ring of the field with $p^s$ elements, $A_s = W(\mathbb{F}_{p^s})$. Equivalently, these are the rings of integers in those local number fields that are unramified extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers. Let $ku = K[0,\infty)$, let $bu = K[2,\infty)$, and let $j$ be the homotopy fiber of

$$\psi^q - \mathrm{id} : ku \longrightarrow bu,$$

where $q$ is a unit mod $p^2$ (equivalently, a topological generator of $\mathbb{Z}_p^\times$); if $p = 2$, we take $q = 3$. As usual, we understand these spectra to be completed at $p$. Their homotopy groups are well-known from classical computations in topological $K$-theory.

**Theorem 3.2** (Bökstedt-Madsen; Tsalidis). *For $p > 2$, there are homotopy equivalences of $p$-complete spectra*

$$K(A_s) \simeq TC(A_s)[0,\infty) \simeq j \vee \Sigma j \vee \Sigma bu \vee \bigvee_{s-1} \Sigma^{-1} bu.$$

Remember that $A_1 = \mathbb{Z}_p$. Here Rognes has carried out the analogous computation at the prime 2. The basic line of argument is the same, but there is considerably more technical work and some new ideas are needed.

**Theorem 3.3** (Rognes). *At $p = 2$, there is a cofibration sequence*

$$K^{red}(\mathbb{Z}_2) \longrightarrow K(\mathbb{Z}_2) \xrightarrow{red} j$$

*such that red induces a split epimorphism on homotopy groups. There is also a cofibration sequence*

$$\Sigma j \xrightarrow{f} K^{red}(\mathbb{Z}_2) \longrightarrow \Sigma ku$$

*such that $f$ induces a split monomorphism on homotopy groups.*

Both cofibration sequences admit precise homotopical characterizations. The map *red* is a Galois reduction map that is constructed using ideas of Dwyer and Mitchell. The starting point is Quillen's work on the $K$-theory of finite fields, especially the identification $K(\mathbb{F}_3) \simeq j$ at $p = 2$.

The bulk of the work in the proof of these theorems is the computation of the homotopy groups of $TC(A_s)[0, \infty)$. Schematically, this works in the same way at all primes, although the technical spectral sequence calculations are different at the prime 2. The identification of homotopy types follows from the homotopy group calculations using a rather substantial amount of algebraic topology. For example, an understanding of the Bousfield localization of spectra at periodic $K$-theory plays an essential role. In the case of odd primes, a not too hard comparison betweeen algebraic $K$-theory and a special case of Waldhausen's algebraic $K$-theory of spaces plays a key role. This doesn't work at $p = 2$ because of the difference between the real and complex image of $J$ spectra, and this accounts for the need to introduce the Galois reduction map.

I will not say anything more about this part, nor will I say anything about the explicit computation of the homotopy groups. I just want to emphasize how remarkably clean and simple the final answers are. This is very much in the same spirit as Quillen's original determination of the $K$-theory of finite fields and their algebraic closures in terms of topological $K$-theory, but it is remarkable how far this kind of reduction goes. This is also in the spirit of, and consistent with, the Lichtenbaum-Quillen conjectures.

## 4. Topological Hochschild homology

In the rest of the talk, I will describe the definition of TC and the inductive scheme for calculations such as these that flows from the definition. The starting point of the construction is topological Hochschild homology, THH, so I must begin with that. It was first defined by Bökstedt and, implicitly, Breen. There are now several quite different constructions, but they have recently been proven to be equivalent. I want to begin abstractly. Let $\mathscr{C}$ be any symmetric monoidal category. Let $\otimes$ denote its product and $u$ its unit object. A monoid $R$ in $\mathscr{C}$ is an object of $\mathscr{C}$ with a unit map $\eta : u \longrightarrow R$ and product $\phi : R \otimes R \longrightarrow R$ that is associative and unital. We define a cyclic object $Cyc_\bullet(R)$ in $\mathscr{C}$. Let $R^n$ denote the $n$-fold tensor power of $R$ and let $C_n$ be the cyclic group of order $n$ with generator $\tau_n$.

**Definition 4.1.** Define $Cyc_n(R) = R^{n+1}$, with its natural $C_{n+1}$-action. Define face and degeneracy operators by

$$d_i = \begin{cases} (\mathrm{id})^i \wedge \phi \wedge (\mathrm{id})^{n-i-1} & \text{if } 0 \leq i < n \\ (\phi \wedge (\mathrm{id})^{n-1}) \circ \tau_{n+1} & \text{if } i = n, \end{cases}$$

$$s_i = (\mathrm{id})^{i+1} \wedge \eta \wedge (\mathrm{id})^{n-i} \qquad \text{if } 0 \leq i \leq n.$$

Algebraically, if $k$ is a commutative ring, the Hochschild homology $HH^k_*(R)$ of a $k$-algebra $R$ is defined to be the homology of the associated $k$-chain complex of $Cyc_\bullet(R)$. Of course, $\mathscr{C}$ here is the category of $k$-modules, with its usual tensor product. There is another way to think about this. We can view $Cyc_\bullet(R)$ as a simplicial abelian group, take its geometric realization $HH^k(R) = |Cyc_\bullet(R)|$, and define

$$HH^k_*(R) = \pi_*(HH^k(R)).$$

Here $HH^k(R)$ is a product of Eilenberg-Mac Lane spaces. With the topology, more structure becomes visible: because $HH_\bullet(R)$ is cyclic and not just simplicial, $HH^k(R)$ is a $\mathbb{T}$-space, where $\mathbb{T} = S^1$ is the circle group.

Another example that we shall need is that of a based topological monoid $\Pi$. This is a based space $\Pi$ with an associative and unital product $\Pi \wedge \Pi \longrightarrow \Pi$. That is, we take $\mathscr{C}$ to be the category of based spaces and $\otimes$ to be the smash product $X \wedge Y = X \times Y / X \vee Y$. In this case, $|Cyc_\bullet(\Pi)|$ is called the cyclic bar construction and is denoted $B^{cyc}(\Pi)$. Again, it is a $\mathbb{T}$-space. There is an evident unbased version of this construction that is obtained by working with Cartesian products in the category of unbased spaces.

By work of Elmendorf, Kriz, Mandell, and myself, there is a category $\mathscr{M}_S$ of spectra that is symmetric monoidal under its smash product $\wedge$. The unit object is the sphere spectrum $S$, and monoids in this category are called $S$-algebras. For a commutative $S$-algebra $k$, there is a symmetric monoidal category of $k$-modules, and this allows the definition of a $k$-algebra $R$. We can take $\mathscr{C}$ to be the category of $k$-modules. There is a geometric realization functor from simplicial $k$-modules to $k$-modules, and this allows us to define

$$THH^k(R) = |Cyc_\bullet(R)|.$$

We then define topological Hochschild homology groups by

$$THH^k_*(R) = \pi_*(THH^k(R)).$$

Here again, the $k$-module spectrum $THH^k(R)$ has a $\mathbb{T}$-action coming from the cyclic structure on $THH_\bullet(R)$.

If we switch back to the algebraic context of a $k$-algebra $R$, we find that the algebraic Hochschild homology groups are special cases of the topological ones. If $R$ is flat as a $k$-module, then

$$HH^k_*(R) \cong THH^{Hk}_*(HR).$$

However, the main interest is not in such relative topological Hochschild homology groups, but rather in the absolute case of general $S$-algebras; that is, we take $k = S$ as our ground ring. Given a ring $R$, in the discrete algebraic sense, we define

$$THH(R) = THH^S(HR).$$

This is equivalent to the original definition of Bökstedt in terms of functors with smash products. The simplicial filtration of $THH(R)$ leads to a spectral sequence of the form

$$E^2_{s,t} = HH^{\mathbb{F}_p}_{s,t}(H_*(HR;\mathbb{F}_p)) \Longrightarrow H_*(THH(R);\mathbb{F}_p).$$

In favorable cases, this allows the computation of the mod $p$ homology of $THH(R)$. When $R$ is commutative, $THH(R)$ is a product of Eilenberg-Mac Lane spectra, and it is not difficult to determine the homotopy type of $THH(R)$ from its mod $p$ homology. Thus we may regard $THH$ as a reasonably calculable functor. For example, a basic computation of Bökstedt and Breen gives that

$$THH_*(\mathbb{F}_p) \cong \mathbb{F}_p[\sigma], \quad \deg(\sigma) = 2.$$

## 5. Witt rings

I need to explain some classical algebra and some equivariant stable homotopy theory in order to give the idea behind the construction of TC from THH. I will first define the Witt ring $W(A)$ of a commutative ring $A$. This is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once.

As a set, $W(A) = A^{\aleph_0}$ is the product of countably many copies of $A$, with elements written $a = (a_0, a_1, \ldots)$. Define a function, called the ghost map,

$$w : W(A) \longrightarrow A^{\aleph_0}$$

with coordinates the Witt polynomials $w_i$:

$$\begin{aligned} &w_0 = a_0 \\ &w_1 = a_0^p + pa_1 \\ &w_2 = a_0^{p^2} + pa_1^p + p^2 a_2 \\ &\ldots\ldots\ldots \\ &w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \ldots + p^n a_n \\ &\ldots\ldots\ldots \end{aligned}$$

The ring structure on $W(A)$ is characterized by the requirement that $w$ be a natural ring homomorphism. It is a theorem that this makes sense. More concretely,

$$a + b = (s_0(a,b), s_1(a,b), \ldots)$$

and

$$ab = (p_0(a,b), p_1(a,b), \ldots)$$

for certain integral polynomials $s_i$ and $p_i$ that depend only on $(a_0, \ldots, a_i)$ and $(b_0, \ldots, b_i)$. The first few $s_i$ and $p_i$ are

$$s_0(a,b) = a_0 + b_0, \quad s_1(a,b) = (a_0^p + b_0^p - (a_0 + b_0)^p)/p$$
$$p_0(a,b) = a_0 b_0, \quad p_1(a,b) = a_1 b_0^p + pa_1 b_1 + a_0^p b_1.$$

$W(A)$ is a commutative ring with unit $1 = (1, 0, \ldots)$.

One way to prove that this makes sense is to show that for any polynomial

$$\Phi \in \mathbb{Z}[x, y],$$

such as $x + y$ and $xy$, there exist unique polynomials

$$\phi_i \in \mathbb{Z}[x_0, x_1, \ldots; y_0, y_1, \ldots]$$

such that

$$w_n(\phi_0, \ldots, \phi_i, \ldots) = \Phi(w_n(x_0, x_1, \ldots), w_n(y_0, y_1, \ldots))$$

for $n \geq 0$. The polynomials $\phi_i$ depend only on the first $i$ variables.

There is a ring homomorphism

$$F : W(A) \longrightarrow W(A) \quad \text{(Frobenius homomorphism)}$$

characterized by

$$F(w_0, w_1, \ldots) = (w_1, w_2, \ldots).$$

When $A$ is an $\mathbb{F}_p$-algebra, $F = W(\phi)$ where $\phi(x) = x^p$ for $x \in A$. There is an additive map

$$V : W(A) \longrightarrow W(A) \quad \text{(Verschiebung map)}$$

characterized by

$$V(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots).$$

"Verschiebung" means "displacement", and we think of it as "transfer". If $A$ is an $\mathbb{F}_p$-algebra, then $V(1) = p$. There is a multiplicative map

$$r : A \longrightarrow W(A) \quad \text{(Teichmüller character)}$$

characterized by

$$r(x) = (x, 0, \ldots).$$

Any $a$ can be reconstructed as

$$a = \sum_n V^n(r(a_n)).$$

These maps satisfy the relations

$$aV(b) = V(F(a)b), \quad FV(a) = pa, \quad VF(a) = V(1)a.$$

We note that $V^n W(A)$ is an ideal of $W(A)$ and define

$$W_n(A) = W(A)/V^n W(A).$$

This is the ring of Witt vectors of length $n$. We have restriction homomorphisms

$$R : W_{n+1}(A) \longrightarrow W_n(A)$$

characterized by

$$R(a_0, a_1, \ldots, a_n) = (a_0, a_1, \ldots, a_{n-1})$$

Clearly

$$W(A) \cong \lim W_n(A).$$

The map $V$ induces maps

$$V : W_n(A) \longrightarrow W_{n+1}(A),$$

and the following sequences are exact:

$$0 \to W_n(A) \xrightarrow{V^r} W_{n+r}(A) \xrightarrow{R^n} W_r(A) \to 0.$$

The map $F$ induces maps

$$F : W_{n+1}(A) \longrightarrow W_n(A).$$

The restriction maps $R$ commute with the maps $F$ and $V$.

By a basic theorem of Witt, if $k$ is a perfect field of characteristic $p$, then $W(k)$ is a complete discrete valuation ring with residue field $k$ and uniformizing element $p$. In particular, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$. This is an amazing and beautiful fact: in principle, we can reconstruct these complete discrete valuation rings directly from their residue fields.

## 6. Fixed points of $G$-spaces and $G$-spectra

And now for something completely different. It turns out that $THH(R)$ is the underlying spectrum of a genuine $\mathbb{T}$-spectrum $T(R)$ whose fixed point spectra under the action of the cyclic groups $C_{p^n}$ have structure that mimics the algebraic structures that I have just described. However, to discuss this intelligibly, I need a digression about fixed points of based $G$-spaces and of $G$-spectra for compact Lie groups $G$. For a based $G$-space $X$, with $G$-fixed basepoint, and for a based space $K$ regarded as a $G$-space with trivial action, we have

$$G\text{-Maps}(K, X) \cong \text{Maps}(K, X^G).$$

For a second $G$-space $Y$, we have

$$X^G \wedge Y^G \cong (X \wedge Y)^G.$$

Now consider $G$-spectra and their associated equivariant cohomology theories. There are two kinds. First, there are $\mathbb{Z}$-graded theories. These satisfy the usual Eilenberg-Steenrod axioms and thus have suspension isomorphisms

$$E^m_G(X) \cong E^{m+n}_G(X \wedge S^n).$$

They are represented by naive $G$-spectra, which are just spectra with $G$-actions. There are also $RO(G)$-graded theories that have suspension isomorphisms

$$E^\alpha_G(X) \cong E^{\alpha+V}_G(X \wedge S^V),$$

where $\alpha \in RO(G)$ and $S^V$ is the one-point compactification of a representation $V$ of $G$. These are represented by genuine $G$-spectra. I don't want to go into explicit definitions. Suffice it to say that Gaunce Lewis and I developed a good stable homotopy category of $G$-spectra in which to do equivariant stable homotopy theory.

There is a suspension spectrum functor $\Sigma^\infty_G$ from based $G$-spaces to genuine $G$-spectra. One would like to have spectrum-level analogues of the things I told you about fixed points of $G$-spaces. However, we need two different functors to achieve this. There is a geometric fixed point functor $\Phi^G$ that has the desirable properties

$$\Phi^G(\Sigma^\infty_G X) \simeq \Sigma^\infty(X^G)$$

and

$$\Phi^G E \wedge \Phi^G F \simeq \Phi^G(E \wedge F).$$

There is a categorical fixed point functor that is characterized by

$$G\text{-Maps}(D, E) \cong \text{Maps}(D, E^G)$$

for a nonequivariant spectrum $D$ regarded as a $G$-trivial $S_G$-spectrum.

These functors are quite different. If $S_G = \Sigma^\infty_G S^0$, then $\Phi^G(S_G) = S$, and of course $\pi_0(S) = \mathbb{Z}$. However $\pi_0((S_G)^G)$ is the Burnside ring $A(G)$. I will say more about the relationship between these functors later. For now, all we need to know is that there is a natural map

$$\beta : E^G \longrightarrow \Phi^G(E).$$

## 7. The definition of $TC(A)$

Any proper subgroup $C$ of $\mathbb{T}$ is cyclic, and $\mathbb{T}/C \cong \mathbb{T}$. Let $\Lambda X$ be the free loop space on a space $X$. Mapping $f$ to $g$, where $g(x) = f(x^q)$, we see that $(\Lambda X)^{C_q} \cong \Lambda X$ as a $\mathbb{T}$-space for any $q$. Using the version of the cyclic bar construction for unbased monoids $\Pi$ and assuming that $\pi_0(\Pi)$ is a group, there is a $G$-map

$$B^{cyc}(\Pi) \xrightarrow{\simeq} \Lambda B\Pi$$

that induces a $\mathbb{T}$-homotopy equivalence on passage to $C$-fixed point spaces for any proper subgroup $C$.

Fix a ring $A$. We assume that $A$ is commutative since we want to emphasize the analogy with $W(A)$. It turns out that $THH(A)$ is the underlying naive $G$-spectrum of a genuine $G$-spectrum, which we shall denote by $T(A)$. Its geometric fixed points mimic the behavior of fixed point spaces of free loop spaces. This should seem plausible in view of the relationship between the cyclic bar construction and $THH$. We say that a $G$-spectrum $E$ is "cyclotomic" if it is a commutative ring $\mathbb{T}$-spectrum and if there are equivalences of $\mathbb{T}$-spectra

$$\Phi^{C_q}(E) \simeq E$$

that are compatible as $q$ varies, in the sense that the following diagrams commute:

$$\begin{array}{ccc} \Phi^{C_q}\Phi^{C_r}(E) & = & \Phi^{C_{qr}}(E) \\ \simeq\downarrow & & \downarrow\simeq \\ \Phi^{C_q}(E) & \xrightarrow[\simeq]{} & E. \end{array}$$

**Theorem 7.1** (Hesselholt-Madsen)**.** *There is a cyclotomic $\mathbb{T}$-spectrum $T(A)$ whose underlying naive $\mathbb{T}$-spectrum is $THH(A)$.*

I find the existing proof, due to Hesselholt and Madsen, a little ad hoc, and I will say nothing about it. I am working on a more conceptually satisfactory construction, but I am not yet ready to talk about it.

Remember that we are completing all spectra at a fixed prime $p$. We need only consider the $C_{p^n}$-fixed point spectra in constructing $TC(A)$ from $T(A)$. Write

$$T_n(A) = T(A)^{C_{p^{n-1}}}, \quad n \geq 1.$$

We think of $T_n(A)$ as analogous to $W_n(A)$. Inclusion of fixed points gives ring maps

$$F : T_{n+1}(A) \longrightarrow T_n(A).$$

There are transfer maps

$$V : T_n(A) \longrightarrow T_{n+1}(A)$$

associated to the quotient map $\mathbb{T}/C_{p^{n-1}} \longrightarrow \mathbb{T}/C_{p^n}$. There are restriction maps

$$R : T_{n+1}(A) \longrightarrow T_n(A),$$

namely the composites

$$T(A)^{C_{p^n}} \cong (T(A)^{C_p})^{C_{p^{n-1}}} \xrightarrow{\beta} (\Phi^{C_p}T(A))^{C_{p^{n-1}}} \simeq T(A)^{C_{p^{n-1}}}.$$

On passage to homotopy groups, these maps satisfy

$$aV(b) = V(F(a)b), \quad FV(a) = pa, \quad VF(a) = V(1)a.$$

$$FR = RF \quad \text{and} \quad VR = RV.$$

These relations should look familiar.

**Theorem 7.2** (Hesselholt-Madsen)**.** *There are natural isomorphisms of rings* $I : W_n(A) \longrightarrow \pi_0(T_n(A))$ *such that* $RI = IR$*,* $FI = IF$*, and* $VI = IV$*.*

The proof depends on the construction of a map

$$B^{cyc}(A) \simeq B^{cyc}(A)^{C_{p^{n-1}}} \longrightarrow \Omega^\infty T_n(A)$$

whose induced map on $\pi_0$ agrees with $r : A \longrightarrow W_n(A)$. Here $A$ is regarded as a based monoid under multiplication, with basepoint 0. It also depends on cofibrations that I will describe shortly. These lead to exact sequences involving $V$ and $R$ that mimic those for the Witt rings and so allow an inductive proof of the theorem.

We have inverse systems of maps

$$\cdots T_{n+1}(A) \xrightarrow{R} T_n(A) \xrightarrow{R} \cdots \xrightarrow{R} T_1(A)$$

and

$$\cdots T_{n+1}(A) \xrightarrow{F} T_n(A) \xrightarrow{F} \cdots \xrightarrow{F} T_1(A).$$

Define $TR(A)$ to be the homotopy limit of the first system and $TF(A)$ to be the homotopy limit of the second system. Because FR=RF, we have induced maps

$$F : TR(A) \longrightarrow TR(A) \quad \text{and} \quad R : TF(A) \longrightarrow TF(A).$$

Define $TC(A)$ to be the homotopy fiber of $F - \mathrm{id} : TR(A) \longrightarrow TR(A)$ or, equivalently, the homotopy fiber of $R - \mathrm{id} : TF(A) \longrightarrow TF(A)$.

## 8. Isotropy separation

We return to equivariant stable homotopy theory to describe what is needed to calculate the homotopy groups of the $T_n(A)$ in positive dimensions. We begin by giving a better understanding of geometric and categorical fixed point spectra.

For a family $\mathscr{F}$ of subgroups of $G$, namely a set closed under passage to subgroups and conjugates, there is a universal $\mathscr{F}$-space $E\mathscr{F}$ characterized by

$$(E\mathscr{F})^H = \begin{cases} \text{contractible if } H \in \mathscr{F} \\ \text{empty if } H \notin \mathscr{F}. \end{cases}$$

We can form the cofiber sequence

$$E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}.$$

Smashing a based $G$-space $X$ with this cofiber sequence gives a cofibration sequence

$$E\mathscr{F}_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathscr{F} \wedge X.$$

This separates $X$ into the part of the space with isotropy groups in $\mathscr{F}$ and not in $\mathscr{F}$. We can do the same for a $G$-spectrum $E$, obtaining a cofibration sequence

$$E\mathscr{F}_+ \wedge E \longrightarrow E \longrightarrow \tilde{E}\mathscr{F} \wedge E.$$

If $\mathscr{P}$ is the family of proper subgroups of $G$, then

$$\Phi^G(E) = (E \wedge \tilde{E}\mathscr{P})^G.$$

The point is that for $G$-spaces $X$, it is immediate that $(X \wedge E\mathscr{P}_+)^G = \{*\}$ and thus $(X \wedge \tilde{E}\mathscr{P})^G = X^G$. That is, concentrating $X$ at the portion with isotropy group $G$ and then taking fixed points is the same as taking fixed points directly.

The analogue for $G$-spectra is false, as the example of $S_G$ makes clear. The map $S^0 \longrightarrow \tilde{E}\mathscr{P}$ induces the natural map

$$\beta : E^G \longrightarrow \Phi^G(E).$$

Note that $EG = E\{e\}$. We have an isotropy separation cofiber sequence

$$EG_+ \wedge E \longrightarrow E \longrightarrow \tilde{E}G \wedge E.$$

If $G = C_p$, then $\mathscr{P} = \{e\}$ and therefore

$$\Phi^{C_p}(E) = (\tilde{E}C_p \wedge E)^{C_p}.$$

This is the starting point for the inductive analysis of the $T_n(A)$.

## 9. Homotopy orbits and fixed points

We next define homotopy orbit and fixed point spectra.

For based $G$-spaces $X$ and $Y$, let $F(X,Y)$ denote the function space of based maps $X \longrightarrow Y$, with $G$ acting by conjugation. Its $G$-fixed points are the based $G$-maps $X \longrightarrow Y$. If $EG$ is a free contractible $G$-space, we adjoin a disjoint basepoint to it and define the homotopy fixed point space of $X$ by

$$X^{hG} = F(EG_+, X)^G.$$

Analogously, the homotopy orbit space of $X$ is

$$X_{hG} = EG_+ \wedge_G X.$$

Similarly, for based $G$-spaces $X$ and $G$-spectra $E$, we have the function $G$-spectrum $F(X,E)$ and we define the homotopy fixed point functor by

$$E^{hG} = F(EG_+, E)^G.$$

Analogously, the homotopy orbit spectrum of $E$ is

$$E_{hG} = EG_+ \wedge_G E.$$

The homotopy fixed point and orbit spectra are often under reasonable calculational control. They can be studied by nonequivariant techniques.

The projection $EG_+ \longrightarrow S^0$ induces a natural map

$$\gamma : E^G \longrightarrow E^{hG}.$$

Analysis of this map in the cases of equivariant $K$-theory, equivariant cohomotopy, and equivariant cobordism, leads to the Atiyah-Segal completion theorem, Carlsson's proof of the Segal conjecture, and a recent completion theorem of Greenlees and myself. Analysis of other cases leads to the explicit calculations of $TC$ that I have described.

## 10. The Tate diagram

In earlier work of Greenlees and myself, we placed the map $\gamma$ in a context of generalized Tate theories by means of a certain fundamental diagram. Analysis of special cases of this diagram is at the heart of the calculations of TC. Write $F(E) =$

$F(EG_+, E)$. Smashing the map $\gamma : E \longrightarrow F(E)$ with the isotropy separation cofiber sequence, we obtain the commutative diagram

$$\begin{array}{ccccc} EG_+ \wedge E & \longrightarrow & E & \longrightarrow & \tilde{E}G \wedge E \\ \simeq \downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge F(E) & \longrightarrow & F(E) & \longrightarrow & \tilde{E}G \wedge F(E). \end{array}$$

The left vertical arrow is a $G$-equivalence. We may apply the $G$-fixed point functor to this diagram. The rows are still cofiber sequences. For finite $G$, we have

$$E_{hG} \simeq (EG_+ \wedge E)^G$$

by a result of Adams, as strengthened by Lewis and myself. We define the $G$-fixed Tate spectrum to be

$$\hat{\mathbb{H}}(G; E) = [\tilde{E}G \wedge F(EG_+, E)]^G.$$

The resulting fixed point diagram can now be rewritten in the form

$$\begin{array}{ccccc} E_{hG} & \longrightarrow & E^G & \longrightarrow & (\tilde{E}G \wedge E)^G \\ \| & & \downarrow \gamma & & \downarrow \hat{\gamma} \\ E_{hG} & \longrightarrow & E^{hG} & \longrightarrow & \hat{\mathbb{H}}(G; E). \end{array}$$

Clearly $\gamma$ is an equivalence, perhaps after suitable completion and passage to connective covers, if and only if $\hat{\gamma}$ is so. This fact leads to the inductive step in the analysis of the $T_n(A)$. The essential calculational point is that, for finite $G$, there are compatible spectral sequences:

$$E^2_{s,t} = H_s(G; \pi_t(E)) \Longrightarrow \pi_{s+t}(E_{hG})$$
$$E^2_{s,t} = H^{-s}(G; \pi_t(E)) \Longrightarrow \pi_{s+t}(E^{hG})$$
$$E^2_{S,t} = \hat{H}^{-s}(G; \pi_t(E)) \Longrightarrow \pi_{s+t}(\hat{\mathbb{H}}(G; E)).$$

Here $\hat{H}$ denotes the classical Tate cohomology theory. Convergence has to be interpreted with caution, since the last is a whole plane spectral sequence in general.

## 11. Sketch proofs and further results

To calculate $\pi_*(TC(A))$, one must calculate the $\pi_*(T_n(A))$ and the maps connecting them. Remember that $T_n(A) = T(A)^{C_{p^{n-1}}}$. Remember too that

$$T(A) \simeq \Phi^{C_p} T(A) = (\tilde{E}C_p \wedge T(A))^{C_p}.$$

Passing to $C_{p^{n-1}}$-fixed spectra, we see that

$$T(A)^{C_{p^{n-1}}} \simeq (\tilde{E}C_{p^n} \wedge T(A))^{C_{p^n}}.$$

We substitute this in the upper right corner of the Tate diagram for the $C_{p^n}$-fixed spectrum of $T(A)$ and obtain a diagram

$$\begin{array}{ccccc} T(A)_{hC_{p^n}} & \longrightarrow & T(A)^{C_{p^n}} & \longrightarrow & T(A)^{C_{p^{n-1}}} \\ \| & & \downarrow \gamma_n & & \downarrow \hat{\gamma}_n \\ T(A)_{hC_{p^n}} & \longrightarrow & T(A)^{hC_{p^n}} & \longrightarrow & \hat{\mathbb{H}}(C_{p^n}; T(A)). \end{array}$$

As a starting point, one calculates directly that, for $A = \mathbb{F}_p$,

$$\pi_*(\hat{\mathbb{H}}(C_p; T(\mathbb{F}_p))) \cong \mathbb{F}_p[\sigma, \sigma^{-1}]$$

and $(\hat{\gamma}_1)_*$ is just the evident inclusion

$$\mathbb{F}_p[\sigma] \longrightarrow \mathbb{F}_p[\sigma, \sigma^{-1}].$$

Consider the induced maps to connective covers

$$\gamma_n : T(A)^{C_{p^n}} \longrightarrow T(A)^{hC_{p^n}}[0, \infty).$$

A general theorem of Tsalidis asserts that if $\gamma_1$ is an equivalence, then $\gamma_n$ is an equivalence for all $n \geq 1$. His argument is modelled on Carlsson's reduction of the Segal conjecture to elementary Abelian $p$-groups, or rather on a modification of Carlsson's proof due to Caruso, Priddy, and myself. Therefore, for $A = \mathbb{F}_p$, we see that the maps $\gamma_n$ and $\hat{\gamma}_n$ in our diagram become equivalences on passage to connective covers for all $n \geq 1$. Following through the calculations in detail, one finds that for all $n \geq 1$,

$$(\hat{\gamma}_n)_* : \pi_*(T(A)^{C_{p^{n-1}}}) \longrightarrow \pi_*(\hat{\mathbb{H}}(C_{p^n}; T(A)))$$

can be identified with the inclusion

$$(\mathbb{Z}/p^n\mathbb{Z})[\sigma_n] \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})[\sigma_n, \sigma_n^{-1}],$$

where $\deg(\sigma_n) = 2$. With $\mathbb{Z}/p^n\mathbb{Z}$ replaced by $W_n(k)$, the calculation works out the same way for any other perfect field $k$ of characteristic $p$. The maps $F$, $R$, and $V$ are given by

$$F(\sigma_n) = \sigma_{n-1}$$
$$R(\sigma_n) = p\lambda_n\sigma_{n-1}$$

for some unit $\lambda_n \in \mathbb{Z}/p^n\mathbb{Z}$, and

$$V(\sigma_{n-1}) = p\sigma_n.$$

This allows one to read off $\pi_*(TC(k))$.

The calculation of $\widetilde{TC}_*(k[x]/(x^q))$ starts from a $\mathbb{T}$-equivalence

$$T(k[x^q]/(x^q)) \simeq T(k) \wedge B^{cyc}(\Pi_n),$$

where $\Pi_n$ is the based monoid $\{0, x, \ldots, x^{q-1}\}$ with $x^q = 0$. Although considerable combinatorial work is involved, especially when $q > 2$, the basic idea is to reduce the analysis of fixed point spectra to the case of $T(k)$.

Hesselholt has generalized the calculation of $\pi_*(T(k)^{C_{p^{n-1}}}$ to obtain

$$\pi_*(T(A)^{C_{p^{n-1}}}) \cong (W_n\Omega^*_A)[\sigma_n]$$

for any smooth $k$-algebra $A$. (Smooth means that $A$ is finitely presented and that any map of $k$-algebras $A \to B/N$, $N$ a nilpotent ideal, lifts to a map of $k$-algebras $A \to B$.) Here $W\Omega^*_A$ is the de Rham-Witt complex of Bloch, Deligne, and Illusie. The maps $F$, $R$, and $V$ are given by the same formulas as above. Passing to limits from these isomorphisms, Hesselholt also proves that

$$W\Omega^*_A \cong \pi_*(TR(A)) \cong C_*(A; p),$$

where $C_*(A; p)$ is Bloch's complex of $p$-typical curves in algebraic $K$-theory. It would take us too far afield to define terms, but an essential and illuminating point is that there is a differential

$$\delta : \pi_n(T(A)) \longrightarrow \pi_{n+1}(T(A))$$

that linearizes to Connes' operator $B$: there is a linearization map

$$\ell : THH(A) \longrightarrow HH(A)$$

and a commutative diagram

$$\begin{array}{ccc} \pi_n(T(A)) & \xrightarrow{\ell} & HH_n(A) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle B} \\ \pi_{n+1}(T(A)) & \xrightarrow{\ell} & HH_{n+1}(A). \end{array}$$

The analysis of the fixed point spectra of $T(\mathbb{Z}_p)$ begins with Bökstedt's calculation of the mod $p$ homotopy groups of $T(\mathbb{Z}_p)$:

$$\pi_*(T(\mathbb{Z}_p); \mathbb{F}_p) \cong E\{e\} \otimes \mathbb{F}_p[f],$$

where $\deg(e) = 2p - 1$, $\deg(f) = 2p$, and the Bockstein is given by $\beta(e) = f$. In outline, the calculation of $\pi_*(T(\mathbb{Z}_p)^{C_{p^n}})$ from here is just like that given above for $k$, but the arguments require a very careful analysis of the differentials in the relevant spectral sequences.

The calculation of $TC(A_s)$, $A_s = W(\mathbb{F}_{p^s})$ follows fairly directly since there is a cofibration sequence

$$TC(A_s) \longrightarrow TF(\mathbb{Z}_p) \xrightarrow{R^s - \mathrm{id}} TF(\mathbb{Z}_p).$$

This is not very hard to obtain, and in fact it generalizes to a cofibration sequence

$$TC(A) \longrightarrow TF(B) \xrightarrow{R^s - \mathrm{id}} TF(B)$$

where $A \longrightarrow B$ is the inclusion of rings of integers associated to any degree $s$ unramified extension of local fields of residual characteristic $p$. Of course this suggests that the calculations generalize accordingly. Lindenstrauss and Madsen are also working on the ramified case and have calculated $THH(A)$ for general number rings $A = \mathbb{Z}[x]/(f(x))$.

This completes my very rough sketch of the present state of the art in this active area. The essential point is how effectively methods of equivariant algebraic topology are being used to make explicit calculations in algebraic $K$-theory, via the intermediary of topological cyclic homology.