Thom Thom Spectra and Other New Brave Algebras

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The Thom spectrum MU was the motivating example that led to the definition of an E_∞ ring spectrum in 1972 [MQR]. The definition proceeded by analogy with a description of BU as an E_∞ space. It was immediately apparent that BU and MU really ought to be part of a single mathematical structure. It has taken 35 years and the serious development of parametrized spectra to understand what that structure is.

My goal today is to describe that structure and to show how common it is. I will start by describing what we understood in 1972, but recasting these structures in modern language.

[This is a report on work in progress with Andrew Blumberg and Johann Sigurdsson]

Let $\mathscr{I}$ be the category of finite dimensional real inner product spaces and linear isometric isomorphisms. (We can complexify.) Note that $\mathscr{I}$ is symmetric monoidal under direct sums.

For now, those who know and love symmetric spectra may replace $\mathscr{I}$ by the category Σ of finite sets and isomorphisms.

Codify structure (commutative case only today):

$\mathscr{I}$-FCP

[Functor with Cartesian Product]

A symmetric monoidal functor B under $*$ from $\mathscr{I}$ to the cartesian monoidal category of spaces:

$$* \longrightarrow B(V), \quad B(V) \times B(W) \longrightarrow B(V \oplus W).$$

The category of $\mathscr{I}$-FCP's has products, and we can define group and monoid $\mathscr{I}$-FCP's G. Monoid homomorphisms

$$G(V) \times G(W) \longrightarrow G(V \oplus W).$$

Examples:

$$O, U, SO, SU, Sp, Spin$$

$$String, Top, STop, F, SF.$$

The classifying space functor B takes a monoid $\mathscr{I}$-FCP to an ordinary $\mathscr{I}$-FCP:

$$BO, BU, BSO, BSU, BSp, BSpin$$

$$BString, BTop, BSTop, BF, BSF.$$

The maps

$$BG(V) \times BG(W) \longrightarrow BG(V \oplus W)$$

classify Whitney sum. Bundle level: let

$$Sph(G)(V) = B(*, G(V), S^V).$$

S^V is the one-point compactification of V (or its complexification, etc, as needed). These give sectioned *universal sphere bundles:*

$$BG(V) \xrightarrow{s} Sph(G)(V) \xrightarrow{p} BG(V).$$

Fiberwise smash product of total spaces

$$Sph(G)(V) \,\overline{\wedge}\, Sph(G)(W) \longrightarrow Sph(G)(V \oplus W),$$

gives a map of sectioned bundles with fiber $S^{V \oplus W}$ over

$$BG(V) \times BG(W) \longrightarrow BG(V \oplus W).$$

Codify structure (again, commutative case only):

$\underline{\mathscr{I} - PFSP}$

[Parametrized Functor with Smash Product]

$\mathscr{I}$-FCP B and a symmetric monoidal functor E under $S = \{S^V\}$ from $\mathscr{I}$ to the symmetric monoidal category of retracts:

$$S^V \longrightarrow E(V)$$

and

$$E(V) \,\bar{\wedge}\, E(W) \longrightarrow E(V \oplus W)$$

over and under

$$B(V) \times B(W) \longrightarrow B(V \oplus W).$$

$\underline{\mathscr{I} - FSP}$

[Functor with Smash Product]

An $\mathscr{I}$-PFSP over $B = *$, such as $S = \{S^V\}$.

Thom space functor:

$$TG(V) = Sph(G)(V)/s(BG(V)) = r_! Sph(G)(V).$$

Here $r\colon BG(V) \longrightarrow *$, and $r_!$ is a *base change functor* from parametrized spaces to spaces. Induced products

$$TG(V) \wedge TG(W) \longrightarrow TG(V \oplus W).$$

In general, $r_!$ takes $\mathscr{I}$-PFSP's to $\mathscr{I}$-FSP's.

First Key Diagram:

$$\begin{tikzcd}
\mathscr{I}\text{-FSP} & \\
\mathscr{I}\text{-PFSP} \arrow[u, "\text{Fiber}"] \arrow[r, "\text{Base}"] \arrow[d, "r_!"'] & \mathscr{I}\text{-FCP} \\
\mathscr{I}\text{-FSP} &
\end{tikzcd}$$

For a PFSP E, let $R = \text{Fiber}(E)$: then E is an "R-PFSP". Have a map $R \longrightarrow r_!(E)$ of FSP's.

Orthogonal spectra:

Functors T from $\mathscr{I}$ to based spaces with structure maps

$$\sigma\colon \Sigma^W T(V) = T(V) \wedge S^W \longrightarrow T(V \oplus W).$$

TG is an example:

$$TG(V) \wedge S^W \longrightarrow TG(V) \wedge TG(W) \longrightarrow TG(V \oplus W).$$

External smash product of orthogonal spectra:

$$(T \mathbin{\bar{\wedge}} T')(V, W) = TV \wedge T'W,$$

a functor on $\mathscr{I} \times \mathscr{I}$.

Left Kan extension along $\oplus\colon \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$ gives the *internal smash product* $T \wedge T'$.

The category of orthogonal spectra is symmetric monoidal with unit $S = \{S^V\}$. Its monoids are the *orthogonal ring spectra*. The TG are examples.

Using Σ instead of $\mathscr{I}$, the symmetric monoidal category of topological *symmetric spectra* is defined similarly. Σ embeds in $\mathscr{I}$ via $n \mapsto \mathbb{R}^n$. The TG restrict to *symmetric ring spectra*.

<u>Model category yoga</u>: restriction along $\Sigma \subset \mathscr{I}$ gives Quillen equivalences relating all types of structured orthogonal spectra to the analogous structured symmetric spectra. [HSS, MMSS].

Extend $\mathscr{I}$ to finite or countably infinite inner product spaces and linear isometries that are not necessarily isomorphisms. We can extend $\mathscr{I}$-FCP's and $\mathscr{I}$-PFSP's to functors defined on the extended $\mathscr{I}$, uniquely up to isomorphism.

Fix $\mathbb{U} \cong \mathbb{R}^\infty$. Define $\mathscr{L}(j) = \mathscr{I}(\mathbb{U}^j, \mathbb{U})$. With evident permutations and structure maps given by $\oplus$ and $\circ$, $\mathscr{L}$ has the structure of an *operad*. It is an E_∞ operad, meaning that the spaces $\mathscr{L}(j)$ are Σ_j-free and contractible.

For an $\mathscr{I}$-FCP B, also write B for $\operatorname{colim} B(V)$, where the colimit runs over the inclusions of the finite dimensional $V \subset \mathbb{U}$ (*not* over the whole category $\mathscr{I}$). There are action maps

$$\mathscr{L}(j) \times_{\Sigma_j} B \times \cdots \times B \longrightarrow B$$

that make B an "$\mathscr{L}$-space" (or E_∞ space).

Digression: An infinite loop space machine is a "group completion" functor from E_∞-spaces to spectra. There is an essentially unique one, and it gives an equivalence between grouplike E_∞-spaces and connective spectra.

Ignoring isometries, a prespectrum T indexed on $V \subset U$ gives a spectrum $E = LT$ [LMS]. When the adjoint structure maps

$$T(V) \longrightarrow \Omega^{W-V}T(W)$$

are inclusions,

$$E(V) = \mathrm{colim}_{V \subset W} \Omega^{W-V}T(W).$$

For $V \subset W$, $E(V) \cong \Omega^{W-V}E(W)$. No non-trivial symmetric or orthogonal spectrum can be such an "honest" spectrum.

$\Omega^\infty E = E_0$ is an E_∞-space. (The relevant E_∞ operad is the infinite little cubes operad). Symmetric and orthogonal spectra cannot have such highly structured zeroth spaces.

Each $f\colon U^j \longrightarrow U$ in $\mathscr{L}(j)$ defines a choice

$$E_1 \wedge \cdots \wedge E_j = f_*(E_1 \,\bar{\wedge} \cdots \bar{\wedge}\, E_j)$$

of internalization of an external smash product

$$E_1 \,\bar{\wedge} \cdots \bar{\wedge}\, E_j = L(\ell E_1 \,\bar{\wedge} \cdots \bar{\wedge}\, \ell E_j),$$

where ℓ views spectra as prespectra. These choices glue to a twisted half smash product

$$\mathscr{L}(j) \ltimes E_1 \,\bar{\wedge} \cdots \bar{\wedge}\, E_j,$$

a canonical j-fold internal smash product.

$BG = \operatorname{colim} BG(V)$. Analogously, $MG = LTG$. The Thom spectra MG were the first examples of E_∞ ring spectra.

An E_∞ ring spectrum E has an action by $\mathscr{L}$ given by action maps

$$\mathscr{L}(j) \ltimes_{\Sigma_j} E \,\bar{\wedge} \cdots \bar{\wedge}\, E \longrightarrow E.$$

<u>Digression</u> Starting point of [EKMM].

Parametrize the identity map of E: $\mathbb{L}$-spectra are spectra with an action $\mathscr{L}(1) \ltimes E \longrightarrow E$ by the monoid $\mathscr{L}(1)$. Hide the operad $\mathscr{L}$ in a smash product on $\mathbb{L}$-spectra:

$$E_1 \wedge \cdots \wedge E_j \equiv \mathscr{L}(j) \ltimes_{\mathscr{L}(1)^j} E_1 \,\bar{\wedge} \cdots \bar{\wedge}\, E_j.$$

This is associative [Mike Hopkins observation about $\mathscr{L}$] and commutative. Not quite unital, but there is a weak equivalence $\lambda\colon S \wedge E \longrightarrow E$.

EKMM S-modules are $\mathbb{L}$-spectra such that λ is an isomorphism, and all $S \wedge E$ are S-modules.

S-modules, $\mathbb{L}$-spectra, and spectra give Quillen equivalent model categories.

EKMM ring spectra are S-algebras, that is, monoids in the category of S-modules.

For an MQR E_∞ ring spectrum E, $S \wedge E$ is a (commutative) S-algebra and $\lambda\colon S \wedge E \longrightarrow E$ is a weak equivalence of E_∞ ring spectra. Thus modern S-algebras are essentially the same thing as E_∞ ring spectra.

Miracle [MM,S] There are Quillen equivalences relating all types of structured symmetric and orthogonal spectra to the analogous structured S-modules.

The calculational information explicit on the zeroth "E_∞ ring spaces" of E_∞ ring spectra is implicit in symmetric and orthogonal ring spectra, which do not have such zeroth spaces.

Second Key Diagram:

$$\begin{tikzcd}
E_\infty \text{ ring spectra} \\
\text{parametrized } E_\infty \text{ ring spectra} \arrow[u, "\text{Fiber}"'] \arrow[r, "\text{Base}"] \arrow[d, "r_!"'] & E_\infty \text{ spaces} \\
E_\infty \text{ ring spectra}
\end{tikzcd}$$

E_∞ ring map $R = \mathrm{Fiber} E \longrightarrow r_!(E)$.

Functor L from first key diagram to second:

$$L = \mathrm{colim}\colon \mathscr{I}\text{-FCP} \longrightarrow E_\infty \text{ spaces}$$

$$L\colon \mathscr{I}\text{-FSP} \longrightarrow E_\infty \text{ ring spectra}$$

$$L\colon \mathscr{I}\text{-PFSP} \longrightarrow \text{parametrized } E_\infty \text{ ring spectra}$$

Theorem 1 *For an R-PFSP E, $r_! LE$ is an E_∞ ring spectrum under LR and therefore gives a commutative LR-algebra in the sense of EKMM.*

Theorem 2 *For an E_∞ map $f\colon X \longrightarrow B$ and a parametrized E_∞ ring spectrum E over B, f^*E is a parametrized E_∞ ring spectrum over X and $r_! f^*E$ is an E_∞ ring spectrum.*

Theorem 1 creates examples that feed into Theorem 2. Theorem 2 generalizes the generalized E_∞ ring Thom spectra of Gaunce Lewis.

Example: Let $\phi \in k^2(X)$ for a spectrum X. Then ϕ is a map of spectra $X \longrightarrow \Sigma^2 k$ and has zeroth map $f\colon X_0 \longrightarrow BU = k_2$, which is an E_∞ map. For any E over BU, we have an E_∞ ring spectrum $r_! f^*E$.

Orthogonal spectra over spaces B as in [MS]? No good theory of E_∞ ring spectra over B:

$$E \wedge_B E' = \Delta^*(E \mathbin{\bar\wedge}_B E')$$

is not well-behaved. [LMS] type parametrized spectra! Model theory problems, but they do give an equivalent homotopy category.

For an [LMS] spectrum E indexed on $\mathbb{U}$ and parametrized over B, there is a twisted half-smash product [ELM, BMS]

$$\mathscr{I}(\mathbb{U}, \mathbb{U}') \ltimes E$$

indexed on $\mathbb{U}'$ and parametrized over

$$\mathscr{I}(\mathbb{U}, \mathbb{U}') \times B.$$

The fiber over (f, b) is $f_*(E_b)$. Parametrized E_∞ ring spectra E have action maps

$$\mathscr{L}(j) \ltimes_{\Sigma_j} E \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} E \longrightarrow E$$

over

$$\mathscr{L}(j) \times_{\Sigma_j} B \times \cdots \times B \longrightarrow B.$$

Bar construction $\mathscr{I}$-FCP's [MQR]

A monoid $\mathscr{I}$-FCP G can act termwise from the left on an $\mathscr{I}$-FCP X and on the right on an $\mathscr{I}$-FCP Y. We then have a two-sided bar construction $\mathscr{I}$-FCP [MQR]

$$B(Y,G,X)(V) = B(Y(V),G(V),X(V)).$$

$$B(Y,G,X) = |B_*(Y,G,X)|$$

$$B_q(Y,G,X) = Y \times G^q \times X$$

Nota bene: When G is a group $\mathscr{I}$-FCP, it acts on $B(Y,G,X)$ via action on $Y \times G^q \times X$:

$$g(y,g_1,\cdots,g_q,x) = (yg^{-1},gg_1g^{-1},\cdots,gg_qg^{-1},gx)$$

Examples of R-PFSP's [MS]

When G maps to F, $F(V) = F(S^V, S^V)$, we can replace X by an $\mathscr{I}$-FSP R with G-action to obtain an R-PFSP

$$B(Y, G, R)(V) = B(Y(V), G(V), R(V))$$

over $B(Y, G, *)$ and an R-FSP $r_{!}B(Y, G, R)$.

(R-FSP's are essentially orthogonal R-algebras).

When $Y = *$ and $R = S$, this gave

$$Sph(G) = B(*, G, S)$$

and the Thom FSP $(MG =)TG = r_{!}Sph(G)$.

Generalized Thom spectra $r_{!}B(Y, G, S)$.

Example: Let $Y = GL_1(E)$ for a ring spectrum E. This is the space of unit components of E_0. Any G mapping to F acts on Y, and $B(Y, G, *)$ classifies E-oriented G-bundles. When E is an E_∞ ring spectrum, Y and $B(Y, G, *)$ are E_∞ spaces, and $B(Y, G, *) \longrightarrow BG$ is an E_∞-map.

Example: Away from 2,

$$M\,Top = r_! Sph(Top) \simeq r_!(BO_\otimes, F, S)$$

as FSP's, or equivalently as S-algebras.

Iterated examples: Thom Thom spectra

Let G be a group FCP that maps to F, Y be a right G-FCP, and R be a left G-FSP.

Theorem 3 $Q = r_! B(Y, G, R)$ *is both a left G-FSP and an R-FSP.*

Can plug in Q instead of R to get $r_! B(Y, G, Q)$, and can iterate. Specialize to $Y = *$. Define

$$M(G; R) = r_! B(*, G, R).$$

$M(G;R)$ is an $MG{\wedge}R$-FSP (=$MG{\wedge}R$-algebra).

Unit $\eta\colon S \longrightarrow R$ induces

$$\alpha\colon MG = M(G;S) \longrightarrow M(G;R).$$

Inclusion of fiber gives

$$\iota\colon R = M(e;R) \longrightarrow M(G;R).$$

Via product ϕ, these give a map S-algebras

$$\xi\colon MG \wedge R \xrightarrow{\alpha\wedge\iota} M(G;R) \wedge M(G;R) \xrightarrow{\phi} M(G;R).$$

Often ξ is an equivalence:

$$\text{``}M(G;S) \wedge_S R \simeq M(G;R)\text{''}$$

Define $M^0G = S$, $M^1G = MG$,

$$M^nG = M(G; M^{n-1}G).$$

M^nG is an $MG \wedge M^{n-1}G$-algebra.

Iterated geometric Thom spectra.

Iterates of ξ give equivalences

Theorem 4 *$M^nU \simeq (MU)^{\wedge n}$.*

For an E_∞ map $f\colon X \longrightarrow BU$,

$$M^nf \equiv r_!f^*M^nU \simeq (Mf)^{\wedge n}.$$

Post talk addendum:

Let B be an FCP (such as BG), T be an FSP (such as TG). Get a new FSP $B_+ \wedge T$ by

$$(B_+ \wedge T)(V) = B(V)_+ \wedge T(V).$$

Idea: "FSP's are tensored over FCP's."

For a group FCP G, the Thom diagonal

$$TG \longrightarrow BG_+ \wedge TG$$

is a map of FSP's. Pass to spectra:

$$\Delta : MG \longrightarrow \Sigma^\infty BG_+ \wedge MG$$

is a map of (commutative) S-algebras.

Let $\mu : MG \longrightarrow E$ be a map of S-algebras, e.g.

id, $MU \to kU$, $MSpin \to kO$, $MString \to tmf$.

Let $\phi \colon E \wedge E \longrightarrow E$ be the product.

The composite Thom isomorphism map

$$\begin{array}{c} MG \wedge E \\ \big\downarrow{\scriptstyle \Delta\wedge\mathrm{id}} \\ \Sigma^\infty BG_+ \wedge MG \wedge E \\ \big\downarrow{\scriptstyle \mathrm{id}\wedge\mu\wedge\mathrm{id}} \\ \Sigma^\infty BG_+ \wedge E \wedge E \\ \big\downarrow{\scriptstyle \mathrm{id}\wedge\phi} \\ \Sigma^\infty BG_+ \wedge E \end{array}$$

is an equivalence of S-algebras.

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