LECTURE 1

NOTES BY SHAWN HENRY IN 2007; REVISED BY PETER MAY IN 2011

1. Introduction

We begin with a definition and a theorem. Do not worry if you do not understand; everything will be defined and explained in due course. Let K be a field.

Definition 1.1. *An n-Topological Quantum Field Theory (n-TQFT) is a symmetric monoidal functor* $F\colon$ ***n-Cob*** $\to$ ***VectK***.

Theorem 1.2. *The category of 2-TQFTs is equivalent to the category of commutative Frobenius K-algebras.*

Proof. Left to the reader. □

Just kidding. Understanding this definition and proving this theorem will be the main subject of this course, but we will meander on the way to getting there. We need to understand at least three things. The category of n-TQFT's, the category of commutative Frobenius K-algebras, and the idea of an equivalence of categories. The first is mostly topology, the second is algebra, and the third is categorical language. However, we already need a fair amount of categorical language to explain the first two. The theorem is a comparison of apples and oranges, and it says that these apples and oranges are in some sense the same. We will start with the categorical language that makes sense of such a comparison between two kinds of mathematical things, but let's first give a quick algebraic definition that may make the target category at least mildly accessible.

Definition 1.3. *An algebra A over a field K is a vector space A over K together with a bilinear associative and unital multiplication $A\times A \to A$, written $(a,b)\to ab$, such that for $a,b\in A$ and $k\in K$, $(ka)b = k(ab) = a(kb)$.*

After the introduction to category theory we will explain the relevant linear algebra, using categorical conceptualization, and only after that will we turn to topology and TQFT's.

2. Categories

Definition 2.1. *A category $\mathfrak{C}$ is a collection of objects $(X,Y,Z,\ldots)$, denoted $Ob(\mathfrak{C})$, together with, for each pair (X,Y) of objects of $\mathfrak{C}$, a set of morphisms (alias maps) $f\colon X\to Y$, denoted $\mathfrak{C}(X,Y)$, satisfying the following: For each object X of $\mathfrak{C}$ there is a given identity morphism $1_X\colon X\to X$ and for each triple (X,Y,Z) of objects of $\mathfrak{C}$ and pair of morphisms $f\colon X\to Y$, $g\colon Y\to Z$ there is given a morphism $g\circ f\colon X\to Z$. This is viewed as a composition law*

$$\circ\colon \mathfrak{C}(Y,Z)\times\mathfrak{C}(X,Y)\to\mathfrak{C}(X,Z).$$

We require $1_Y \circ f = f = f \circ 1_X$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphism h with domain Z. Remark: We do not require that $Ob(\mathfrak{C})$ be a set; it may be a proper class. If it is a set, we say that the category is small.

Example: The collection of all sets is a category denoted **SET**. Its morphisms are functions.

Example: The collection of all groups is a category denoted **GRP**. Its morphisms are group homomorphisms.

Example: The collection of all topological spaces is a category denoted **TOP**. Its morphisms are continuous functions.

Example: A monoid is a set M with an associative binary operation and an identity element. Note that in a category $\mathfrak{C}$ the composition law $\circ$ on the set $\mathfrak{C}(X, X)$ is just such a binary operation with identity element 1_X. Therefore a monoid is a category with one object. A category can be thought of as a "monoid with many objects".

In any category, there is a notion of isomorphism. It answers the sensible version of the question "when are two things the same". The nonsensical version would have the answer "when they are equal". The sensible version interprets "things" to mean objects of a category" and the sensible answer is that we think of two objects as essentially the same when they are isomorphic.

Definition 2.2. *A morphism $f\colon X \to Y$ in a category $\mathfrak{C}$ is called an isomorphism if there is a morphism $g\colon Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.*

Exercise: If a morphism f has a left inverse and a right inverse then it is an isomorphism and the left and right inverses coincide.

Definition 2.3. *A groupoid is a category in which every morphism is an isomorphism. Just as a monoid can be defined to be a category with just one object, a group can be defined to be a groupoid with just one object. Similarly, a groupoid can be thought of as a "group with many objects".*

3. Functors

A morphism of categories is called a functor.

Definition 3.1. *Let $\mathfrak{C}, \mathfrak{D}$ be categories. A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ consists of a rule that assigns to each object X of $\mathfrak{C}$ an object FX of $\mathfrak{D}$, together with, for each pair (X,Y) of objects of $\mathfrak{C}$, a function $F\colon \mathfrak{C}(X,Y) \to \mathfrak{D}(FX, FY)$, written $f \mapsto Ff$, such that $F(1_X) = 1_{FX}$ and $F(g \circ f) = Fg \circ Ff$.*

Exercise: If f is an isomorphism in $\mathfrak{C}$, then Ff is an isomorphism in $\mathfrak{D}$.

Example: The collection of all small categories is a category denoted **CAT**. Its morphisms $F\colon \mathfrak{C} \to \mathfrak{D}$ are the functors. Remark: we insist that categories be small for the purposes of this definition to ensure that we have a well-defined set and not just a proper class of functors between any two categories.

Example: The abelianization of a group G is the group G/[G,G] where [G,G] is the commutator subgroup, that is, the subgroup generated by the set $\{ghg^{-1}h^{-1} \mid g, h \in G\}$. Abelianization defines a functor $A\colon$ **GRP** $\to$ **AB** where **AB** is the category of abelian groups.

Definition 3.2. *A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is said to be faithful if the function*

$$F\colon \mathfrak{C}(X, Y) \to \mathfrak{D}(FX, FY)$$

is injective for every pair (X,Y) of objects of $\mathfrak{C}$.

Definition 3.3. *A functor* $F\colon \mathfrak{C} \to \mathfrak{D}$ *is said to be full if the function*

$$F\colon \mathfrak{C}(X,Y) \to \mathfrak{D}(FX,FY)$$

is surjective for every pair (X,Y) of objects of $\mathfrak{C}$.

Definition 3.4. *A functor* $F\colon \mathfrak{C} \to \mathfrak{D}$ *is said to be an isomorphism of categories if there is a functor* $G\colon \mathfrak{D} \to \mathfrak{C}$ *such that* FG *is the identity functor on* $\mathfrak{D}$ *and* GF *is the identity functor on* $\mathfrak{C}$.

Definition 3.5. *A functor* $F\colon \mathfrak{C} \to \mathfrak{D}$ *is said to be essentially surjective if, for every object* Y *of* $\mathfrak{D}$, *there is an object* X *of* $\mathfrak{C}$ *and an isomorphism* $FX \cong Y$.

Definition 3.6. *A functor* $F\colon \mathfrak{C} \to \mathfrak{D}$ *is said to be an equivalence of categories if it is full, faithful, and essentially surjective.*

Definition 3.7. *A subcategory of a category* $\mathfrak{C}$ *is a category that consists of some of the objects and some of the morphisms of* $\mathfrak{C}$; *it is a full subcategory if it contains all of the morphisms in* $\mathfrak{C}$ *between any two of its objects. The skeleton of a category* $\mathfrak{C}$ *is a a full subcategory which contains exactly one object from each isomorphism class of objects of* $\mathfrak{C}$.

Proposition 3.8. *The inclusion of a skeleton of* $\mathfrak{C}$ *in* $\mathfrak{C}$ *is an equivalence of categories.*

Proof. We understand a skeleton to be a full subcategory, so the inclusion is full and faithful, and it is essentially surjective by definition. □

4. Natural Transformations

Naturally, there are also morphisms of functors.

Definition 4.1. *Let* $F, F'\colon \mathfrak{C} \to \mathfrak{D}$ *be functors. A natural transformation*

$$\eta\colon F \to F'$$

is a collection of maps $\eta_X\colon FX \to F'X$, *one for each object* X *of* $\mathfrak{C}$, *such that the following diagram commutes for each map* $f\colon X \to Y$ *in* $\mathfrak{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ F'X & \xrightarrow[F'f]{} & F'Y. \end{array}$$

Definition 4.2. *A natural transformation* η *is said to be a natural isomorphism if each of the maps* η_X *is an isomorphism.*

Example: A finite dimensional vector space V over K is naturally isomorphic to its double dual DDV, where $DV = \mathrm{Hom}(V,K)$. That is, there is a natural isomorphism $\mathrm{Id} \to DD$ on the category of finite dimensional vector spaces over K.

Definition 4.3. *A functor* $F\colon \mathfrak{C} \to \mathfrak{D}$ *is said to be an equivalence of categories if there is a functor* $G\colon \mathfrak{D} \to \mathfrak{C}$ *and there are natural isomorphisms* $FG \to Id_{\mathfrak{D}}$ *and* $GF \to Id_{\mathfrak{C}}$. *Note that an isomorphism of categories is an equivalence, but not conversely.*

These notes were revised by somebody with a silly liking for theorems of the following form.

Proposition 4.4. *An equivalence of categories is an equivalence of categories.*

That is, the two definitions of what it means for a functor to be an equivalence of categories are equivalent. It is easy to show that if F is an equivalence of categories in our second sense, then F is certainly full, faithful, and essentially surjective. The converse requires a little work and a use of the axiom of choice that the fastidious set-theoretically minded reader may find distasteful: the first step is to choose an object $G(D)$ in $\mathfrak{C}$ such that $FG(D)$ is isomorphic to D for each object D of $\mathfrak{D}$. The second is to choose an isomorphism $\eta\colon FG(D) \to D$ for each D. We then define G on morphisms so as to make η a natural isomorphism by definition, using that

$$F\colon \mathfrak{C}(G(D), G(D')) \longrightarrow \mathfrak{D}(FG(D), FG(D'))$$

is a bijection. For a morphism $g\colon D \to D'$ in $\mathfrak{D}$, we define $Gg\colon G(D) \to G(D')$ to be F^{-1} of the composite

$$FG(D) \xrightarrow{\ \eta\ } D \xrightarrow{\ f\ } D' \xrightarrow{\ \eta^{-1}\ } FG(D').$$

The reader can see how composition must be defined in order to complete the proof.

Note that the proof of Proposition 3.8 is easy using our first definition of an equivalence of categories, but not so easy using the second. Proposition 4.4 has real force: it makes it easy to recognize equivalences of categories (in the second sense) when we see them. We shall eventually construct a functor from the category of 2-TQFT's to the category of commutative Frobenius algebras over K and prove that it is full, faithful, and essentially surjective.

5. The fundamental groupoid of a space

We illustrate the idea of translating topology into algebra by explaining the fundamental groupoid. These quick notes will leave the diagrams presented in the talk to the reader's imagination.

We construct a functor $\Pi\colon \mathbf{TOP} \to \mathbf{GPD}$, where $\mathbf{GPD}$ is the full subcategory of $\mathbf{CAT}$ whose objects are groupoids. For a topological space X, the objects of the category ΠX are the points of the space X. Let $I = [0,1]$ be the unit interval. A path $p\colon x \to y$ is a continuous map $p\colon I \to X$ such that $p(0) = x$ and $p(1) = y$. Two paths p and p' from x to y are said to be equivalent if there is a map $h\colon I \times I \to X$ such that, for all $t \in I$,

$$h(t,0) = x, \quad h(t,1) = y, \quad h(0,t) = p(t) \quad \text{and} \quad h(1,t) = p'(t).$$

h is said to be a homotopy from p to p' through paths from x to y. The set of morphisms $x \to y$ in ΠX is the set of equivalence classes of paths $x \to y$. For a path $q\colon y \to z$, the composite $q \circ p$ is defined by

$$(q \circ p)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le 1/2 \\ q(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Define id_x to be the constant path at x, $\mathrm{id}_x(t) = x$. Define $p^{-1}(t) = p(1-t)$. Composition is not associative or unital, but it becomes so after passage to equivalence classes. Verifications that we leave to the reader (or the first chapter of "A concise course in algebraic topology"[1]) show that ΠX is a well-defined groupoid. For a

[1] [95] at http://www.math.uchicago.edu/ may/PAPERSMaster.html

map $f\colon X \to Y$, we define Πf on objects by sending x to $f(x)$ and on morphisms by sending the equivalence class $[p]$ to the equivalence class $[f \circ p]$. Then Π is a well-defined functor.

If we fix basepoints, we get a functor that is perhaps more familiar. The fundamental group of X at the basepoint x is the group $\pi_1(X, x)$ given by the morphisms $x \to x$ in the groupoid ΠX. If we define $\mathbf{TOP}_*$ to be the category of spaces X with a chosen basepoint x and maps $f\colon X \to Y$ that preserve basepoints, $f(x) = y$, then π_1 gives a functor from based spaces to groups, called the fundamental group functor. Its construction is the first step towards algebraic topology.

Exercise: By definition, $\pi_1(X, x)$, regarded as a category with a single object x, is a full subcategory of ΠX. Show that if X is path connected, then $\pi_1(X, x)$ is a skeleton of ΠX. Thus the essential information in ΠX is captured by the fundamental group.