# LECTURE 2

NOTES BY SHAWN HENRY IN 2007, REVISED BY PETER MAY IN 2011

## 1. Finitely Generated Abelian Groups

We discuss the fundamental theorem of abelian groups to give a concrete illustration of when something that seems natural is not.

**Definition 1.1.** *An abelian group $A$ is said to be finitely generated if there are finitely many elements $a_1, ..., a_q \in A$ such that, for any $x \in A$, there are integers $k_1, ..., k_q$ such that $x = \sum_{i=1}^{q} k_i a_i$. The finitely generated group $A$ is said to be free of rank $q$ if the $a_i$ can be so chosen that $x = \sum_{i=1}^{q} k_i' a_i$ implies $k_i = k_i'$ for $1 \leq i \leq q$. That is, $x$ can be expressed uniquely as a linear combination of the $a_i$; the set $\{a_i\}$ is then said to be a basis for $A$.*

Let $\mathbb{N}$ denote the positive integers (natural numbers).

**Definition 1.2.** *Let $A$ be an abelian group. The torsion subgroup of $A$, denoted $T(A)$, is the set $T(A) = \{a \in A \mid \exists n \in \mathbb{N}$ such that $na = 0\}$.*

**Definition 1.3.** *An abelian group $A$ is said to be torsion-free if $T(A) = \{0\}$.*

**Lemma 1.4.** *Let $A$ be an abelian group. Then $A/T(A)$ is torsion-free.*

*Proof.* We leave this as an exercise for the reader. □

**Theorem 1.5.** *If $A$ is a finitely generated torsion-free abelian group that has a minimal set of generators with $q$ elements, then $A$ is isomorphic to the free abelian group of rank $q$.*

*Proof.* By induction on the minimal number of generators of $A$. If $A$ is cyclic (that is, generated by one non-zero element), the conclusion is clear. Suppose that the result holds for all finitely generated torsion-free abelian groups with a minimal set of generators having less than $q$ elements. Suppose $A$ is torsion-free and has a minimal set of $q$ generators, say $a_1, \cdots, a_q$. If $mb = a_i$ for any $i$ and some $m \in \mathbb{N}$, we may replace $a_i$ by $b$ and still have a set of generators, so we may as well assume that $mb = a_i$ implies $m = 1$. We claim that $\{a_i\}$ is a basis for $A$. Let $a = a_1$. We claim that the quotient $A/\mathbf{Z}a$ is torsion free. Suppose not, say $m\bar{c} = 0$ where $\bar{c}$ is the image of $c \in A$ in $A/\mathbf{Z}a$. Then $mc = na$ for some $n \in \mathbf{Z}$. We can assume that $n \geq 1$ and that $m$ and $n$ are relatively prime, say $rm + sn = 1$ for integers $r$ and $s$. Let $b = ra + sc$. Then

$$mb = mra + msc = rma + sna = a,$$

hence $m = 1$ and $\bar{c} = 0$. Now $A/\mathbf{Z}a_1$ is torsion-free, and the set of $q - 1$ elements $\bar{a}_i$, $i \geq 2$, is a minimal set of generators and hence a basis for $A/\mathbf{Z}a$. To check our first claim, it suffices to show that $\sum k_i a_k = 0$ implies $k_i = 0$ for each $i$, and this is now an immediate verification. □

**Definition 1.6.** *Let $A$ be an abelian group, and let $B$ and $C$ be subgroups of $A$. We say that $A$ is the internal direct sum of $B$ and $C$, denoted $A = B \oplus C$, if $A = B + C$ and $B \cap C = \{0\}$ where $B + C = \{b + c | b \in B \text{ and } c \in C\}$.*

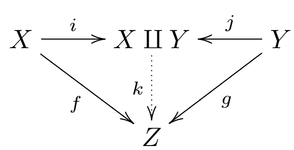
**Example**: A free abelian group of rank $q$ is the internal direct sum of $q$ cyclic subgroups, each isomorphic to $\mathbf{Z}$.

**Definition 1.7.** *Let $\mathfrak{C}$ be a category and let $X$ and $Y$ be objects of $\mathfrak{C}$. A morphism $f : X \to Y$ is said to be a monomorphism when, for any object $Z$ of $\mathfrak{C}$ and any pair of morphisms $i, j : Z \to X$, if $f \circ i = f \circ j$ then $i = j$.*

**Definition 1.8.** *Let $\mathfrak{C}$ be a category and let $X$ and $Y$ be objects of $\mathfrak{C}$. A morphism $f : X \to Y$ is said to be an epimorphism when, for any object $Z$ of $\mathfrak{C}$ and any pair of morphisms $i, j : Y \to Z$, if $i \circ f = j \circ f$ then $i = j$.*

**Exercise**: Show that monomorphisms and epimorphisms of abelian groups are what you think they are: homomorphisms that are one to one or onto.

**Definition 1.9.** *Let $X$ and $Y$ be objects of a category $\mathfrak{C}$. The coproduct of $X$ and $Y$, is an object $X \amalg Y$ together with homomorphisms $i \colon X \to X \amalg Y$ and $j \colon Y \to X \amalg Y$ with the universal property that for any object $Z$ and morphisms $f : X \to Z$ and $g : Y \to Z$, there is a unique homomorphism $k : X \amalg Y \to Z$ making the following diagram commute:*




*It follows that $i$ and $j$ are monomorphisms.*

**Remark**: This is an example of a definition by a universal property. Anything defined in such a fashion is unique up to isomorphism if it exists, but such an object need not exist in a given category; one must give a construction of such an object and prove that it satisfies the universal property.

**Definition 1.10.** *Let $A$ and $B$ be abelian groups. The coproduct of $A$ and $B$ is called the (external) direct sum of $A$ and $B$ and again written $A \oplus B$. It is constructed explicitly so that if we identify $A$ and $B$ with their images under $i$ and $j$, then $A \oplus B$ is the internal direct sum of $A$ and $B$.*

**Theorem 1.11.** *Let $A$ be a finitely generated abelian group. Then there is an isomorphism $f \colon T(A) \oplus A/T(A) \to A$.*

*Proof.* We have what is called an exact sequence

$$0 \longrightarrow T(A) \xrightarrow{i} A \xrightarrow{q} A/T(A) \longrightarrow 0,$$

where $i$ is the inclusion (a monomorphism) and $q$ is the quotient homomorphism (an epimorphism). Since $A/T(A)$ is torsion free and finitely generated, it is free on a finite set of generators $q(a_i)$. Mapping $q(a_i)$ to $a_i$ specifies a homomorphism $j \colon A/T(A) \to A$. Then $A$ is the internal direct sum of the image $T(A)$ of $i$ and the image $B$ of $j$. Indeed, $T(A) \cap B = 0$ since $B$ is torsion free. For $a \in A$, $q(a) = \sum_i k_i q(a_i)$ and $a - \sum_i k_i a_i$ is in $T(A)$, hence $A = T(A) + B$. $\square$

**Corollary 1.12.** *Every finitely generated abelian group is the direct sum of a finite group and a free abelian group of rank $q$ for some $q \geq 0$.*

*Proof.* Since $A$ is finitely generated, $T(A)$ is a finite group. □

**Exercise**: Let $A$ be a finite abelian group. Then $T(A) = \bigoplus_{\text{p prime}} T_p(A)$ where $T_p(A) = \{a \in A | p^q a = 0 \text{ for some } q \in \mathbb{Z}\}$.

The description of a finitely generated abelian group as the direct sum of a free abelian subgroup and the finite subgroups $T_p(A)$ is a version of the fundamental theorem of abelian groups. In fact, one can go further and prove that each $T_p(A)$ is a finite direct sum of cyclic groups of order a power of $p$.

## 2. Naturality

The short exact sequence above is a functor of $A$. If $f\colon A \to B$ is a homomorphism of abelian groups, we have an induced map of short exact sequences

$$\begin{CD}
0 @>>> T(A) @>i>> A @>q>> A/T(A) @>>> 0 \\
@. @VfVV @VfVV @V\bar{f}VV \\
0 @>>> T(B) @>i>> B @>q>> B/T(B) @>>> 0
\end{CD}$$

In more detail $T(A)$, $A$, and $A/T(A)$ are functors of $A$ and $i$ and $q$ are natural transformations. However, the splitting of $A$ as a direct sum is *not* natural. To explain this, we shall use the following result about the identity functor on the category of abelian groups or of finitely generated abelian groups.

**Lemma 2.1.** *Every natural transformation $\mu\colon Id \to Id$ is multiplication by some $n \in \mathbf{Z}$.*

*Proof.* Let $A$ be an abelian group and $a \in A$. Let $f\colon \mathbf{Z} \to A$ be the homomorphism given by $f(1) = a$. Naturality gives the commutative diagram

$$\begin{CD}
\mathbf{Z} @>f>> A \\
@V\mu_{\mathbf{Z}}VV @VV\mu_AV \\
\mathbf{Z} @>>f> A
\end{CD}$$

Applied to $1 \in \mathbf{Z}$, this gives

$$\mu_A(a) = \mu_A(f(1)) = f(\mu_{\mathbf{Z}}(1)) = \mu_{\mathbf{Z}}(1)f(1) = \mu_{\mathbf{Z}}(1) \cdot a.$$

The promised integer $n$ is $\mu_{\mathbf{Z}}(1)$. □

**Theorem 2.2.** *There is no non-zero natural homomorphism $\nu\colon A/T(A) \to A$, let alone a natural monomorphism.*

*Proof.* Assume $\nu$ is a natural homomorphism. The composite $\mu = \eta \circ q\colon Id \to Id$ is also natural. By the lemma, there exists $n$ such that $\nu$ is multiplication by $n$. We claim that $n = 0$. If not, consider $A = \mathbf{Z}/2n\mathbf{Z}$. Multiplication by $n$ is a nonzero homomorphism $A \to A$, but $A/T(A) = 0$ and therefore $\nu\colon A \to A$ is zero. This is a contradiction. □

Naturality is not as obvious a notion as it might appear at first sight!