ASSOCIATIVITY CONSTRAINTS IN MONOIDAL CATEGORIES

MITYA BOYARCHENKO

Abstract. In this note we explain a few ways of thinking about and working with the "associativity constraints" in monoidal categories. It is not uncommon to ignore these constraints, i.e., to pretend that all monoidal categories one is working with are *strictly associative*. Nevertheless, for some people (including myself) it might be psychologically comforting to know that associativity constraints are in fact something rather simple and understandable, as well as to learn the precise reason why they can often be ignored in practice.

Contents

1. Introduction/overview

1.1. For clarity of exposition, in this note we will ignore all foundational questions having to do with unit objects. In practice, it is often important to know what the unit object in a given monoidal category is, and to understand its structure as explicitly as possible. On the other hand, the focus of this note is on the *axioms* defining a monoidal category. The "rule of thumb" in the axiomatics underlying the theory of monoidal categories, braided monoidal categories, symmetric monoidal categories, etc., is that the axioms involving the unit object are usually much easier to understand and spell out than the other axioms (such as the axioms for the associativity constraint and the braiding/symmetry constraint).

1.2. In accordance with this comment, in the part of the text having to do with associativity constraints themselves, we will work not with monoidal categories, but rather with *semigroupal categories* (their definition will be given in §3). To explain the terminology, let me recall the notion of a *semigroup*: it is simply a set M equipped with an associative binary operation $M \times M \longrightarrow M$. Thus the definition of a semigroup[1] is obtained from that of a monoid by discarding the requirement of the existence of a multiplicative identity. Likewise, the definition of a semigroupal category is obtained from the definition of a monoidal category by discarding the requirement of the existence of a unit object and all the axioms involving it.

1.3. As usual, I will begin the discussion of the associativity constraints with a toy model, namely, a thorough discussion of the associativity requirement in the definition of a semigroup, see §2. While this requirement seems to be completely trivial and standard, we will see that a careful analysis of it allows one to define the associativity constraints in a semigroupal category in a very natural way.

1.4. Based on this toy model, the definition of semigroupal categories (or, what more or less amounts to the same thing, the definition of associativity constraints for bifunctors on categories) will be given and discussed in §3.

1.5. In §5 I will explain the main reason why people often ignore the associativity constraints in practice, i.e., why they pretend that all monoidal (or semigroupal) categories are strictly associative. Namely, every monoidal category $\mathscr{M}$ is equivalent to a strictly associative one, constructed from $\mathscr{M}$ in a very natural manner. I will sketch a proof of this fact that is different from (and, in many ways, more enlightening than) the argument one finds in many textbooks on this subject. A toy model for this proof is presented in §4, which also contains some elementary preliminary remarks about a process in algebra known as "categorification".

1.6. Finally, in §6, I will explain one reason why sometimes thinking about associativity constraints is important, from my personal (hence biased) point of view. Namely, in at least one of the approaches to the theory of quantum groups, associativity constraints on certain monoidal categories turn out to be highly nontrivial.

1.7. Acknowledgement. The approach to explaining and motivating the definition of associativity constraints in monoidal categories that I use in this note has been very much inspired by the book "Lectures on Tensor Categories and Modular Functors" by B. Bakalov and A.A. Kirillov, Jr. The discussion of the relationship between associativity and symmetry constraints on a monoidal category, and maps from a free semigroup (respectively, a free commutative semigroup) into this category, has been borrowed from §1.4 of Deligne's exposé "La formule de dualité globale" (Exp. XVIII) in SGA 4, vol. 3. Finally, the idea of the proof of the fact that every monoidal category is equivalent to a strictly associative and strictly unital one, presented in §5, was explained to me by D. Kazhdan.

[1]Unfortunately, this might lead the reader to confusion: sometimes people use the term "semigroup" to describe what we call a "monoid". For example, as far as I understand, for an expert in functional analysis, semigroups are required to have units. On the other hand, as far as I know, in the part of algebra that has to do with monoidal categories, the definitions of the terms "monoid" and "semigroup" that I gave are the universally accepted ones.

2. Associativity in semigroups

2.1. In this section we will explain several points of view on the definition of associativity for a binary operation on a set, starting from the most elementary one and proceeding to the more sophisticated ones. In the next section we will see that each of these viewpoints has a counterpart in the context of semigroupal categories.

2.2. Let us recall the most standard sequence of definitions. If M is a set, a *binary operation* on M is a map of sets $M\times M \longrightarrow M$, which we will denote by $(a,b)\longmapsto ab$. We will think of it, and refer to it, as "multiplication" in M. Such an operation is said to be *associative* if $(ab)c=a(bc)$ for all $a,b,c\in M$. A *semigroup* is a set equipped with an associative binary operation.

2.3. In practice, however, one uses (almost always without explicitly mentioning it) the following consequence of associativity. Suppose M is a set with a binary operation, and let $a_1,\dots,a_n$ be any finite collection of (not necessarily distinct) elements of M, where $n\geq 1$. Consider the string of symbols $a_1a_2a_3\cdots a_n$. We do not know how to make sense of this string (if $n>2$), because we only know how to multiply *two* elements of M. Thus, in order to make sense out of this expression, we need to insert some parentheses in a meaningful way. It is clear that we need precisely $n-2$ pairs of parentheses in order to be able to evaluate the result in M.

Here are two examples of meaningful arrangements of parentheses when $n=5$:

$$(((a_1a_2)a_3)a_4)a_5 \qquad \text{and} \qquad (a_1a_2)((a_3a_4)a_5)$$

Here is an example of a meaningless arrangement of parentheses:

$$(a_1(a_2)a_3)a_4(a_5)$$

Going back to the general case, observe that the associativity requirement for the given binary operation on M is equivalent to the requirement that *for any two meaningful arrangements of parentheses in the string $a_1a_2\cdots a_n$, the results of multiplication using these two arrangements yield <u>the same</u> element of M.*

2.4. If you think about it, you will realize that, in fact, in all the computations with semigroups and monoids, one always uses precisely this property of multiplication. Of course, the original associativity axiom is a special case of this one for $n=3$. The fact that the special case $n=3$ implies all other cases is, of course, almost trivial; nevertheless, it is a prototype of many "coherence theorems" in the theory of monoidal categories and related areas.

2.5. I will now explain a more sophisticated point of view on the associativity axiom for a binary operation. It might seem completely unjustified at the moment. However, this point of view leads to the cleanest and most transparent approach to associativity constraints in semigroupal categories, as we will see in §3.

2.6. The key idea is that the notion of a *free semigroup* can be defined without first defining what a semigroup is! Namely, let $x_1, \dots, x_n$ be symbols, and let FS_n denote the set of all (nonempty) words formed by these symbols. In other words, elements of FS_n are strings of the form $x_{k_1} x_{k_2} x_{k_3} \cdots x_{k_t}$, where $t \geq 1$ is arbitrary and $1 \leq k_i \leq n$ for all $1 \leq i \leq t$. Here are some examples of elements of the set FS_7: x_5, $x_1 x_1 x_3 x_1$, $x_7 x_7 x_7$, $x_6 x_7 x_7 x_4 x_2 x_3 x_1 x_2 x_2 x_2 x_5$.

The multiplication in FS_n is given by concatenation of strings, e.g.: $(x_1 x_1 x_3 x_1) \cdot (x_7 x_7 x_7) = x_1 x_1 x_3 x_1 x_7 x_7 x_7$. Of course, associativity of multiplication is built into our definition of FS_n, which is precisely why we do not have to mention it while defining FS_n. We will call FS_n the *free semigroup on the symbols* $x_1, \dots, x_n$.

2.7. Next, observe that the definition of a homomorphism of semigroups (or, if you prefer, monoids, or groups, etc.) does not involve the associativity property of multiplication. Therefore, if we have two sets, M and N, each of which is equipped with a binary operation, whether associative or not, it is not unreasonable to define a *homomorphism* from M to N to be a map of sets $f : M \longrightarrow N$ satisfying $f(ab) = f(a)f(b)$ for all $a, b \in M$.

2.8. Main claim. Let M be any set equipped with a binary operation. This binary operation is associative (i.e., makes M into a semigroup) if and only if, for every finite collection of elements $a_1, \dots, a_n \in M$, there exists a (necessarily unique) homomorphism $f : FS_n \longrightarrow M$ such that $f(x_j) = a_j$.

2.9. Remark. This application of a construction of a “free something” looks rather different from the applications you may be used to. Namely, FS_n is only free in the usual (universalproperty) sense in the category of semigroups, not in the category of all sets with binary operations. Instead, we are using FS_n to “test” whether a given object of the larger category lies within the smaller category.

2.10. The proof of the claim above is essentially trivial. The “only if” direction follows from the universal property of the free semigroup FS_n which we invite the reader to formulate. The “if” direction, which is the one we care about, is left as an instructive exercise for the reader who does not think it is obvious.

3. Associativity in semigroupal categories

3.1. Bifunctors. Let us fix a category $\mathscr{M}$. We define a *bifunctor* on $\mathscr{M}$,

$$\otimes : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}, \qquad \text{written} \qquad (X, Y) \longmapsto X \otimes Y,$$

to be the following collection of data. First, a rule which to every pair of objects X, Y of $\mathscr{M}$ assigns an object $X \otimes Y$ of $\mathscr{M}$. Second, a rule which to every pair of morphisms $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ in $\mathscr{M}$ assigns a morphism $f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$.

These data are required to satisfy axioms analogous to those appearing in the definition of a functor, namely, we insist that $\mathrm{id}_X \otimes \mathrm{id}_Y : X \otimes Y \longrightarrow X \otimes Y$ equals the identity morphism $\mathrm{id}_{X\otimes Y}$, and, if we have morphisms $X \xrightarrow{f} X' \xrightarrow{f'} X''$ and $Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$ in $\mathscr{M}$, then $(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$.

Of course, this is where the term "bifunctor" comes from; in fact, it is easy to define the notion of a *direct product* of two categories, and with this terminology, a bifunctor on $\mathscr{M}$ is the same thing as a functor from the direct product category $\mathscr{M} \times \mathscr{M}$ to the category $\mathscr{M}$.

3.2. We think of the bifunctor $\otimes$ as some sort of "generalized tensor product" (hence the notation). In fact, one of the main examples for us is the bifunctor $\mathrm{Vect}_k \times \mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k$ given by $(V, W) \longmapsto V \otimes_k W$, where Vect_k is the category of vector spaces over a given field k (with linear maps as morphisms), and $\otimes_k$ denotes the usual tensor product of vector spaces over k. If you have not seen this example before, you should use either the universal property of tensor products, or the explicit construction of tensor products, to verify that $\otimes_k$ can indeed be made a bifunctor in a canonical way. (In other words, describe the action of $\otimes_k$ on pairs of linear maps between pairs of vector spaces.)

3.3. Let us return to an abstract category $\mathscr{M}$ and a bifunctor $\otimes : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$. What should it mean for $\otimes$ to be associative? As a first approximation, we could require "associativity on the nose", i.e., that for any triple of objects $X, Y, Z \in \mathscr{M}$, we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, and for any triple of morphisms $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$, $Z \xrightarrow{h} Z'$ in $\mathscr{M}$, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$. If this condition is satisfied, we will call $\mathscr{M}$ a *strictly associative semigroupal category*. Observe that if $\mathscr{M}$ is also small, then, in particular, the set of objects of $\mathscr{M}$ becomes a semigroup[2] with respect to the operation $\otimes$.

3.4. The experienced reader will object that this condition is too strong: when working with categories, one usually should not require that two objects be *equal*. (This is analogous to the distinction between an isomorphism of categories and an equivalence of categories, the latter being a much more practical notion.) This is almost true; nevertheless, there are some interesting examples of strictly associative monoidal categories. Some types of such examples are described and used in §§4,5. Here we mention a more trivial, but still important, type of examples. Let M be a semigroup. Let us denote by $\underline{M}$ the category whose set of objects is the underlying set of M, and which has no non-identity morphisms. The multiplication on M induces a bifunctor $\otimes : \underline{M} \times \underline{M} \longrightarrow \underline{M}$ (we do not have to specify what this

[2] An analogous statement is false if $\mathscr{M}$ is merely a semigroupal category. In this case, what does inherit a natural semigroup structure is the set of *isomorphism classes* of objects of $\mathscr{M}$.

bifunctor does to morphisms because there aren't any nontrivial ones). It is obvious that this bifunctor is strictly associative.

Conversely, if $\mathscr{M}$ is any small strictly associative semigroupal category which is *discrete*, i.e., has no non-identity morphisms, then $\mathscr{M}$ arises in this way, namely, if M is the set of objects of $\mathscr{M}$, which is a semigroup, as explained before, then $\mathscr{M}$ is naturally isomorphic (not just equivalent!) to $\underline{M}$, in a way compatible with $\otimes$.

While all of this might seem like a trivial and pointless exercise, we will see that, in fact, this construction is very useful in giving a clear formulation of the axioms for an associativity constraint.

3.5. Let us now discuss another type of associativity requirement which, in particular, applies to the category of vector spaces with their usual tensor product. If $\mathscr{M}$ is a category equipped with a bifunctor $\otimes : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$, then, if we wish to say that $\otimes$ is associative in some sense, we must at least require that $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathscr{M}$. This condition alone suffices to ensure that if $\mathscr{M}$ is small, then the set of isomorphism classes of objects of $\mathscr{M}$ becomes a semigroup with respect to the operation induced by $\otimes$.

However, in practice it is almost always insufficient to require the existence of *some* isomorphism between $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$: the notion one obtains is too non-rigid to be useful. Instead, what one does is requires that the desired isomorphisms be specified as part of the data.

3.6. Take one. Let us make the following provisional definition. If $(\mathscr{M}, \otimes)$ is a pair consisting of a category $\mathscr{M}$ and a bifunctor on $\mathscr{M}$, a *pre-associativity constraint*[3] for $\otimes$ is a collection of isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z),$$

one for each triple of objects $X, Y, Z \in \mathscr{M}$, which are *trifunctorial* in the following sense. Given morphisms $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ and $Z \xrightarrow{h} Z'$ in $\mathscr{M}$, the diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "(f \otimes g) \otimes h"'] & X \otimes (Y \otimes Z) \arrow[d, "(f' \otimes g') \otimes h'"] \\
(X' \otimes Y') \otimes Z' \arrow[r, "\alpha_{X',Y',Z'}"] & X' \otimes (Y' \otimes Z')
\end{tikzcd}$$

is commutative.

[3]This terminology is probably non-standard. I chose this term simply to emphasize that an associativity constraint is a pre-associativity constraint satisfying certain extra conditions.

3.7. Remark. Some readers may prefer to use the word “natural” instead of “functorial” or “trifunctorial” in this situation, since α is really a natural transformation between the two obvious trifunctors. However, the term “functorial” seems to be more standard in modern literature, perhaps due to the fact that the word “natural” has many different meanings (including the colloquial one).

3.8. Take two. It turns out, however, that the notion of a pre-associativity constraint is still not strong enough to be useful in real life. From the point of view of our approach to the notion of associativity, it is easy to understand the reason why. Suppose that we consider four or more objects of $\mathscr{M}$, call them $X_1, \dots, X_n$. There are many different ways of inserting parentheses in the expression $X_1 \otimes \cdots \otimes X_n$ in a meaningful way so that the result can be evaluated in $\mathscr{M}$. Just as for semigroups, we would like the results obtained from two different arrangements of parentheses to be isomorphic in $\mathscr{M}$. It is clear that this is forced by the existence of a pre-associativity constraint. However, there is now an extra twist: *the results of different arrangements of parentheses could be isomorphic in many different ways!*

3.9. Example. Let us consider one example of this phenomenon. It will turn out (cf. §3.12) that this is *the only* example one needs to worry about; however, this fact is a rather nontrivial theorem. Fix four objects $X, Y, Z, W \in \mathscr{M}$. Here are two of the five possible meaningful arrangements of parentheses in $X \otimes Y \otimes Z \otimes W$:

$$((X \otimes Y) \otimes Z) \otimes W \qquad \text{and} \qquad X \otimes (Y \otimes (Z \otimes W)).$$

Using a pre-associativity constraint α for $\otimes$, one can construct two different isomorphisms between these two objects, which are captured by the diagram

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "\mathrm{id}_X \otimes \alpha_{Y,Z,W}"] \\
((X \otimes Y) \otimes Z) \otimes W \arrow[u, "\alpha_{X,Y,Z} \otimes \mathrm{id}_W"] \arrow[dr, "\alpha_{X\otimes Y,Z,W}"'] & & X \otimes (Y \otimes (Z \otimes W)) \\
& (X \otimes Y) \otimes (Z \otimes W) \arrow[ur, "\alpha_{X,Y,Z\otimes W}"'] &
\end{tikzcd} \tag{1}$$

More precisely, the two isomorphisms between $((X \otimes Y) \otimes Z) \otimes W$ and $X \otimes (Y \otimes (Z \otimes W))$ that we mentioned are the composition of the three “top” arrows in the diagram (1), and the composition of the three “bottom” arrows in the diagram (1).

3.10. It is natural, however, to require that the isomorphism between two different arrangements of parentheses in the expression $X_1 \otimes \cdots \otimes X_n$ should not depend on the particular order in which we compose the associativity constraints. Thus, for instance, one should insist that the diagram (1) commutes. This requirement is known as the *pentagon axiom*, for obvious reasons.

3.11. Semigroupal categories. With these comments in mind, the following definitions should now be completely natural.

(a) If $(\mathscr{M}, \otimes)$ is a pair consisting of a category $\mathscr{M}$ and a bifunctor $\otimes$ on $\mathscr{M}$, an *associativity constraint* for $\otimes$ is a pre-associativity constraint α for $\otimes$ satisfying the following axiom. For any finite collection $X_1, \dots, X_n$ of objects of $\mathscr{M}$, let us consider two (not necessarily different) meaningful arrangements of parentheses in the expression $X_1 \otimes \cdots \otimes X_n$, and let $Y, Z \in \mathscr{M}$ denote the results of evaluating the corresponding expressions in $\mathscr{M}$. Then, any two isomorphisms between Y and Z obtained by composing the various isomorphisms coming from α, as well as their inverses, are equal to each other.

(b) A *semigroupal category* is a triple $(\mathscr{M}, \otimes, \alpha)$ consisting of a category $\mathscr{M}$, a bifunctor $\otimes$ on $\mathscr{M}$, and an associativity constraint α for $\otimes$. In this setup, $\otimes$ is sometimes called the *semigroupal structure*, or the *semigroupal functor*, on $\mathscr{M}$.

3.12. MacLane's coherence theorem. As mentioned before, it turns out that in the definition of a semigroupal category one only needs to impose the pentagon axiom: all other consistency requirements follow from it. This is a nontrivial theorem due to S. MacLane. We state it as follows.

Suppose $\mathscr{M}$ is a category, $\otimes$ is a bifunctor on $\mathscr{M}$, and α is a pre-associativity constraint for $\otimes$. If the diagram (1) *commutes for all quadruples X, Y, Z, W of objects of $\mathscr{M}$, then α is an associativity constraint for $\otimes$.*

We briefly sketch one way of proving this result, in the presence of unit objects, in §5.7. Note that this is NOT how MacLane proved the theorem. MacLane's proof, while more sophisticated, also yields important extra information. Unfortunately, a proper explanation of MacLane's approach is beyond the scope of these notes.

`THE REST OF THIS SECTION CAN BE SKIPPED AT THE FIRST READING`

3.13. Take three. We now change our point of view completely and forget about the notion of a (pre-)associativity constraint for a moment. Let us return to the idea explained at the end of §2. Recall, in particular, that we introduced the free semigroup FS_n on n symbols $x_1, \dots, x_n$. Let us apply the construction of §3.4 to it. Thus we obtain a discrete strictly associative semigroupal category $\underline{FS}_n$.

Next, let us introduce another ugly and non-standard term. We will call a *pre-semigroupal category* any pair $(\mathscr{M}, \otimes)$ consisting of a category $\mathscr{M}$ and a bifunctor $\otimes$ on it. We would like to define a "homomorphism" of pre-semigroupal categories, $\Phi : (\mathscr{M}, \otimes) \longrightarrow (\mathscr{N}, \otimes')$. As before, it would be "wrong" (both philosophically and practically) to require Φ to be a functor which is compatible with the two bifunctors "on the nose", i.e., that $\otimes' \circ (\Phi \times \Phi) = \Phi \circ \otimes$ as bifunctors $\mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{N}$. Instead, we require the existence of a natural isomorphism between the two compositions.

More concretely, we define a 1-*morphism between pre-semigroupal categories* $(\mathscr{M}, \otimes)$ and $(\mathscr{N}, \otimes')$ to be the following collection of data. First, we need to specify a functor $\Phi : \mathscr{M} \longrightarrow \mathscr{N}$ between the underlying categories. Second, we need to specify a collection of bifunctorial isomorphisms $\Phi_{X,Y} : \Phi(X) \otimes' \Phi(Y) \xrightarrow{\simeq} \Phi(X \otimes Y)$ for all pairs of objects X, Y of $\mathscr{M}$. At this moment we impose no further axioms.

3.14. Comments on the terminology. The reader is probably used to the philosophy that "a morphism between categories is called a functor". However, I am against using the term "functor between (pre-)semigroupal categories", because it is ambiguous, precisely for the reason that we are trying to define something which is more than just a functor between the underlying categories. On the other hand, I decided to use the term "1-morphism" rather than just "morphism" to emphasize the idea that semigroupal categories form something more interesting than just a category. (This is the point where one starts moving on to "higher categories".)

A comment of a different nature is that in the definition I gave above, it is sometimes useful to relax the requirement that $\Phi_{X,Y}$ is an isomorphism, and simply require that it is a morphism in the category $\mathscr{N}$. In this case, what we called a 1-morphism above becomes a "strong 1-morphism". However, in this text we are only interested in the strong 1-morphisms, so we omit the adjective "strong".

3.15. A new approach to associativity constraints. Let $(\mathscr{M}, \otimes)$ be a pre-semigroupal category. In this subsection we explain a different way of thinking about associativity constraints on $\mathscr{M}$, motivated by the discussion at the end of §2.

3.15.1. Consider first a triple of objects $X_1, X_2, X_3 \in \mathscr{M}$. Suppose we wish to find a 1-morphism of pre-semigroupal categories $\Phi : \underline{FS}_3 \longrightarrow \mathscr{M}$ such that $\Phi(x_j) = X_j$ for $j = 1, 2, 3$. What kind of extra structure would we need?

3.15.2. First of all, Φ must come equipped with isomorphisms

$$\Phi_{x_1,x_2} : X_1 \otimes X_2 \xrightarrow{\simeq} \Phi(x_1x_2), \quad \Phi_{x_1x_2,x_3} : \Phi(x_1x_2) \otimes X_3 \xrightarrow{\simeq} \Phi(x_1x_2x_3),$$

$$\Phi_{x_2,x_3} : X_2 \otimes X_3 \xrightarrow{\simeq} \Phi(x_2x_3), \quad \Phi_{x_1,x_2x_3} : X_1 \otimes \Phi(x_2x_3) \xrightarrow{\simeq} \Phi(x_1x_2x_3).$$

3.15.3. Now, the following composition

$$\alpha_{X_1,X_2,X_3} := (\mathrm{id}_{X_1} \otimes \Phi_{x_2,x_3})^{-1} \circ \Phi^{-1}_{x_1,x_2x_3} \circ \Phi_{x_1x_2,x_3} \circ (\Phi_{x_1,x_2} \otimes \mathrm{id}_{X_3})$$

yields an isomorphism $\alpha_{X_1,X_2,X_3} : (X_1 \otimes X_2) \otimes X_3 \xrightarrow{\simeq} X_1 \otimes (X_2 \otimes X_3)$.

3.15.4. In addition, suppose that for any triple X_1, X_2, X_3 of objects of $\mathscr{M}$, we are given a 1-morphism $\Phi^{X_1,X_2,X_3} : \underline{FS}_3 \longrightarrow \mathscr{M}$ of pre-semigroupal categories such that $x_j \longmapsto X_j$ for $j = 1, 2, 3$, which is *trifunctorial* with respect to X_1, X_2, X_3 in the obvious sense. Then α is trifunctorial as well, i.e., α is a pre-associativity constraint.

3.15.5. The moral of the story is that if we want to extend this picture to all $n \geq 3$, i.e., if we want to specify, for each finite collection $X_1, \ldots, X_n$ of objects of $\mathscr{M}$, a 1-morphism of pre-semigroupal categories $\Phi^{X_1,\ldots,X_n} : \underline{FS}_n \longrightarrow \mathscr{M}$ such that $x_j \longmapsto X_j$ for $1 \leq j \leq n$, so that the 1-morphisms $\Phi^{X_1,\ldots,X_n}$ are functorial with respect to the X_j's and are compatible with each other for various n's in a suitable sense, then what we need is precisely an *associativity* constraint for $\otimes$. Unfortunately, a precise formulation of these ideas would take us too far afield.

3.16. What about symmetry constraints? We end this section with a few comments about symmetric/braided monoidal categories. Once again, due to the lack of space, we will not even be able to give a definition of these terms. However, the reader somewhat familiar with this theory will know that a *symmetry constraint*[4] on a semigroupal category $(\mathscr{M}, \otimes, \alpha)$ is a collection β of bifunctorial isomorphisms $\beta_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X$, one for each pair of objects X, Y of $\mathscr{M}$, satisfying suitable conditions of compatibility with α, known as the *hexagon axioms* (whereas there is only one pentagon axiom, there are two different hexagon axioms). If this is the only requirement one imposes, one obtains the notion of a *braided semigroupal category.* If, in addition, one requires $\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathscr{M}$, one obtains the more restrictive notion of a *symmetric semigroupal category.*

3.17. Remarks.

(1) Of course, these notions have variants for monoidal categories, where one requires additional conditions of compatibility with the unit object and the left and right unit constraints. Thus one obtains the notions of braided monoidal categories and symmetric monoidal categories. However, as I mentioned before, the axioms involving the unit object are far less important than the pentagon axiom and the hexagon axioms.

(2) The terms "braided" and "symmetric" are partially justified by the following remark, on which, unfortunately, we will not be able to elaborate. Let $\mathscr{M}$ be a braided semigroupal category, and choose $X \in \mathscr{M}$. Write $X^{\otimes n} = X \otimes X \otimes \cdots \otimes X$ (n factors). Note that we are implicitly using the associativity constraint to suppress the parentheses in this expression. One can use the symmetry constraint for $\otimes$ to define a natural action of the *braid group* B_n in n strands on the object $X^{\otimes n}$. If $\mathscr{M}$ is a *symmetric* monoidal category, this action factors through the quotient $B_n \twoheadrightarrow S_n$, where S_n is the symmetric group on n letters.

(3) The previous remark becomes especially useful when the category $\mathscr{M}$ has more structure; for instance, when we have a faithful functor from $\mathscr{M}$ to the category of vector spaces over a field k. In this way one obtains interesting linear representations of braid groups. This story is closely related to quantum groups.

[4]Some people prefer to use the term "braiding" in this situation.

3.18. A word of caution. One might have a natural desire to repeat the story explained in §3.15 for symmetric semigroupal (or monoidal) categories. Namely, for each $n \in \mathbb{N}$, let FCS_n denote the free *commutative* semigroup on n symbols $x_1, \dots, x_n$. We get the corresponding strictly associative semigroupal category $\underline{FCS}_n$, and we could ask for a structure on a given pre-semigroupal category $(\mathscr{M}, \otimes)$ which would allow us to define, for every finite collection $X_1, \dots, X_n$ of objects of $\mathscr{M}$, a 1-morphism of pre-semigroupal categories $\underline{FCS}_n \longrightarrow \mathscr{M}$ which takes $x_j \mapsto X_j$ for all $1 \leq j \leq n$, and which is natural in the appropriate sense.

Naively, one would expect the answer to be related to symmetry constraints on $\mathscr{M}$. This is indeed the case, but the following remark is very important.

The answer to the question raised above involves not merely symmetric monoidal categories, but <u>*strictly symmetric*</u> *monoidal categories. The strict symmetry requirement is much stronger than the symmetry requirement, and it fails for most of the interesting monoidal categories one encounters with practice, such as the category of vector spaces with the monoidal structure given by the tensor product, or even with the one given by the direct sum of vector spaces.*

3.19. Let us briefly explain what is going on here. Suppose we choose two objects $X_1, X_2 \in \mathscr{M}$ and a 1-morphism $\Phi : \underline{FCS}_2 \longrightarrow \mathscr{M}$ taking $x_1 \longmapsto X_1$ and $x_2 \longmapsto X_2$. Then we get an isomorphism $\beta_{X_1,X_2} : X_1 \otimes X_2 \xrightarrow{\simeq} X_2 \otimes X_1$ as the composition

$$X_1 \otimes X_2 \xrightarrow{\Phi_{x_1,x_2}} \Phi(x_1 x_2) = \Phi(x_2 x_1) \xrightarrow{\Phi^{-1}_{x_2,x_1}} X_2 \otimes X_1.$$

Now suppose that for any finite collection of objects $X_1, \dots, X_n \in \mathscr{M}$, we are given a 1-morphism $\Phi^{X_1,\dots,X_n} : \underline{FCS}_n \longrightarrow \mathscr{M}$ which takes $x_j \longmapsto X_j$ for all $1 \leq j \leq n$, and, moreover, the morphisms $\Phi^{X_1,\dots,X_n}$ are natural with respect to the X_j's and are compatible with each other for various n's, in the appropriate sense[5]. Then the isomorphisms β_{X_1,X_2} certainly define a symmetry constraint for $\otimes$, but it makes $\mathscr{M}$ into a *strictly symmetric* monoidal category, which means that not only is the composition $\beta_{Y,X} \circ \beta_{X,Y}$ always equal to $\mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathscr{M}$, but also $\beta_{X,X} = \mathrm{id}_{X \otimes X}$ for all $X \in \mathscr{M}$. The reason is simply that, by definition, if $\Phi : \underline{FCS}_1 \longrightarrow \mathscr{M}$ is the 1-morphism taking x_1 to X, then $\beta_{X,X} = \Phi^{-1}_{x_1,x_1} \circ \Phi_{x_1,x_1} = \mathrm{id}_{X \otimes X}$.

3.20. The requirement that $\beta_{X,X} = \mathrm{id}_{X \otimes X}$ for all $X \in \mathscr{M}$ is extremely strong, as we will see momentarily. On the other hand, the previous discussion implies that we cannot really think of non-strict symmetry constraints in any way that is similar to how we think of associativity constraints. In some sense, this is one of the reasons why the theory of symmetry constraints is related to some interesting and nontrivial mathematics (e.g., to invariants of knots).

[5]For our purposes, it is not necessary to understand the precise definition of these words.

3.21. To get some feeling for what it means for a monoidal category to be strictly symmetric, let us first consider the category of vector spaces over a given field k, with direct sum as the monoidal structure. This category is symmetric. For any pair of vector spaces V, W, the symmetry isomorphism $\beta_{V,W} : V \oplus W \xrightarrow{\simeq} W \oplus V$ is given by $(v, w) \longmapsto (w, v)$. If V is any nonzero vector space and $v \in V \setminus \{0\}$, then $\beta_{V,V}(v, 0) = (0, v) \neq (v, 0)$, which shows that $\beta_{V,V} \neq \mathrm{id}_{V \oplus V}$, and hence this category is not strictly symmetric.

As another example, consider the category of vector spaces over k with tensor product as the monoidal structure. The symmetry isomorphism $\beta_{V,W} : V \otimes_k W \xrightarrow{\simeq} W \otimes_k V$ is induced by $v \otimes w \longmapsto w \otimes v$. If $\dim_k V \geq 2$ and $v, w \in V$ are linearly independent elements of V, then $v \otimes w$ and $w \otimes v$ are linearly independent in $V \otimes_k V$ (because they are easily seen to be a part of a basis!), and hence, *a fortiori*, $v \otimes w \neq w \otimes v$. Thus, again, $\beta_{V,V} \neq \mathrm{id}_{V \otimes V}$ and the category is not symmetric.

Note, on the other hand, that if $\dim_k V \leq 1$, then, in fact, we do have $v \otimes w = w \otimes v$ in $V \otimes_k V$ for all $v, w \in V$. Hence the category of 1-*dimensional* vector spaces over k, with tensor product as the monoidal structure, is in fact *strictly symmetric*. For a generalization of this example, look at the category of line bundles on a real or complex manifold, or on an algebraic variety. In each case take the monoidal functor to be the tensor product of line bundles, and take the associativity and symmetry constraints to be the obvious ones. You get a strictly symmetric monoidal category. Examples of this sort, while being far less common than more general braided and symmetric monoidal categories, are still important.

4. Another approach: categorification

4.1. The prototypical example of a monoid is the set of all maps $X \longrightarrow X$, where X is a given set, where the binary operation is given by composition of maps. Note that now we do have a unit element, namely, the identity map $\mathrm{id}_X : X \longrightarrow X$. A variant of this construction can be obtained as follows. If X is equipped with some kind of "extra structure", then in most of the meaningful situations, the set of all maps $X \longrightarrow X$ preserving this structure is closed under composition and contains id_X; thus, this set of maps is also a monoid[6].

4.2. A trivial but important observation is that this example is universal, in the sense that *every* monoid M arises in this way. Namely, if M is a monoid, let us consider the right action of M on itself by right multiplication[7], and let E be the set of all maps $f : M \longrightarrow M$ that commute with this action. More concretely,

[6] The standard examples one should keep in mind are: X is a group (or a ring, or an algebra, etc.) and we look at all group (respectively, ring or algebra) homomorphisms $X \longrightarrow X$; or, X is a topological space (respectively, a manifold) and we look at all continuous (respectively, smooth) maps $X \longrightarrow X$; and so on.

[7] This action is the "extra structure" mentioned in the previous paragraph.

the requirement is that $f(xm) = f(x)m$ for all $m, x \in M$. It is obvious that E is a monoid under composition of maps. *We claim that E is canonically isomorphic to M as a monoid.* Indeed, we have a homomorphism $M \xrightarrow{\phi} E$ defined by $m \mapsto \lambda_m$, where $\lambda_m(x) = mx$. *Note that both the verification of the fact that $\lambda_m \in E$ and the verification of the identity $\lambda_{m_1 m_2} = \lambda_{m_1} \circ \lambda_{m_2}$ use the associativity of the multiplication in M.* Moreover, we have $\lambda_m(1) = m$, and if $f \in E$ is arbitrary, then $f(x) = f(1 \cdot x) = f(1) \cdot x$ for all $x \in M$. This shows that the map $\psi : E \longrightarrow M$, $f \mapsto f(1)$, is a two-sided inverse to ϕ. Thus ϕ is an isomorphism, as desired.

4.3. There is a game[8] which many people like to play, called "categorification". At the most basic level it consists of starting with some sort of algebraic structure, replacing the sets appearing in the definition of this structure with categories, replacing maps of sets with functors, and so on, and changing the original axioms to get a meaningful and useful notion. Sometimes this turns out to be nontrivial to accomplish. The most basic "obstruction" to playing this game is that whereas equality of two elements of a set is a perfectly reasonable requirement, equality of two objects is a category is often the wrong condition to impose. On the other hand, simply requiring two objects to be isomorphic, without specifying what the isomorphism should look like, is often too weak and just does not do the job.

4.4. In practice, the most typical "compromise" consists of specifying the necessary isomorphisms as part of the data, and then one has to formulate additional axioms which, roughly speaking, describe how these isomorphisms relate to each other. The reason this is *not* a trivial exercise is that these additional axioms have no counterparts in the "uncategorified" situation.

Of course, the main example to keep in mind is precisely the definition of a semigroupal category introduced in §3 above.

4.5. Let us try to categorify our discussion of the universal example of a monoid. Thus we first need to fix a category $\mathscr{C}$, as the replacement for our set X. Next, self-maps of X should be replaced by $\mathrm{Funct}(\mathscr{C})$, the collection of all functors $F : \mathscr{C} \longrightarrow \mathscr{C}$. Note that, whereas self-maps of X merely form a set, $\mathrm{Funct}(\mathscr{C})$ is a category, the morphisms being natural transformations of functors. Of course, this is consistent with the philosophy of categorification. Finally, we still know how to compose functors, which defines the "composition bifunctor"

$$\mathrm{Funct}(\mathscr{C}) \times \mathrm{Funct}(\mathscr{C}) \longrightarrow \mathrm{Funct}(\mathscr{C}), \qquad (F, G) \longmapsto F \circ G.$$

[8]Don't get me wrong: this "game" has often lead to extremely important advances and discoveries in various parts of algebra. It is also often played by very brilliant mathematicians.

4.6. Exercise. If you are not very comfortable with such constructions yet, write out the precise definition of the bifunctor structure, i.e., describe how composition acts on natural transformations in each of the arguments, and verify that the axioms defining a bifunctor hold in this situation.

4.7. Main point. The composition of functors turns $\mathrm{Funct}(\mathscr{C})$ into a *strictly associative* and *strictly unital* monoidal category. This is clear, because composition of functors is associative "on the nose", in the sense that if $F, G, H : \mathscr{C} \longrightarrow \mathscr{C}$ are functors, then the functors $(F \circ G) \circ H$ and $F \circ (G \circ H)$ are not just canonically isomorphic: they are *equal* in the literal sense. Similarly, we have the identity functor $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}$, and for any functor $F : \mathscr{C} \longrightarrow \mathscr{C}$, we have $\mathrm{Id}_{\mathscr{C}} \circ F = F = F \circ \mathrm{Id}_{\mathscr{C}}$.

4.8. The monoidal category $\mathrm{Funct}(\mathscr{C})$ is in fact the prototypical example of a strictly associative and strictly unital monoidal category. In a way, this is one of the reasons why the definition of and the axioms for associativity constraints in a general monoidal category are not completely transparent. In fact, it is not so easy to give a "universal" example of a monoidal category without strict associativity.

4.9. The good news is that we explore this construction in a slightly different way to prove something very useful. Namely, let us recall the game we played to show that every monoid is canonically isomorphic to the monoid of endomorphisms of a certain set with extra structure. If we try to play the "categorified" version of this game, replacing a monoid with an arbitrary (not necessarily strictly associative) monoidal category, we will "discover" (as if by accident) the proof of the fact that every monoidal (or semigroupal) category is equivalent to a strictly associative one. This is explained in §5. As far as I understand, this proof is due to J. Bernstein.

5. How to make a monoidal category strictly associative?

5.1. In this section we will be more sketchy than in the previous ones. On one hand, writing out all the definitions very precisely would take up too much space, and would also obscure, to some extent, the understanding of the underlying ideas, which, I hope to convince you, are very simple and natural. On the other hand, this section is meant to be read by those who are unwilling to assume that all monoidal categories are strictly associative, in particular, by those who want to actually understand monoidal categories. Such a reader should work out the technical details on their own, as a very instructive exercise.

5.2. As we already mentioned, the main goal of this section is to sketch a proof of the following theorem: *Every monoidal category is equivalent, as a monoidal category, to a strictly associative and strictly unital one.*

5.3. The reader who is unfamiliar with unit objects should concentrate on the "strict associativity" part of the theorem. The construction we will explain works for any semigroupal category, even though to prove that it yields the correct result one has to use the unit object. Still, this is not essential for understanding the main ideas.

5.4. We pause for a moment to briefly discuss the notion of an equivalence for semigroupal (or monoidal categories). In §3 we defined the notion of a 1-morphism between pre-semigroupal categories $(\mathscr{M},\otimes)$ and $(\mathscr{N},\otimes')$. Now suppose we have semigroupal categories $(\mathscr{M},\otimes,\alpha)$ and $(\mathscr{N},\otimes',\beta)$, where α and β are the respective associativity constraints. A *1-morphism of semigroupal categories* between $(\mathscr{M},\otimes,\alpha)$ and $(\mathscr{N},\otimes',\beta)$ is a 1-morphism between the underlying pre-semigroupal categories, $(\mathscr{M},\otimes)\longrightarrow(\mathscr{N},\otimes')$, which satisfies the additional condition of being compatible with the associativity constraints α and β. The formulation of this condition is left as an exercise for the reader; there is only one way to state it that makes sense.

Now, for our purposes, an *equivalence of semigroupal categories* between $(\mathscr{M},\otimes,\alpha)$ and $(\mathscr{N},\otimes',\beta)$ is a 1-morphism of semigroupal categories $(\mathscr{M},\otimes,\alpha)\longrightarrow(\mathscr{N},\otimes',\beta)$ such that the underlying functor between the underlying categories, $\mathscr{M}\longrightarrow\mathscr{N}$, is an equivalence of categories.

Of course, the reader should immediately object that this notion does not look right. For example, how do we know that in this situation we can get a 1-morphism $(\mathscr{N},\otimes',\beta)\longrightarrow(\mathscr{M},\otimes,\alpha)$ in the opposite direction? Fortunately, all these issues are easily resolved. Namely, if $\Phi:(\mathscr{M},\otimes,\alpha)\longrightarrow(\mathscr{N},\otimes',\beta)$ is any 1-morphism which is an equivalence of semigroupal categories in the sense of our definition, then for *any* quasi-inverse to Φ, i.e., a functor $\Psi:\mathscr{N}\longrightarrow\mathscr{M}$ such that $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are isomorphic to the appropriate identity functors merely as functors, there exists a *unique* structure of a 1-morphism on Ψ such that $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are isomorphic to the appropriate identity functors *as 1-morphisms*.

As promised, we leave the formulation of the appropriate definitions and the proof of this result to the diligent reader. A similar statement is true in the context of monoidal categories.

5.5. Key idea. Let us fix a semigroupal category $(\mathscr{M},\otimes,\alpha)$. Following the discussion in the first half of §4, we would like to define the "category of endo-functors of $\mathscr{M}$ which commute with the right action of $\mathscr{M}$ on itself". As usual, we need to be careful when defining the word "commute". The correct definition looks as follows.

5.5.1. We will describe a certain semigroupal category, $\widetilde{\mathscr{M}}$, and leave the details of its construction to the reader. The objects of $\widetilde{\mathscr{M}}$ are pairs (F,δ), where $F:\mathscr{M}\longrightarrow\mathscr{M}$ is a functor and δ is a collection of bifunctorial isomorphisms

$$\delta_{X,Y}:F(X)\otimes Y\xrightarrow{\simeq}F(X\otimes Y),$$

one for each pair of objects $X, Y \in \mathscr{M}$. These isomorphisms are required to be compatible with the bifunctor $\otimes$ on $\mathscr{M}$ and with the associativity constraint α, in the sense that for $X, Y, Z \in \mathscr{M}$, the two possible isomorphisms between $F(X) \otimes (Y \otimes Z)$ and $F((X \otimes Y) \otimes Z)$ should coincide. We leave it to the reader to draw the appropriate pentagon-shaped diagram to express this requirement.

5.5.2. A morphism between two objects, (F, δ) and (F', δ'), in the category $\widetilde{\mathscr{M}}$, is a natural transformation $F \longrightarrow F'$ of the underlying functors which is compatible with δ and δ' in the obvious sense. The precise definition is left to the reader.

5.5.3. The monoidal structure on $\widetilde{\mathscr{M}}$ is induced by composition of functors. We leave it to the reader to formulate the precise definition and to verify that $\widetilde{\mathscr{M}}$ is a strictly associative and strictly unital monoidal category. The unit object in this category is the pair $(\mathrm{Id}, \mathrm{id})$, where Id is the identity functor on $\mathscr{M}$ and id is the collection of identity morphisms $X \otimes Y \longrightarrow X \otimes Y$.

5.5.4. Finally, note that we have a natural 1-morphism $\mathscr{M} \longrightarrow \widetilde{\mathscr{M}}$. At the level of objects it is defined as follows. Given $A \in \mathscr{M}$, the corresponding functor $F_A : \mathscr{M} \longrightarrow \mathscr{M}$ is given by $X \longmapsto A \otimes X$, and the corresponding collection of isomorphisms $\delta_{X,Y} := (A \otimes X) \otimes Y \xrightarrow{\simeq} A \otimes (X \otimes Y)$ is defined by $\delta_{X,Y} = \alpha_{A,X,Y}$, where α is the associativity constraint. To check that this pair (F_A, δ) is indeed an object of $\mathscr{M}$ one uses the pentagon axiom (cf. §3). Finally, it is not difficult to turn this assignment $A \longmapsto (F_A, \delta)$ into a functor between categories, and then into a 1-morphism between semigroupal categories.

5.6. Main point. As a very instructive exercise, prove that if we apply the construction above to a *monoidal* category $\mathscr{M}$, then the resulting 1-morphism $\mathscr{M} \longrightarrow \widetilde{\mathscr{M}}$ is an equivalence of categories. This will prove the theorem stated at the beginning of this section. Note that the existence of a unit object in $\mathscr{M}$ is necessary for this argument to work. (If $\mathscr{M}$ does not have a unit object, the functor $\mathscr{M} \longrightarrow \widetilde{\mathscr{M}}$ may even fail to be faithful, in general.)

5.7. Bonus! Use the ideas explained in this section to give a proof of MacLane's coherence theorem (§3.12), at least for monoidal categories. This is not hard. The key point is that in the construction of an equivalence between an arbitrary monoidal category and a strictly associative one that we presented, we only need to use the pentagon axiom. On the other hand, for a strictly associative monoidal category, there is clearly nothing to prove.

6. Why should we care about associativity constraints?

6.1. If you ask people for a reason why one should worry about associativity constraints in the first place, one of the most typical responses is that the tensor product of vector spaces is not associative "on the nose". In other words, if U, V, W are vector spaces over a field k, then the vector spaces $(U \otimes_k V) \otimes_k W$ and $U \otimes_k (V \otimes_k W)$ are *not equal* to each other; rather, one can construct a "canonical" isomorphism between them, namely, the one induced by the map $(u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w)$.

6.2. However, if this were literally the only reason, experts in the theory of monoidal categories would probably not attach very much importance to the definition of associativity constraints in general. In fact, there are certain areas of mathematics, in which monoidal categories play a permanent role, where something deeper is going on. For instance, in some cases one has a monoidal structure with an obvious associativity constraint, such as the one defined above for tensor products of vector spaces, but then one would like to change this constraint to a nontrivial one. In such situations, it becomes crucial to know precisely which axioms one needs to impose on associativity constraint in order to get meaningful objects.

6.3. Unfortunately, I do not know of any useful examples of this phenomenon that can be explained in an elementary way. Therefore, a proper understanding of the rest of this section will require much more background and/or mathematical maturity than the previous sections.

6.4. Let $\mathfrak{g}$ be a complex simple Lie algebra. The theory of "quantum groups", in one of its most basic forms, has to do with certain "deformations" of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. (At this moment, it is not necessary to understand the precise meaning of the word "deformation".) One approach to this theory, due to V. Drinfeld, consists of constructing directly the monoidal category of representations of the "quantum group" $\mathcal{U}_q(\mathfrak{g})$ (where q is the deformation parameter), in the following way. As a category, it coincides with the category of (finite dimensional) representations of $\mathfrak{g}$, or, equivalently, of $\mathcal{U}(\mathfrak{g})$. The monoidal structure is still given by the tensor product of representations of $\mathfrak{g}$. However, the *associativity constraint* is different: it is obtained from the "obvious" associativity constraint for tensor products, defined above, by a deformation depending on the parameter q.

6.5. WARNING. Due to the lack of space, I completely omitted the much more interesting part of the story having to do with deforming the *symmetry constraint* in a way compatible with the deformation of the associativity constraint. Indeed, the category of representations of $\mathcal{U}_q(\mathfrak{g})$ is not merely monoidal: it is *braided monoidal* (but almost never symmetric!). This part of the story has a lot to do with other interesting and important areas of mathematics, such as 3-dimensional topology, invariants of knots, etc. See, e.g., the book "Quantum Groups" by C. Kassel.

6.6. Further study. To conclude these notes, let me draw your attention to a beautiful expository article "Sackler Lectures" by J. Bernstein, available from the following URL: `http://www.arxiv.org/abs/q-alg/9501032`

In this article, Bernstein discusses three topics: tensor categories (which are, roughly speaking, braided monoidal categories equipped with a little bit of extra structure), quantum groups, and topological quantum field theories. He also explains the interplay between these three subjects. Understanding the article requires a lot of background in algebra, but if you are interested in any of these three subjects, I strongly recommend reading this article, since Bernstein is well known to be one of the absolutely best expositors of modern algebra. To learn more about these subjects, experts (for instance, P. Etingof) recommend the two books I mentioned previously (namely, the one about tensor categories and modular functors by Bakalov and Kirillov, and the one about quantum groups by Kassel).