# NOTES ON AND AROUND TQFT'S

## Contents

This is a collation of notes taken by Anna Marie Bohmann, John Lind, and Shawn Henry of REU talks given by Peter May in 2007. They were collated and slightly edited by May in 2011. All errors are to be ascribed to May. The note takers are identified in the section titles.

The notes are leisurely and non-linear. The same topics are returned to with increasing detail and differing emphases as the notes proceed. The individual sections are very short, and the table of contents above gives a guide. Since the order of sections has been rearranged, there may be inconsistencies of internal references.

The material in Sections 32–36 comes from lecture notes taken by Bohmann on a lecture given by Mitya Boyarchenko that was based on ideas of Graeme Segal. It attempts to relate the axioms for a TQFT to physics. This is purely motivational and plays no role in the rest of the course.

## 1. Introduction (Henry)

We begin with a definition and a theorem. Do not worry if you do not understand; everything will be defined and explained in due course. Let $K$ be a field.

**Definition 1.1.** An n-Topological Quantum Field Theory (n-TQFT) is a symmetric monoidal functor $F\colon \mathbf{n\text{-}Cob} \longrightarrow \mathbf{VectK}$.

**Theorem 1.2.** *The category of 2-TQFTs is equivalent to the category of commutative Frobenius K-algebras.*

*Proof.* Left to the reader. □

Just kidding. Understanding this definition and proving this theorem will be the main subject of this course, but we will meander on the way to getting there. We need to understand at least three things. The category of $n$-TQFT's, the category of commutative Frobenius $K$-algebras, and the idea of an equivalence of categories. The first is mostly topology, the second is algebra, and the third is categorical language. However, we already need a fair amount of categorical language to explain the first two. The theorem is a comparison of apples and oranges, and it says that these apples and oranges are in some sense the same. We will start with the categorical language that makes sense of such a comparison between two kinds of mathematical things, but let's first give a quick algebraic definition that may make the target category at least mildly accessible.

**Definition 1.3.** An algebra A over a field K is a vector space A over K together with an associative and unital bilinear multiplication $A \times A \longrightarrow A$, written $(a, b) \longrightarrow ab$, such that for $a, b \in A$ and $k \in K$, (ka)b = k(ab) = a(kb).

After the introduction to category theory we will explain the relevant linear algebra, using categorical conceptualization, and only after that will we turn to topology and TQFT's.

## 2. Categories (Henry)

**Definition 2.1.** A category $\mathfrak{C}$ is a collection of objects (X,Y,Z,...), denoted $Ob(\mathfrak{C})$, together with, for each pair (X,Y) of objects of $\mathfrak{C}$, a set of morphisms (alias maps) $f\colon X \longrightarrow Y$, denoted $\mathfrak{C}(X,Y)$, satisfying the following: For each object X of $\mathfrak{C}$ there is a given identity morphism $1_X\colon X \longrightarrow X$ and for each triple (X,Y,Z) of objects of $\mathfrak{C}$ and pair of morphisms $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ there is given a morphism $g \circ f\colon X \longrightarrow Z$. This is viewed as a composition law

$$\circ\colon \mathfrak{C}(Y,Z) \times \mathfrak{C}(X,Y) \longrightarrow \mathfrak{C}(X,Z).$$

We require $1_Y \circ f = f = f \circ 1_X$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphism h with domain Z. Remark: We do not require that $Ob(\mathfrak{C})$ be a set; it may be a proper class. If it is a set, we say that the category is small.

**Example**: The collection of all sets is a category denoted **SET**. Its morphisms are functions.

**Example**: The collection of all groups is a category denoted **GRP**. Its morphisms are group homomorphisms.

**Example**: The collection of all topological spaces is a category denoted **TOP**. Its morphisms are continuous functions.

**Example**: A monoid is a set M with an associative binary operation and an identity element. Note that in a category $\mathfrak{C}$ the composition law $\circ$ on the set $\mathfrak{C}(X,X)$ is just such a binary operation with identity element $1_X$. Therefore a monoid is a category with one object. A category can be thought of as a "monoid with many objects".

In any category, there is a notion of isomorphism. It answers the sensible version of the question "when are two things the same". The nonsensical version would have the answer "when they are equal". The sensible version interprets "things" to mean objects of a category" and the sensible answer is that we think of two objects as essentially the same when they are isomorphic.

**Definition 2.2.** A morphism $f\colon X \longrightarrow Y$ in a category $\mathfrak{C}$ is called an isomorphism if there is a morphism $g\colon Y \longrightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

**Exercise**: If a morphism f has a left inverse and a right inverse then it is an isomorphism and the left and right inverses coincide.

**Definition 2.3.** A groupoid is a category in which every morphism is an isomorphism. Just as a monoid can be defined to be a category with just one object, a group can be defined to be a groupoid with just one object. Similarly, a groupoid can be thought of as a "group with many objects".

## 3. Functors (Henry)

A morphism of categories is called a functor.

**Definition 3.1.** Let $\mathfrak{C}, \mathfrak{D}$ be categories. A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ consists of a rule that assigns to each object X of $\mathfrak{C}$ an object FX of $\mathfrak{D}$, together with, for each pair (X,Y) of objects of $\mathfrak{C}$, a function $F\colon \mathfrak{C}(X,Y) \longrightarrow \mathfrak{D}(FX,FY)$, written $f \mapsto Ff$, such that $F(1_X) = 1_{FX}$ and $F(g \circ f) = Fg \circ Ff$.

**Exercise**: If $f$ is an isomorphism in $\mathfrak{C}$, then $Ff$ is an isomorphism in $\mathfrak{D}$.

**Example**: The collection of all small categories is a category denoted **CAT**. Its morphisms $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ are the functors. Remark: we insist that categories be small for the purposes of this definition to ensure that we have a well-defined set and not just a proper class of functors between any two categories.

**Example**: The abelianization of a group G is the group G/[G,G] where [G,G] is the commutator subgroup, that is, the subgroup generated by the set $\{ghg^{-1}h^{-1} \mid g, h \in G\}$. Abelianization defines a functor $A\colon \mathbf{GRP} \longrightarrow \mathbf{AB}$ where $\mathbf{AB}$ is the category of abelian groups.

**Definition 3.2.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be faithful if the function

$$F\colon \mathfrak{C}(X,Y) \longrightarrow \mathfrak{D}(FX,FY)$$

is injective for every pair (X,Y) of objects of $\mathfrak{C}$.

**Definition 3.3.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be full if the function

$$F\colon \mathfrak{C}(X,Y) \longrightarrow \mathfrak{D}(FX,FY)$$

is surjective for every pair (X,Y) of objects of $\mathfrak{C}$.

**Definition 3.4.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be an isomorphism of categories if there is a functor $G\colon \mathfrak{D} \longrightarrow \mathfrak{C}$ such that FG is the identity functor on $\mathfrak{D}$ and GF is the identity functor on $\mathfrak{C}$.

**Definition 3.5.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be essentially surjective if, for every object Y of $\mathfrak{D}$, there is an object X of $\mathfrak{C}$ and an isomorphism $FX \cong Y$.

**Definition 3.6.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be an equivalence of categories if it is full, faithful, and essentially surjective.

**Definition 3.7.** A subcategory of a category $\mathfrak{C}$ is a category that consists of some of the objects and some of the morphisms of $\mathfrak{C}$; it is a full subcategory if it contains all of the morphisms in $\mathfrak{C}$ between any two of its objects. A skeleton of a category $\mathfrak{C}$ is a full subcategory which contains exactly one object from each isomorphism class of objects of $\mathfrak{C}$.

**Proposition 3.8.** *The inclusion of a skeleton of $\mathfrak{C}$ in $\mathfrak{C}$ is an equivalence of categories.*

*Proof.* We understand a skeleton to be a full subcategory, so the inclusion is full and faithful, and it is essentially surjective by definition. ☐

## 4. Natural transformations (Henry)

Naturally, there are also morphisms of functors.

**Definition 4.1.** Let $F, F'\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ be functors. A natural transformation

$$\eta\colon F \longrightarrow F'$$

is a collection of maps $\eta_X\colon FX \longrightarrow F'X$, one for each object $X$ of $\mathfrak{C}$, such that the following diagram commutes for each map $f\colon X \longrightarrow Y$ in $\mathfrak{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ F'X & \xrightarrow[F'f]{} & F'Y. \end{array}$$

**Definition 4.2.** A natural transformation $\eta$ is said to be a natural isomorphism if each of the maps $\eta_X$ is an isomorphism.

**Example**: A finite dimensional vector space $V$ over $K$ is naturally isomorphic to its double dual $DDV$, where $DV = \mathrm{Hom}(V, K)$. That is, there is a natural isomorphism $\mathrm{Id} \longrightarrow DD$ on the category of finite dimensional vector spaces over $K$.

**Definition 4.3.** A functor $F\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is said to be an equivalence of categories if there is a functor $G\colon \mathfrak{D} \longrightarrow \mathfrak{C}$ and there are natural isomorphisms $FG \longrightarrow \mathrm{Id}_{\mathfrak{D}}$ and $GF \longrightarrow \mathrm{Id}_{\mathfrak{C}}$. Note that an isomorphism of categories is an equivalence, but not conversely.

These notes were revised by somebody with a silly liking for theorems of the following form.

**Proposition 4.4.** *An equivalence of categories is an equivalence of categories.*

That is, the two definitions of what it means for a functor to be an equivalence of categories are equivalent. It is easy to show that if $F$ is an equivalence of categories in our second sense, then $F$ is certainly full, faithful, and essentially surjective. The converse requires a little work and a use of the axiom of choice that the fastidious set-theoretically minded reader may find distasteful: the first step is to choose an object $G(D)$ in $\mathfrak{C}$ such that $FG(D)$ is isomorphic to $D$ for each object $D$ of $\mathfrak{D}$. The second is to choose an isomorphism $\eta\colon FG(D) \longrightarrow D$ for each $D$. We then define $G$ on morphisms so as to make $\eta$ a natural isomorphism by definition, using that

$$F\colon \mathfrak{C}(G(D), G(D')) \longrightarrow \mathfrak{D}(FG(D), FG(D'))$$

is a bijection. For a morphism $g\colon D \longrightarrow D'$ in $\mathfrak{D}$, we define $Gg\colon G(D) \longrightarrow G(D')$ to be $F^{-1}$ of the composite

$$FG(D) \xrightarrow{\eta} D \xrightarrow{f} D' \xrightarrow{\eta^{-1}} FG(D').$$

The reader can see how composition must be defined in order to complete the proof.

Note that the proof of Proposition 3.8 is easy using our first definition of an equivalence of categories, but not so easy using the second. Proposition 4.4 has real force: it makes it easy to recognize equivalences of categories (in the second sense) when we see them. We shall eventually construct a functor from the category of 2-TQFT's to the category of commutative Frobenius algebras over $K$ and prove that it is full, faithful, and essentially surjective.

## 5. The fundamental groupoid of a space (May)

We illustrate the idea of translating topology into algebra by explaining the fundamental groupoid. These quick notes will leave the diagrams presented in the talk to the reader's imagination.

We construct a functor $\Pi\colon \mathbf{TOP} \longrightarrow \mathbf{GPD}$, where $\mathbf{GPD}$ is the full subcategory of $\mathbf{CAT}$ whose objects are groupoids. For a topological space $X$, the objects of the category $\Pi X$ are the points of the space $X$. Let $I = [0, 1]$ be the unit interval. A path $p\colon x \longrightarrow y$ is a continuous map $p\colon I \longrightarrow X$ such that $p(0) = x$ and $p(1) = y$. Two paths $p$ and $p'$ from $x$ to $y$ are said to be equivalent if there is a map $h\colon I \times I \longrightarrow X$ such that, for all $t \in I$,

$$h(t, 0) = x, \quad h(t, 1) = y, \quad h(0, t) = p(t) \quad \text{and} \quad h(1, t) = p'(t).$$

$h$ is said to be a homotopy from $p$ to $p'$ through paths from $x$ to $y$. The set of morphisms $x \longrightarrow y$ in $\Pi X$ is the set of equivalence classes of paths $x \longrightarrow y$. For a path $q\colon y \longrightarrow z$, the composite $q \circ p$ is defined by

$$(q \circ p)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le 1/2 \\ q(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Define $\mathrm{id}_x$ to be the constant path at $x$, $\mathrm{id}_x(t) = x$. Define $p^{-1}(t) = p(1-t)$. Composition is not associative or unital, but it becomes so after passage to equivalence classes. Verifications that we leave to the reader (or the first chapter of "A concise course in algebraic topology"[1]) show that $\Pi X$ is a well-defined groupoid. For a map $f\colon X \longrightarrow Y$, we define $\Pi f$ on objects by sending $x$ to $f(x)$ and on morphisms by sending the equivalence class $[p]$ to the equivalence class $[f \circ p]$. Then $\Pi$ is a well-defined functor.

If we fix basepoints, we get a functor that is perhaps more familiar. The fundamental group of $X$ at the basepoint $x$ is the group $\pi_1(X, x)$ given by the morphisms $x \longrightarrow x$ in the groupoid $\Pi X$. If we define $\mathbf{TOP}_*$ to be the category of spaces $X$ with a chosen basepoint $x$ and maps $f\colon X \longrightarrow Y$ that preserve basepoints, $f(x) = y$, then $\pi_1$ gives a functor from based spaces to groups, called the fundamental group functor. Its construction is the first step towards algebraic topology.

**Exercise**: By definition, $\pi_1(X, x)$, regarded as a category with a single object $x$, is a full subcategory of $\Pi X$. Show that if $X$ is path connected, then $\pi_1(X, x)$ is a skeleton of $\Pi X$. Thus the essential information in $\Pi X$ is captured by the fundamental group.

## 6. Natural transformations and homotopies (Bohmann)

It was remarked in class that the definition of a categorical equivalence is similar to that of a homotopy equivalence. We will formalize this intuition.

**Definition 6.1.** Let $X$ and $Y$ be topological spaces, and let $I = [0, 1]$ be the unit interval. Let $f, g : X \longrightarrow Y$ be continuous maps. A *homotopy* from $f$ to $g$ is a continuous map $h : X \times I \longrightarrow Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

That is, a homotopy is a continuous deformation of $f$ to $g$. When there exists a homotopy from $f$ to $g$, we say $f$ is homotopic to $g$ and write $f \simeq g$.

**Definition 6.2.** Two spaces $X$ and $Y$ are *homotopy equivalent* if there exist maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$.

*Example.* Euclidean space $\mathbb{R}^n$ is homotopy equivalent to a point just by contraction. Details left to the reader.

Now, why is this notion of homotopy equivalence like an equivalence of categories? Recall that an equivalence of two categories $\mathcal{C}$ and $\mathcal{D}$ is given by two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such there are natural isomorphisms $\eta : F \circ G \longrightarrow \mathrm{Id}_{\mathcal{D}}$ and $\mu : G \circ F \longrightarrow \mathrm{Id}_{\mathcal{C}}$. We can make this definition resemble the definition of a homotopy equivalence by turning natural transformations into "homotopies of categories".

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we define a category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $(X, Y)$ for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and whose morphisms are given by $(f, g) : (X, Y) \longrightarrow (X', Y')$ where $f : X \longrightarrow X'$ is a morphism in $\mathcal{C}$ and

[1] [95] at http://www.math.uchicago.edu/ may/PAPERSMaster.html

$g : Y \longrightarrow Y'$ is a morphism in $\mathcal{D}$. Let $\mathcal{I}$ be a category with two objects, 0 and 1, and one non-identity morphism $I : 0 \longrightarrow 1$. This category $\mathcal{I}$ will be analogous to the unit interval in our definition of homotopy.

Suppose we have functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\eta : F \longrightarrow G$. We get a functor $H : \mathcal{C} \times \mathcal{I} \longrightarrow \mathcal{D}$ by letting $H(X, 0) = F(X)$ and $H(X, 1) = G(X)$ for an object $X \in \mathcal{C}$, and by letting $H(f, \mathrm{Id}_0) = F(f)$ and $H(f, \mathrm{Id}_1) = G(f)$ for a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$. We further let $\eta_X = H(\mathrm{Id}_X, I) : F(X) \longrightarrow G(X)$. Composition of maps in the category $\mathcal{C} \times \mathcal{I}$, which is given component-wise, implies that $(f, \mathrm{Id}_1) \circ (\mathrm{Id}_X, I) = (f, I) = (\mathrm{Id}_Y, I)(f, \mathrm{Id}_0)$, and so in order to define $H(f, I)$, the following diagram must commute:

$$\begin{array}{ccc} H(X,0) & \xrightarrow{H(f,\mathrm{Id}_0)} & H(Y,0) \\ {\scriptstyle H(\mathrm{Id}_X,I)}\downarrow & \searrow & \downarrow{\scriptstyle H(\mathrm{Id}_Y,I)} \\ H(X,1) & \xrightarrow{H(f,\mathrm{Id}_1)} & H(Y,1) \end{array}$$

Then we can define $H(f, I)$ to be the diagonal map. But if we pass to our definitions in terms of the natural transformation $\eta$, the above diagram becomes:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\downarrow & \overset{H(f,I)}{\searrow} & \downarrow{\scriptstyle \eta_Y} \\ GX & \xrightarrow{Gf} & GY \end{array}$$

This is the commutivity diagram that defines a natural transformation, and so we see that the well-definedness of a functor $H : \mathcal{C} \times \mathcal{I} \longrightarrow \mathcal{D}$ is equivalent to the existence of a natural transformation $\eta : F \longrightarrow G$. This allows us to view natural transformations as "homotopies" from one functor to another, and thus we see that an equivalence of categories has the same form as a homotopy equivalence of spaces.

In fact, if we look only at small categories, there is a way of transforming a category into a topological space so that a functor goes to a continuous map and a natural transformation goes to a homotopy. If we view the cyclic group of order 2 as a category with one object and one nonidentity morphsim, this construction takes this group to $\mathbb{R}P^\infty$.

## 7. Adjoint functors (Bohmann)

We define the notion of adjoint functors, one of the crucial ideas in category theory.

**Definition 7.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $L : \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{D} \longrightarrow \mathcal{C}$ be functors. We say that $L$ is *left-adjoint* to $R$ and $R$ is *right-adjoint* to $L$ if, for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there exists a natural isomorphism

$$\mathcal{D}(LX, Y) \cong \mathcal{C}(X, RY).$$

Recall here that $\mathcal{C}(X, RY)$ is the set of morphisms $X \longrightarrow RY$ in the category $\mathcal{C}$, and an isomorphism in the category of sets is just a bijection.

By naturality of the isomorphism $\mathcal{D}(LX, Y) \cong \mathcal{C}(X, RY)$, we mean that for any maps $f : X' \longrightarrow X$ and $g : Y \longrightarrow Y'$, the diagram

$$\begin{array}{ccc} \mathcal{D}(LX, Y) & \xrightarrow{\cong} & \mathcal{C}(X, RY) \\ {\scriptstyle \mathcal{D}(Lf,g)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(f,Rg)} \\ \mathcal{D}(LX', Y') & \xrightarrow{\cong} & \mathcal{C}(X', RY') \end{array}$$

commutes, where the vertical maps are given by pre- and post-composition with $f$ and $g$. That is, for $\varphi : LX \longrightarrow Y$ in $\mathcal{D}(LX, Y)$, we let $\mathcal{D}(Lf, g)(\varphi) = g \circ \varphi \circ Lf : LX' \longrightarrow LX \longrightarrow Y \longrightarrow Y'$, and similarly, for $\psi : X \longrightarrow RY \in \mathcal{C}(X, RY)$, we let $\mathcal{C}(f, Rg)(\psi) = Rg \circ \psi \circ f : X' \longrightarrow X \longrightarrow RY \longrightarrow RY'$. We can then re-phrase our definition of what it means for $L$ and $R$ to be adjoint in terms of a natural transformation.

We define a functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}ets$ by $\mathrm{Hom}_{\mathcal{C}}(W, Z) = \mathcal{C}(W, Z)$ for any objects $W, Z$ in $\mathcal{C}$, and by defining $\mathrm{Hom}_{\mathcal{C}}(f, g)$, for morphisms $f : W' \longrightarrow W$ and $g : Z \longrightarrow Z'$, by letting $\mathrm{Hom}_{\mathcal{C}}(f, g)(\varphi) = f \circ \varphi \circ g$ for any $\varphi \in \mathcal{C}(W, Z)$. Then the condition that $L$ and $R$ be adjoint functors is the same as having a natural isomorphism between the functors $\mathrm{Hom}_{\mathcal{C}}(L(-), -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, R(-))$. You can check that the diagram above is precisely the diagram that defines a natural transformation.

Here is an alternative version of the definition of an adjoint pair of functors.

**Definition 7.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $L : \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{D} \longrightarrow \mathcal{C}$ be functors. We say that $L$ is *left-adjoint* to $R$ and $R$ is *right-adjoint* to $L$ if, for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there exist natural transformations $\eta\colon \mathrm{Id} \longrightarrow RL$ and $\epsilon\colon LR \longrightarrow \mathrm{Id}$ such that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the composites

$$LX \xrightarrow{L\eta} LRLX \xrightarrow{\epsilon} LX$$

and

$$RY \xrightarrow{\eta} RLRY \xrightarrow{R\epsilon} RY$$

are identity maps.

The maps $\eta$ and $\epsilon$ are called the unit and counit of the adjunction. We can write the required conditions on composites as triangular diagrams, and they are then called the triangular identities.

**Proposition 7.3.** *An adjoint pair of functors is an adjoint pair of functors.*

That is, the two definitions are equivalent. If we are given the natural isomorphism

$$\mathcal{D}(LX, Y) \cong \mathcal{C}(X, RY),$$

we define $\eta\colon X \longrightarrow RLX$ to be the map corresponding to the identity map of $X$ and define $\epsilon\colon LRY \longrightarrow Y$ to be the map corresponding to the identity map of $RY$. The triangular identities follow from naturality. If we are given $\eta$ and $\epsilon$ satisfying the triangular identities, we define the map $X \longrightarrow RY$ corresponding to $f\colon LX \longrightarrow Y$ to be the composite

$$X \xrightarrow{\eta} RLX \xrightarrow{Rf} RY$$

and we define the map $LX \longrightarrow Y$ corresponding to $g\colon X \longrightarrow RY$ to be the composite

$$LX \xrightarrow{Lg} LRY \xrightarrow{\epsilon} Y.$$

We leave it as an exercise for the reader to check that these give inverse isomorphisms as required to verify the first definition of an adjoint pair of functors.

As a first example, consider the functor $F : \mathcal{S}ets \longrightarrow \mathcal{A}b$ defined by taking the free abelian group on a set. Consider also the forgetful functor $U : \mathcal{A}b \longrightarrow \mathcal{S}ets$ that takes an abelian group to its underlying set. Then $U$ is a right-adjoint to $F$, and $F$ is a left adjoint to $U$. That is, for any abelian group $A$ and set $S$, we have a bijection $\mathcal{A}b(FS, A) \cong \mathcal{S}ets(S, UA)$. This can be seen as follows.

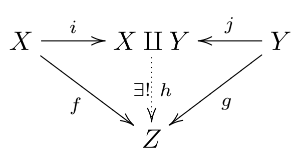
Recall that $FS = \{\sum_{i=1}^k n_i s_i \,|\, s_i \in S,\ n_i \in \mathbb{Z}\}$. Then given any map of sets $f : S \longrightarrow A$ (which is the same as a map $f : S \longrightarrow UA$), the map $f$ induces a homomorphism of abelian groups $\widetilde{f} : FS \longrightarrow A$ by extending linearly, so that $\widetilde{f}(\sum_{i=1}^k n_i s_i) = \sum_i^k n_i f(s_i)$. Conversely, any homomorphism of abelian groups $FS \longrightarrow A$ restricts to a map $S \longrightarrow A$. In other words, this adjunction is equivalent to the fact that a homomorphism from a free group is uniquely determined by where it sends basis elements. In fact, we have a similar adjunction between the free and forgetful functors for vector spaces and, more generally, for modules over rings.

This example is illustrative. Free and forgetful functors always come as adjoint pairs of this general form. We shall encounter many other examples of adjoint functors as we go on.

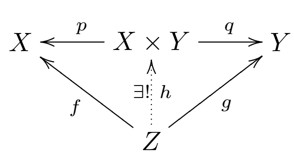
## 8. Coproducts, products, pushouts, and pullbacks (Bohmann)

We recall the definitions of the dual notions of coproduct and product in an arbitrary category $\mathcal{C}$.

**Definition 8.1.** The *coproduct* of objects $X$ and $Y$ in $\mathcal{C}$ is an object $X \amalg Y$ together with maps $i : X \longrightarrow X \amalg Y$ and $j : Y \longrightarrow X \amalg Y$ satisfying the following universal property. For any $Z \in \mathcal{C}$ and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, there exists a unique map $h : X \amalg Y \longrightarrow Z$ so that the following digram commutes:

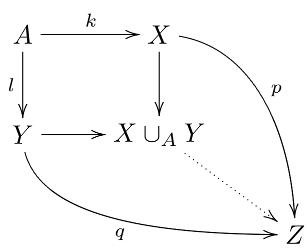



**Definition 8.2.** The *(cartesian) product* of objects $X$ and $Y$ in $\mathcal{C}$ is an object $X \times Y$ together with maps $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ satisfying the following universal property. For any $Z \in \mathcal{C}$ and maps $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$, there exists a unique map $h : Z \longrightarrow X \times Y$ so that the following digram commutes:




In general, the coproduct and product of two objects are different, but as we mentioned last time, they are isomorphic in the category of abelian groups. Generalizing these notions, we further define pushout and pullback in a category $\mathcal{C}$.

**Definition 8.3.** Let $A$ be an object of $\mathcal{C}$ and let $k : A \longrightarrow X$ and $l : A \longrightarrow Y$ be morphisms. The *pushout* of $X$ and $Y$ under $A$ is an object $X \cup_A Y$ with maps $X \longrightarrow X \cup_A Y$ and $Y \longrightarrow X \cup_A Y$ such that for any $Z \in \mathcal{C}$ and morphisms $p : X \longrightarrow Z$ and $q : Y \longrightarrow Z$ such that $p \circ k = q \circ l$, there exists a unique map $X \cup_A Y \longrightarrow Z$ such that the following diagram commutes:




Note that despite the notation, $X \cup_A Y$ depends not just on $A$ but also on the maps $k$ and $l$. Here are some examples of pushouts in familiar categories:

Sets: In the category of sets, $X \cup_A Y = X \amalg Y/(\sim)$, where $X \amalg Y$ is the disjoint union and $\sim$ is the equivalence relation given by setting $k(a) \sim l(a)$ for all $a \in A$.

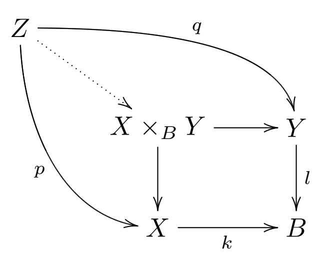
Spaces: In the category of topological spaces, $X \cup_A X$ is again $X \amalg Y/(\sim)$, where $X \amalg Y$ is again disjoint union and $\sim$ is the same equivalence relation. Here we give $X \cup_A Y$ the quotient topology.

Groups: $X \cup_A Y$ is the amalgamated free product $X \amalg_A Y$. Recall that the free product is given by the set of all words in the elements of $X$ and $Y$; the amalgamated free product is given by gluing together the images of $A$ in $X$ and $Y$.

Abelian Groups: $X \cup_A Y = X \oplus Y/\langle k(a) - l(a)\rangle$

The dual notion to pushout is that of pullback:

**Definition 8.4.** Let $B$ be an object of $\mathcal{C}$ and let $k : X \longrightarrow B$ and $l : Y \longrightarrow B$ be morphisms. The *pullback* of $X$ and $Y$ over $B$ is an object $X \times_B Y$ with maps $X \times_B Y \longrightarrow X$ and $X \times_B Y \longrightarrow Y$ such that for any $Z \in \mathcal{C}$ and morphisms $p : Z \longrightarrow X$ and $q : Z \longrightarrow Y$ such that $k \circ p = l \circ q$, there exists a unique map $Z \longrightarrow X \times_B Y$ such that the following diagram commutes:




Again, the pullback depends on the maps $k$ and $l$. In the category of sets, the pullback is given by $X \times_B Y = \{(x, y) \mid k(x) = l(y)\} \subset X \times Y$.

Note that coproducts, pushouts and the like do not always exist for all objects or morphisms in a given category. In fact, if a category has all coproducts, it has extra structure. Consider the coproduct of the empty set of objects. This is actually just an initial object!

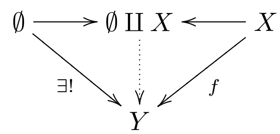
**Definition 8.5.** An *initial object* $\emptyset$ is an object that has a unique map to any other object.

In the category of sets, the initial object is just the empty set. In the category of spaces, it is the empty space. Note we have already used the empty space when we talked about cobordisms: the creation of a string is a cobordism from the empty 1-manifold and the annihilation is a cobordism to the empty 1-manifold.

We now assume we are in a category $\mathcal{C}$ that has all coproducts, and let $\emptyset$ be an initial object of $\mathcal{C}$.

**Proposition 8.6.** *For all $X \in \mathcal{C}$, $\emptyset \amalg X$ is isomorphic to $X$.*

*Proof.* Since $\emptyset$ is initial, there exists a unique map $\emptyset \longrightarrow Y$ for any $Y \in \mathcal{C}$. By the universal property of coproducts, $\emptyset \amalg X$ is the object (unique up to isomorphism) making the following diagram commute for any $f : X \longrightarrow Y$ in $\mathcal{C}$:

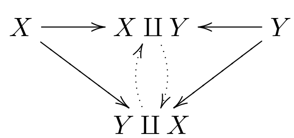



If we replace $\emptyset \amalg X$ with $X$, letting the top right map $X \longrightarrow X$ be the identity and the map $\emptyset \longrightarrow X$ be the unique map given by the initialness of $\emptyset$, we do indeed get a commutative diagram. Hence $\emptyset \amalg X \cong X$. $\square$

**Proposition 8.7.** *For $X, Y \in \mathcal{C}$, $X \amalg Y$ is naturally isomorphic to $Y \amalg X$.*

In particular, note that $\emptyset \amalg X \cong X \cong X \amalg \emptyset$, so that $\emptyset$ is a left and right unit for the coproduct operation.

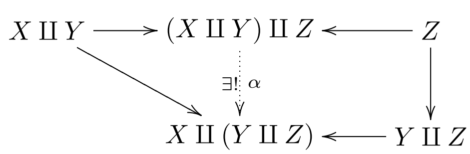
*Proof.* Applying the universal property of coproducts to $X \amalg Y$ and $Y \amalg X$, we get the maps indictated in the diagram, and their composites are the identity.




Details of naturality are left to the reader. $\square$

**Proposition 8.8.** *For $X, Y, Z \in \mathcal{C}$, $(X \amalg Y) \amalg Z \cong X \amalg (Y \amalg Z)$.*

*Proof.* A similar sort of diagrammatic argument: We have maps from $Y \longrightarrow Y \amalg Z$ and $Z \longrightarrow Y \amalg Z$. By the universal property, the maps $X \longrightarrow X \amalg (Y \amalg Z)$ and $Y \longrightarrow Y \amalg Z$ give a map $X \amalg X \amalg (Y \amalg Z)$ such that the following diagram commutes:




The same argument backwards gives us that $\alpha$ is an isomorphism. $\square$

These three propositions imply that in our category $\mathcal{C}$ that has all coproducts, coproduct is a symmetric and associative operation with a unit $\emptyset$. In a category that contains all cartesian products, we can do the same thing. Here the unit object is a terminal object.

**Definition 8.9.** An object $* \in \mathcal{C}$ is a *terminal object* if there exists a unique map $X \longrightarrow *$ for any object $X \in \mathcal{C}$.

In the category of abelian groups, the terminal and initial objects are both the trivial group. For products, the duals of the above three propositions can be summed up in the three idenities

$$
\begin{gathered}
* \times X \cong X \cong X \times * \\
X \times Y \cong Y \times X \\
(X \times Y) \times Z \cong X \times (Y \times Z)
\end{gathered}
$$

## 9. Symmetric monoidal categories (Bohmann)

Consider a category $\mathcal{C}$ with a product $\square$ that is symmetric, associative, and unital. That is, $\square$ is a functor $\square : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that for all $X, Y, Z \in \mathcal{C}$ there exists a (natural) commutativity isomorphism $\gamma : X \square Y \longrightarrow Y \square X$, an associativity isomorphism $\alpha : X \square (Y \square Z) \longrightarrow (X \square Y) \square Z$, and a unit object $I$ with natural isomorphisms $\lambda : I \square X \longrightarrow X$ and $\rho : X \square I \longrightarrow X$. The following definition is informal and incomplete. We shall make it categorically precise later on.

**Definition 9.1.** Such a category $(\mathcal{C}, \square, I, \alpha, \gamma, \lambda, \rho)$ is a *symmetric monoidal category.* A *monoid* in $\mathcal{C}$ is an object $M \in \mathcal{C}$ with maps $\eta : I \longrightarrow M$ and $\varphi : M \square M \longrightarrow M$ such that the following two diagrams commute:

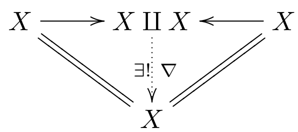
$$
\begin{array}{ccccc}
I \square M & \xrightarrow{\eta \square \mathrm{Id}} & M \square M & \xleftarrow{\mathrm{Id} \square \eta} & M \square I \\
& {\scriptstyle \lambda} \searrow & \downarrow {\scriptstyle \varphi} & \swarrow {\scriptstyle \rho} & \\
& & M & &
\end{array}
\qquad
\begin{array}{ccc}
M \square M \square M & \xrightarrow{\varphi \square \mathrm{Id}} & M \square M \\
{\scriptstyle \mathrm{Id} \square \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\
M \square M & \xrightarrow{\varphi} & M
\end{array}
$$

Note that we have suppressed the associativity isomorphism in the second diagram. In the category of sets, a monoid is, as before, a group without inverses. That is, a set with a unital and associative operation. A *comonoid* is, unsurprisingly, the dual of a monoid and we can obtain a formal definition by reversing the arrows in the above diagrams.

The results of the previous section prove that a category $\mathcal{C}$ that has all coproducts is a symmetric monoidal category with the product given by $\amalg$, and similarly that a category that has all products is a symmetric monoidal category under cartesian product. If we consider such a category under $\square = \times$ cartesian product, every object is uniquely a comonoid, but not necessarily a monoid, as the following diagram illustrates:

$$
\begin{array}{ccccc}
X & \longleftarrow & X \times X & \longrightarrow & X \\
& \mathbin{\rotatebox{-45}{$=$}} & \uparrow {\scriptstyle \exists! \, \Delta} & \mathbin{\rotatebox{45}{$=$}} & \\
& & X & &
\end{array}
$$

The map $\Delta$ is the *diagonal map*. Similarly, under $\square = \amalg$ coproduct, every object is uniquely a monoid, but not necessarily a comonoid:




The map $\nabla$ is the *codiagonal* or *folding map*.

*Example.* Letting $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, for which $n$ is $S^n$ a monoid? That is, when is there a map $S^n \times S^n \longrightarrow S^n$ with the appropriate properties? Answer: For $n = 0$, $S^0 = \mathbb{Z}/2\mathbb{Z}$; $n = 1$, $S^1$ is the unit complex numbers; $n = 3$, $S^3$ is the unit quaternions. $S^7$ is the unit Cayley numbers, which is not a monoid because the Cayley numbers are not associative. However, $S^0, S^1, S^3$ and $S^7$ are all examples of Hopf spaces. Note that $S^2$ is *not* a monoid or even a Hopf space.

*Example.* For abelian groups, recall that cartesian product and coproduct are the same, so $(\mathcal{A}b, \times, 0) = (\mathcal{A}b, \oplus, 0)$. Thus for $A \in \mathcal{A}b$, we have both the diagonal map $\Delta : A \longrightarrow A \times A$ and the codiagonal map $\nabla : A \oplus A \longrightarrow A$. The codiagonal map is the unique map that is the identity on each copy of $A$, and is simply given by addition. Thus each abelian group is both a monoid and a comonoid in the category $(\mathcal{A}b, \times, 0)$.

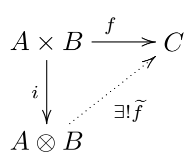
## 10. Rings and tensor products (Bohmann)

Although every abelian group is a monoid and comonoid in the category $(\mathcal{A}b, \times, 0)$, this does not capture all the structure it is possible for an abelian group to have. We know that abelians groups can have operations other than addition; for example, the integers under multiplication. We want a category whose monoids are rings.

**Definition 10.1.** Let $A, B$ and $C$ be abelian groups. A map $f : A \times B \longrightarrow C$ is *bilinear* if $f(a, b + b') = f(a, b) + f(a, b')$ and $f(a + a', b) = f(a, b) + f(a', b)$ for all $a, a' \in A$ and $b, b' \in B$.

Note that this immediately implies that $f$ is $\mathbb{Z}$-bilinear.

**Definition 10.2.** For $A$ and $B$ abelian groups, the *tensor product* $A \otimes B$ is an abelian group with a bilinear map $i : A \times B \longrightarrow A \otimes B$ satisfying the following universal property: For any abelian group $C$ and bilinear map $f : A \times B \longrightarrow C$, there exists a unique map $\widetilde{f} : A \otimes B \longrightarrow C$ such that




commutes.

The intuition is that every element of $A \otimes B$ is of the form $\sum_{i=1}^{q} a_i \otimes b_i$, where we write $i(a, b) = a \otimes b$.

We show that there exists an abelian group satisfying the above universal property. Recall the explicit definition of the free abelian group generated by a set $S$:

$\mathbb{Z}[S] = \{\sum_{i=1}^{q} n_i s_i\}$ with addition in the usual way. Let $U : \mathcal{A}b \longrightarrow \mathcal{S}ets$ be the forgetful functor that takes a group to its underlying set. Then

$$A \otimes B = \mathbb{Z}[U(A \times B)]/\left\langle \begin{smallmatrix} (a,b+b')-(a,b)-(a,b') \\ (a+a',b)-(a,b)-(a',b) \end{smallmatrix} \right\rangle$$

That is, $A \otimes B$ is the free abelian group on $U(A \times B)$, modulo the relations imposed by bilinearity. We leave it to the reader to check that this construction satisfies the universal property of tensor product.

The tensor product gives us a product on $\mathcal{A}b$ that is associative, commutative and unital. The unit object is $\mathbb{Z}$: $A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A$, since in, for example, $A \otimes \mathbb{Z}$, $a \otimes n = na \otimes 1$, and we have an isomorphism given by $a \otimes 1 \mapsto a$. Tensor product is also associative: $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ via the map $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$. It is commutative, with $A \otimes B \cong B \otimes A$, since $A \otimes B$ and $B \otimes A$ both satisfy the defining universal property. All of this means that $(\mathcal{A}b, \otimes, \mathbb{Z})$ is a symmmetric monoidal category.

In fact, the category $(\mathcal{A}b, \otimes, \mathbb{Z})$ is exactly the category whose monoids are rings: the maps $\varphi : R \otimes R \longrightarrow R$ and $\eta : \mathbb{Z} \longrightarrow R$ define a ring structure on an abelian group $R$. Note that the bilinearity takes care of itself since we defined our maps from the tensor product.

*Exercise.* Compute $\mathbb{Z}/m \otimes \mathbb{Z}/n \cong \mathbb{Z}/?$ Hint: $\mathbb{Z}/p \otimes \mathbb{Z}/q = 0$ when $p$ and $q$ are distinct primes, as $1 \cdot q \otimes r = 1 \otimes qr = 1 \otimes 0 = 0$.

Here is another general definition in an arbitrary symmetric monoidal category $(\mathcal{C}, \Box, I)$.

**Definition 10.3.** Let $M \in \mathcal{C}$ be a monoid, and $V \in \mathcal{C}$ be an object. A *(left) action* of $M$ on $V$ is a map $\mu : M \Box V \longrightarrow V$ such that the following diagrams commute:

$$\begin{array}{ccc} M\Box V & \xrightarrow{\mu} & V \\ {\scriptstyle \eta\Box \mathrm{Id}}\uparrow & \nearrow{\scriptstyle \lambda} & \\ I\Box V & & \end{array} \qquad \begin{array}{ccc} M\Box M\Box M & \xrightarrow{\mathrm{Id}\,\Box\mu} & M\Box V \\ {\scriptstyle \varphi\Box \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ M\Box V & \xrightarrow[\mu]{} & V \end{array}$$

In the category $(\mathcal{A}b, \otimes, \mathbb{Z})$, where the monoid is a ring $R$, the object $V$ is a (left) $R$-module. For commutative rings, left- and right-modules are the same. If the ring happens to be a field, a module is just a vector space.

We generalize tensor product to modules over commutative rings.

**Definition 10.4.** Let $A, B$ and $C$ be $R$-modules, where $R$ is a commutative ring. The tensor product $A \otimes_R B$ of $A$ and $B$ over $R$ is an $R$-module $A \otimes_R B$ and an $R$-bilinear map $i : A \otimes B \longrightarrow A \otimes_R B$ such that for all $R$-bilinear maps $f : A \otimes B \longrightarrow C$, there exists a unique map of $R$-modules $\widetilde{f}$ such that

$$\begin{array}{ccc} A \otimes B & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \exists!\widetilde{f}} & \\ A \otimes_R B & & \end{array}$$

commutes.

Here, by saying that $f$ is $R$-bilinear, we mean that

$$f(ra \otimes b) = rf(a \otimes b) = f(a \otimes rb)$$

for all $r \in R$, $a \in A$, and $b \in B$.

If $R$ is a field, this specializes to define the tensor product of vector spaces.

We sketch the construction of $A \otimes_R B$. Let $\mu : A \otimes R \longrightarrow A$ and $\nu : R \otimes B \longrightarrow B$ be the maps of Abelian groups that define the action of $R$ on $A$ and $B$. (We choose a right action on $A$ for symmetry). Then $A \otimes_R B$ is the cokernel of the map

$$A \otimes R \otimes B \xrightarrow{\mu \otimes \mathrm{Id}_B - \mathrm{Id}_A \otimes \nu} A \otimes B.$$

Recall that the cokernel is obtained by identifying elements in the image. This forces $ar \otimes b = r(a \otimes b) = a \otimes rb$.

We remark parenthetically that when $R$ is a non-commutative ring, a very similar definition and construction give a tensor product $A \otimes_R B$ of a right $R$-module $A$ and a left $R$-module $B$, but it is only an Abelian group, not an $R$-module.

**Definition 10.5.** The *free $R$-module* on a set $S$ is $R[S] = \{\sum_{i=1}^{q} r_i s_i\}$ for $r_i \in R$, $s_i \in S$.

It is immediate from the universal property that for free $R$-modules $R[S]$ and $R[T]$, $R[S] \otimes_R R[T] \cong R[S \times T]$. Note that if the ground ring $R$ is a field, then every $R$-module is free.

Let $(\mathcal{M}_R, \otimes_R, R)$ be the category of $R$-modules for a commutative ring $R$. This is also a symmetric monoidal category! A monoid in $(\mathcal{M}_R, \otimes_R, R)$ is an *$R$-algebra*, and a comonoid is an *$R$-coalgebra*. That is, for $A$ an $R$-algebra, we have maps $\varphi : A \otimes_R A \longrightarrow A$ and $\eta : R \longrightarrow A$ such that the diagrams in definition 9.1 commute.

*Example.* The complex numbers $\mathbb{C}$ form an $\mathbb{R}$-algebra, and so do the quaternions $\mathbb{H}$. The octonians $\mathbb{O}$ are not an $\mathbb{R}$-algebra because they are not associative.

An algebra $A$ is commutative if the diagram

$$\begin{array}{ccc} A \otimes_R A & \xrightarrow{\gamma} & A \otimes_R A \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

commutes, where $\gamma$ is the twist isomorphism $\gamma : a \otimes a' \mapsto a' \otimes a$.

A *Frobenius* algebra is an $R$-module that is simultaneously an algebra and a coalgebra, together with an as yet unspecified compatibility condition. We will be interested primarily in commutative Frobenius algebras, since those are the ones that arise from 2-TQFT's.

Suppose $A$ is a free $R$-module, $A = R[S]$. Then a map $R[S] \otimes R[S] \longrightarrow R[S]$ is defined by where it takes $(s, t)$. This gives a multiplication table; thinking of a map this way is like thinking about a linear transformation in terms of a matrix.

## 11. Adjoint functors in representation theory (Bohmann)

As a digression, we turn to representation theory to illustrate ideas and give another example of adjoint functors. This is a classic example, but if you are not familiar with all the algebraic notions involved, don't worry too much about it. We won't be using this. Let $G$ be a group. We can define a representation of $G$ as a vector space $V$ over some field $k$, together with a $G$-action on $V$. A $G$-action means that for every $g \in G$, we have a linear transformation $g : V \longrightarrow V$

such that $g(g'(v)) = (gg')(v)$ and $1(v) = v$ for all $g, g' \in G$ and $v \in V$. We can define $G$-$\mathcal{R}ep$, the category of representations of $G$, to be the category whose objects are representations of $G$, and whose morphisms are $G$-linear transformations $T : V \longrightarrow V'$ such that $T(gv) = g(Tv)$ for all $v \in V$ and $g \in G$.

For a subgroup $H \subset G$, we have a forgetful functor $U : G\text{-}\mathcal{R}ep \longrightarrow H\text{-}\mathcal{R}ep$ given by restricting a $G$-action on a vector space $V$ to elements in $H$. We'd like to find a left-adjoint $\operatorname{ind}_H^G$ to this functor $U$ so that we have an isomorphism

$$G\text{-}\mathcal{R}ep(\operatorname{ind}_H^G(W), V) \cong H\text{-}\mathcal{R}ep(W, UV)$$

for representation $V$ of $G$ and $W$ of $H$. Such an adjoint allows us to study representations of a group via representations of its subgroups. We can construct such a functor $\operatorname{ind}_H^G$ by using the notion of the group ring.

**Definition 11.1.** The *group ring* $k[G]$ of a group $G$ over a field $k$ is the set of $k$-linear combinations $\{\sum_{i=1}^n k_i g_i \mid k_i \in k,\ g_i \in G\}$, with the multiplication induced by the multiplication in $G$. That is, $k[G]$ is the free $k$-module on the underlying set of $G$ with multiplication on the basis elements $g \in G$ given by the multiplication table of $G$.

Note that a $G$-action on a vector space $V$ gives $V$ a $k[G]$-module structure.

For a subgroup $H \subset G$, we can define a (right) $k[H]$-module structure on $k[G]$ via the inclusion $k[H] \hookrightarrow k[G]$ and the multiplication $k[G] \otimes_k k[G] \longrightarrow k[G]$:

$$\begin{array}{ccc} k[G] \otimes_k k[H] & \longrightarrow & k[G] \otimes_k k[G] \\ & \searrow & \downarrow \\ & & k[G] \end{array}$$

Now, given a representation $W$ of $H$, we define $\operatorname{ind}_H^G(W)$ by

$$\operatorname{ind}_H^G(W) = k[G] \otimes_{k[H]} W.$$

This has a $k[G]$-module structure given by the left action of $k[G]$ on itself, so $\operatorname{ind}_H^G$ is a representation of $G$. This construction makes ind the left adjoint of the forgetful functor $U$.

We can also define a right-adjoint $\operatorname{coind}_H^G$ to $U$ so that

$$G\text{-}\mathcal{R}ep(V, \operatorname{coind}_H^G(W)) \cong H\text{-}\mathcal{R}ep(UV, W).$$

This construction is given by defining $\operatorname{coind}_H^G(W) = \operatorname{Hom}^l_{k[H]}(k[G], W)$, the set of morphisms $k[G] \longrightarrow W$ in the category of left $k[H]$-modules. We have an induced left-action of $G$ on $\operatorname{Hom}^l_{k[H]}(k[G], W)$ from the right action of $k[G]$ on itself, so that for $T : k[G] \longrightarrow W$, $(gT)(x) = T(xg)$ for all $x \in k[G]$. This makes $\operatorname{Hom}_{k[H]}(k[G], W)$ a representation of $G$, and then $\operatorname{coind}_H^G$ is a right-adjoint to the functor $U : G\text{-}\mathcal{R}ep \longrightarrow H\text{-}\mathcal{R}ep$. It is a theorem that, if we restrict to finite dimensional vector spaces $V$ and $W$, then the functors $\operatorname{ind}_H^G$ and $\operatorname{coind}_H^G$ are the same, but in general the left- and right-adjoints of a given functor need not coincide.

## 12. Adjoint functors relating $\otimes$ and Hom; duality (Bohmann)

For yet another example, we show that tensor products in vector spaces are an adjunction with Hom. Given vector spaces $V, W$ and $Z$ over a field $k$, we have bijections

$$\eta : \mathrm{Vect}_k(V \otimes_k W, Z) \longrightarrow \mathrm{Vect}_k(V, \mathrm{Hom}_k(W, Z))$$

and

$$\epsilon : \mathrm{Vect}_k(V, \mathrm{Hom}_k(W, Z)) \longrightarrow \mathrm{Vect}_k(V \otimes_k W, Z).$$

For $f \in \mathrm{Vect}_k(V \otimes_k W, Z)$, we define $\eta(f)(v)$ for $v \in V$ to be the map $W \longrightarrow Z$ given by $\big(\eta(f)(v)\big)(w) = f(v \otimes w)$. Conversely, for $g \in \mathrm{Vect}_k(V, \mathrm{Hom}_k(W, Z))$, we define $\epsilon(g)(v \otimes w)$ to be $\big(g(v)\big)(w)$ for $v \in V$ and $w \in W$. This gives an adjuction between tensor product and Hom in the category of vector spaces. Notice that we are using the fact that $\mathrm{Vect}_k(W, Z)$ is actually a vector space, which we denote by $\mathrm{Hom}_k(W, Z)$ to remember that additional structure. That is, we are using that our hom-sets have extra structure in this category. This example is related to the previous example because $k[G] \otimes_{k[H]} W \cong \mathrm{Hom}_{k[H]}(k[G], W)$ when $k[G]$ is finite dimensional, making the left- and right-adjoint functors to $U$ the same.

This example is also related to duality between vector spaces. We shall give more detail on duality later on. Recall that we say that a (finite dimensional) vector space $X$ over $k$ is dual to a vector space $Y$ if we have maps $\epsilon : Y \otimes X \longrightarrow k$ and $\eta : k \longrightarrow X \otimes Y$ such that the triangle identities are satisfied. We can rephrase this in terms of adjoints by saying that, for any other vector spaces $W$ and $Z$, we have inverse isomorphisms

$$\mathrm{Vect}_k(W \otimes_k X, Z) \underset{\eta_\sharp}{\overset{\epsilon_\sharp}{\rightleftarrows}} \mathrm{Vect}_k(W, Z \otimes_k Y)$$

where for $g \in \mathrm{Vect}_k(W, Z \otimes_k Y)$ and $f \in \mathrm{Vect}_k(W \otimes_k X, Z)$, $\epsilon_\sharp(g)$ and $\eta_\sharp(f)$ are defined by

$$\epsilon_\sharp(g) : \ W \otimes_k X \xrightarrow{g \otimes \mathrm{Id}} Z \otimes_k Y \otimes_k X \xrightarrow{\mathrm{Id} \otimes \epsilon} Z \otimes k \cong Z$$
$$\eta_\sharp(f) : \ W \otimes k \xrightarrow{\mathrm{Id} \otimes \eta} W \otimes_k X \otimes_k Y \xrightarrow{f \otimes \mathrm{Id}} Z \otimes_k Y$$

Then, with $X$ and $Y$ fixed, this gives an adjoint pair of functors $- \otimes_k X$ and $- \otimes_k Y$ on $W$ and $Z$. Moreover, since we're dealing with vector spaces, we also have another adjunction

$$\mathrm{Vect}_k(W \otimes_k X, Z) \cong \mathrm{Vect}_k(W, \mathrm{Hom}_k(X, Z)),$$

hence a composite isomorphism

$$\mathrm{Vect}_k(W, Z \otimes_k Y) \cong \mathrm{Vect}_k(W, \mathrm{Hom}_k(X, Z)).$$

As a formal consequence, using either the uniqueness of adjoints or something called the Yoneda Lemma, this implies an isomophism

$$Z \otimes_k Y \cong \mathrm{Hom}_k(X, Z).$$

## 13. Duality in symmetric monoidal categories (Bohmann)

The construction of adjoint functors from a duality pairing is so formal that it applies not just to the category of vector spaces over a field but to any symmetric monoidal category. In a symmetric monoidal category $\mathcal{C}$, we say that an object $X$ is dual to an object $Y$ if there exist $\epsilon : Y \otimes X \longrightarrow I$ and $\eta : I \longrightarrow X \otimes Y$ such that the triangle identities hold. We can define $\epsilon_\sharp$ and $\eta_\sharp$ in the same way as for vector spaces to get isomorphisms

$$\mathcal{C}(W\Box X, Z) \underset{\eta_\sharp}{\overset{\epsilon_\sharp}{\rightleftarrows}} \mathcal{C}(W, Z\Box Y)\ .$$

Explicitly, we define $\epsilon_\sharp$ and $\eta_\sharp$ for $g \in \mathcal{C}(W, Z\Box Y)$ and $f \in \mathcal{C}(W\Box X, Y)$ by

$$\epsilon_\sharp(g):\ W\Box X \xrightarrow{g\otimes \mathrm{Id}} Z\Box Y\Box X \xrightarrow{\mathrm{Id}\,\Box\epsilon} Z\Box I \cong Z$$

$$\eta_\sharp(f):\ W\Box I \xrightarrow{\mathrm{Id}\,\Box\eta} W\Box X\Box Y \xrightarrow{f\Box\,\mathrm{Id}} Z\Box Y$$

Fixing $X$ and $Y$, this again gives us a pair of adjoint functors in $W$ and $Z$. We then get the following proposition.

**Proposition 13.1.** *Given $\epsilon : Y\Box X \longrightarrow I$, there exists $\eta : I \longrightarrow X\Box Y$ such that $(\eta, \epsilon)$ is a duality pairing if and only if $\epsilon_\sharp$ is a bijection.*

*Proof.* Given a duality pairing $(\eta, \epsilon)$, chasing the definitions of $\eta_\sharp$ and $\epsilon_\sharp$ through the diagram defining the triangle identities show that $\eta_\sharp$ and $\epsilon_\sharp$ are inverses. Conversely, if $\epsilon_\sharp$ is a bijection, we can take $W = I$ and $Z = X$ to get an isomorphism

$$\mathcal{C}(I\Box X, X) \xleftarrow{\epsilon_\sharp} \mathcal{C}(I, X\Box Y).$$

We then take $\eta$ to be $\epsilon_\sharp^{-1}$ applied to the unit isomorphism $I\Box X \cong X$. □

*Remark* 1. Let $A$ be a $k$-algebra, let $M$ be a right $A$-module and $N$ be a left $A$-module. Then we have isomorphisms of $k$-modules

$$\mathrm{Hom}^r_A(M, \mathrm{Hom}_k(N, k)) \cong \mathrm{Hom}_k(M \otimes_A N, k) \cong \mathrm{Hom}^l_A(N, \mathrm{Hom}_k(M, k))$$

where $\mathrm{Hom}^r_A$ denotes the morphisms in the category of right $A$-modules and $\mathrm{Hom}^l_A$ denotes the morphisms in the category of left $A$-modules. Here, the left $A$-module structure on $N$ gives a right $A$-module structure on $\mathrm{Hom}_k(N, k)$ via $(ga)(n) = g(an)$, and we can similarly define a left $A$-module structure on $\mathrm{Hom}_k(M, k)$. In our discussion of Frobenius algebras, we will later apply this with $M = A$. In that case we also have a left action of $A$ on $M$, which induces a right action on the first two Hom's and a right action of $A$ on $N$, which induces a left action on the second two Hom's. Here, writing $A^* = \mathrm{Hom}_k(A, k)$, we obtain isomorphisms

$$\mathrm{Hom}^r_A(A, A^*) \cong A^* \cong \mathrm{Hom}^l_A(A, A^*)$$

of left and of right $A$-modules.

## 14. Monoidal categories and functors (Henry)

We recall the following definition from a previous lecture.

**Definition 14.1.** A monoidal category $\mathfrak{C}$ is a category $\mathfrak{C}$ together with a distinguished object $I \in Ob(\mathfrak{C})$, a functor $\Box : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, for each triple $(X, Y, Z)$ of objects of $\mathfrak{C}$ an associativity isomorphism $\alpha : (X\Box Y)\Box Z \cong X\Box(Y\Box Z)$, and for each object X of $\mathfrak{C}$ unit isomorphisms $\lambda : I\Box X \cong X$ and $\rho : X\Box I \cong X$. A monoidal category is said to be symmetric if, in addition, for each pair $(X, Y)$ of objects of $\mathfrak{C}$ there is a commutativity isomorphism $\gamma : X\Box Y \cong Y\Box X$.

We are still being informal and incomplete since we have omitted the "coherence diagrams" relating $\alpha$, $\lambda$, $\rho$, and $\gamma$. We will come back to that later.

We have now defined monoidal categories, and we have seen examples of them in the previous lectures. We must next define functors between them.

**Definition 14.2.** Let $(\mathfrak{C}, \Box, I)$ and $(\mathfrak{D}, \Box, J)$ be monoidal categories. A (lax) monoidal functor $F : (\mathfrak{C}, \Box, I) \to (\mathfrak{D}, \Box, J)$ is a functor $F : \mathfrak{C} \to \mathfrak{D}$ together with a morphism $\iota : J \to FI$, and for each pair (X,Y) of objects of $\mathfrak{C}$ a morphism $\varphi : FX \Box FY \to F(X \Box Y)$, such that the following diagrams commute:

$$\begin{CD}
(FX\Box FY)\Box FZ @>{\alpha}>> FX\Box(FY\Box FZ) \\
@V{\varphi\Box id}VV @VV{id\Box\varphi}V \\
F(X\Box Y)\Box Z @. FX\Box F(Y\Box Z) \\
@V{\varphi}VV @VV{\varphi}V \\
F((X\Box Y)\Box Z) @>>{F\alpha}> F(X\Box(Y\Box Z))
\end{CD}$$

$$\begin{CD}
FX\Box J @>{id\Box\iota}>> FX\Box FI \\
@V{\rho}VV @VV{\varphi}V \\
FX @<<{F\rho}< F(X\Box I)
\end{CD}
\qquad
\begin{CD}
J\Box FY @>{\iota\Box id}>> FI\Box FY \\
@V{\lambda}VV @VV{\varphi}V \\
FY @<<{F\lambda}< F(I\Box Y)
\end{CD}$$

A monoidal functor is said to be symmetric if, in addition, for each pair (X,Y) of objects of $\mathfrak{C}$ the following diagram commutes:

$$\begin{CD}
FX\Box FY @>{\gamma}>> FY\Box FX \\
@V{\varphi}VV @VV{\varphi}V \\
F(X\Box Y) @>>{F\gamma}> F(Y\Box X)
\end{CD}$$

Monoidal functors actually come in three flavors: lax, strong, and strict. We have defined the lax flavor; for the strong and strict flavors, $\varphi$ and $\iota$ are required to be isomorphisms or identity maps. The strict notion is rarely encountered. The lax notion is encountered most frequently. However, for our study of TQFT's, it is the strong notion that we shall most be concerned with.

**Example**: Let R be a commutative ring. In a previous lecture we defined R-modules, morphisms of R-modules, and the tensor product of R-modules. Recall that if X and Y are R-modules, then $X \otimes_R Y = X \otimes Y / < xr \otimes y - x \otimes ry >$, with $xr \otimes y = r(x \otimes y) = x \otimes ry$ giving the $R$-module structure. Let **R-Mod** be the category of R-modules. It is symmetric monoidal. The forgetful functor $U$ : **R-Mod** $\to$ $\mathbb{Z}$**-Mod** = **Ab** that sends an R-module to its underlying abelian group is a lax monoidal functor.

## 15. Cobordism (Henry)

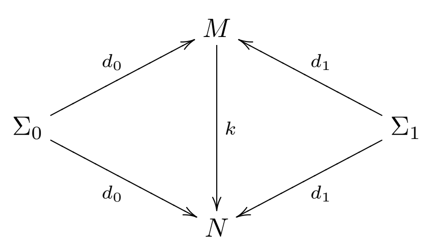
We now know what at least two terms in the definition of an n-TQFT mean: We know what a vector space is and therefore we know what the category **VectK** is, and now we know what it means for a functor $F$ : **n-Cob** $\to$ **VectK** to be a symmetric monoidal functor, granted that the mysterious category **n-Cob** is, in fact, a symmetric monoidal category. Note that we already know **VectK** is a symmetric monoidal category; it is the special case of **R-Mod** when R is is the field K. Next we will define the category **n-Cob**. From professor Farb's class you should be familiar with manifolds and the classification of surfaces. For convenience we regard the empty manifold as as an n-manifold for every n.

**Definition 15.1.** A manifold M is said to be closed if it is compact and without boundary.

We shall use the language of smooth manifolds, but don't worry if you have not seen it. A quick review is given later (starting in §28). In dimension 2, we shall not need this and can work just with compact topological manifolds, not necessarily smooth. These are compact surfaces with boundaries, and the boundaries are homeomorphic to disjoint unions of circles. We shall later return to the case $n = 2$ and give more detail.

**Definition 15.2.** Let M and N be smooth closed n-manifolds. M and N are said to be cobordant if there is a smooth compact n+1-manifold W such that the boundary of W is the disjoint union of M and N, $\partial W = M \amalg N$.

**Definition 15.3.** (**n-Cob**) The objects of the category **n-Cob** are smooth closed oriented $(n-1)$-manifolds. The morphisms of **n-Cob** are diffeomorphism classes, relative to the boundary, of cobordisms. A cobordism M of smooth closed oriented $(n-1)$-manifolds $\Sigma_0$ and $\Sigma_1$ comes equipped with given diffeomorphisms $d_0 : \Sigma_0 \to \partial M$ and $d_1 : \Sigma_1 \to \partial M$ onto the parts of the boundary "pointing into the cobordism" and "pointing out of the cobordism". These are the incoming and outgoing parts of the boundary, where these notions are determined by the orientations of the boundary components and their relationship to the orientation of the cobordism. Thus two corbordisms M and N represent the same morphism in **n-Cob** if there is a diffeomorphism $k : M \to N$ making the following diagram commute:




We are, however, only interested in the 2-dimensional case.

**Proposition 15.4.** *A (smooth) closed 1-manifold is either empty or a finite disjoint union of circles.*

We regard the empty 1-manifold as the disjoint union of 0 circles. Thus we see that the objects of **2-Cob** are the finite disjoint unions of circles. We claimed before that **2-Cob** was a symmetric monoidal category. Now it should begin to be obvious what the operation $\Box$ should be. The disjoint union of two objects of **2-Cob** gives us another object of **2-Cob** and the disjoint union of two cobordisms is a cobordism; passing to equivalence classes, we obtain the required operation $\Box$, which of course we now write as $\amalg$.

**Proposition 15.5.** $(\textbf{\textit{2-Cob}}, \amalg, \emptyset)$ *is a symmetric monoidal category.*

Of course the proof requires us to first define composition and show that we have a well-defined category. [These notes omit the discussion that was given in class.] Briefly, composition is defined by gluing together two (representative) cobordisms and then passing to equivalence classes, where two cobordisms can be glued if the outgoing boundary components of the first coincide with the incoming boundary components of the second. To make this rigorous we have to be careful about diffeomorphism classes, but the pictures are clear from the handout. [See the file CollatedPictures.pdf] In understanding the pictures, the essential point is that we are entitled to smoothly deform the interiors of cobordisms, leaving the boundary components fixed, without changing the equivalence class and thus without changing the morphism (which, we repeat, is an equivalence class of cobordisms rather than a cobordism).

## 16. 2-TQFTs and Frobenius algebras (Henry)

We now know what a 2-TQFT is, since all the words have been defined: a 2-TQFT is a (strong) symmetric monoidal functor $F\colon \textbf{2-Cob} \to \textbf{VectK}$. Let F be a 2-TQFT and write $F(S^1) = A \in Ob(\textbf{VectK})$. Then we have maps $\mu : A \otimes A \to A$, $\psi : A \to A \otimes A$, $\nu : K \to A$, and $\epsilon : A \to K$ corresponding to the top row of four pictured cobordisms in the the handout [CollatedPictures.pdf]. They correspond to the following commutative diagrams in $\textbf{VectK}$ that come from our requirement that $F$ is a strong monoidal functor.

$$\begin{array}{ccc} F(S^1)\otimes F(S^1) & \xrightarrow{\mu} & F(S^1) \\ \cong\downarrow & \nearrow & \\ F(S^1 \amalg S^1) & & \end{array} \qquad \begin{array}{ccc} F(S^1) & \xrightarrow{\psi} & F(S^1)\otimes F(S^1) \\ & \searrow & \downarrow\cong \\ & & F(S^1 \amalg S^1) \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{\nu} & F(S^1) \\ \cong\downarrow & \nearrow & \\ F(\emptyset) & & \end{array} \qquad \begin{array}{ccc} F(S^1) & \xrightarrow{\epsilon} & K \\ & \searrow & \downarrow\cong \\ & & F(\emptyset) \end{array}$$

The pictures in the second and third row of the handout say that the product $\mu$ is associative with unit $\eta$ and the coproduct $\psi$ is coassociative with counit $\epsilon$. The picture in the fifth row says that the product is also commutative. The pictures in the fourth row give the commutativity of the following crucial diagram.

$$\begin{array}{ccccc} & & A\otimes A\otimes A & & \\ & \overset{id\otimes\psi}{\nearrow} & & \overset{\mu\otimes id}{\searrow} & \\ A\otimes A & \xrightarrow{\mu} & A & \xrightarrow{\psi} & A\otimes A \\ & \underset{\psi\otimes id}{\searrow} & & \underset{id\otimes\mu}{\nearrow} & \\ & & A\otimes A\otimes A & & \end{array}$$

For a vector space $V$ and a $K$-algebra $A$, $A \otimes V$ is a left and $V \otimes A$ is a right $A$-module via the multiplication of $A$. The diagram says that the coproduct is a map of left and right $A$-modules. This is called the "Frobenius relation".

**Definition 16.1.** A Frobenius algebra is an algebra and a coalgebra A such that the coproduct is a map of left and right A-modules.

The pictures have told us the following result, which is part of our main theorem.

**Proposition 16.2.** *If $F$ is a 2-TQFT, then $A = F(S^1)$ is a commutative Frobenius algebra.*

We shall later give details of the following non-commutative examples of Frobenius algebras.

**Example**: Let K be a field and G be a group. Then the group algebra K[G] (the vector space whose basis elements are the elements of G and whose multiplication is given by the multiplication in G) is a Frobenius algebra.

**Example**: The matrix algebra $M_n(K)$ of $n \times n$ matrices of elements of $K$ is a Frobenius algebra.

The composites $\alpha = \psi \circ \eta \colon K \to A \otimes A$ and $\omega = \epsilon \circ \mu \colon A \otimes A \to K$ play a key role in understanding the structure of Frobenius algebras. They are pictured in the next to last row of the handout. If you flip a coin, with heads up, the cobordism traced out by the coin is (equivalent to) the identity if the coin lands heads down and is (equivalent to) $\alpha$ if the coin lands heads up. Similarly, if one starts tails up and lands tails up, the cobordism traced out is (equivalent to) $\omega$. Think about it!

## 17. Coherence in topology (Lind)

Historically, the notion of coherence first arose in algebraic topology, so we'll begin there. Recall that we may define a "multiplication" on $S^1, S^3$, and $S^7$. To do so, we associate these spheres with the unit-length complex numbers, quaternions, and Cayley numbers, respectively. In the first two cases, $S^1$ and $S^3$ become groups under this multiplication. However, the Cayley numbers are not associative under multiplication, so $S^7$ is not associative under this product structure: for some $x, y, z \in S^7$,

$$(x \cdot y) \cdot z \neq x \cdot (y \cdot z).$$

Thus we would like to understand spaces with a product structure that is not strictly associative, but rather associative "up to homotopy". (Although $S^7$ is not even homotopy associative, other interesting examples are so).

Suppose that $X$ is a topological space with a continuous multiplication $\mu$:

$$\mu \colon X \times X \longrightarrow X.$$

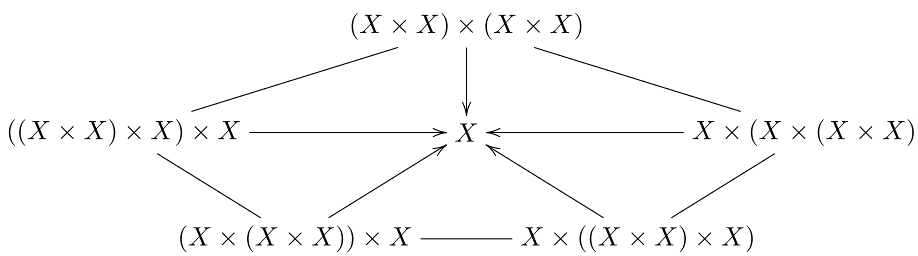
We will usually write $x \cdot y$ for $\mu(x, y)$. Also assume that $\mu$ has a left and right unit $e \in X$, so $\mu(x, e) = \mu(e, x) = x$ for all $x \in X$. A space with such a unital product is called an "$H$-space". Suppose that $(x \cdot y) \cdot z \neq x \cdot (y \cdot z)$, but that we can draw a path in $X$ connecting the points $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$. If we can do this for every triple of points $x, y, z \in X$, then the multiplication $\mu$ is *homotopy associative*. More formally, there exists a continuous map

$$\alpha \colon X \times X \times X \times I \longrightarrow X$$

such that for all $x, y, z \in X$,

$$\alpha(x, y, z, 0) = (x \cdot y) \cdot z \quad \text{and} \quad \alpha(x, y, z, 1) = x \cdot (y \cdot z).$$

One might hope that guaranteeing the associativity of three-fold products of elements of $X$ suffices to ensure that higher order products all behave well up to homotopy as well. Unfortunately, this is not the case. Consider all possible ways of multiplying four elements of $X$. The five possibilities are arranged in the following pentagon:




The parentheses are meant to suggest the order of multiplication carried out in each map to $X$ in the center. Thus, the vertical map is:

$$(X \times X) \times (X \times X) \xrightarrow{\mu\times\mu} X \times X \xrightarrow{\mu} X$$
$$(x, y, z, w) \longmapsto (x \cdot y) \cdot (z \cdot w).$$

The solid lines comprising the boundaries of the pentagon represent homotopies between the two maps from either vertex. Thus we can think of the edge connecting two vertices as the domain of the homotopy relating the two multiplication maps from the vertices. For example, the bottom edge is the domain in the following homotopy:

$$h\colon X^4 \times I \longrightarrow X,$$
$$h(x, y, z, w, 0) = (x \cdot (y \cdot z)) \cdot w, \qquad h(x, y, z, w, 1) = x \cdot ((y \cdot z) \cdot w).$$

Topologically, the boundary of the pentagon is a circle $S^1$, and since all five homotopies agree at the vertices they have in common, they can be collated into a single map:

$$\beta\colon X^4 \times S^1 \longrightarrow X.$$

We may now ask whether this map extends to the interior of the pentagon. Letting $D^2$ denote the unit disk in $\mathbb{R}^2$, we are asking whether there exists a map

$$\overline{\beta}\colon X^4 \times D^2 \longrightarrow X$$

agreeing with $\beta$ on $\partial D^2 = S^1$. Such an extension does not always exist. Intuitively, for a fixed point $(x, y, z, w) \in X^4$, the image of $S^1$ under $\beta(x, y, z, w, -)$ may go around a "hole" in $X$ such that there is no way to extend to a map from the entire unit disk $D^2$ without filling in the hole.

It is precisely this failure which forces us to look at higher homotopies of four-fold associativities, then five-fold associativies, and $n$-fold associativities in general; at each step there is no guarantee that success in the extension problem for $n$-fold associativities will imply that we can extend to $(n+1)$-fold associativities. In his thesis, Stasheff was able to characterize those spaces for which this extension problem can be solved for all $n$.

Let $X$ be an H-space with multiplication $\mu$, associative up to homotopy. We make the following inductive definition: assuming that $X$ is $(n-1)$-fold homotopy associative, we say that $X$ is *$n$-fold homotopy associative*, or an "$A_n$-space", if the map

$$X^n \times S^{n-3} \longrightarrow X,$$

determined on the regular polytope whose vertices are determined by all possible orders of $n$-fold multiplication and whose faces are determined inductively by $(n-1)$-fold homotopies, can be extended to a map

$$X^n \times D^{n-2} \longrightarrow X$$

agreeing with the original map on the boundary. Notice that the original definition of homotopy associativity is precisely the base case of 3-fold homotopy associativity: an $H$-space is an $A_2$-space, and a homotopy associative $H$-space is an $A_3$-space.

**Theorem 17.1** (Stasheff)**.** *$X$ is $n$-fold homotopy associative for all $n$ (an "$A_\infty$-space") if and only if $X$ is homotopy equivalent to a topological monoid.*

Recall that a topological monoid is a monoid in the symmetric monoidal category of topological spaces under Cartesian product $\times$. Thus it is a space $X$ with a multiplication $\mu$ that is *strictly* associative and unital. It is thus a strict $A_\infty$-space, one for which no non-constant homotopies are required: we can fill in the disc to get $X^n \times D^{n-2} \to X$ by sending $(x_1, \cdots, x_n, u)$ to the product $x_1 \cdots x_n$ for all $u \in D^{n-2}$; this is well-defined since all ways of associating give the same answer.

## 18. Coherence in category theory (Lind)

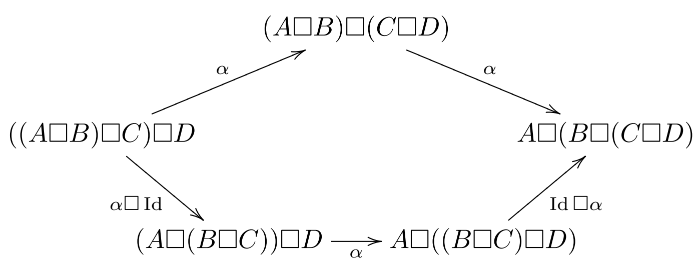
In analogy to topology, we will consider higher associativies over the product in a monoidal category. Consider a monoidal category $(\mathcal{C}, \Box, I, \alpha, \lambda, \rho)$, with associativity and left and right unit isomorphisms:

$$\alpha\colon (A\Box B)\Box C \xrightarrow{\cong} A\Box(B\Box C),$$

$$\lambda\colon I\Box A \xrightarrow{\cong} A, \qquad \rho\colon A\Box I \xrightarrow{\cong} A.$$

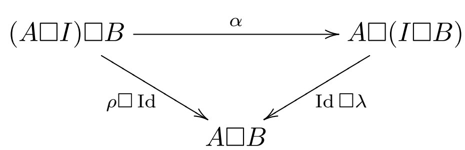
Here is the pentagon for four-fold associativities over objects $A, B, C, D$ of $\mathcal{C}$:

(1)




The following diagram is necessary to relate the left and right unit isomorphisms apropriately:

(2)




The definition of a monoidal category given earlier is not complete until we stipulate that the coherence diagrams (1) and (2) commute. Here is an example that may convince the reader of the need for coherence diagrams:

*Example.* Suppose that we are working in the monoidal category of commutative rings with product $\otimes_{\mathbb{Z}}$ and unit object $\mathbb{Z}$. A clever but nefarious interloper may propose twisting the associativity isomorphism by a minus sign:

$$\alpha((x \otimes y) \otimes z) = -x \otimes (y \otimes z).$$

All of the properties of a monoidal category go through with this definition, except that diagrams (1) and (2) will not commute, as the two routes differ by a sign. To avoid incoherence such as this, we must include these diagrams.

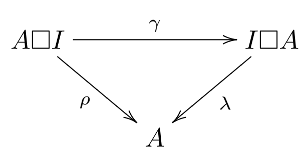
There is a remarkable coherence theorem due to MacLane which states roughly that given the commutativity of (1) and (2), all "sensible" diagrams in a monoidal category must commute, and furthermore the category is equivalent to a *strict* monoidal category, meaning that the associativity isomorphism $\alpha$ and unit isomorphisms $\lambda$ and $\rho$ are strict identities. To provide meaning to the term sensible, MacLane details a recursive way to define well-formed diagrams out of the "atomic words" $\alpha$, $\lambda$, and $\rho$, similar to the syntactic definition of well-formed formulae in formal logic.

Now suppose that $\mathcal{C}$ is a symmetric monoidal category, so that we additionally have a symmetry isomorphism for each pair of objects $A$ and $B$:

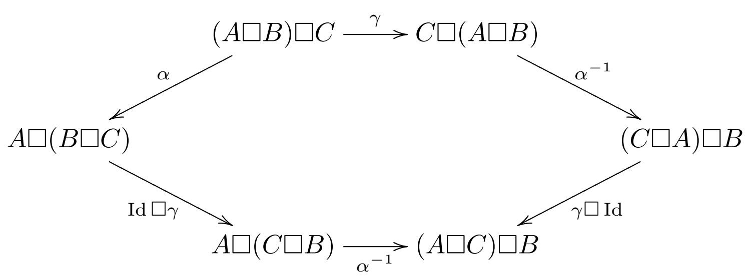
$$\gamma\colon A\Box B \xrightarrow{\cong} B\Box A.$$

The following two coherence diagrams are non-negotiable; we must insist that they commute as part of the definition of a symmetric monoidal category:

(3)




(4)




*Example.* Supply new definitions of $\alpha$ and $\gamma$ in commutative rings (or your favorite symmetric monoidal category) such that the above two diagrams do not commute, and convince yourself that we should exclude such examples.

There is another remarkable coherence theorem due to MacLane which states roughly that given the commutativity of (3) and (4), all "sensible" diagrams in a symmetric monoidal category must commute, and furthermore the category is equivalent to a *strict* symmetric monoidal category, or "permutative category", meaning that the associativity isomorphism $\alpha$ and unit isomorphisms $\lambda$ and $\rho$ are strict identities. The commutativity isomorphism $\gamma$ cannot be made strict.

There is a third diagram for which we are presented with a choice:

(5)
$$\begin{array}{ccc} A\square B & \xrightarrow{\gamma} & B\square A \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \gamma} \\ & & A\square B \end{array}$$

We will include the commutativity of (5) in the definition of a symmetric monoidal category. If this diagram does not commute, $\mathcal{C}$ is called a *braided* monoidal category. This is because of its relation to the braid group $B_n$ on $n$-letters. This is the group consisting of "braids" of $n$ strands, where multiplication of two braids is defined by identifying the $n$ loose strands on the bottom of one braid with the $n$ loose strands on the top of the next. Notice that there is a surjective homomorphism of groups $B_n \longrightarrow \Sigma_n$ from the braid group onto the symmetric group on $n$ letters, defined by forgetting how strands were crossed over each other and only retaining the induced permutation of strands. There is a similar relation between braided monoidal categories and symmetric monoidal categories.

There is one final coherence diagram to include. Notice that if (5) commutes, the following diagram automatically commutes, so we need not include it. However, it should be included to complete the definition of a braided monoidal category:

(6)
$$\begin{array}{ccccc} & A\square(B\square C) & \xrightarrow{\gamma} & (B\square C)\square A & \\ {\scriptstyle \alpha^{-1}}\swarrow & & & & \searrow{\scriptstyle \alpha} \\ (A\square B)\square C & & & & B\square(C\square A) \\ {\scriptstyle \gamma\square \mathrm{Id}}\searrow & & & & \swarrow{\scriptstyle \mathrm{Id}\,\square\gamma} \\ & (B\square A)\square C & \xrightarrow[\alpha]{} & B\square(A\square C) & \end{array}$$

To learn more about these topics in category theory, see MacLane's book *Categories for the Working Mathematician*, particularly chapter 7 (for coherence) and chapter 11 (for symmetric/braided monoidal categories and braid groups).

## 19. The necessity of coherence (Lind)

The following discussion is meant to impress upon the reader the necessity of thinking about coherence. One might hope that by reducing the category in question to its skeleton, in effect identifying all isomorphic objects, all associativity isomorphisms would become *strict* equalities, thereby circumventing any need for coherence diagrams. We will construct a paradox resulting from this approach.

Let $\mathcal{C}$ be a category with a (categorical) product $\times$. Recall that $sk(\mathcal{C})$, a *skeleton* of $\mathcal{C}$, is a category with with one object from each isomorphism class of objects in $\mathcal{C}$ and morphisms the full set of morphisms in $\mathcal{C}$ between its objects. There is an equivalence of categories:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} sk(\mathcal{C})$$

between the category $\mathcal{C}$ and it skeleton $sk(\mathcal{C})$, where the functor $F$ assigns each object of $\mathcal{C}$ to the unique object of $sk(\mathcal{C})$ representing its isomorphism class, and the functor $G$ sends an object of $sk(\mathcal{C})$ to the same object in $\mathcal{C}$. From the definitions, it is clear that $F$ and $G$ are full and faithful functors. Also, notice that for any morphism $\varphi\colon X \to Y$ of $sk(\mathcal{C})$, we have a strict equality of objects $F(G(X)) = X$, $F(G(Y))$ and of morphisms $F(G(\varphi)) = \varphi$, given by the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \| & & \| \\ F(G(X)) & \xrightarrow[F(G(\varphi))]{} & F(G(Y)) \end{array}$$

Since we have a product $\times$ on $\mathcal{C}$, we can define a product, denoted by $\times_{sk}$, on $sk(\mathcal{C})$: given objects $X, Y$ of $sk(\mathcal{C})$, let $X \times_{sk} Y = F(G(X) \times G(Y))$, i.e. the unique object of $sk(\mathcal{C})$ isomorphic to the product of $X$ and $Y$ as objects of $\mathcal{C}$. To verify that this indeed defines a product on $sk(\mathcal{C})$, consider a pair of maps $Z \to X$, $Z \to Y$. By the universal mapping property for the product $G(X) \times G(Y)$ in $\mathcal{C}$, there exists a unique vertical map making the following diagram commute:

$$\begin{array}{ccccc} & \swarrow & G(Z) & \searrow & \\ & & \downarrow \exists! & & \\ G(X) & \longleftarrow & G(X) \times G(Y) & \longrightarrow & G(Y) \end{array}$$

Applying the functor $F$ and noting the strict equality among objects and morphisms, we have a vertical map satisfying the universal property for the product in $sk(\mathcal{C})$:

$$\begin{array}{ccccc} & \swarrow & Z & \searrow & \\ & & \downarrow \exists! & & \\ X & \longleftarrow & F(G(X) \times G(Y)) & \longrightarrow & Y \end{array}$$

Of course, we usually don't bother writing the functors $F$ and $G$ explicitly, identifying the skeleton $sk(\mathcal{C})$ with the subcategory it defines in $\mathcal{C}$.

Now consider $\mathcal{C} = Set$, the category of sets and functions. Let $D$ be the unique object of the skeleton $sk(Set)$ that has countably infinite cardinality. Then there is a strict equality of objects: $D = D \times D$. Now suppose that the cartesian product $\times$ is strictly associative in $sk(Set)$. Let $f, g, h\colon D \to D$ be functions on $D$. Then by the naturality of the equality

$$D \times (D \times D) = (D \times D) \times D,$$

we must have the equality of morphisms:

$$f \times (g \times h) = (f \times g) \times h\colon D \longrightarrow D.$$

Letting $\pi_1\colon D = D \times D \to D$ be the projection onto the first factor from the product $D \times (D \times D) = (D \times D) \times D$, consider the following commutative diagram:

$$\begin{array}{ccccccc} D & \xleftarrow{\pi_1} & D \times (D \times D) & = & (D \times D) \times D & \xrightarrow{\pi_1} & D \times D \\ \downarrow f & & \downarrow f\times(g\times h) & = & (f\times g)\times h \downarrow & & \downarrow f\times g \\ D & \xleftarrow{\pi_1} & D \times (D \times D) & = & (D \times D) \times D & \xrightarrow{\pi_1} & D \times D \end{array}$$

Reading off along the outermost rectangle, there is an equality of functions

$$f \circ \pi_1 = (f \times g) \circ \pi_1.$$

Since the projection $\pi_1$ is surjective in the category of sets, this means that $f = f \times g$. Similarly, by using the equality $(h \times f) \times g = h \times (f \times g)$ and the second projection $\pi_2$, deduce that $g = f \times g$. Therefore, $f = g$ for any pair of functions $f, g \colon D \to D$, which is absurd.

## 20. Duality (Lind)

We have talked about duality before, but it is such an important notion that it seems worthwhile to consider it in more detail.

We first reconsider the familiar notion of the dual to a vector space. Let $V$ be a finite dimensional vector space over a field $\mathbf{k}$. The dual space of $V$ is by definition $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$. All tensor products will be over the coefficient ring $\mathbf{k}$, so we will drop this from the notation. Consider the *evaluation map*:

$$\begin{aligned} \epsilon \colon V^* \otimes V &\longrightarrow \mathbf{k} \\ T \otimes v &\longmapsto T(v). \end{aligned}$$

Notice that it suffices to define $\epsilon$ on simple tensors of the form $T \otimes v$, as they span the tensor product $V^* \otimes V$.

In order to define the map dual to $\epsilon$, we must first choose a basis $\{v_1, \ldots, v_n\}$ for $V$. There is then a dual basis $\{T_1, \ldots, T_n\}$ for $V^*$ defined by the relations $T_i(v_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta. We may now define the *coevaluation map*:

$$\begin{aligned} \eta \colon \mathbf{k} &\longrightarrow V \otimes V^* \\ \eta(1) &= \sum_{i=1}^{n} v_i \otimes T_i. \end{aligned}$$

Notice that in order to describe the $\mathbf{k}$-linear map $\eta$ it suffices to specify its value on the basis element 1.

Consider the composite map:

$$V \xrightarrow{\cong} \mathbf{k} \otimes V \xrightarrow{\eta \otimes \mathrm{Id}} V \otimes V^* \otimes V \xrightarrow{\mathrm{Id} \otimes \epsilon} V \otimes \mathbf{k} \xrightarrow{\cong} V. \tag{7}$$

Explicitly, this map is given by:

$$v \longmapsto 1 \otimes v \longmapsto \sum_i v_i \otimes T_i \otimes v \longmapsto \sum_i v_i \otimes T_i(v) \longmapsto \sum_i T_i(v) v_i.$$

If we express $v$ in the given basis, say as $v = \sum_j k_j v_j$, we then compute that:

$$\sum_i T_i(v) v_i = \sum_i \sum_j k_j T_i(v_j) v_i = \sum_i k_i v_i = v.$$

Therefore, the composition (7) is the identity. We may also consider the map:

$$V^* \xrightarrow{\cong} V^* \otimes \mathbf{k} \xrightarrow{\mathrm{Id} \otimes \eta} V^* \otimes V \otimes V^* \xrightarrow{\epsilon \otimes \mathrm{Id}} \mathbf{k} \otimes V^* \xrightarrow{\cong} V^*. \tag{8}$$

Since a linear functional $T \in V^*$ is sent under this map to $\sum_i T(v_i) T_i$, by expressing $T$ in terms of the basis $\{T_i\}$ we compute that (8) is the identity map of $V^*$.

Notice that the above discussion hinges on the fact that the dual space $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ is itself a vector space. For an object $X$ in an arbitrary monoidal category $\mathcal{C}$ with unit object $I$, there is no guarantee that the collection of morphisms $\mathcal{C}(X, I)$ is itself an object of the category $\mathcal{C}$, and very often it is not. Hence we are led to take the relationship between the evaluation and coevaluation maps as the defining property of duality:

**Definition 20.1.** Let $\mathcal{C}$ be a monoidal category with product $\square$ and unit object $I$. Two objects $X$ and $Y$ of $\mathcal{C}$ form a *dual pair* (we say that $X$ is *dual* to $Y$, and that $Y$ is *dual* to $X$) if there exist *evaluation* and *coevaluation* maps

$$\epsilon\colon Y\square X \longrightarrow I \qquad \text{and} \qquad \eta\colon I \longrightarrow X\square Y$$

such that the following two composites are the respective identity maps:

$$X \xrightarrow{\cong} I\square X \xrightarrow{\eta\square\mathrm{Id}} X\square Y\square X \xrightarrow{\mathrm{Id}\square\epsilon} X\square I \xrightarrow{\cong} X,$$
$$Y \xrightarrow{\cong} Y\square I \xrightarrow{\mathrm{Id}\square\eta} Y\square X\square Y \xrightarrow{\epsilon\square\mathrm{Id}} I\square Y \xrightarrow{\cong} Y.$$

We will soon see that the circle $S^1$ is a *self-dual* object in 2-$\mathcal{C}ob$.

## 21. Associativity in 2-$\mathcal{C}ob$ (Lind)

There are two different kinds of associativity to keep track of in 2-$\mathcal{C}ob$. First of all, the associativity of composition of morphisms is strict, as in any category: we have equalities $f\circ(g\circ h) = (f\circ g)\circ h$ for morphisms $f, g, h$. Recall that morphisms in 2-$\mathcal{C}ob$ are equivalence classes of cobordisms between the objects, collections of disjoint circles. In particular, if we have objects $\Sigma_m$ and $\Sigma_n$, two cobordisms $M$ and $N$ are equivalent if there exists a diffeomorphism $\varphi\colon M \to N$ making the following diagram commute:

$$\begin{array}{ccccc} & & M & & \\ & \nearrow & & \nwarrow & \\ \Sigma_m & & \downarrow{\scriptstyle\varphi} & & \Sigma_n \\ & \searrow & & \swarrow & \\ & & N & & \end{array}$$

Here the maps from $\Sigma_m$ and $\Sigma_n$ are diffeomorphisms onto the inward and outward boundary components, respectively, of $M$ and $N$. We compose morphisms by glueing cobordisms along the $k$ common circles in the middle, after first taking a tubular neighborhood (that is, a neighborhood of the boundary diffeomorphic to $(S^1)^k\times I$) on either cobordism to identify neighborhoods. Therefore the composition of three morphisms results in the same cobordism, up to diffeomorphism, regardless of which order we choose to glue the three pieces together.

There is also the associativity of the monoidal structure in 2-$\mathcal{C}ob$. Recall that 2-$\mathcal{C}ob$ is a symmetric monoidal category with product $\amalg$, the disjoint union of objects. Explicitly, if $\Sigma_m = \coprod_{i=1}^m S^1$ and $\Sigma_n = \coprod_{i=1}^n S^1$ are two objects of 2-$\mathcal{C}ob$, their product is:

$$\Sigma_m \amalg \Sigma_n = \coprod_{i=1}^{m+n} S^1,$$

the disjoint collection of $m+n$ circles. It is clear that this defines a strict associativity on objects, since the order of adding disjoint collections of circles is irrelevent. This means that the associativity isomorphism $\alpha$ for 2-$\mathcal{C}ob$ is strict equality. The induced product on morphisms is again strictly associative, as the disjoint union of cobordisms gives the same result in whatever order you choose to perform it.

The symmetry isomorphism $\gamma\colon X\amalg Y \to Y\amalg X$ is not a strict equality, however. In fact, it never is in any nontrivial (or at least interesting) symmetric monoidal category. You can see this by noticing that $\gamma$, as a cobordism, is a single tube on each circle comprising $X \amalg Y$, connecting to the same circle in $X$ and $Y$ within $Y \amalg X$. There is an inherent "twisting" necessary to connect the appropriate circles with a tube. Thus 2-$\mathcal{C}ob$ is a *permutative* symmetric monoidal category, meaning that the structure maps $\alpha, \lambda$, and $\rho$ are strict identities, although the symmetry isomorphism $\gamma$ is not.

*Example.* For objects $X, Y$ of 2-$\mathcal{C}ob$, consider the composite:

$$X \amalg Y \xrightarrow{\gamma} Y \amalg X \xrightarrow{\gamma} X \amalg Y.$$

Recalling the definitions in §2 of Lecture 6, check that this composite is the identity, so that 2-$\mathcal{C}ob$ is symmetric monoidal and not just braided monoidal.

## 22. Duality and Frobenius algebras (Lind)

We will first show that a single circle $S^1$ is a self-dual object in 2-$\mathcal{C}ob$, then prove the equivalence of two different definitions of a Frobenius algebra.

Recall that to show that two objects $X$ and $Y$ of a monoidal category are dual, we must define evaluation and coevaluation maps $\epsilon$ and $\eta$, then demonstrate that the interlacing maps involving $\epsilon$ and $\eta$ are identities. To show that $S^1$ is self dual in 2-$\mathcal{C}ob$, we must define an evaluation map $\epsilon\colon S^1 \amalg S^1 \to \emptyset$ (note that the empty circle is the unit object in 2-$\mathcal{C}ob$) and a coevaluation map $\eta\colon \emptyset \to S^1 \amalg S^1$. The appropriate cobordisms are displayed in the next to last row of the file CollatedPictures.pdf. [Tex error prevents incorporation into this file.]

We must show that the composites

$$S^1 \cong S^1 \amalg \emptyset \xrightarrow{\mathrm{Id}\amalg\eta} S^1 \amalg S^1 \amalg S^1 \xrightarrow{\epsilon\amalg\mathrm{Id}} \emptyset \amalg S^1 \cong S^1.$$

$$S^1 \cong \emptyset \amalg S^1 \xrightarrow{\eta\amalg\mathrm{Id}} S^1 \amalg S^1 \amalg S^1 \xrightarrow{\mathrm{Id}\amalg\epsilon} S^1 \amalg \emptyset \cong S^1.$$

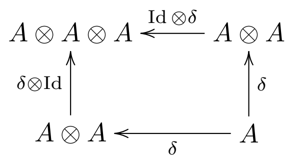
are the identity morphisms. These morphisms are displayed in the last row of the file CollatedPictures.pdf, where it is clear that the left and right cobordisms are diffeomeorphic to the single middle tube — the identity map.

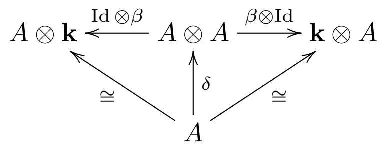
This proves that $S^1$ is dual to $S^1$ in 2-$\mathcal{C}ob$. Since symmetric monoidal functors preserve structure involving monoidal products, and the image of $S^1$ under a symmetric monoidal functor $F\colon$ 2-$\mathcal{C}ob \to \mathcal{V}ect_{\mathbf{k}}$ is the Frobenius algebra associated to the functor $F$, it is not surprising that Frobenius algebras are also self dual objects. This will be made explicit in the following series of definitions.

Recall that an associative algebra with unit $(A, \mu, \alpha)$ over a field $\mathbf{k}$ is a $\mathbf{k}$-module $A$ (i.e. a vector space over $\mathbf{k}$) with a product $\mu\colon A \otimes A \to A$ and unit $\alpha\colon \mathbf{k} \to A$ (all tensor product here and onwards being taken over the ground field $\mathbf{k}$), such that the associativity and unit diagrams commute:

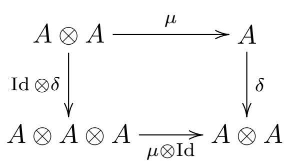
$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{Id}\otimes\mu} & A \otimes A \\ \downarrow{\scriptstyle \mu\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} A \otimes \mathbf{k} & \xrightarrow{\mathrm{Id}\otimes\alpha} & A \otimes A & \xleftarrow{\alpha\otimes\mathrm{Id}} & \mathbf{k} \otimes A \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array}$$

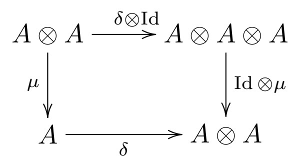
Dually, a coassociative coalgebra with counit $(A, \delta, \beta)$ over $\mathbf{k}$ is a $\mathbf{k}$-module $A$ with a coproduct $\delta\colon A \to A \otimes A$ and counit $\beta\colon A \to \mathbf{k}$ such that the coassociativity and counit diagrams commute:




**Definition 22.1.** A *Frobenius algebra* $A$ is both an associative algebra with unit $(A, \mu, \alpha)$ and a coassociative coalgebra with counit $(A, \delta, \beta)$ for which the coproduct $\delta$ satisfies the Frobenius relation, meaning that $\delta$ is a map of left and right $A$-modules. This means that the following diagrams commute:



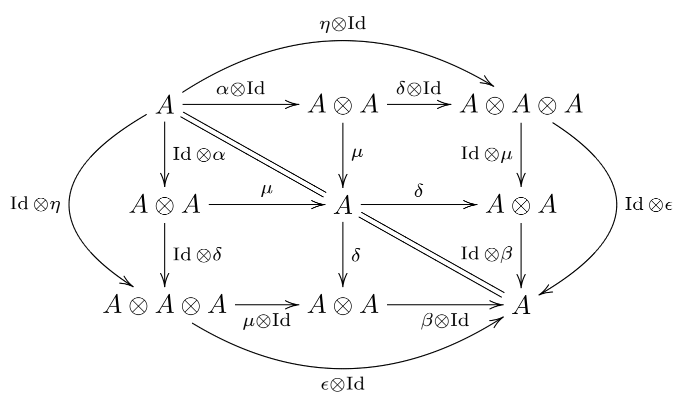




**Definition 22.2.** A *Frobenius algebra* is an associative algebra with unit $(A, \mu, \alpha)$ and a map $\beta \colon A \to \mathbf{k}$ such that $\epsilon := \beta \circ \mu \colon A \otimes A \to \mathbf{k}$ is a duality map. By definition, this means that there exists a map $\eta \colon \mathbf{k} \to A \otimes A$ such that the two interlacing composites for $\eta$ and $\epsilon$ are the identity, thus making $A$ a self-dual object in the category of vector spaces.

Naturally, we must provide a proof of the following statement:

**Theorem 22.3.** *A Frobenius algebra is a Frobenius algebra.*

*Proof.* Assume that $A$ satisfies the requirements of Definition 22.1. To prove duality, the evaluation map is $\epsilon = \beta \circ \mu$, while the coevaluation map is $\eta = \delta \circ \alpha$. Consider the following diagram, where we often leave out implicit tensor products by $\mathbf{k}$ in order to relieve clutter:

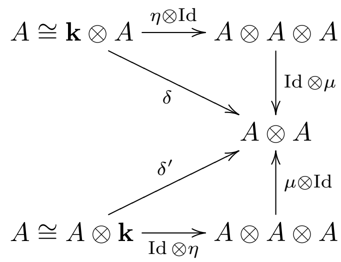



The upper left pair of triangles commute by the unit diagram for $A$, while the lower right pair of triangles commute by the counit diagram. The upper right square commutes because $\delta$ is a map of right $A$-algebras, while the lower left square commutes because $\delta$ is a map of left $A$-modules. Notice that the outer triangles commute by the definition of $\eta$ and $\epsilon$, so the entire diagram commutes. Therefore the composites

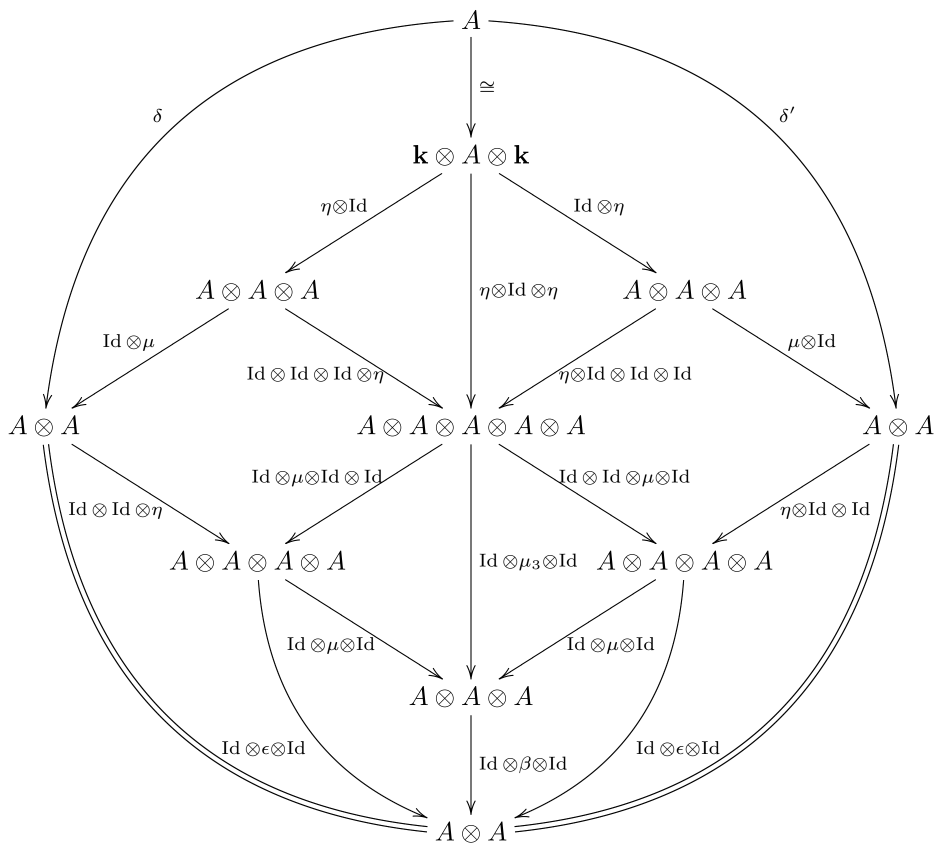
$$A \cong \mathbf{k} \otimes A \xrightarrow{\eta \otimes \mathrm{Id}} A \otimes A \otimes A \xrightarrow{\mathrm{Id} \otimes \epsilon} A \otimes \mathbf{k} \cong A,$$
$$A \cong A \otimes \mathbf{k} \xrightarrow{\mathrm{Id} \otimes \eta} A \otimes A \otimes A \xrightarrow{\epsilon \otimes \mathrm{Id}} \mathbf{k} \otimes A \cong A$$

are identities, as required, so $A$ satisfies Definition 22.2.

Conversely, suppose that A satisfies the requirements of Definition 22.2. We are given a coevaluation map $\eta$ as part of the definition of duality, so we may define maps $\delta, \delta' \colon A \to A \otimes A$ so that the following diagram commutes:

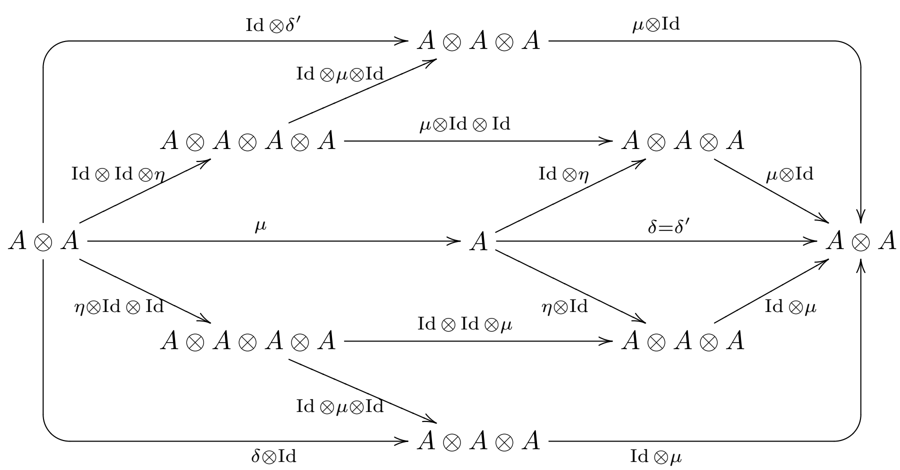



First we will show that $\delta = \delta'$. Consider the following diagram:




The top regions commute by definition of $\delta$ and $\delta'$. The top, left and right diamond shapes commute because either path expresses the same composition. The bottom diamond commutes by the associativity of $\mu$. Of the triangles on the bottom of the diagram, the inner pair commute by definition of $\epsilon = \beta \circ \mu$, while the outer pair commute because they contain the interlacing composite maps for duality, which are identity maps by assumption. Therefore the whole diagram commutes, so $\delta = \delta'$. Also notice that the composite from $A$ to the lower $A \otimes A \otimes A$ is $(\mathrm{Id} \otimes \delta') \circ \delta$ on the left and $(\delta \otimes \mathrm{Id}) \circ \delta'$ on the right. Thus the coassociativity of $\delta = \delta'$ can also be read off of this diagram. The counit conditions are embedded in the lower portion of the diagram, and in particular follow from the identity composite maps resulting from duality. Therefore $\delta$ defines a coassociative and counital coproduct on $A$.

Finally, we must prove that $\delta$ that satisfies the Frobenius relation. The following is the required diagram:




The regions to the left and the middle trapezoids commute by definition. The diamond on the right commutes by the definitions of $\delta = \delta'$. The final two regions commute by the associativity of $\mu$, so the diagram commutes. Comparing the center composition with the uppermost and lowermost paths yields the fact that $\delta$ is a map of left and right $A$-modules. □

*Example.* Draw pictures of composite cobordisms that are equivalent to the previous two diagrams, but in the category 2-$\mathcal{C}ob$!!!

## 23. Duality and internal Hom functors (Bohmann)

We discuss a conceptual point about symmetric monoidal categories with internal hom objects. A symmetric monoidal category $\mathcal{C}$ is said to have internal hom objects if for all objects $Y$ and $Z$, there exists an object $\mathrm{Hom}(Y, Z)$ in $\mathcal{C}$ such that there is a natural isomorphism

$$\mathcal{C}(X \Box Y, Z) \cong \mathcal{C}(X, \mathrm{Hom}(Y, Z))$$

for all $X \in \mathcal{C}$. This is true, for example, when $\mathcal{C} = \mathrm{Vect}_k$ and $\Box = \otimes$, as we discussed before. The important point here is that in a category with internal hom sets, there is a canonical choice of a dual object. That is, if there exists a dual to an object $X \in \mathcal{C}$, it will be isomorphic to $\mathrm{Hom}(X, I)$. We then denote $\mathrm{Hom}(X, I) = X^*$, and note that for any map $\epsilon : Y \Box X \longrightarrow I$, the isomorphism above gives a unique map $\widetilde{\epsilon} : Y \longrightarrow X^*$. This is an isomorphism when the given $\epsilon$ is a duality pairing.

To illustrate the use of the canonical dual, we prove the following proposition. Say that $C$ is a *retract* of $X$ if there exist maps $i, r$ such that the composition

$$C \xrightarrow{i} X \xrightarrow{r} C$$

is the identity.

**Proposition 23.1.** *Suppose $\mathcal{C}$ has internal hom objects. Let $C \in \mathcal{C}$ be a retract of $X \in \mathcal{C}$. Then if $X$ is dualizable, so is $C$.*

*Proof.* Let $X^* = \mathrm{Hom}(X, I)$. The definition of internal hom gives us an isomorphism $\mathcal{C}(X^*\Box X, I) \cong \mathcal{C}(X^*, X^*)$. Thus, the identity map $\mathrm{Id} : X^* \longrightarrow X^*$ gives us a map $\epsilon : X^*\Box X \longrightarrow I$. (This is the counit of the adjunction.) This map $\epsilon$ is natural, by the naturality of internal hom, and so for any map $f : X \longrightarrow X'$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{C}(W, Z\Box X^*) & \xrightarrow{\epsilon_\sharp} & \mathcal{C}(W\Box X, Z) \\
{\scriptstyle \mathcal{C}(\mathrm{Id}, \mathrm{Id}\,\Box f^*)}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\mathrm{Id}\,\Box f, \mathrm{Id})} \\
\mathcal{C}(W, Z\Box X'^*) & \xrightarrow{\epsilon_\sharp} & \mathcal{C}(W\Box X', Z)
\end{array}$$

where here $\epsilon_\sharp$ is as in yesterday's notes.

Taking $f = i$ and $f = r$, we get

$$\begin{array}{ccc}
\mathcal{C}(W, Z\Box C^*) & \xrightarrow{\epsilon_\sharp} & \mathcal{C}(W\Box C, Z) \\
{\scriptstyle \mathcal{C}(\mathrm{Id}, \mathrm{Id}\,\Box i^*)}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\mathrm{Id}\,\Box i, \mathrm{Id})} \\
\mathcal{C}(W, Z\Box X^*) & \xrightarrow{\epsilon_\sharp} & \mathcal{C}(W\Box X, Z) \\
{\scriptstyle \mathcal{C}(\mathrm{Id}, \mathrm{Id}\,\Box r^*)}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\mathrm{Id}\,\Box r, \mathrm{Id})} \\
\mathcal{C}(W, Z\Box C^*) & \xrightarrow{\epsilon_\sharp} & \mathcal{C}(W\Box C, Z)
\end{array}$$

The two vertical composites are the identity since $i^*r^* = (ri)^* = \mathrm{Id}$, and $\epsilon_\sharp$ for $X$ is a bijection. Therefore $\epsilon_\sharp$ for $C$ is a bijection since a retract of a bijection is a bijection. Therefore $C^*$ is dual to $C$. $\Box$

## 24. A few examples, and morphisms of Frobenius algebras (Bohmann)

Both group algebras and matrix algebras are Frobenius algebras, as we will show shortly. These are not commutative, and so they are not examples of the sorts of Frobenius algebras that arise as 2-TQFTs. However, the following proposition allows us to find commutative Frobenius algebras given non-commutative ones.

[At this point, the notes followed a mistake in one of May's talks. He claimed to prove that the center of a Frobenius algebra is a Frobenius algebra. That may be true, but the proof May gave is not correct. Symmetric algebras will be defined and discussed shortly.]

**Proposition 24.1.** *At least when $k = \mathbb{R}$, The center of a symmetric algebra is a Frobenius algebra.*

*Proof.* Let $A$ be a Frobenius algebra, and $C = \{c \,|\, ca = ac \;\forall\, a \in A\}$ be its center. The center of any algebra is a subalgebra. Indeed, the unit $\alpha\colon k \longrightarrow A$ has image in $C$ and the product $\mu\colon A \otimes A \longrightarrow A$ restricts to a product $C \otimes C \longrightarrow C$ since the product of two elements in the center of $A$ is again in the center of $A$. Since $A$ is Frobenius, there exists $\beta : A \longrightarrow k$ such that $\epsilon = \beta \circ \mu : A \otimes A \longrightarrow k$ is a duality map. Letting $i : C \hookrightarrow A$ be the inclusion, we get maps

$$C \overset{i}{\hookrightarrow} A \overset{\beta}{\longrightarrow} k$$

$$C \otimes C \overset{i \otimes i}{\longrightarrow} A \otimes A \overset{\epsilon}{\longrightarrow} k.$$

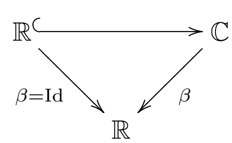
which we again denote by $\beta$ and $\epsilon$. When $\epsilon$ is a nondegenerate pairing on $C$, $C$ is a Frobenius algebra. Let $D = \{a \,|\, \epsilon(c, a) = 0 \ \forall\, c \in C\}$. Then $D$ is a vector subspace of $A$. We would like to say that $A$ is the direct sum of $C$ and $D$. If $A$ is a symmetric algebra, then, by definition, $\epsilon$ on $A$ is a nondegenerate symmetric bilinear form. When $k = \mathbb{R}$, that means that $\epsilon$ is an inner product, and the reader will likely have seen the proof that $A = C \oplus D$, a result that depends only on the fact that $C$ is a sub vector space of the inner product space $A$. Thus assume that $A = C \oplus D$. Then if $c \in C$ and $\epsilon(c, c') = 0$ for all $c' \in C$, $\epsilon(c, a) = 0$ for all $a \in A$ and thus $c = 0$. Therefore $C$ is a Frobenius algebra. $\square$

Now we provide some concrete examples of Frobenius algebras.

*Example.* Take $k = \mathbb{R}$. Then $\mathbb{C}$ is a Frobenius $\mathbb{R}$-algebra. The multiplication $\mu$ and inclusion $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ are the usual multiplication on the complex numbers and inclusion of the reals. We define $\beta : \mathbb{C} \longrightarrow \mathbb{R}$ by $\beta(a + bi) = a$, and $\delta : \mathbb{C} \longrightarrow \mathbb{C} \otimes \mathbb{C}$ by $\delta(1) = 1 \otimes 1 - i \otimes i$ and $\delta(i) = i \otimes 1 + 1 \otimes i$. We leave as an exercise to the reader that these maps make $\mathbb{C}$ into a Frobenius $\mathbb{R}$-algebra.

An interesting and subtle point about this structure on $\mathbb{C}$ illustrates an important difference between the algebra and coalgebra definition of a Frobenius algebra and the definition using a duality map. Using the first definition — that is, the definition requiring an algebra and coalgebra structure on $A$ so that $\delta \circ \mu$ is a module map — we defined the morphisms in the category of Frobenius algebras to be maps preserving the structure $(\mu, \alpha, \delta, \beta)$. But if we used the other definition of a Frobenius algebra, which only involves maps $(\mu, \alpha, \beta)$, the most obvious definition of morphisms in the category of Frobenius algebras would only preserve the structure given by these maps. Hence the two equivalent definitions of Frobenius algebras suggest different notions of the category of Frobenius algebras.

The above example is an excellent illustration of this point. Taking $\mathbb{C}$ to be a Frobenius $\mathbb{R}$ algebra with the above structure, it is easy enough to check that the diagram




commutes. Hence, according to the second definition of the category of Frobenius algebras, the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ should be a morphism of Frobenius algebras. But this inclusion does not commute with $\delta$. Thus we see that the second definition of Frobenius algebras gives rise to an insufficient notion of morphisms between Frobenius algebras, and we therefore prefer the algebra and coalgebra definition of a Frobenius algebra.

In fact, using the algebra and coalgebra definition of a Frobenius algebra and defining morphisms of Frobenius algebras as maps that preserve $(\mu, \alpha, \delta, \beta)$, we have the following proposition, which says that the category of Frobenius algebras is a groupoid.

**Proposition 24.2.** *Every map of Frobenius algebras is an isomorphism.*

*Proof.* We first prove that every map of Frobenius algebras is a monomorphism, i.e. injective; the result then follows by the self-duality of the definition of Frobenius algebras.

Let $f : A \longrightarrow B$ be a map of Frobenius algebras. Since $f$ preserves the Frobenius algebra structure $(\mu, \alpha, \delta, \beta)$, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{\beta}\searrow \quad \swarrow_{\beta} & \\ & k & \end{array}$$

We claim that the map $\beta : A \longrightarrow k$ has no nonzero ideals in its kernel. Commutivity then implies that $\ker f$ also contains no nonzero ideals, but since $f$ is an algebra homomorphism, $\ker f$ is itself an ideal and thus $\ker f = 0$. The bilinear form $\epsilon = \beta \circ \mu : A \otimes A \longrightarrow k$ must be nondegenerate, since it is a duality map. Suppose $b \in A$ is contained in some ideal in $\ker \beta$. Then for all $a \in A$, $ab$ is also contained in this ideal, and so $ab \in \ker \beta$. Now, $\epsilon(a \otimes b) = \beta(ab)$ by definition, and so if $ab \in \ker \beta$ for all $a \in A$, the nondegenerateness of $\epsilon$ implies that $b = 0$. Hence we see that the kernel of $\beta$ contains no nonzero ideals, and so $\ker f = 0$. Thus $f$ is a monomorphism. Since the definition of a Frobenius algebra is self-dual, the dual map $f^*$ must be a map of Frobenius algebras as well. This implies that $f^*$ is a monomorphism. Therefore $f$ is an epimorphism since the dual $f^*$ of a map of vector spaces can only be a monomorphism when $f$ is an epimorphism. □

## 25. More examples of Frobenius algebras (Bohmann)

We give some important examples of Frobenius algebras.

*Example.* The matrix algebra $M_n(k)$ of $n \times n$ matrices with entries in a field $k$ is a Frobenius algebra. We take $\mu : M_n(k) \otimes M_n(k) \longrightarrow M_n(k)$ to be the usual matrix multiplication, and $\alpha : k \longrightarrow M_n(k)$ to be the usual inclusion of scalar matrices, so that $\alpha(1) = I$. For the co-algebra structure on $M_n(k)$, we define $\beta : M_n(k) \longrightarrow k$ to be the trace function, and $\delta : M_n(k) \longrightarrow M_n(k) \otimes M_n(k)$ to be defined by $\delta(T) = \sum_{i,j} TE_{ij} \otimes E_{ji}$ where the $E_{ij}$ are the elementary symmetric matrices. That is, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, so $E_ij$ has 1 in the $(i, j)$th entry and all other entries zero.

We check that $\epsilon = \beta \circ \mu$ and $\eta = \delta \circ \alpha$ satisfy the triangle identies, so that this structure indeed gives a Frobenius algebra structure on $M_n(k)$. Note that $\eta(1) = \delta(I) = \sum_{ij} E_{ij} \otimes E_{ji}$, and $\epsilon(S \otimes T) = \mathrm{tr}(ST)$. Then we have

$$M_n(k) \cong k \otimes M_n(k) \xrightarrow{\eta \otimes \mathrm{Id}} M_n(k) \otimes M_n(k) \otimes M_n(k) \xrightarrow{\mathrm{Id} \otimes \epsilon} M_n(k) \otimes k \cong M_n(k)$$

$$T \longmapsto \sum_{ij} E_{ij} \otimes E_{ji} \otimes T \longmapsto \sum_{ij} \mathrm{tr}(E_{ji}T) E_{ij} = T$$

so we see that this composition is the identity.

We make a definition.

**Definition 25.1.** A Frobenius algebra $A$ is a *symmetric algebra* if

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\quad\gamma\quad} & A\otimes A \\ & {}_{\epsilon}\searrow \quad \swarrow_{\epsilon} & \\ & k & \end{array}$$

commutes, where $\gamma$ is the twist map.

The matrix algebra $M_n(k)$ is an example of a noncommutative symmetric algebra, since $\epsilon(S\otimes T) = \mathrm{tr}(ST) = \mathrm{tr}(TS) = \epsilon(T\otimes S)$. Note that being symmetric depends on the choice of the map $\beta : A \longrightarrow k$; we can twist $\beta$ by any unit and still get a well-defined Frobenius algebra, but unless the unit is in the center of $A$, the new Frobenius algebra structure won't be symmetric.

*Example.* A field extension gives a trivial example of a Frobenius algebra. If $\alpha : K \hookrightarrow L$ is a field extension with $[L : K] < \infty$, we can get a Frobenius algebra by taking any map of vector spaces $\beta : L \longrightarrow K$. Since $L$ is a field, it has no nontrivial ideals, and so any $\beta : L \longrightarrow K$ has no nontrivial ideals in the kernel. This guarantees that $\epsilon$ is nondegenerate, and makes $L$ a Frobenius algebra. Note that if the extension is seperable, trace gives a canonical choice of maps $L \longrightarrow K$.

*Example.* A group algebra is a Frobenius algebra. Let $G$ be a finite group, and let $k[G]$ be the group algebra. We have a multiplication $\mu : k[G]\otimes k[G] \longrightarrow k[G]$ defined on basis elements $g, h \in G$ by $\mu(g\otimes h) = gh$, and we define $\alpha : k \longrightarrow k[G]$ by $\alpha(1) = e$, where $e$ is the identity of $G$. We then define a coalgebra structure $(\beta, \delta)$ on $k[G]$ by

$$\beta(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

and, $\delta(g) = \sum_i gg_i \otimes g_i^{-1}$, where $\{g_i\}$ is some ordering of the elements of $G$. Then $\eta(1) = \sum_i g_i \otimes g_i^{-1}$ and

$$\epsilon(g\otimes h) = \beta(gh) = \begin{cases} 1 & h = g^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

We check the triangle identity again:

$$k[G] \cong k\otimes k[G] \xrightarrow{\eta\otimes \mathrm{Id}} k[G]\otimes k[G]\otimes k[G] \xrightarrow{\mathrm{Id}\otimes\epsilon} k[G]\otimes k \cong k[G].$$

$$g \longmapsto \sum g_i \otimes g_i^{-1} \otimes g \longmapsto \sum \epsilon(g_i^{-1}\otimes g) g_i$$

so since $\epsilon(g_i^{-1}, g) = 0$ unless $g_i = g$, we see that this composition is the identity. Hence this $\beta$ and diagonal map $\delta : k[G] \longrightarrow k[G]\otimes k[G]$ give a Frobenius algebra structure on the group algebra $k[G]$. Observe that it too is a symmetric algebra.

It may seem more natural to take the diagonal map $\psi : k[G] \longrightarrow k[G]\otimes k[G]$ defined by $\psi(g) = g\otimes g$. Together with the map $\zeta : k[G] \longrightarrow k$ given by $\zeta(g) = 1$ for all $g \in G$, this does indeed define a coalgebra structure on $k[G]$. But this cannot be a Frobenius algebra structure, because $\zeta$ is a map of algebras and its kernel, which is the ideal generated by all elements $e - g$, is a non-zero ideal. This algebra and coalgebra structure $(\mu, \alpha, \psi, \zeta)$ give a *Hopf algebra* structure on $k[G]$. Although the proof is beyond the scope of this course, the following theorem holds.

**Theorem 25.2.** *A finite dimensional Hopf algebra admits the structure of a Frobenius algebra.*

When generalized to graded Hopf algebras, a topological interpretation of this theorem says that the cohomology of an $H$-space satisfies Poincaré duality.

## 26. Generators and relations in algebra (Lind)

Suppose that $A$ is an algebra over a field $\mathbf{k}$ generated by a set of elements $\{x_1, \ldots, x_q\}$. This means that every element of $A$ can be written as a linear combination of products of the $x_i$:

$$\sum_{k=1}^{p} c_k\, x_1^{e_{k1}} \cdots x_q^{e_{kq}}, \qquad \text{where } c_k \in \mathbf{k}, e_{ki} \geq 0.$$

There may be many relations that the $x_i$ satisfy which one must know in order to use the generators for computation in $A$. For example, consider the complex numbers $\mathbb{C}$ as an algebra over $\mathbb{R}$. Since $i^2 = -1$, we know that $\mathbb{C}$ is generated by the single element $i$, and indeed this relation suffices to describe $\mathbb{C}$ in terms of the generator $i$:

$$\mathbb{C} \cong \mathbb{R}[i]/(i^2+1).$$

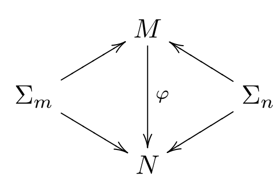
Quotienting out by the ideal generated by $i^2+1$ is equivalent to the relation $i^2 = -1$ holding in the quotient algebra. More generally, a set of relations for a generating set $\{x_1, \ldots, x_q\}$ is a set of elements $\{\rho_j\}_{j \in J}$ of $\mathbf{k}[x_1, \ldots, x_q]$ such that:

$$A \cong \mathbf{k}[x_1, \ldots, x_q]/(\rho_j)_{j \in J}. \tag{9}$$

The equations $\rho_j = 0$ are thus all of the information necessary to construct $A$ from the generators $x_i$. Of course, we could ask for a minimal set of relations, and so on, but as long as (9) holds, we have identified enough relations to describe $A$.

## 27. Generators for the morphisms in 2-$\mathcal{C}ob$ (Lind)

We would like to do the analogous sort of description in terms of generators and relations for the morphisms in 2-$\mathcal{C}ob$. Recall that morphisms in 2-$\mathcal{C}ob$ are equivalence classes of cobordisms between the objects, collections of disjoint circles. In particular, if we have objects $\Sigma_m$ and $\Sigma_n$, two cobordisms $M$ and $N$ are equivalent if there exists a diffeomorphism $\varphi \colon M \to N$ making the following diagram commute:




Here the maps from $\Sigma_m$ and $\Sigma_n$ are diffeomorphisms onto the inward and outward boundary components, respectively, of $M$ and $N$.

We will take for granted the classification theorem of closed (i.e. compact and without boundary) oriented surfaces. Essentially, this states that the genus of a surface completely classifies it, up to diffeomorphism, and in fact up to homeomorphism as well. Thus if the genus of a surface $S$ is $g$, then $S$ is diffeomorphic to the connected sum of $g$ copies of the torus $T^2$:

$$S \approx T^2 \# \cdots \# T^2.$$

In the case of cobordisms, we have to worry about boundary components as well. Given a *connected* cobordism $M$ between $\Sigma_m$ and $\Sigma_n$, we glue $m$ disks onto the inward oriented boundary components and $n$ disks onto the outward oriented boundary components. The resulting surface $\overline{M}$ is closed, hence uniquely determined up to diffeomorphism by its genus $g$. Notice that an equivalence of cobordisms fixes the boundary components, and hence fixes the disks after we add them on, so defines a diffeomorphism of the resulting closed surface. This means that a connected morphism $[M]$ in 2-$\mathcal{C}ob$ is uniquely determined by:

- the genus $g$ of any representative and completed surface $\overline{M}$,
- the number (and labelling) of inward oriented boundary components $m$.
- the number (and labelling) of outward oriented boundary components $n$.

The labelling, or ordering, of the boundary components helps keep track of the domain and codomain of $[M]$ as a morphism in 2-$\mathcal{C}ob$: the object $\Sigma_n$ of 2-$\mathcal{C}ob$ is the (ordered) disjoint union of $n$ circles. Thinking of disjoint union as categorical coproduct, the labelling is essential. Observe, however, that different orderings give diffeomorphic cobordisms. This relates to the commutativity and cocommutativity of the monoid and comonoid $S^1$ in 2-$\mathcal{C}ob$, as we shall see. Since we may take the disjoint union of cobordisms under the monoidal product $\amalg$, we also have to account for cobordisms with many connected components, and again we have to keep track of the labelling or ordering of these components. Suppose that a morphism in 2-$\mathcal{C}ob$ is represented by a cobordism $M$ with $k$ components. Then after gluing disks onto the boundary components, $\overline{M}$ is the disjoint union of $k$ closed surfaces, so the morphism is determined by:

- the genera $g_1, \ldots, g_k$ of the components of $\overline{M}$,
- the number of inward oriented boundary components $m_1, \ldots, m_k$ of each component,
- the number of outward oriented boundary components $n_1, \ldots, n_k$ of each component,
- the labelling (or ordering) of the components.

This classification will allow us to find a complete set of generators for the morphisms of 2-$\mathcal{C}ob$.

Let $\mu_m$ be the cobordism corresponding to $m$-fold multiplication, with $m$ circles combining into one outgoing circle [Blank space is provided for the reader to supply the missing pictures. Doing so is a highly recommended exercise!]:

Similarly, let $\delta_n$ be the cobordism corresponding to $n$-fold comultplication:

To create a given genus, notice that $\sigma = \delta\mu$ is a cobordism with genus one (and one inward, one outward boundary component):

Set $\sigma_g = \sigma^g$, i.e. the gluing of $g$ copies of $\sigma$ in a chain. Then $\sigma_g$ has genus $g$. To keep track of permutations of order, identify each element $\tau$ of the symmetric group on $a$ letters with the cobordism from $a$ circles to $a$ circles which realizes $\tau$ by permuting the order of the circles. Then we may write an arbitrary cobordism in the following form:

$$\tau\Big(\coprod_{i=1}^{k} \mu_{m_i}\sigma_{g_i}\delta_{n_i}\Big)\omega,$$

where $\tau$ permutes the $\sum_i m_i$ inward boundary circles and $\omega$ permutes the $\sum_i n_i$ outward boundary circles. This is our normal form for cobordisms. We still have to determine how to manipulate an arbitrary cobordism into this form through equivalences of cobordisms. This will be accomplished in lectures to come.

Notice that all of the elements of the normal form for cobordisms may be built out of the following atoms: $\mu$, $\delta$, $\alpha$, $\beta$, Id, and $\gamma$. These cobordisms are displayed below [see also CobordismPictures.pdf]:

Recall that all permutations may be written as products of transpositions, so it suffices to include the single transposition cobordism $\gamma$ to generate all permutation cobordisms. Following the rubric outlined in the previous section, having identified a set of generators for the morphisms of 2-$\mathcal{C}ob$, we must determine a complete set of relations among them. For example, the Frobenius relation states (in part) that the following cobordisms are equivalent:

Diagramatically, this may be expressed by the commutativity of the following:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle \mathrm{Id} \otimes \delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \otimes A \otimes A & \xrightarrow[\mu \otimes \mathrm{Id}]{} & A \otimes A \end{array}$$

Relations such as these will be accounted for in the following lectures.

In a slightly different direction, consider the following diagram:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\psi \otimes \psi} & A \otimes A \otimes A \otimes A \\ {\scriptstyle \varphi}\downarrow & & \\ A & & \downarrow{\scriptstyle \mathrm{Id} \otimes \gamma \otimes \mathrm{Id}} \\ {\scriptstyle \psi}\downarrow & & \\ A \otimes A & \xleftarrow[\varphi \otimes \varphi]{} & A \otimes A \otimes A \otimes A \end{array}$$

Letting $\varphi$ be multiplication and $\psi$ comultiplication, this expresses the defining property of Hopf algebras, another type of algebra and coalgebra that arises with even more frequency than Frobenius algebras in mathematics. Translating this into topology, the diagram would state the equivalence of the following two cobordisms:

By our classification, it is *impossible* for these to be equivalent, since the cobordism on the left has genus zero while the cobordism on the right has genus one. Of course, this illustrates that topological quantum field theories do not give rise to Hopf algebras: the algebraic diagrams do not correspond to the topology of surfaces. However, Hopf algebras do arise in a quite different way in algebraic topology: the homology and cohomology of $H$-spaces give examples of Hopf algebras.

## 28. Differentiation on manifolds (Henry)

Let us begin by recalling a few definitions. For many this will be a review.

**Definition 28.1.** Roughly speaking, a manifold is a topological space with the property that every point has a neighborhood homeomorphic to Euclidean space. Specifically, an n-dimensional manifold M is a (second countable Hausdorff) topological space M, together with, for each point $m \in M$, a neighborhood $U_m$ of m and a homeomorphism $\varphi_m : U_m \to \mathbb{R}^n$. The maps $\varphi_m$ are called the coordinate charts of the manifold M.

**Definition 28.2.** Let f be a map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. We say that f is of class $C^k$ if all of the partial derivatives $\frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}$ exist and are continuous, where $i_1, ..., i_k$ are integers between 1 and n. We say that f is smooth if f is of class $C^k$ for all $k \in \mathbb{N}$.

**Definition 28.3.** An atlas of an n-dimensional manifold M is a set of pairs A $= \{(U_\alpha, \varphi_\alpha)\}$ where the $U_\alpha$ are open sets that cover M and the $\varphi_\alpha$ are homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The transition maps of A are the maps $\psi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$. Note that the transition maps of an atlas of $M$ are maps from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$.

**Definition 28.4.** A manifold M is said to be differentiable if it has an atlas whose transition maps are all differentiable. More specifically we say that M is of class $C^k$ if M has an atlas whose transition maps are all of class $C^k$, and we say that M is smooth if it is of class $C^k$ for every $k \in \mathbb{N}$.

**Definition 28.5.** Let M be an n-dimensional smooth manifold. A map $f : M \to \mathbb{R}$ is said to be differentiable at a point $m \in M$ if there is a neighborhood U of m and a coordinate chart $\varphi : U \to \mathbb{R}^n$ such that the map $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(m)$. The map f is said to be differentiable if it is differentiable at every $m \in M$. The map f is said to be smooth if the map $f \circ \varphi^{-1}$ above is smooth. This definition appears to depend on the the choice of of coordinate chart, but this is not the case. One may check that, applying the chain rule to the transition maps between coordinate charts, if f is differentiable in one chart at m it is differentiable in every chart at m.

Let $M$ be a smooth manifold and consider all of the smooth maps $f : M \to \mathbb{R}$. The idea of Morse theory is to use particularly nice such maps to see what $M$ looks like. Interesting things happen when all of the first partial derivatives of f are zero.

**Definition 28.6.** Let $f : M \to \mathbb{R}$ be a smooth map. The points of M where all of the first partial derivatives of f are zero are called critical points. Their images under f are called critical values.

One important theorem of Morse theory is that if $M$ is compact then there are only finitely many critical points of $f$. The reader is welcome to prove this result. Let $m$ be a critical point of $M$ and consider the $n \times n$ matrix $T$ whose $ij^{th}$ entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}(m)$. We can ask whether this matrix is invertible. If its determinate is non-zero then the critical point $m$ is called non-degenerate, otherwise it is called degenerate. Such a matrix is diagonalizable with real eigenvalues. We define the index of the non-degenerate critical point $m$ to be the number of negative eigenvalues the matrix $T$.

**Lemma 28.7.** *(Morse) Let m be a non-degenerate critical point of the smooth map $f : M \to \mathbb{R}$ and suppose m has index k. Then there is a neighborhood U of m and a coordinate chart $\varphi : U \to \mathbb{R}^n$ such that $f \circ \varphi^{-1}(u_1, ..., u_n) = f(m) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2$ on $\varphi(U)$.*

In dimension 2 there are precisely three choices for $k$, namely 0, 1, or 2, when $m$ is a local minimum, a saddle point, or a local maximum.

## 29. A short digression on the twist map (Henry)

Recall that if $V$ is a vector space and $V_1, V_2 \subseteq V$ are subspaces such that $V_1 \cap V_2 = 0$ and $V_1 + V_2 = V$ then $V$ is the internal direct sum $V_1 \oplus V_2$ of $V_1$ and $V_2$, and the order in which we write the sum doesn't matter. This is no longer true when we consider the "external" direct sum, the categorical coproduct, of two given vector spaces $V_1$ and $V_2$; here the order in which we write the sum does matter. Similarly, the order matters for tensor products. The analogous dichotomy occurs with our cobordisms. When considered as manifolds with boundary, the order of the components of the (incoming and outgoing parts) of the boundary does not matter. but when we consider cobordisms as morphisms in the category **n-Cob**, we use homeomorphisms from external disjoint unions to the incoming and outgoing parts of the boundary and the ordering of components must be kept track of.

Remember that **n-Cob** is a symmetric monoidal category, so that the symmetry $\gamma$ is *natural.* Let $A$ be a Frobenius algebra. Then the following diagram commutes:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\gamma} & A\otimes A \\ {\scriptstyle \delta\otimes 1}\downarrow & & \downarrow{\scriptstyle 1\otimes\delta} \\ (A\otimes A)\otimes A & \xrightarrow[\gamma]{} & A\otimes(A\otimes A) \end{array}$$

But we also have coherence relations for $\gamma$, so that the following diagram commutes where, in the bottom triangle, we are thinking of the top two entries as unparenthesized:

$$\begin{array}{ccccc} A\otimes A & & \xrightarrow{\gamma} & & A\otimes A \\ {\scriptstyle \delta\otimes 1}\downarrow & & & & \downarrow{\scriptstyle 1\otimes\delta} \\ (A\otimes A)\otimes A & & \xrightarrow[\gamma]{} & & A\otimes(A\otimes A) \\ & {\scriptstyle 1\otimes\gamma}\searrow & & \nearrow{\scriptstyle \gamma\otimes 1} & \\ & & A\otimes A\otimes A & & \end{array}$$

In class we translated these diagrams into pictures in **2-Cob**. If you took notes, good for you. If not, you should try to replicate them yourself.

## 30. Back to Morse theory (Henry)

If we have a smooth compact oriented manifold $M$ with boundary and a smooth map $f : M \to \mathbb{R}$, by suitable composition of smooth maps we can arrange so that the target of $f$ is the interval $[0, 1]$, $f^{-1}(0)$ is the incoming boundary of $M$, and $f^{-1}(1)$ is the outgoing boundary of $M$. Since $M$ is compact, as we remarked above, $f$ has finitely many critical points, and so we may order the critical points of $f$ in such a way that there is a unique critical point that maps to each critical value.

**Definition 30.1.** Recall that a smooth real-valued function on M is called a Morse function if it has no degenerate critical points.

We have been discussing Morse functions for a while now. But how do we know that there are any? Here is a sketch proof: Look at the set of maps $f : M \to \mathbb{R}$ and give it a nice topology. (The $C^2$ topology for those who are interested.) Apply the Baire Category Theorem to show that the Morse functions are dense in this space. The details are left to the reader (to look up in a suitable text, for example Hirsch's Differential Topology).

## 31. A little algebraic topology (Henry)

Suppose we have a connected 2-cobordism $M$ in normal form. Let us look at the generators. Suppose that we have $a$ generators that fork to the left, $b$ generators that fork to the right, $p$ generators that cap off to the right, and $q$ generators that cap off to the left. Suppose further that $M$ has genus $g$, has $m$ incoming circles, and $n$ outgoing circles. We now introduce a topological invariant called the Euler characteristic $\chi(M)$, defined to be the number of vertices (0-cells) of $M$ minus the number of edges (1-cells)of $M$ plus the number of discs (2-cells) of $M$ etc...

**Example**: Consider the circle $S^1$ decomposed as two vertices and two edges. It follows that $\chi(S^1) = 0$. Consider the sphere $S^2$ decomposed as two discs. It follows that $\chi(S^2) = 2$.

**Example**: Consider the torus. Recall that a torus can be constructed from the disjoint union of four vertices, four edges, and one disc by identifying opposite edges. It follows that the torus has one vertex, two edges, and one disc, hence its Euler characteristic is zero.

A little thought will show that a closed surface of genus $g$ can be constructed from the disjoint union of $4g$ vertices, $4g$ edges, and a disc by identifying the appropriate pairs of edges. All of the vertices will end up being identified, hence the Euler characteristic of a closed surface of genus $g$ is $1 - 4g/2 + 1 = 2 - 2g$. If we have a 2-manifold with boundary, we can think of each boundary component as having arisen from removing a cap consisting of a vertex, an edge, and a disc from a closed surface. Such a cap evidently has Euler characteristic one, so each such removal lowers the Euler characteristic by one. Thus the Euler charactersistic of the our connected 2-cobordism $M$ is $2 - 2g - m - n$. Think about gluing two 2-manifolds $M$ and $N$ together: generally $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$. In our case $M \cap N$ is always a circle (or rather a disjoint union of circles), which has Euler characteristic zero, hence the Euler characteristic is additive with respect to gluing. It follows that the Euler characteristic of $M$ is also $p + q - a - b$. We also have $a + q + n = b + p + m$, so, solving the resulting system of equations, we have

$$a = m + g - 1 + p \quad \text{and} \quad b = n + g - 1 + p.$$

## 32. General ideas (Bohmann)

The material in the rest of these notes comes from lecture notes taken by Bohmann on a lecture given by Mitya Boyarchenko, who reported having shamelessly stolen the material from notes by Graeme Segal that are available on the web**See `http://www.cgtp.duke.edu/ITP99/segal/`. We attempt to relate the topological quantum field theories that are the subject of this course–namely, symmetric monoidal functors $n\text{-}\mathfrak{Cob} \longrightarrow \mathrm{Vect}_k$–to physics. In particular, we will try to explain where the axioms of TQFT theory come from, and why we care about oriented cobordisms, time permitting.

There are two general pictures or ideas to keep in mind:

1.) Computing areas of polygons in $\mathbb{R}^2$.

We can compute the area of a polygon in $\mathbb{R}^2$ by cutting it into right triangles, and using the area formula $A = \frac{1}{2}ab$ for a right triangle with legs of length $a$ and $b$. We could in fact define the area of a polygon this way, but it is unclear a priori that this is a well-defined concept: different ways of cutting up a polygon might give different answers. You should think about why they don't.

2.) Cohomology.

We consider cohomology in as much as it gives us a way of assigning numbers to compact manifolds. For a compact smooth manifold $X$, we can define the Betti numbers $b^i(X) = \dim_{\mathbb{R}}(H^i(X, \mathbb{R}))$. In order to compute these numbers, we have to enlarge the set of spaces we're considering by at least allowing non-compact manifolds. This gives us tools like the Mayer-Vietoris sequence, which, if we write $X = U \cup V$, where $U, V \subset X$ are open subsets, relates the cohomology of $X$ to that of $U$, $V$ and $U \cap V$.

In a TQFT, something similar to (2) happens. Fix a natural number $n$, the dimension of the theory. A TQFT in dimension $n$ will assign certain "invariants" to smooth compact $n$-dimensional manifolds. These invariants will not be integers. Rather, to a smooth compact $n$-manifold $X$, a TQFT will assign a complex number depending only on the diffeomorphism class of $X$. The key property of a TQFT is that these invariants can be computed by cutting $X$ into "pieces" where are manifolds with boundary. However, if one of these pieces has a non-empty boundary, the TQFT will assign to it not a number, but an element in some vector space that gets attached to the boundary by the TQFT.

*Remark* 2. If we think of a manifold without boundary as a cobordism from the empty manifold to itself, it gets assigned a linear map $\mathbb{C} \longrightarrow \mathbb{C}$ which is just multiplication by some complex number.

## 33. Fields on manifolds (Bohmann)

The invariants of a TQFT come from "fields" on manifolds. What are fields? Well, roughly speaking, fields[2] are "objects that live on manifolds, which can be defined locally". To a manifold $M$, one attaches the "space of fields" $\mathcal{F}(M)$ on $M$.

*Example.*

1.) $\mathcal{F}(M) = \{C^\infty \text{ functions } M \longrightarrow \mathbb{R}\}$
2.) $\mathcal{F}(M) = \{\text{space of all Riemannian metrics on } M\}$
3.) Let $G$ be a finite group. Then we can take $\mathcal{F}(M) = \{G\text{-torsors on } M\}$, where a $G$-torsor on $M$ is a finite covering space $N \xrightarrow{\pi} M$ together with a $G$-action on $N$ that is simply transitive along the fibers. We can in fact take $G$ to be any Lie group, and let $\mathcal{F}(M)$ be principal $G$-bundles, but we will only use the finite group example.

   In fact, the TQFT we get from this example in dimension 2 gives rise to the commutative Frobenius algebra $Z(k[G])$, the center of the group algebra $k[G]$ under the equivalence of categories that is the main theorem of this course. The TQFT itself is defined at the end of the lecture.

Suppose we have a "field theory" as in one of the examples above. We can get the invariant $z(M)$ associated to a complex $n$-dimensional manifold $M$ without

[2] There is no reasonable all-encompassing mathematical definition of the word "field", just as there is no such definition of the word "particle". Both fields and particles are physical concepts.

boundary by setting

$$z(M) = \int_{\mathcal{F}(M)} e^{S(\varphi)}\, D\varphi$$

where $D\varphi$ is some measure on $\mathcal{F}(M)$ and $S : \mathcal{F}(M) \longrightarrow \mathbb{C}$ is the "action functional". Any field theory worth its salt comes with this measure $D\varphi$ and the functional $S$.

Given a field theory, the idea is to compute the numbers $z(M)$. Certain properties of fields make it possible to compute these numbers by cutting $M$ into simpler manifolds with boundary. If we let $M = M_1 \cup M_2$, where $M_1$ and $M_2$ are $n$-dimensional manifolds with boundary such that $\partial M_1 = M_1 \cap M_2 = \partial M_2$, then the locality of fields implies that to give a field $\varphi$ on $M$ is the same as to give fields $\varphi_i$ on $M_i$, $i = 1, 2$, such that $\varphi_1 | M_1 \cap M_2 = \varphi_2 | M_1 \cap M_2$. What's more, in this case, the action function $S$ has the property that $S(\varphi) = S(\varphi_1) + S(\varphi_2)$. These properties allow us to make the following "calculation":

$$\begin{aligned} z(M) &= \int_{\mathcal{F}(M)} e^{-S(\varphi)}\, D\varphi \\ &= \int_{\substack{\varphi_1 \in \mathcal{F}(M_1),\, \varphi_2 \in \mathcal{F}(M_2) \\ \varphi_1|_{M_1 \cap M_2} = \varphi_2|_{M_1 \cap M_2}}} e^{-S(\varphi_1) - S(\varphi_2)}\, D\varphi \\ &= \iint_{\substack{(\varphi_1, \varphi_2) \in \mathcal{F}(M_1) \times \mathcal{F}(M_2) \\ \text{s.t. } \varphi_1|_{M_1 \cap M_2} = \varphi_2|_{M_1 \cap M_2}}} e^{-S(\varphi_1)} e^{-S(\varphi_2)}\, D\varphi_1\, D\varphi_2 \\ &= \int_{\varphi' \in \mathcal{F}(M_1 \cap M_2)} \int_{\substack{(\varphi_1, \varphi_2) \in \mathcal{F}(M_1) \times \mathcal{F}(M_2) \\ \text{s.t. } \varphi_1|_{M_1 \cap M_2} = \varphi_2|_{M_1 \cap M_2} = \varphi'}} e^{-S(\varphi_1)} e^{-S(\varphi_2)}\, D\varphi_1\, D\varphi_2 \end{aligned}$$

Now, the point here is that the set $\{(\varphi_1, \varphi_2) \in \mathcal{F}(M_1) \times \mathcal{F}(M_2) \,|\, \varphi_i|_{M_1 \cap M_2} = \varphi'\}$ is the same as $\{\varphi_1 \in \mathcal{F}(M) \,|\, \varphi_1|_{M_1 \cap M_2} = \varphi'\} \times \{\varphi_2 \in \mathcal{F}(M) \,|\, \varphi_2|_{M_1 \cap M_2} = \varphi'\}$. Recalling that $M_1 \cap M_2 = \partial M_1 = \partial M_2$, this allows us to rewrite the final integral above as

$$\begin{aligned} &\int_{\varphi' \in \mathcal{F}(M_1 \cap M_2)} \int_{\substack{(\varphi_1, \varphi_2) \in \mathcal{F}(M_1) \times \mathcal{F}(M_2) \\ \text{s.t. } \varphi_1|_{M_1 \cap M_2} = \varphi_2|_{M_1 \cap M_2} = \varphi'}} e^{-S(\varphi_1)} e^{-S(\varphi_2)}\, D\varphi_1\, D\varphi_2 \\ &\quad = \int_{\varphi' \in \mathcal{F}(M_1 \cap M_2)} \left( \int_{\substack{\varphi_1 \in \mathcal{F}(M_1) \\ \varphi_1|_{\partial M_1} = \varphi'}} e^{-S(\varphi_1)}\, D\varphi_1 \right) \left( \int_{\substack{\varphi_2 \in \mathcal{F}(M_2) \\ \varphi_2|_{\partial M_2} = \varphi'}} e^{-S(\varphi_2)}\, D\varphi_2 \right). \end{aligned}$$

Notice that the integrals in the parenthesis depend either on $M_1$ or on $M_2$, but not on both. Thus we have computed

$$z(M) = \int_{\mathcal{F}(M_1 \cap M_2)} (\text{contribution from } M_1)\, (\text{contribution from } M_2)\,.$$

This ability to decompose makes the theory work. Note here that the contributions from $M_1$ and $M_2$ are no longer numbers, but rather functions $\mathcal{F}(\partial M_i) \longrightarrow \mathbb{C}$. More generally, we can think of our TQFT as attaching a function $\mathcal{F}(\partial X) \longrightarrow \mathbb{C}$ to every $n$-dimensional compact manifold $X$ with boundary. Namely, for $\varphi' \in \mathcal{F}(\partial X)$, this function is

$$\varphi' \mapsto \int_{\substack{\varphi \in \mathcal{F}(X) \\ \varphi|_{\partial X}}} e^{-S(\varphi)}\, D\varphi$$

Taking a slightly more general viewpoint, for a compact $n$-dimensional manifold $X$ with boundary, we decompose the boundary $\partial X$ as a disjoint union $\partial X =$

$\partial^- X \amalg \partial X^+ X$, where each of $\partial^\pm X$ is a union of some subset of the connected components of the boundary $\partial X$. We call $\partial^- X$ the *incoming* part of the boundary and $\partial^+ X$ the *outgoing* part. Then the manifold $X$ gives a map

$$\{\text{functions on } \mathcal{F}(\partial^- X)\} \longrightarrow \{\text{functions on } \mathcal{F}(\partial^+ X)\}.$$

For a function $f : \mathcal{F}(\partial^- X) \longrightarrow \mathbb{C}$, and $\psi \in \mathcal{F}(\partial^+ X)$, this map is given by

$$(T_X f)(\psi) := \int_{\psi' \in \mathcal{F}(\partial^- X)} \Big( f(\psi') \int_{\substack{\varphi \in \mathcal{F}(X) \text{s.t.} \\ \varphi|_{\partial^- X} = \psi',\, \varphi|_{\partial^+ X} = \psi}} e^{-S(\varphi)}\, D\varphi \Big)\, D\psi'.$$

The previous example is just a special case obtained by taking $\partial^+ X = \partial X$ and $\partial^- X = \emptyset$; by convention, the set $\mathcal{F}(\varnothing)$ of fields on the empty manifold consists of 1 element, so the space of functions on $\mathcal{F}(\varnothing)$ is $\mathbb{C}$.

## 34. Properties: why a TQFT is a symmetric monoidal functor (Bohmann)

The upshot here is that to a pair of closed $(n-1)$-dimensional manifolds $Y_1, Y_2$ and a cobordism $X$ between them (i.e. a compact $n$-dimensional manifold with boundary so that we can give identifications $Y_1 \simeq \partial^- X$ and $Y_2 \simeq \partial^+ X$), our TQFT assigns a linear map $T_X : \mathrm{Fun}(\mathcal{F}(Y_1)) \longrightarrow \mathrm{Fun}(\mathcal{F}(Y_2))$. This map has the following properties:

0.) $T_X$ is an integral operator with kernel

$$K_X(\varphi_1, \varphi_2) = \int_{\substack{\varphi \in \mathcal{F}(X) \\ \varphi|_{\partial^- X} = \varphi_1,\, \varphi|_{\partial^+ X} = \varphi_2}} e^{-S(\varphi)}\, D\varphi$$

for $\varphi_1 \in \mathcal{F}(\partial^- X)$ and $\varphi_2 \in \mathcal{F}(\partial^+ X)$. This just means that $(T_X f)(\varphi_2) = \int_{\mathcal{F}(\partial^- X)} f(\varphi_1) K_X(\varphi_1, \varphi_2)$ for $f : \mathcal{F}(\partial^- X) \longrightarrow k$.

1.) Suppose $Y_1$, $Y_2$ and $Y_3$ are closed $(n-1)$-dimensional manifolds, and $X$ is a cobordism $Y_1 \rightsquigarrow Y_2$ and $X'$ is a cobordism $Y_2 \rightsquigarrow Y_3$. Let $X''$ be the cobordism $Y_1 \rightsquigarrow Y_3$ obtained by gluing $X$ and $X'$ by identifying points of $Y_2$. Then

*Exercise.* Use the locality property of fields to show that

$$K_{X''}(\varphi_1, \varphi_3) = \int_{\varphi_2 \in \mathcal{F}(Y_2)} K_X(\varphi_1, \varphi_2) K_{X'}(\varphi_2, \varphi_3).$$

This gives immediately that $T_{X''} = T_{X'} \circ T_X$. That is, the "gluing axiom"

$$\mathrm{Fun}(\mathcal{F}(Y_1)) \xrightarrow{T_X} \mathrm{Fun}(\mathcal{F}(Y_2)) \xrightarrow{T_{X'}} \mathrm{Fun}(\mathcal{F}(Y_3)), \qquad T_{X''} : \mathrm{Fun}(\mathcal{F}(Y_1)) \to \mathrm{Fun}(\mathcal{F}(Y_3))$$

This property corresponds to functoriality, the axiom that says that a TQFT is a functor from $n$-$\mathcal{C}$ob to $\mathrm{Vect}_k$.

2.) Locality of fields implies that $\mathcal{F}(Y \amalg Y') = \mathcal{F}(Y) \times \mathcal{F}(Y')$, and this in turn implies that

$$\mathrm{Fun}(\mathcal{F}(Y \amalg Y')) \cong \mathrm{Fun}(\mathcal{F}(Y)) \otimes_{\mathbb{C}} \mathrm{Fun}(\mathcal{F}(Y')).$$

We can see this from the universal property of tensor products: the bilinear map

$$\mathrm{Fun}(\mathcal{F}(Y)) \times \mathrm{Fun}(\mathcal{F}(Y')) \longrightarrow \mathrm{Fun}(\mathcal{F}(Y \amalg Y'))$$

given by

$$(f,g) \mapsto \big(\varphi \mapsto f(\varphi|_Y) \cdot g(\varphi|_{Y'})\big)$$

satisfies the universal property.

Now suppose we have two cobordisms $Y_1 \overset{X}{\rightsquigarrow} Y_2$ and $Y_1' \overset{X'}{\rightsquigarrow} Y_2'$. Then $X \amalg X'$ is a cobordism $(Y_1 \amalg Y_1') \rightsquigarrow (Y_2, Y_2')$.

*Exercise.* Check that $T_{X \amalg X'} = T_X \otimes T_{X'}$, that is, that

$$T_{X\amalg X'} : \mathrm{Fun}(\mathcal{F}(Y_1 \amalg Y_1')) \longrightarrow \mathrm{Fun}(\mathcal{F}(Y_2 \amalg Y_2'))$$

is given by

$$T_X \otimes T_{X'} : \mathrm{Fun}(\mathcal{F}(Y_1)) \otimes_{\mathbb{C}} \mathrm{Fun}(\mathcal{F}(Y_1')) \longrightarrow \ Fun(\mathcal{F}(Y_2)) \otimes_{\mathbb{C}} \mathrm{Fun}(\mathcal{F}(Y_2'))$$

This property corresponds to the requirement that a TQFT is a symmetric monoidal functor.

## 35. Final Remarks (Bohmann)

A few final words on orientations, and a concrete example.

First, recall that we defined an $n$-TQFT to be a functor on the category of oriented $n$ cobordisms. The axioms for a TQFT make sense without the orientation requirement; this comes from the physical interpretation. Given, for example, a 2-manifold $X$ whose boundary $\partial^- X \amalg \partial^+ X$ is a disjoint union of circles, we can regard $X$ as a picture of the evolution of the closed strings, or circles, of $\partial^- X$ in time. The orientation allows us to consistently label which way the circles of any cross section are flowing in time.

Finally, we return to the example of a fields on a manifold given by $G$-torsors for a finite group $G$. That is, for a finite group $G$ and a manifold $M$, let $\mathcal{F}(M)$ be the set of isomorphism classes of $G$-torsors on $M$. For a compact surface $X$ with boundary $\partial X = \partial^- X \amalg \partial^+ X$, we have a formula for the kernel $K_X$ that defines the map $\mathrm{Fun}(\mathcal{F}(\partial^- X)) \longrightarrow \mathrm{Fun}(\mathcal{F}(\partial^+ X))$:

$$K_X(\varphi_1, \varphi_2) = \sum_{\substack{\varphi \in \mathcal{F}(X) \\ \varphi|_{\partial^- X} = \varphi_1,\, \varphi|_{\partial^+ X} = \varphi_2}} \frac{1}{|\mathrm{Aut}(\varphi)|},$$

where $\mathrm{Aut}(\varphi)$ is the (finite) group of automorphisms of the torsor $\varphi$. This is a finite sum, since $G$-torsors correspond to homomorphisms from the finitely generated fundamental group to the finite $G$, and so there is no question of convergence and everything is well-defined. Under the equivalence of categories between TQFT and commutative Frobenius algebras, the corresponding Frobenius algebra to this TQFT is $Z(k[G])$, the center of the group algebra $k[G]$.

## 36. Further comments (added by Mitya Boyarchenko for clarification)

I should emphasize that all the "calculations" performed with integrals of functions on spaces of fields in the notes above are by no means mathematically rigorous or precise. The sole purpose of these calculations is to explain the original motivation behind the axioms of a TQFT. In other words, while the calculations have no mathematical value, they do have some historical value.

On the other hand, the definition of a TQFT associated to a finite group $G$ given at the end of the lecture **is** completely rigorous, and the statement that this

TQFT in dimension 2 corresponds to the commutative Frobenius algebra $Z(k[G])$ **is** a precise mathematical result.

The appearance of the "weights" $|\mathrm{Aut}(\varphi)|^{-1}$ in the formula for the kernels defining the linear operators in this TQFT is related to the fact that if we naively define the "fields" to be isomorphism classes of $G$-torsors, these fields *do not* satisfy the locality property mentioned during the derivation of the TQFT axioms. Also, both of these phenomena are related to the fact that $G$-torsors on a given manifold form a groupoid in which objects have nontrivial automorphisms. However, a careful explanation of this comment would take up another lecture.