

NATURAL TRANSFORMATIONS $\text{Id} \rightarrow \text{Id}$

Here is a categorical way to think about natural transformations $\eta: \text{Id} \rightarrow \text{Id}$, where Id is the identity functor of a category \mathcal{C} . Let $\text{Nat}(\text{Id}, \text{Id})$ denote the set of such natural transformations. It is a monoid under composition, its unit being the identity natural transformation (Warning to set watchers: this may not actually be a set when \mathcal{C} is large: we shall ignore the question). For any object X , $\eta_X: X \rightarrow X$ is an element of the monoid $\text{End}(X) = \mathcal{C}(X, X)$ of endomorphisms of X , and passage from η to η_X specifies a homomorphism of monoids

$$\varepsilon: \text{Nat}(\text{Id}, \text{Id}) \rightarrow \text{End}(X).$$

Now suppose that $(\mathcal{C}, \square, I)$ is a monoidal category. We might expect $\text{End}(I)$ to play a special role, and it does.

Lemma 1. *There is a homomorphism of monoids*

$$\zeta: \text{End}(I) \rightarrow \text{Nat}(\text{Id}, \text{Id})$$

such that $\varepsilon \circ \zeta$ is the identity homomorphism of $\text{End}(I)$.

Proof. Let $\lambda: I \square X \rightarrow X$ and $\rho: X \square I \rightarrow X$ be the unit isomorphisms that are part of the structure of a monoidal category. We have not yet specified the additional coherence properties that are part of the definition of a monoidal category, but one of them is that

$$\lambda = \rho: I \square I \rightarrow I.$$

For an endomorphism $f: I \rightarrow I$, define $\zeta(f)_X$ to be the composite

$$(0.2) \quad X \xrightarrow{\lambda^{-1}} I \square X \xrightarrow{f \square \text{id}} I \square X \xrightarrow{\lambda} X.$$

Certainly this gives a natural transformation. Because $\lambda = \rho$ on $I \square I$, the naturality of ρ implies that this composite coincides with f when $X = I$. \square

It is natural to ask whether or not ζ and ε are inverse isomorphisms. If we start with a natural transformation η , then $(\zeta \circ \varepsilon)(\eta)_X = \lambda \circ (\eta_I \square \text{id}_X) \circ \lambda^{-1}$, and the question is whether or not that coincides with $\eta_{I \square X}$. To see that that is not true in general, consider the category $\mathcal{A}b$, regarded as a monoidal category under \oplus . Then the unit object is 0, and $\text{End}(0) = 0$. However, multiplication by an integer n gives a non-trivial natural transformation $n: \text{Id} \rightarrow \text{Id}$.

There are general circumstances when it is true. Suppose that we have a faithful forgetful functor $\mathbb{U}: \mathcal{C} \rightarrow \mathcal{S}$, where \mathcal{S} is the category of sets. We might have free objects in \mathcal{C} , denoted $\mathbb{F}(S)$ for a set S . These come with an inclusion $S \rightarrow \mathbb{U}\mathbb{F}(S)$ of sets such that the set $\mathcal{C}(F(S), X)$ of morphisms in \mathcal{C} coincides under restriction with the set of functions $S \rightarrow \mathbb{U}X$. When S is the one-point set $*$, such a function is given by a point of $\mathbb{U}X$ and we may as well identify $\mathbb{U}X$ with the set of functions $* \rightarrow X$. In turn, we may identify this set with the set of maps $F(*) \rightarrow X$ in \mathcal{C} . With these assumptions and notations, we have the following observation.

Lemma 3. *If $I = F(*)$, then ε and ζ are inverse isomorphisms of monoids.*

Proof. Let $\eta: \text{Id} \rightarrow \text{Id}$ be a natural transformation and let $\eta_I = f$. Each element of each object X is determined by a map $g: I = F(*) \rightarrow X$, and the naturality of η gives a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{g} & X \\ \eta_I \downarrow & & \downarrow \eta_X \\ I & \xrightarrow{g} & X. \end{array}$$

Applying the functor \mathbb{U} and restricting to $* \in \mathbb{UF}(*)$, this tells us that the element $\eta_X(x) \in \mathbb{U}X$ is determined by the element $f(*) = \eta_I(*) \in \mathbb{U}(I)$. To be precise about how this determination works, observe that the monoid $\text{End}(I)$ acts on the set $\mathbb{U}X$. The action of f on X is obtained by applying \mathbb{U} to the map in \mathcal{C} displayed in (0.2). Writing the action by juxtaposition, we have $\eta_X(x) = \eta_I(*)x$. To be really pedantic about this, apply \mathbb{U} to the following commutative diagram in \mathcal{C} and recall that we have already observed that the composite on the top line coincides with f .

$$\begin{array}{ccccccc} I & \xrightarrow{\lambda^{-1}} & I \square I & \xrightarrow{f \square \text{id}} & I \square I & \xrightarrow{\lambda} & I \\ g \downarrow & & \downarrow \text{id} \square g & & \downarrow \text{id} \square g & & \downarrow g \\ X & \xrightarrow{\lambda^{-1}} & I \square X & \xrightarrow{f \square \text{id}} & I \square X & \xrightarrow{\lambda} & X. \end{array}$$

Example 4. Let R be a ring, for the moment not necessarily commutative, and let \mathcal{M}_R be the category of left R -modules. Regarding R as a left module over itself using its product, observe that $\text{End}(R)$ is the set R , regarded as a monoid under multiplication. The point is that since R is free on one generator 1, an endomorphism f is determined by the element $f(1)$ via $f(r) = f(r1) = rf(1)$; the image of f is the left ideal $Rf(1)$.

Example 5. Let R be a commutative ring and consider \mathcal{M}_R as a (symmetric) monoidal category under \otimes , with unit object R . The discussion above applies, and the only natural transformations of the identity functor are those given by scalar multiplication by an element $r \in R$. If for every $r \neq 0$ in R , there exists $s \in R$ such that $rs \neq 0$, then no natural transformation of the identity functor can annihilate all of the quotient modules $R/(rs)$. This generalizes Mitya's version of the counterexample that I chose to present in class by crude direct calculation. We're getting more sophisticated now!

Example 6. There are no non-identity natural transformations of the identity functor on the category of sets or on the category of spaces. We regard these categories as monoidal under the cartesian product, with unit object $*$. The discussion above applies directly, and $\text{End}(*)$ consists only of the identity morphism of $*$.

Here is an elementary observation with interesting implications.

Lemma 7. *If $(\mathcal{C}, \square, I)$ is a monoidal category, then $\text{End}(I)$ is a commutative monoid under composition.*

As Mitya pointed out to me, this is immediate from Lemma 1 and the following observation.

Lemma 8. *For any category \mathcal{C} , $\text{Nat}(\text{Id}, \text{Id})$ is a commutative monoid.*

Proof. For natural transformations $\eta, \xi: I \rightarrow I$, $\eta_X \xi_X = \xi_X \eta_X$ by the naturality of either η or ξ . \square

There is a more interesting proof when \mathcal{C} is *symmetric* monoidal that shows how to actually compute the composition product of $\text{End}(I)$ in terms of the \square product on morphisms, without reference to $\text{Nat}(\text{Id}, \text{Id})$. To see this, we note that one of the “coherence conditions” that we have not yet specified in the definition of a symmetric monoidal category is that the commutativity isomorphism

$$\gamma: I \square I \rightarrow I \square I$$

must be the identity morphism. This is closely related to the requirement that $\lambda = \rho: I \square I \rightarrow I$ in any monoidal category.

Let f and g be endomorphisms of I . Since γ is natural,

$$f \square g = (f \square \text{id}) \circ (\text{id} \square g) = (\text{id} \square f) \circ (g \square \text{id}) = g \square f.$$

We have already seen that $\lambda \circ (\text{id} \square f) \circ \lambda^{-1} = f$, and we can just as well replace $\text{id} \square f$ by $f \square \text{id}$. Thus, “conjugating” with $\lambda = \rho$, we see that

$$f \circ g = \lambda \circ (f \square g) \lambda^{-1} = \lambda \circ (g \square f) \lambda^{-1} = g \circ f.$$

Corollary 9. *If $\text{End}(I)$ is a non-commutative monoid, then I is not the unit object for any monoidal structure on \mathcal{C} ; in particular, if R is a non-commutative ring, then R is not the unit object for any monoidal structure on \mathcal{M}_R .*

We observed in class that we can define and construct the tensor product $M \otimes_R N$ of a right R -module M and a left R -module N , but it is only an Abelian group, not an R -module.