NOTES ON PETER MAY'S JULY 11 LECTURE

1. Differential Forms on Manifolds

First, let M be a surface (possibly living in $\mathbb{R}^3$), or if you prefer let M be an arbitrary manifold (for example, M could be all of $\mathbb{R}^3$.)

Definition 1.1. A **0-form** on M is just a function $f : M \to \mathbb{R}$.

A **1-form** on M is a function $\omega : T_pM \to \mathbb{R}$ for each $p \in M$, which "fit together smoothly as p varies in M," such that ω is linear, i.e. whenever $v, w \in T_pM$ are tangent vectors at p and a, b are scalars in $\mathbb{R}$, we have

$$\omega(av + bw) = a\omega(v) + b\omega(w) \quad .$$

A **2-form** on M is a function $\eta : T_pM \times T_pM \to \mathbb{R}$ for each $p \in M$, which again "fit together smoothly as p varies in M." We require η to be bilinear, i.e. linear in each variable, and to be skew-symmetric, i.e. $\eta(v, w) = -\eta(w, v)$.

A **3-form** on M is a function $\zeta : T_pM \times T_pM \times T_pM \to \mathbb{R}$ which is trilinear, i.e. linear in each variable separately, and is alternating. To be alternating means that switching two of the variables introduces a minus sign.

An imaginative person might be able to generalize this to define **n-forms** on M.

Remark 1.2. Any 3-form on a surface is zero. Why? Can you generalize this statement to higher dimensions?

We can differentiate forms. This works as follows: If f is a 0-form, alias function, we can define a 1-form df, the differential of f, by the formula $(df)(v) = v[f]$, the directional derivative of f in the direction of the tangent vector v. In case you forgot the definition of that, I'll recall it: suppose α is a curve on M such that $\alpha(0) = p$ and $\alpha'(0) = v$. Here $v \in T_pM$. Then $v[f] = \frac{d}{dt}(f \circ \alpha)|_{t=0}$ by definition.

Next, if ω is a 1-form, we can define a 2-form $d\omega$, its differential, by the formula

$$d\omega(\phi_u, \phi_v) = \frac{\partial}{\partial u}(\omega(\phi_v)) - \frac{\partial}{\partial v}(\omega(\phi_u)) \quad .$$

Here $\phi : \mathbb{R}^2 \to M$ is a coordinate patch of M, so ϕ_u and ϕ_v are elements of the tangent space of M.

Date: July 11, 2002.

Lemma 1.3. *This definition does not depend on the choice of coordinate patch ϕ in any imaginable way.*

We can "multiply" forms (technically they form what is known as an algebra). Of course everyone knows how to multiply 0-forms. To multiply a 0-form f and a 1-form ω, you define $f\omega(v) = f(p) \cdot \omega(v)$ whenever $v \in T_pM$. This product defines a 1-form. To multiply a 1-form ω and another 1-form ν, one uses the "wedge product" and the fancy symbol $\wedge$. The definition is:

$$\nu \wedge \omega(v, w) = \nu(v)\omega(w) - \nu(w)\omega(v) \quad .$$

The careful reader will notice that $\nu \wedge \omega$ is a 2-form. We have the following proposition relating the wedge product and the differential:

Proposition 1.4. $d(fg) = (df)g + f(dg)$ $d(f\omega) = df \wedge \omega + fd\omega$ $d(\nu \wedge \omega) = d\nu \wedge \omega - \nu \wedge d\omega$. *And so on ....*

It turns out that the differential satisfies a very important property: when applied twice, it gives zero. In other words, $d^2 = 0$. You are invited to check this for functions (i.e. show that $d(df) = 0$ for a 0-form f), the hint being that it follows from equality of mixed partials.

2. Working with the tangent space of $\mathbb{R}^3$

Definition 2.1. A **framing** of $\mathbb{R}^3$ is the data of three vector fields $E_1, E_2, E_3 : \mathbb{R}^3 \to T(\mathbb{R}^3)$ such that at every point $p \in \mathbb{R}^3$, the vectors $E_1(p), E_2(p), E_3(p)$ form an orthonormal basis of $T_p(\mathbb{R}^3) \simeq \mathbb{R}^3$.

In case you've forgotten (or never knew), an orthonormal basis of $\mathbb{R}^3$ is a basis e_1, e_2, e_3 of $\mathbb{R}^3$ with the additional property that $e_i \cdot e_j = \delta_{ij}$, i.e. is 1 if $i = j$ and 0 if $i \neq j$.

Using a choice of framing (and there are many, many such choices!), we can define the "connection 1-forms" ω_{ij}, $1 \leq i, j \leq 3$ as follows. if $v \in T_p(\mathbb{R}^3)$, then we define

$$\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p) \quad .$$

Note that $\omega_{ij} = -\omega_{ji}$ and in particular $\omega_{ii} = 0$. To prove this, simply differentiate the equation $E_i \cdot E_j = \delta_{ij}$ directionally.

Now if $V : \mathbb{R}^3 \to T(\mathbb{R}^3)$ is any old vector field, we discover the formula

$$\nabla_{V(p)} E_i = \Sigma \omega_{ij}(V(p)) E_j(p) \quad .$$

Definition 2.2. The **standard framing** of $\mathbb{R}^3$ is the framing $\bar{x_1}, \bar{x_2}, \bar{x_3}$ defined by $\bar{x_j} = (p, e_j)$, where e_j is the vector in $\mathbb{R}^3$ whose j entry is 1 and all other entries are 0. The p means we've "translated" e_j to "start" at the point p, so it can be an element of $T_p(\mathbb{R}^3)$.

Since the $\bar{x_j}$ form a basis for the tangent space at each point, we may write $E_i = \Sigma a_{ij}\bar{x_j}$, where $a_{ij} = E_i \cdot \bar{x_j} : \mathbb{R}^3 \to \mathbb{R}$. Then $A = (a_{ij})$ is an orthogonal matrix since it describes the change of basis from $\bar{x_j}$ to E_j, and both of these bases are orthonormal. That means $AA^t = Id$. Note that A is a matrix of 0-forms.

Define $dA = (da_{ij})$, a matrix of 1-forms. Does it have any special properties? Think about differentiating the orthogonality equation of A .... Define $\omega = (\omega_{ij})$, another matrix of 1-forms. ω is skew-symmetric, i.e. $\omega = -\omega^t$.

Completing the assembly of every object previously discussed into a matrix, define

$$\bar{x} = \begin{pmatrix} \bar{x_1} \\ \bar{x_2} \\ \bar{x_3} \end{pmatrix},$$

$$E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

and finally

$$dx = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}.$$

Here $x_i : \mathbb{R}^3 \to \mathbb{R}$ is the function which projects a vector onto its ith coordinate.

We have the equations $E = A\bar{x}$ and $\omega = dA \cdot A^t$, which follow from the definitions of A and ω.

3. An intrinsic definition of the Gaussian curvature

We wish to define the Gaussian curvature K in a way that does not make use of the actual embedding of our surface in 3-space. In other words, we want a definition that only takes into account the way our surface looks to the eyes of someone living on the surface. If we can define K in this way, it will be obvious that it is an invariant of the shape of the surface; in other words, it will be invariant under isometries.

To make such a definition, we require a "duality" between forms and vector fields, which is similar to (in fact, is just a fancy version of) duality for vector spaces.

Define a 1-form θ_i by $\theta_i(v) = v \cdot E_i(p)$. If we had chosen $E_i = \bar{x_i}$, then θ_i would be equal to dx_i. Now if V is any vector field, then $V = \Sigma f_i E_i$, where $f_i = V \cdot E_i = \theta_i(V)$.

Dually, if ν is any 1-form, we may write $\nu = \Sigma \nu(E_i)\theta_i$.

We have $\theta_i = \Sigma a_{ij} dx_j$ by the corresponding formula for E_i in terms of the $\bar{x}_j$.

Finally, we present some important relations between some of the strange characters we've presented thus far.

Theorem 3.1 (Cartan Structural Equations)**.**

$$d\theta = \omega \wedge \theta \quad \text{i.e.} \quad d\theta_i = \Sigma \omega_{ij} \wedge \theta_j \tag{1}$$

$$d\omega = \omega \wedge \omega \quad \text{i.e.} \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \tag{2}$$

Proof. Just play games with the fact that $AA^t = Id$. □

Now let's get down to business. Let $M \subset \mathbb{R}^3$ be an orientable surface. Since M is orientable, we can choose a global unit normal vector field E_3 on M. In other words, at every point p of M, $E_3(p)$ points normal to M and has unit length. Now comes the tricky part. Let's choose a point $p \in M$ and choose a coordinate patch of M centered at p. Now on this coordinate patch, we can find vector fields E_1 and E_2 which are tangent to M at every point and such that at each point q of the coordinate patch, $E_1(q), E_2(q)$ and $E_3(q)$ form an orthonormal basis of $T_q(\mathbb{R}^3)$. It is important to remember that the vector fields E_1 and E_2 do NOT in general extend to globally defined vector fields on M, but this is a rather deep theorem.

Still thinking locally, i.e. on the coordinate patch above, given $v \in T_q(M)$, define 1-forms on M by $\omega_{ij}(v) = \nabla_v E_i \cdot E_j(q)$. These are (I think!) globally defined (just patch all the local forms together and it works, hopefully ....) We have the familiar relations $\omega_{ij} = -\omega ji$, so in particular $\omega_{ii} = 0$.

Next define vector fields on M by $\theta_i(v) = v \cdot E_i(q)$. Note that $\theta_3 = 0$ since the normal vector $E_3(q)$ is perpendicular to the tangent vector v.

We have $\nabla_v E_i = \sum_j \omega_{ij}(v) E_j(p)$.

Proposition 3.2. *The shape operator is given by*

$$S(v) = \omega_{13}(v) E_1(p) + \omega_{23}(v) E_2(p) \quad .$$

Further, $S(v) = -\nabla_v E_3 = -\sum_j \omega_{3j} E_j(p)$.

Now, on application of the Cartan structural equations, we get the following equations.

The first structural equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2. \tag{3}$$

$$d\theta_2 = \omega_{21} \wedge \theta_1. \tag{4}$$

Symmetry:

$$\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0. \tag{5}$$

Gauss Equation:

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}. \tag{6}$$

Codazzi Equation:

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}. \tag{7}$$

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}. \tag{8}$$

Curvature Equations:

$$-d\omega_{12} = \omega_{13} \wedge \omega_{23} = K[\theta_1 \wedge \theta_2]. \tag{9}$$

$$\frac{1}{2}(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23}) = H[\theta_1 \wedge \theta_2]. \tag{10}$$

To see at least some of these, consider the matrix

$$\begin{pmatrix} \omega_{13}(E_1) & \omega_{23}(E_1) \\ \omega_{13}(E_2) & \omega_{23}(E_2) \end{pmatrix}.$$

It has determinant $\det(S) = K$, and is $\omega_{13} \wedge \omega_{23}$.

Lemma 3.3. $\omega_{12} = -\omega_{21}$ *is the only 1-form satisfying (3).*

This is satisfying, because it tells us that once we know the θ_i, ω_{12} is forced on us. This allows us to **define** the Gaussian curvature K by the formula $d\omega_{12} = K[\theta_1 \wedge \theta_2]$, and this definition, by the lemma, only depends on the θ_i.

Theorem 3.4. *[Theorema Egrigium] Suppose $F : M \to \bar{M}$ is an isometry between two surfaces (i.e. F preserves dot products on the tangent spaces of the surfaces). Then, for any $p \in M$, $K(p) = \bar{K}(F(p))$. In other words, the Gaussian curvature K is an intrinsic invariant of the isometry type of a surface.*

Remark 3.5. This is **not true** for S and H: they are not preserved by isometries.

Proof of 3.4. We can define $\bar{E_1} = F_*(E_1)$ and $\bar{E_2} = F_*(E_2)$. Since F is an isometry, these form an orthonormal basis of the tangent space of $\bar{M}$ at each point. By the uniqueness lemma 3.3, $F^*\bar{\omega_{12}} = \omega_{12}$. If we apply F^* to the curvature equation (9) for M and $\bar{M}$, the result follows. $\square$