# ABSTRACTS: 2004 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

## DISCRETE MATHEMATICS

Weeks 1 – 8 Laci Babai

The course will cover gems in a number of areas of mathematics; topics will be built up through solving sequences of challenging problems. Observing striking interactions between seemingly distant areas will be the central theme. Topics will include graph theory, combinatorics, finite fields, discrete geometry, number theory, group theory, analysis (orthogonal polynomials), finite models in the theory of computing. Discrete probability and linear algebra, and multivariate polynomials will be central to the methodology. Advanced knowledge of the subject areas listed is not required.
Returning students will not be bored.
PQ: Consent of instructor. Basic linear algebra.
CMSC-17400 (Discrete Math) helpful but not required.

## FOURIER ANALYSIS AND TOPICS IN APPLIED MATHEMATICS

In this self-contained course, students will be exposed to mathematical models arising in condensed-matter physics. The course starts with basic analytical and numerical tools, continues with a mathematical introduction to the basic Ginzburg-Landau model for superconductivity, and ends with an introduction to a fundamental class of models describing soft matter.

Weeks 1 and 2: Robert Fefferman

This will be a two week introduction to Fourier Series. We shall explain what a Fourier Series is, and describe properties of Fourier Series. Orthogonality, methods of summability, the Poisson integral and Abel means, Cesaro means and uniform convergence to a continuous function, Parseval's Identity, Riemann-Lebesgue Lemma, etc. This will be accessible to all.

Weeks 2 and 3: Fausto Cattaneo

How long does it take to mix two fluids? The answer, as everybody knows from everyday experience is: not very long if you stir them. But, why does stirring speed up mixing?
We shall address this question by considering a simple (partial) differential equation describing the process of mixing (an advection diffusion equation), and by adopting two approaches. In one we will use some simple arguments from dimensional analysis to derive scaling relationships between the mixing rate and the physical parameters of the problem (system size, stirring velocity, diffusivity, etc.). In the other we shall solve the "mixing" equation on a computer by a method based on truncated Fourier series (pseudo-spectral transform method), and use the numerical results to check our scaling relationships.
At the end of the course, if we are successful, we should understand a bit better the following things:
1) how differential equations describe physical processes
2) how computers can help us to solve equations
3) why James Bond always has his vodka martini shaken

Weeks 5 and 6: Peter Gordon

The Ginzburg Landau model is one of the fundamental models of the phenomenon of superconductivity. In two dimensions, this model has a type of solution which is close in magnitude to a nonzero constant everywhere except for a small neighborhood of a point where the solution vanishes. This type of solution is called a "vortex". There are also solutions consisting of several vortices. In this short course, we will study minimizers of the functional associated with the Ginzburg Landau model. In particular we will discuss the quantization effect. If time permits we will also discuss the dynamics of the vortices.

Weeks 7 and 8: Peter Constantin

The Ginzburg Landau example introduced the issue of minimizing free energy functionals. We'll discuss the connection between convexity and existence of minimizers and give simple examples where the lack of convexity prevents the existence of minimizers. We will then study a class of functionals, their minimizers, and the associated Euler-Lagrange equations, which are "nonlinear Fokker-Planck equations". We will discuss the question of counting the number of solutions for a particular class of such equations, the Smoluchowski equations.

ALSO featuring, in weeks five and six:

Course title: **OPTIMAL CONTROL AND VISCOSITY SOLUTIONS TO HAMILTON-JACOBI EQUATIONS**

Instructor: **MARTA LEWICKA**
Course dates: July 19 - July 30, 2004

The objective of this course is to give a compact introduction to optimal control theory and Hamilton-Jacobi equations. These are important and lively fields of mathematical analysis: We will show the close relationship between these fields, as well as applications to differential games theory, calculus of variation and systems of conservation laws.

No prerequisites beyond some standard analysis in $R^n$ and basic ODEs are necessary. The course should thus be accessible to everybody. Many examples and exercises will be provided. The tentative outlay is the following.

**1. Basics on optimal control theory.**

a) Motivation and examples.
b) The Pontryagin Maximum Principle - the derivation and geometric explanation.
c) The Mayer problem with terminal constraints. The cones of profitable directions and of feasible directions.

**2. Basics on calculus of variations.**

a) Motivation and examples.
b) The Pontryagin Maximum Principle for the Lagrange problem.
c) Derivation of the Euler-Lagrange equations.

**3. Basics on the viscosity solutions to the Hamilton-Jacobi equations.**

a) Motivation and examples.
b) Definition using the one-sided differentials.
c) Comparison and existence theorems.

**4. Dynamic programming.**

a) The setting of the problem. Dynamic programming principle of Bellman.
b) The value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation.
c) The trajectories satisfying the Pontryagin Maximum Principle provide the characteristic curves for the Hamilton-Jacobi equations of dynamic programming.

**5. Some applications to differential games.** - if time permits.

**Primary references:**

- Bressan: Viscosity solutions of Hamilton-Jacobi equations and optimal control problems.
- Bardi and Capuzzo-Dolcetta: Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.
- Friedman: Differential games.

## GROUP THEORY AND NUMBER THEORY

Weeks 1 and 2: Miklos Abert and Jim Borger

Introduction to groups and number theory

This condensed introduction will provide background material for the YSP, and attendance by people teaching in that program will be required. This will start from scratch with the definition of groups and some basic facts about prime numbers and will go as far as the Nielsen-Schreier theorem and the basics of number fields.

Weeks 3 and 4: Miklos Abert

Groups and words

— Is it possible to take a ball, cut it up into finitely many pieces, rearrange them using only rotations and translations, and reassemble them into two balls, both of exactly the same size as the original? Surprisingly, the answer is "yes".
— Can you do the same trick with a circle? Surprisingly, the answer is "no". Still, it is possible to square the circle.
— How can one construct a graph where the girth (the size of the smallest cycle) is as large as possible?
— What are the groups generated by two parabolic matrices?
Far apart as they may seem, these questions are all related. The course covers topics in infinite permutation and matrix groups, amenability, the ping-pong lemma and random methods in group theory. Several challenging and open problems will be offered, some involving computer-based experiments.

Weeks 5 and 6: Jim Borger

Quadratic numbers

Abstract: We will look at the arithmetic properties of domains of numbers that contain more numbers than just the integers. Sometimes properties of the integers (such as unique factorization into primes) hold in these more general contexts, but sometimes they fail in interesting ways. We will also see how we can use these extended versions of integers to prove things about the usual integers.

## LOGIC, DYNAMICS

Weeks 1 and 2: Benson Farb

The goal of this 2-week segment will be to give, starting from scratch, a proof of Godel's famous compactness, completeness, and incompleteness theorems. It will mostly following Paul Cohen's amazing book "Set Theory and the Continuum Hypothesis".

Weeks 3 and 4: Chris Hruska

This 2-week segment will focus on algorithmicaly undecidable problems, such as the undecidability of the word problem for finitely presented groups, the undecidability of the homeomorphism problem for manifolds, etc. It will start out with an explanation of Turing machines and the halting problem. While this segment is conceptually close to the first segment, it will not directly use any material from it.

weeks 7 and 8: Roman Muchnik

Topological Dynamics and Diophantine Approximations This course will be a brief introduction to topological dynamics. We will start with the definition of group actions on compact spaces and compact group extensions of dynamical systems. An application to diophantine approximation will be presented. In the second week, we will discuss symbolic systems. We will learn about recurrence and minimal systems and prove the Birkoff recurrence and Van der Waerden Theorems.

Pre-requisites: Knowledge of metric spaces, limits, and compactness. Knowledge of groups will be very helpful, but is not required.