1. Simplicial Spaces

Informally, we can think of simplicial spaces as spaces built out of oriented lines, triangles, tetrahedra...


Let \( K_n \) be the set of \( n+1 \) point simplices. Recall that the free abelian group \( \mathbb{Z}[K_n] \) consists of formal sums of elements of \( K_n \) (i.e. \( \sum_{i=1}^{n} k_i \sigma_i \in \mathbb{Z}[K_n] \) where \( \sigma_i \in K_n \) and \( k_i \in \mathbb{Z} \) for each \( i \)). We also recall that we were given maps \( d_i : K_n \rightarrow K_{n-1} \) and \( s_i : K_n \rightarrow K_{n+1} \) for \( i = 0 \ldots n \), that satisfy certain relations. Using these maps we define \( d : \mathbb{Z}[K_n] \rightarrow \mathbb{Z}[K_{n-1}] \) as \( d(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma) \).

We compute

\[
(dd)(\sigma) = d \left( \sum_{i=0}^{n} (-1)^i d_i(\sigma) \right)
= \sum_{0 \leq j < n} \sum_{0 \leq i < n} (-1)^{i+j} d_j d_i(\sigma)
= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_j d_i(\sigma) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_j d_i(\sigma)
= - \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{i-1} d_j(\sigma) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_j d_i(\sigma)
= - \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_i d_j(\sigma) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_j d_i(\sigma)
= 0
\]

Using this we define \( B_n(K) = d(\mathbb{Z}[K_{n+1}]) \) and \( Z_n(K) = \{ \sigma \in \mathbb{Z}[K_n] \mid d(\sigma) = 0 \} \).

The \( nth \) homology group is defined to be \( H_n(K) = Z_n(K)/B_n(K) \).

**Example 2.1.** If we let \( K \) be the triangle with ordered points \( a < b < c \) and edges \( f \), connecting \( a \) to \( b \), \( g \) connecting \( b \) to \( c \) and \( h \) connecting \( a \) to \( c \). We can compute \( H_0(K) = H_1(K) = \mathbb{Z} \). Since \( d(\mathbb{Z}[K_0]) \) is by definition 0 we see that \( Z_0(K) = \mathbb{Z}^3 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \). We have that \( d(f) = b - a \), \( d(g) = c - b \) and \( d(h) = c - a = d(f) + d(h) \). So \( B_0(K) = d(\mathbb{Z}[K_1]) = \mathbb{Z}(b - a) \oplus \mathbb{Z}(c - b) \). Rewriting \( Z_0(K) \) as \( \mathbb{Z}(b - a) \oplus \mathbb{Z}(c - b) \oplus \mathbb{Z}b \) we see that \( H_0(K) = \mathbb{Z} \). Since there are no 2-simplices we have \( B_1(K) = 0 \) which implies \( H_1(K) = Z_1(K) = \mathbb{Z}(h - (f + g)) = \mathbb{Z} \).
3. The functoriality of homology.

Suppose we have a map of chain complexes \( f : C_\ast \rightarrow C'_\ast \). That is we have that \( df_n = f_n d \). We see that if \( x \in Z_n(C) \) then \( df_n(x) = f_n(dx) = f_n(0) = 0 \) so \( f(x) \in Z_n(C') \). And we see that if \( x \in B_n(C) \), so \( x = dy \) for some \( y \in C_{n+1} \), then \( f(x) = f(dy) = d(f(y)) \) and we have \( f(x) \in B_n(C') \). These two facts imply that we have a well defined map \( H_n(C) \rightarrow H_n(C') \).

4. The homotopy invariance of homology

We return to one of our favorite simplicial complexes \( I \). Recall \( I \) has two 0-simplices, \( I_0 = \{[0],[1]\} \), and one 1-simplex, \( I_1 = \{[I]\} \). Let \( K \) be a simplicial space, we can check that \( \mathbb{Z}([K \times I]) = \mathbb{Z}[K_\ast \times I_\ast] = \mathbb{Z}[K_\ast] \otimes \mathbb{Z}[I_\ast] \). The \( n \)-chains \( C_n(K \times I) \) then are given by \( \mathbb{Z}[K_\ast] \otimes [0] \oplus \mathbb{Z}[K_\ast] \otimes [1] \oplus \mathbb{Z}[K_{n-1}] \otimes [I] \). For \( x \in C_n(K) \) we set

\[
\begin{align*}
  d(x \otimes [0]) &= d(x) \otimes [0] \\
  d(x \otimes [1]) &= d(x) \otimes [1] \\
  d(x \otimes [I]) &= d(x) \otimes [I] + (-1)^n x \otimes [1] - (-1)^n x \otimes [0].
\end{align*}
\]

Now we see two chain maps \( f, g : C \rightarrow C' \) are homotopic if there exists a map \( h : C \otimes I \rightarrow C' \) such that \( h(x \otimes [0]) = f(x \otimes [0]) \) and \( h(x \otimes [1]) = g(x \otimes [1]) \) for all \( x \in C \).

**Theorem 4.1.** If \( f \) and \( g \) are homotopic then \( f_* = g_* \).