1 The Fundamental Theorem of Algebra - Proof 2 (Topological)

Theorem 1. The Fundamental Theorem of Algebra

Proof. Suppose that \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \). We can assume without loss of generality that \( a_n = 1 \). Furthermore, we can assume that \( a_0 \neq 0 \) otherwise \( z = 0 \) would be a root. Now, \( f(x) \) is a continuous complex polynomial, mapping \( \mathbb{C} \to \mathbb{C} \). We also know that

\[
\lim_{z \to \infty} \frac{z^n}{f(x)} = 1
\]

and so for a circle with sufficiently large radius \( r \), we have

\[
|z^n - f(x)| \leq \alpha r^n
\]

with \( 0 < \alpha < 1 \) and \( z \) on the circle, \( C \).

For any \( r > 0 \), \( z^n \) winds \( C \) around the origin \( n \) times. Therefore, \( f(z) \) will also wind a sufficiently large \( C \), \( n \) times around the origin. For a sufficiently small radius \( r \), \( f(z) \equiv a_0 \) and will not wind around the origin at all. Since \( f(z) \) is continuous for any given \( r, f(C_r) \) will depend on \( r \) continuously. Since \( f(C_r) \) has winding number of 0 for sufficiently small \( r \), and winding number of \( n \) for sufficiently large \( r \), it follows that there exists a radius, say \( r_1 \), such that \( f(C_{r_1}) \) passes through the origin. Thus, \( \exists z_1 \) on \( C_{r_1} \) such that \( f(z_1) = 0 \). This proves the theorem.

\[ \square \]