AN INTRODUCTION TO THE LEBESGUE INTEGRAL

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The history of integration began with Archimedes around the 2nd century B.C., but did not start to gain rigor until the 17th century. Bonaventura Cavalieri began working with indivisibles in the computation of areas by the early 1700s. Leibniz and Newton toward the end of the century discovered the relationship between the integral and the derivative, and really invented integral calculus. It was not until Cauchy that the integral was actually defined rigorously. But it was Riemann who developed the integral frequently used today. This integral is constructed by taking the limit of approximations, called Riemann sums, which are based on partitioning the domain of the function.

However, it was realized quickly that it is not too difficult to find functions which are not Riemann integrable. In particular, the study of fourier series requires an integral such that $\int [\sum_k f_k(x)]dx = \sum_k \int f_k(x)dx$, but it is rather easy for this to fail for the Riemann integral. For instance, consider $A = \mathbb{Q} \bigcap [0,1]$ and 1_A, the indicator function for A. It is clear that 1_A is not Riemann integrable, since for any interval contained in $[0,1]$, the maximum of 1_A is 1 and the minimum is 0 (since both the rationals and irrationals are dense in $\mathbb{R}$), so the upper sum will be 1 and the lower will be 0, always. But since A is countable (since $\mathbb{Q}$ is countable), its elements can be enumerated by a_k, and we can define functions $g_k(x) = \begin{cases} 1 & x = a_k \\ 0 & otherwise \end{cases}$ such that $1_A = \sum_{k=1}^{\infty} g_k$. Note that each partial sum on the right has only finitely many discontinuities, and is thus Riemann integrable, whereas, as noted earlier, the left side is not.

Thus, in his thesis in 1902, Henri Lebesgue developed a new integral in which the focus was on the range of the function, instead of on the domain. The distinction between the two approaches can be seen by envisioning the graph of a real function f whose range is the reals. Whereas Riemann focuses on partitioning the x-axis, Lebesgue's integral partitions the y-axis instead. That is, Riemann partitions the domain of f into a finite number of intervals and on each interval approximates the values that f takes. Using the rectangles generated by the product of the value of the function on each interval and the length of that interval, Riemann approximates the area under the function. Lebesgue, on the other hand, partitions the range of the function into a finite number of intervals, and for each partition chooses a value to "represent" the function for that partition on his approximation (call it s) (so for all x in the partition, $s(x)$ equals the representative). s is called a simple function, which means it has a finite range.

Now, it should be clear that Lebesgue needs to construct some sort of manner of meausuring the area of very complicated sets. Whereas Riemann sums approximate by just using rectangles, a set which Lebesgue needs to approximate, such as $\{x|f(x) \in [a, b]\}$ for some a,b, may be much more complicated than a rectangle. So initially lets define m, on rectangles in $\mathbb{R}^n$: let $m(\prod_i [a_i, b_i]) = \prod_i (b_i - a_i)$. Then it is easy to see that this measure extends onto elementary sets (finite unions of intervals), and with some work Lebesgue shows that in fact this extends to a countably additive, regular, nonnegative function on a subset of the powerset of $\mathbb{R}^n$ (I will still refer to this extension as m.) This subset is actually closed under taking countable intersections, complements, and unions. and it is actually these closures that give the measure its characteristics. We also need to define which functions are "nice," where a "nice" function f is called a measurable function. This means it has the characteristic that the set $\{x|f(x) > a\}$ is measurable for all a. Also, "$>$" can be replaced by "$<$," "$\leq$," or "$\geq$" in the above condition.

The definition of the integral is first given for simple functions:

Definition Let g be a simple measurable function on a m-measurable set E, where $E = \bigsqcup E_i$ and $g(x) = a_i \quad x \in E_i$. Then define the integral of g over E:

$$\int_E g = \sum_i a_i * m(E_i)$$

Then for $f \geq 0$, a measurable function on a measurable set E,

$$\int_E f = \sup_{0 \leq s \leq f} \int_E s$$

For general f, $\int f = \int_{x|f(x) \geq 0} f - \int_{x|f(x) < 0} (-f)$. (Recall that f measurable $\Rightarrow \{x|f(x) \geq 0\}$ and $\{x|f(x) < 0\}$ are both measurable sets).

Note that from now on, the symbol $\int f$ refers to the Lebesgue integral, not the Riemann integral.

Now note that by the countable additivity of m, any countable subset must have measure 0, since a point has measure 0. This means that $\mathbb{Q}$

is of measure zero. It is not too hard to show that if $A = B \bigcup C$, A measurable, then for measurable f $\int_A f = \int_B f + \int_C f$. In particular, if C is of measure zero, then $\int_A f = \int_B f$

This new Lebesgue integral has very important properties. It turns out that the set of Lebesgue integrable functions is actually a superset of the set of Riemann integrable functions, and when a function is both Lebesgue integrable and Riemann integrable, then the two integrals have the same value. Furthermore, Lebesgue's convergence theorems about the "niceness" of being able to pass limits through the integral in certain conditions make this integral much more useful to the theory of fourier series. Here are the convergence results, without proof:

Theorem 0.1. (Lebesgue's Monotone Convergence Theorem) *Let* $\{f_i\}$ *be a sequence of measurable functions such that* $0 \leq f_1 \leq f_2 \leq \dots$ *Then*

$$\lim_{i\to\infty} \int f_i = \int \lim_{i\to\infty} f_i$$

(note: f will be measurable).

Theorem 0.2. (Lebesgue's Bounded Convergence Theorem) *Let* $\{f_i\}$ *be a sequence of measurable functions on a measurable set E, such that* $\exists$ *a Lebesgue integrable function* g *on* E *s.t.* $|f_i(x)| \leq g(x) \forall x \in E$. *Then*

$$\lim_{i\to\infty} \int f_i = \int \lim_{i\to\infty} f_i$$

As stated above, the set of Riemann integrable functions is a subset of the set of Lebesgue integrable functions. Also, the Lebesgue integral actually provides a neat proof of necessary and sufficient conditions for a function to be Riemann integrable. These two theorems will be proven below. First a lemma will be proved:

Lemma 0.1. *If* $f(x) \geq 0$ *and* $\int_E f dm = 0$ *Then* $f(x) = 0$ *almost everywhere on* E.

Proof. Define $E_n = \{x \in E | f(x) < 1/n\}$, and let $E_0 = \bigcup_n E_n = \{x | f(x) > 0\}$. If $\exists n$ s.t $m(E_n) > 0$ then to arrive at a contradiction, define

$$s(x) = \begin{cases} \frac{1}{2n} & x \in E \\ 0 & otherwise \end{cases}$$

so $0 \leq s \leq f$, so $\int_E sdm \leq \int_E fdm$ but $\int_E sdm = m(E_n) * \frac{1}{2n} > 0$, since $m(E_n) > 0 \Rightarrow \int_E fdm > 0$, a contradiction. Thus $m(E_0) = 0$ since $m(E_n) = 0$ $\forall n$ by the additivity of measure. $\square$

For these next theorems, it will be expedient to label the Riemann integral as $\Re \int f$.

Theorem 0.3. *(I) A function f is Riemann integrable over $E = [a, b]$ then it is Lesbegue integrable, and*

$$\int_a^b f = \Re \int_a^b f$$

(II)Further, f is Riemann integrable if and only if it is continous almost everywhere on $[a, b]$.

Proof. Assume f is bounded on E. Now we will define a simple function associated with each partition whose behavior will mimic that of the Riemann sums. Let $U_i(a) = f(a)$ and let $U_i(x) = M_i$ for $x \in [x_i, x_{i+1}]$ where M_i is the maximum value of f on $[x_i, x_{i+1}]$. Define similarly $L_i(x)$, but replace M_i with m_i, the minimum value of f on the same domain. Notice $\int_a^b U_k = \mathfrak{U}(P_k, f)$ and $\int_a^b L_k = \mathfrak{L}(P_k, f)$ Also, note $L_1 \leq L_2 \leq \ldots \leq f \leq \ldots U_2 \leq U_1$. So let $\lim_{k\to\infty} U_k(x) = U(x)$ and $\lim_{k\to\infty} L_k(x) = L(x)$. Both U and L are defined finitely by the above string of inequalities. Note that they are both bounded, and they are measurable since they are the limits of measurable functions (this result was not proved, but is not hard to see). So by the monotone convergence theorem,

$$T = \lim_{k\to\infty} \mathfrak{U}(P_k, f) = \lim_{k\to\infty} \int U_k = \int U \tag{1}$$

$$B = \lim_{k\to\infty} \mathfrak{L}(P_k, f) = \lim_{k\to\infty} \int L_k = \int L$$

Now, if f is Riemann integrable, then we know there exist partitions P_k such that $\lim_{k\to\infty} \mathfrak{L}(P_k, f) = B = T = \lim_{k\to\infty} \mathfrak{U}(P_k, f)$ By equation (1) , this means that $\int U = \int L$. Note that $L \leq f \leq U$, but by lemma 0.1, $U(x) = f(x) = L(x)$ almost everywhere (by regarding $U - L \geq 0$). Thus f is measurable and by equation (1) the two integrals are equal. So (I) of the theorem is proved.

So, as noted above, f Riemann integrable $\iff$ $\int U = \int L$ $\iff$ $U(x) = L(x)$ except on a set F of measure zero. $\Rightarrow f$ continuous almost everywhere on $E - \bigcup_k P_k \Rightarrow f$ continuous almost everywhere, since $\bigcup_k P_k$ is countable, so of measure zero. This is true since for $x \notin P_k, U(x) = L(x) \iff f$ is continuous at x. And f continuous almost everywhere $\Rightarrow U(x) = L(x)$ almost everywhere $\Rightarrow f$ Riemann integrable. So f Riemann integrable $\iff$ f continuous almost everywhere. So (II) is proved. □