A BRIEF INTRODUCTION TO GAUGE INTEGRATION

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1. Introduction

Traditional Riemann integration, while powerful, leaves us with much to be desired. The class of functions that can be evaluated using Riemann's technique, for example, is very small. Another problem is that a convergent sequence of Riemann integrable functions (we will denote this class of functions as R-integrable) does not necessarily converge to an R-integrable function, and furthermore the fundamental theorem of calculus is not general- that is to say when integrating a function f we often find a function F such that $F' = f$ and $\int_a^b f = F(b) - F(a)$ to evaluate. The problem with the Riemann technique is that f may have a primitive F but that does not guarantee that it is R-integrable which prevents us from applying the above equation. There have been many steps to cover and fix the holes left by Riemann. Lebesgue, Perron, and Denjoy all made major advancements in the theory of integration; the later two generalized the fundamental theorem of calculus, fixing the latter problem. The techniques they used, however, were inaccessible and complicated.

In the 1960's Kurzweil and Henstock came up with a new integration technique that is so powerful it includes every function the others can integrate. The technique had the added advantage of being simple, requiring only slightly more effort to learn than the Riemann integral. There was in fact a (failed) movement to replace the teaching of the Riemann integral with that of the Kurzweil-Henstock integral (also called generalized Riemann integral and gauge integral). This paper will serve as a brief introduction to the power and simplicity behind this relatively modern idea that simplifies and strengthens of one of the corner stones of analysis.

Before continuing it is important to acknowledge and note that all of the ideas of this paper are drawn from Robert G. Bartle's A Modern Theory of Integration. The purpose of the paper is to spread awareness of gauge theory and Bartle's book which is a wonderful resource for learning this new method.

2. What Are Gauges Anyways?

Definition 2.1. Let $I = [a, b]$ a nondegenerate subinterval of the number line (that is to say $a \neq b$). We say $\delta : I \mapsto \mathbb{R}$ is a **gauge** on I if $\delta(x) > 0$ for all x in I. The interval around $t \in I$ **controlled by the gauge** δ is the interval $B[t; \delta(t)] := [t - \delta(t), t + \delta(t)]$.

Definition 2.2. Let $I = [a, b]$, nondegenerate, and let $\dot{\mathcal{P}}$ be a tagged partition of I (that is to say for each subinterval I_i comprising the partition $\mathcal{P}$ choose a point

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$t_i \in I_i$- denote $\dot{\mathcal{P}} := \{(I_i, t_i) | i = 1, ..., n\} = \{(I_i, t_i)\}_{i=1}^n$). If δ is a gauge on I, then we say $\dot{\mathcal{P}}$ is **δ-fine** if

$$I_i \subseteqq [t - \delta(t), t + \delta(t)] \text{ for all } i = 1, ..., n; \tag{2.1}$$

that is to say if I_i is contained in the interval controlled by t_i for each i. If $\dot{\mathcal{P}}$ is δ-fine write $\dot{\mathcal{P}} \ll \delta$.

Examples of Gauges (a) $\delta_f : I \mapsto \mathbb{R}$ and $\delta_f(x) = \delta_c$ for all $x \in I$, where δ_c is some number greater than zero. We call this a **constant gauge**. This is the type of gauge used in Riemann integration and makes it clear (or soon will make it clear) that if a function is R-integrable it is R^*-integrable.

(b) If we have two gauges, δ_1 and δ_2, on I define a new gauge δ to be the minimum of the two at each point. If a partition is δ-fine it will also be δ_1 and δ_2-fine. This method also works for finitely many gauges on I.

(c) Gauges can also be used to force particular point to be tagged for any δ-fine partition. Let $I := [0, 1]$, $\delta(0) := 0,$ and $\delta(t) := 1/2t$ for $t \in I \backslash \{0\}$. Then any δ-fine partition of I must have 0 as a tagged point. This fact is left as an exercise.

Existence of δ-fine partitions

Theorem 2.3. *If $I := [a, b]$ is a nondegenerate compact interval in $\mathbb{R}$ and δ is a gauge on I, then there exists a partition of I that is δ-fine.*

Proof. The idea is to prove by contradiction assuming that the interval I is not δ-fine for any partition. Break the interval into two halves. Then at least one of the halves must not be δ-fine. Pick a non-δ-fine half and break it into halves. At least one of these quarter length intervals must also not be δ-fine. Continuing this process of halving we will get an arbitrarily small interval $E = [a, b]$ which is not δ-fine. Pick a point t in E. Then $|\delta(t) - t| > 0$ and is some real value. Since E is of arbitrary length we can make a and b as close to t as we choose, namely closer than $|\delta(t) - t| > 0$ thus E is δ-fine which contradicts the statement that I is not δ-fine. □

The Generalized Reimann Integral

Definition 2.4. A function $f : I \mapsto \mathbb{R}$ is said to be **generalized Riemann integrable (or gauge integrable)** on I if there exists a number $A \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a gauge δ_ϵ on I such that if $\dot{\mathcal{P}} := (I_i, t_i)_{i=1}^n$ is any tagged partition of I that is δ-fine, then

$$|\left(\sum_{i=1}^{n} f(t_i)(t_{i+1} - t_i)\right) - A| \leqslant \epsilon \tag{2.2}$$

The collection of all funtions that are generalized Riemann integrable on an interval I will be denoted by $R^*(I)$.

Theorem 2.5. Uniqueness Theorem. *There is at most one number A that satisfies the property in Definition Definition* 2.4.

We say A is the integral of f evaluated on I.

3. Applications of Gauges

There are several reasons that non-constant gauges work better than constant ones. First, one can enclose a finite or countable set of points in an null interval (an interval of arbitrarily small length) (see Dirichlet's function below). The fact that we can force particular points to be tagged also comes in handy (see the first example in section 4).

Dirichlet's Funtion. The function is defined as $f : [0,1] \mapsto \mathbb{R}$ where

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \tag{3.1}$$

This function is not R-integrable, but we will show it is R^*-integrable. Since the rational numbers are countable let $\{r_k : k \in \mathbb{N}\}$ be an enumeration of them. Take $\epsilon > 0$ and define the gauge

$$\delta_\epsilon(t) := \begin{cases} \epsilon/2^{k+1} & \text{if } t = r_k \\ 1 & \text{if } t \text{ is irrational} \end{cases} \tag{3.2}$$

Let $\dot{\mathcal{P}} := (I_i, t_i)_{i=1}^n$ be a δ_ϵ-fine partition of [0,1]. If the tag t_i is irrational then $f(t_i) = 0$ and the contribution to the sum is zero. Other wise the contribution is 1 times the length of the interval I_i which is less than $\epsilon/2^k$ since $\dot{\mathcal{P}}$ is δ_ϵ-fine. Since only the rational tags will contribute, and in this way, we have

$$|\left(\sum_{i=1}^{n} f(t_i)(t_{i+1} - t_i)\right)| \leqslant \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon. \tag{3.3}$$

Since ϵ is arbitrary the function is integrable and $\int_0^1 f = 0$.

Absolute Integration. In a more impressive display of power, we can show that every convergent series gives rise to an R^*-integrable function. The function is as follows:

Let $\sum_{k=1}^{\infty} a_k$ be and convergent series in $\mathbb{R}$ and let it converge to B. Let $c_n := 1 - 1/2^n$ for $n \in \mathbb{N} \cup \{0\}$. Define now $g(x) : [0,1] \mapsto \mathbb{R}$ as

$$g(x) := \begin{cases} 2^k a_k & \text{for } x \in [c_{k-1}, c_k), k \in \mathbb{N} \\ 0 & x = 1 \end{cases} \tag{3.4}$$

We claim that this function is indeed in $R^*([0,1])$ and that $\int_0^1 g = B$. The key observation is that the length of the intervals cancels out 2^k term in each of the constant intervals. We get $\sum_{k=1}^{\infty} g(x_k)(c_k - c_{k-1}) = \sum_{k=1}^{\infty} 2^k a_k((1 - 1/2^k) - (1 - 1/2^{k-1}))$ where $x_k \in [c_{k-1}, c_k)$. Finding an appropriate gauge and completing the proof is left to the reader as an exercise (or can be found in Bartle's book).

Once this fact is established it is clear that if a function f is R^*-integrable, then we do not know if $|f|$ is, since a convergent sequence need not converge absolutely. For example, the function arising as above as the harmonic series multiplied by negative one every other term

$$h_{-1}(x) := \begin{cases} (-1)^k 2^k 1/k & \text{for } x \in [c_{k-1}, c_k), k \in \mathbb{N} \\ 0 & x = 1 \end{cases} \tag{3.5}$$

is in $R^*([0,1])$, while the function arising from the harmonic series is not

$$h(x) := \begin{cases} 2^k 1/k & \text{for } x \in [c_{k-1}, c_k), k \in \mathbb{N} \\ 0 & x = 1 \end{cases} \tag{3.6}$$

This is a sharp change from Riemann and Lesbegue integration where we can always assume that "f is integrable" implies "$|f|$ is integrable."

4. FUNDAMENDTAL

One of the most beautiful and immediate results of applying gauges is the direct consequence of incredibly powerful and generalized Fundamental Theorems of calculus. This is the problem that motivated and was solved by Perron and Denjoy. Amazingly enough we already have (almost) everything we need to prove the more general and weaker fundamental theorem. First we will look at an example of the problems we run into when working with the Riemann integral. Then, with a few more definitions we will briefly explore the fundamental theorems concerning integrating derivatives and differentiating integrals.

Example. Recall that the function $m : [0,1] \mapsto \mathbb{R}]$ defined as $m(x) := 1/\sqrt{x}$ for $x \in (0,1]$ and $m(0) := 0$ is not in $R([0,1])$. This function has the primitive $M(x) = 2\sqrt{x}$ on the entire interval but 0. The fundamental theorem for Riemann integration, which attempts to assert that $M(1) - M(0) = 2\sqrt{1} - 0 = 2 = \int_0^1 m$, is meaningless here since the integral does not exist.

Definition 4.1. **(a)** A set $Z \subset \mathbb{R}$ is a **null set** (or a **set of meassure zero** if for every $\epsilon > 0$ there exists a countable collection $\{J_k\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty} J_k \text{ and } \sum_{k=1}^{\infty} l(J_k) < \epsilon \tag{4.1}$$

where if $J_k = (a,b)$ then $l(J_k) = |b-a|$.

(b) If $A \subseteq \mathbb{R}$, then a function $f : A \mapsto \mathbb{R}$ is said to be a **null function** if the set $\{x \in A | f(x) \neq 0\}$ is a null set.

(c) If $Q(x)$ is a statement about the point $x \in I$ and if $E \subset I$, we say that E is an **exceptional set** for Q if the satement $Q(x)$ holds for all $x \in I - E$.

(d) If above E is a null set, we say $Q(x)$ holds **almost everywhere (a.e.)** on I.

(e) If E is a countable set (or finite), we say that $Q(x)$ holds **with countably (finitely) many exceptions (c.e. (f.e.))**.

Definition 4.2. Let $I : [a,b] \subset \mathbb{R}$ and let $F, f : I \mapsto \mathbb{R}$.

(a) We say F is a **primitive of f on I** if the derivative of $F'(x)$ exists and $F'(x) = f(x)$ for all $x \in I$.

(b) We say F is an **a-primitive (or c-primitive, or f-primitive)** of f on I if F is continuous on I, and there exists a null (or countable, or finite) set E of points $x \in I$ where either $F'(x)$ does not exist, or does not equal $f(x)$. We call E the exceptional set of f.

(c) If $f \in R^*(I)$ and $u \in I$, then the function defined as

$$F_u(x) := \int_u^x f \text{ where } x \in I \tag{4.2}$$

is called the **indefinite integral of f with base point u.** Any function that differs by a constant from one of these functions is call an **indefinite integral.**

The Straddle Lemma

We will use this lemma to prove one of the following theorems.

Lemma 4.3. Straddle Lemma. *Let $F : I \mapsto \mathbb{R}$ be differentiable at a point $t \in I$. Given ϵ there exists $\delta_\epsilon(t) > 0$ such that if $u, v \in I$ satisfy*

$$t - \delta_\epsilon(t) \leq u \leq t \leq v \leq t + \delta_\epsilon(t) \tag{4.3}$$

then we have

$$|F(v) - F(u) - F'(t)(u - v)| \leq \epsilon(v - u) \tag{4.4}$$

Proof. Since the derivative of F at t exists, by the definition of the derivative we have that given $\epsilon > 0$ there exists a δ_ϵ such that if $0 < |z - t| \leq \delta_\epsilon(t)$, $z \in I$, then

$$|(F(z) - F(t))/z - t - F'(t)| \leq \epsilon \tag{4.5}$$

We pick $u < t$ and $v > t$ to be points controlled by the interval created by δ_ϵ around t. The proof can be completed from here using the above equation and finding the appropriate bound $\epsilon(v - u)$. □

Integrating Derivatives

Theorem 4.4. Fundamental Theorem I. *If $f : [a, b] \mapsto \mathbb{R}$ has a primitive F on [a,b], then $f \in R^*([a, b])$ and*

$$\int_a^b f = F(b) - F(a).$$

Proof. The proof follows from the straddle lemma. We apply the same gauge that arises from the differentiablity of F at each point in the interval. □

The example with $m(x) = 1/\sqrt{x}$ has the f-primitive $M(x) := 2\sqrt{x}$ where E, the exceptional set, is $\{0\}$. By explicitly calculating using gauge integration we get $\int_0^1 m = 2$ we also clearly have $M(1) - M(0) = 2\sqrt{1} - 0 = 2$. This leads the question of wether the fundamental theorem will hold for finitely many points, or even countably many. It in fact does:

Theorem 4.5. Fundamental Theorem I*. *If $f : [a, b] \mapsto \mathbb{R}$ has a c-primitive F on $[a, b]$, then $f \in R^*([a, b])$ and*
$\int_a^b f = F(b) - F(a)$.

Proof. The idea is to contain the countable exceptional set in a gauge like the one we used for Dirichlet's function and to use the gauge arising from diffentiability on the rest of the interval. For more detail see Bartle. □

Now we can apply this theorem to the example: $\int_0^1 m = 2 = 2\sqrt{1} - 0 = M(1) - M(0)$.

Differentiating Integrals

Theorem 4.6. Fundamental Theorem II. *Let $f \in R^*([a, b])$ have a right (left) hand limit A at a point $c \in [a, b)$. Then the indefinite integral*

$$F_u(x) := \int_u^x f \tag{4.6}$$

has a right hand derivative at c equal to A.

Theorem 4.7. Fundamental Theorem II*. *Let $f \in R^*([a,b])$. Then any indefinite integral F is continuous on $[a,b]$ and is an a-primitive of f.*

The proofs of the second set of fundamental equations will be omitted yet stating them leads to two natural questions. First, can we extend the first theorem to require that f have only an a-primitive? Secondly, can we tighten the second theorem to ensure an indefinite integral is a c-primitive? The answer to both these questions is no. To illustrate this by counter example we turn to the Cantor set and the Cantor-Lebesgue singular function (also known as the Devil's staircase").

First recall the Cantor set, or middle thirds set, which is constructed recursively by starting with the interval $[0,1]$ and removing the middle third. We now have two intervals of length one third and on the next step we remove the middle third of these two intervals. Continue infinitely many times. We define the interval $[0,1]$ as Γ_0 and the nth step as Γ_n. The cantor set is the intersection of these sets for $n \in \mathbb{N}$.

Theorem 4.8. *The Cantor set Γ is an uncountable null set.*

Proof. Notice that Γ_n has length $1/3^n$ so that for any ϵ we can choose an n such that $l(\Gamma) < l(\Gamma_n) < \epsilon$. Thus Γ is null.

Next we show it is uncountable. Assume it is countable and that $\{x_n : n \in \mathbb{N}\}$ is an enumeration of Γ. Let I_1 be one of the 2^1 closed intervals in Γ_1 such that $x_1 \ni I_1$. If $n \geq 2$, let I_n be the first interval in $I_{n-1} \cap \Gamma_n$ with length $1/3^n$ such that $x_n \ni I_n$. In this way we obtain a point $z = \bigcap_{k=1}^{\infty} I_k$ such that $z \in \Gamma$. Thus $x_k \neq z$ for all $k \in \mathbb{N}$ and the set is not countable. □

Cantor-Lebesgue singular function (The Devil's staircase). Notice that Γ_k is comprised of 2^k closed intervals, and that its compliment is comprised of $2^k - 1$ closed intervals Define the linear function $\Lambda_k : [0,1] \mapsto \mathbb{R}$ as $\Lambda_k(0) = 0$, $\Lambda_k(1) = 1$, and $\Lambda_k(x) = i/2^k$ for $x \in \Gamma_k^c$ with x in the ith subinterval comprising of Γ_k^c. If x is in the jth subinterval of Γ_k, let $\Lambda_k(x)$ be the straight lines linking the $(j-1)$th constant segment in Γ_k^c to the jth constant segment in Γ_k^c. If $j-1$ is 0, we link the line on the left to $0(=\Lambda_k(0))$; if $j = 2^k$ we link the line on the right to $1(=\Lambda_k(1))$. Exercise: Sketch Λ_k for the first few values of k.

Finally we claim that the sequence of functions $\{\Lambda_k(x)\}_{k=1}^{\infty}$ converges to a limit function which we call the **Cantor-Lebesgue singular function** and we denote it as $\Lambda(x)$.

Theorem 4.9. *The Cantor-Lebesgue singular function $\Lambda : [0,1] \mapsto \mathbb{R}$ is continuous and increasing on $[0,1]$ and its derivative is zero for all points $x \in [0,1] - \Gamma$.*

Now we return to the previous questions. We want the first theorem to state: "If $f : [a,b] \mapsto \mathbb{R}$ has an a-primitive F on $[a,b]$, then $f \in R^*([a,b])$ and $\int_a^b f = F(b) - F(a)$." Applying Λ as the a-primitive to Λ' we would have $\int_0^1 \Lambda' = 0 \neq 1 = \Lambda(1) - \Lambda(0)$.

Turning now to the second theorem, we want to assert: "Let $f \in R^*([a,b])$. Then any indefinite integral F is continuous on $[a,b]$ and is a c-primitive of f." Let $\psi(x) = 1$ for $x \in \Gamma$ and $\psi(x) := 0$ for $x \in [0,1] - \Gamma$. This is a null function implying $\psi \in R^*([0,x])$ and $\Psi(x) := \int_0^x \psi = 0$ for $x \in [0,1]$. This implies $\Psi' = 0$ for all $x \in [0,1]$. But then $\psi(x) \neq \Psi(x)$ for $x \in \Gamma$ which is an uncountable null set.

5. What Next?

So far we have seen the potential power that applying nonconstant gauges has in integration theory. For further study we still have to see the extent of the class of functions that gauge theory is capable of handling. As it will turn out the class of R^*-integral functions is closely related to the collection of measurable functions. We also have only studied compact intervals and have left infinite intervals for further study. There is also more work required to show that the class of Lebesgue-integral functions is contained in the class of R^*-integral functions. The interested reader is referred to A Modern Theory of Integration by Robert G. Bartle to continue study in this topic.